CHAPTER 5: PROBABILITY

Introduction to probability

Probability is the science of studying the outcomes of *random phenomena*.

A phenomenon is called *random* if individual outcomes are uncertain but the long-term pattern of many outcomes is predictable.

To determine the probability of a specific outcome of a probability experiment (random phenomenon) the experiment is repeated a large number of times and we look at the ratio of times that the outcome occurred.

- *Example:* The outcome of the experiment of tossing a coin is a random phenomenon. The probability that the outcome is "Heads" can be found by tossing the coin a large number of times.

Probability Models

A **probability model** is a mathematical representation of a random phenomenon. It is defined by its sample space, events within the sample space, and **probabilities** associated with each event. The sample space S for a **probability model** is the set of all possible outcomes.

<u>*Terminology*</u>:

For any probability experiment (or a random phenomenon)

- ✓ The collection of all the possible outcomes of the experiment is called the **Sample Space**.
- ✓ An **event** is any sub-collection of outcomes in the sample space.
- ✓ A **simple event** is any single outcome.
- ✓ The **complement** of an event *A*, denoted $\overline{A}$, is the set of all outcomes not in *A*.
- ✓ A **probability model** (or a **probability distribution**) is a process of assigning to the outcomes and events in a sample space a value that represents the probability for that outcome to occur. If x is an outcome, the probability of x will be denoted by $P(x)$. If E is an event, the probability of E will be denoted by $P(E)$.
- ✓ The assignment in a probability model must satisfy the following rules:
  - For any event or outcome, **the probability is a number between 0 and 1**.
  - **The probability of the sample space is 1.**
  - **If two events A and B do are disjoint,** i.e. have no outcome in common, **then the probability P(A or B)** that one or the other occurs **is the sum *P(A) + P(B)*** of the individual probabilities of the events.
  - The probability of an impossible (empty) event is 0.

For a finite (discrete) probability space, these rules guarantee that **the probabilities of the individual events must add up to 1.**

Example:

Suppose we have the following proportions for the marital status of a people in this country between the ages of 30 and 40.

marital status	Never married	Married	Widowed	Divorced
probability	.273	.603	.004	

A random experiment is drawing a person aged 30 to 40 at random. What is the sample space for this experiment?

1) What is the event: "The person is not currently married"

2) What is the complement of this event?

3) If you draw a person aged 30 to 40 at random, what is the probability that:

a) the person is divorced?

b) The person is not currently married?

c) The person has never married, or is widowed?

Equally likely outcomes

If a probability experiment has N outcomes, and all are equally likely to occur, the probability of an individual outcome is 1/N.

The probability of an event A is

P(A) = (The number of outcomes that satisfy A) **)** / (Total number of outcomes)

Examples:

1) The experiment of rolling a fair die has six equally likely outcomes (1, 2, 3, 4, 5, or 6).

   The probability of each of them to occur is 1/6.

   a) Let *A* be the event "The outcome is a multiple of 3". List the elements in *A*.

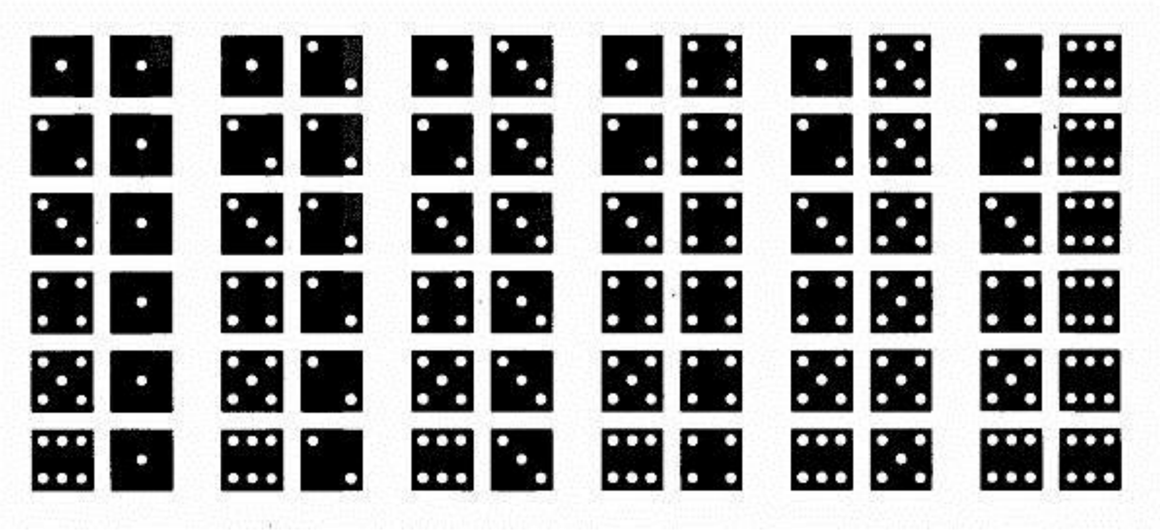
   b) List the elements in $\overline{A}$.

   c) What is *P*(*A*)?

2) Now consider the experiment of rolling two dice, and recording the number of dots on top,

where the order is important.

a) How many outcomes are there? What is the sample space?



b) What is the probability of rolling a six?

c) In the board game *MONOPOLY* you need to "doubles" to get out of jail. What is the probability of getting out in a single roll?

d) Let A be the event that the sum of the number is a multiple of 3. What is A?

e) What is $P(A)$?

f) Let E be the event " The sum of the two dice is 4". What is $P(E)$?

g) What is $\overline{E}$? What is $P(\overline{E})$?

The last two examples illustrate the following rule involving the complement of an event.

- The Complement Rule

$P(\overline{A}) = 1 - P(A)$

Examples:

1) A statistics student who takes the bus to school every day uses relative frequency to find the probabilities that the bus will arrive before a given time. The probability that the bus will come before 8:15 is estimated to be .43. What is the probability that the bus will not come before 8:15?

20 a) A computer assigns a you a PIN at random by assigning 4 symbols, the first is a letter of

the alphabet, the other three are digits from 0 - 9 (with repetition allowed) . Your favorite number is 6. What is the probability of getting a pin with at least one 6?

b) What is the probability of getting a 6 if repeated digits are not allowed?

The Sum Rule

If two events have no elements in common, they follow the rule: $P(A or B) = P(A) + P(B)$

Definition: If two events have no elements in common, they are called **disjoint**

Example

In the example of rolling two die and recording the pair of numbers, let *A* be the event "the sum of the two numbers is a 3", and let *B* be the event "the sum of the two numbers is a 5". List the elements in *A*; list the elements in *B*; and list all the elements in the event " the sum of the two numbers is a 3 or a 5". Find *P*(*A*), *P*(*B*), and the probability that the sum is a 3 or a 5.

But if the events have elements in common this doesn't work.

Example:

In the example of rolling two die and recording the pair of numbers, let *C* be the event "one of the two numbers is a 3", and let *B* be the event "the sum of the two numbers is a 5". List the elements in *C*; list the elements in *B*; and list all the elements in the event " one of the two numbers is a 3 or the sum of the two numbers is a 5". Find *P*(*C*), *P*(*B*), and the probability that one of the numbers is a 3 or the sum is a 5.

This illustrates the sum rule: $P(A or B) = P(A) + P(B) - P(A and B)$, where $P(A and B)$ is the probability that both A and B will happen.

Examples

1) A card is drawn at random from a standard deck of cards. What is the probability it will be a king or a heart?

2) The data for a group of women from a survey is given below:

Status	age 20-34	age 35-49	total
never married	152	51	203
married	226	312	538
widowed	18	69	87
divorced	76	96	172
Total	472	528	1000

If you select a woman at random, what is the probability that she is either married, or over 35?

Random Variables

A random variable gives a value to each outcome in a sample space. A probability model for a random variable assigns a probability value to each variable value.

Examples:

1) In the dice rolling experiment: If we are only interested in the sum of the die, the random variable would be the sum of the two numbers.

2) In the experiment of rolling one die, the random variable could be the number of dots that lands on top.

3) In an experiment of tossing a coin, a random variable could assign the number 1 to *heads* and 0 to *tails*.

A probability model (or probability distribution) satisfies the rules:

- ✓ For any variable, **the probability is a number between 0 and 1**.
- ✓ The probability of the sum of all the variables is 1.

The Mean (Expected Value) of a Probability model

Suppose that the possible values for the random variables are $s_1, s_2, s_3, \ldots, s_k;$ and suppose that the probability of the variable s_j is p_j. Then

the *mean* (also called *expected value*) F of the probability distribution is

$$F = s_1 p_1 + s_2 p_2 + \ldots + s_k p_k$$

Example: The following represents a table of grades given to a class for a term paper. An experiment consists of drawing a student at random and recording that student's grade. The random variable is the value of the grade. The probability distribution is given in the table.

Grade	0	1	2	3	4
Probability	0.10	0.15	0.30	0.30	0.15

Find the mean of the distribution.

Example:

In a gambling game, the player is paid $3 if s/he draws a queen, $5 if s/he draws a king, and $10 if s/he draws an ace. Otherwise, the player will pay the casino the bet amount of $3.00.

1. Find the probability model representing the amount of money the player wins (negative number for loss!)

x				
$P(x)$				

2. Find the mean of this model. Does the game favor the casino or the player?

3. What should the amount of the bet be so that the game is fair?

The Law of Large Numbers

As a random phenomenon is repeated a large number of times

< The proportion of trials on which an outcome occurs gets closer and closer to the probability of that outcome.

< The average of the observed values gets closer and closer to *F*.

An American roulette wheel has 38 slots numbered 0, 00, and 1 to 36. The ball is equally likely to rest in any of these slots when the wheel is spun. One way to place a bet is to bet that the ball will rest on an odd number. Sam places a $10 bet that pays out $20 if an odd number comes up.

a. What is expected value for one play, taking into account the $10 cost of each play? On any individual play, will he make the expected value?

b. Sam plays roulette every day for 10 years. What does the law of large numbers tell us about his results?

Probability of Single or Combined Events

SINGLE EVENT **PROBABILITY**. The chance of some event happening; such as flipping a coin and having it land with the head side up, or rolling a "four" on a **single** die, is called **probability**. **Probability** is expressed as a fraction. ... Therefore, the **probability** of a "four" showing on the die after one throw is 1/6.

Multiple Probabilities. The **probability** of 2 things happening: The **probability** of 2 independent things happening either one after the other or together is the **probability** of the first thing happening multiplied by the **probability** of the second thing happening.

Independent and Dependent Events

Events can be placed into two major categories dependent or Independent events.

Independent - When two events are said to be independent of each other, what this means is that the probability that one event occurs in no way affects the probability of the other event occurring. An example of two independent events is as follows; say you rolled a die and flipped a coin. The probability of getting any number face on the die in no way influences the probability of getting a head or a tail on the coin.

Dependent - When two events are said to be dependent, the probability of one event occurring influences the likelihood of the other event.

For example, if you were to draw a two cards from a deck of 52 cards. If on your first draw you had an ace and you put that aside, the probability of drawing an ace on the second draw is greatly changed because you drew an ace the first time. Let's calculate these different probabilities to see what's going on.

There are 4 Aces in a deck of 52 cards

$$\mathrm{P(Ace)} = \frac{\text{number of Aces in a deck of cards}}{\text{number of cards in a deck}}$$

On your first draw, the probability of getting an ace is given by:

$$\mathrm{P(Ace)} = \frac{4}{52} = \frac{1}{13}$$

If we don't return this card into the deck, the probability of drawing an ace on the second pick is given by

$$\mathrm{P(Ace)} = \frac{\text{number of Aces remaining in the deck of cards}}{\text{number of cards remaining in a deck}}$$

$$\mathrm{P(Ace)} = \frac{4-1}{52-1}$$

$$\mathrm{P(Ace)} = \frac{3}{51}$$

As you can clearly see, the above two probabilities are different, so we say that the two events are dependent. The likelihood of the second event depends on what happens in the first event.

Conditional Probability

Conditional probability deals with further defining dependence of events by looking at probability of an event given that some other event first occurs.

Conditional probability is denoted by the following:

$$P(B|A)$$

The above is read as **the probability that B occurs given that A has already occurred**.

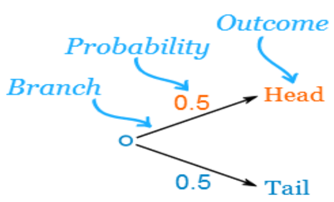
The above is mathematically defined as:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Probability Tree Diagrams

Calculating probabilities can be hard, sometimes we add them, sometimes we multiply them, and often it is hard to figure out what to do ... **tree diagrams to the rescue!**

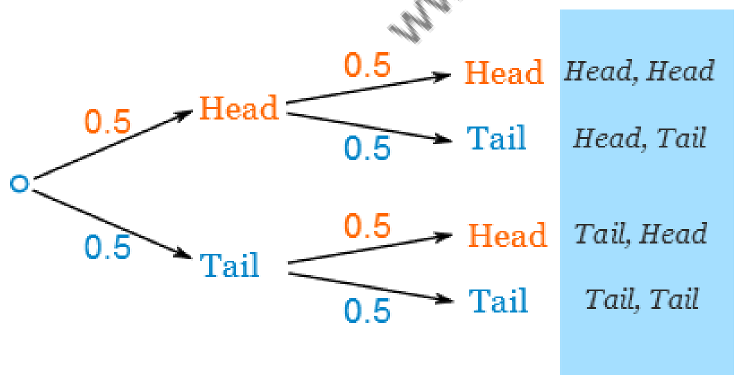
Here is a tree diagram for the toss of a coin:



There are two "branches" (Heads and Tails)

- The probability of each branch is written on the branch
- The outcome is written at the end of the branch

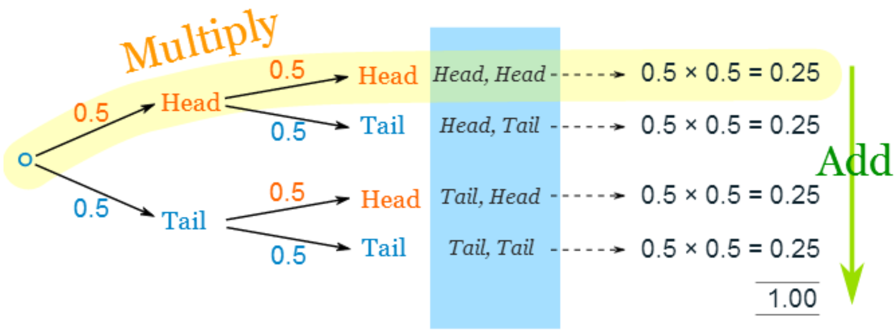
We can extend the tree diagram to two tosses of a coin:



How do we calculate the overall probabilities?

- We **multiply** probabilities **along the branches**

- We **add** probabilities down **columns**



Now we can see such things as:

- The probability of "Head, Head" is 0.5×0.5 = **0.25**
- All probabilities add to **1.0** (which is always a good check)
- The probability of getting at least one Head from two tosses is 0.25+0.25+0.25 = **0.75**
- ... and more

That was a simple example using independent events (each toss of a coin is independent of the previous toss), but tree diagrams are really wonderful for figuring out dependent events (where an event **depends on** what happens in the previous event) like this example:

Example: Soccer Game

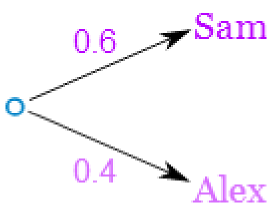
You are off to soccer, and love being the Goalkeeper, but that depends who is the Coach today:

- with Coach Sam the probability of being Goalkeeper is **0.5**
- with Coach Alex the probability of being Goalkeeper is **0.3**

Sam is Coach more often ... about 6 out of every 10 games (a probability of **0.6**).

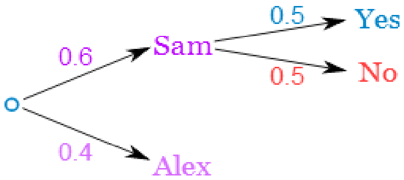
So, what is the probability you will be a Goalkeeper today?

Let's build the tree diagram. First we show the two possible coaches: Sam or Alex:

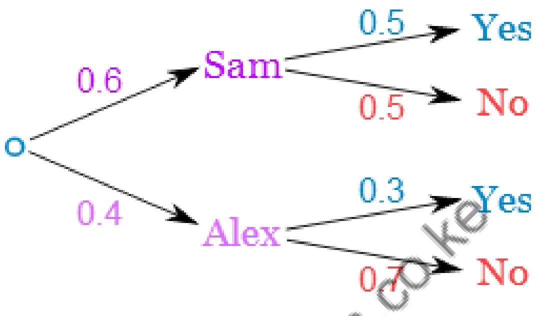


The probability of getting Sam is 0.6, so the probability of Alex must be 0.4 (together the probability is 1)

Now, if you get Sam, there is 0.5 probability of being Goalie (and 0.5 of not being Goalie):

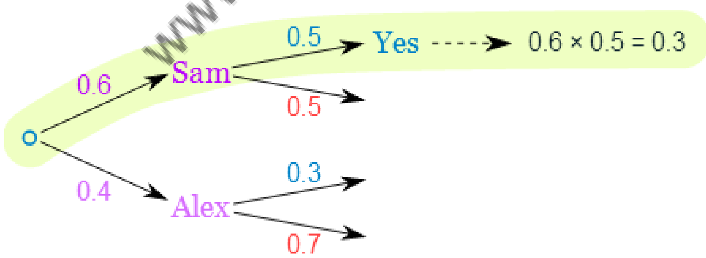


If you get Alex, there is 0.3 probability of being Goalie (and 0.7 not):



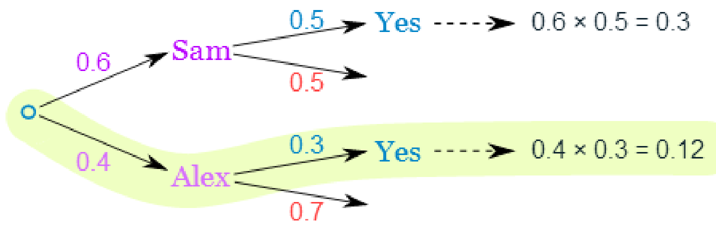
The tree diagram is complete, now let's calculate the overall probabilities. This is done by multiplying each probability along the "branches" of the tree.

Here is how to do it for the "Sam, Yes" branch:



(When we take the 0.6 chance of Sam being coach and include the 0.5 chance that Sam will let you be Goalkeeper we end up with an 0.3 chance.)

But we are not done yet! We haven't included Alex as Coach:



An 0.4 chance of Alex as Coach, followed by an 0.3 chance gives 0.12.

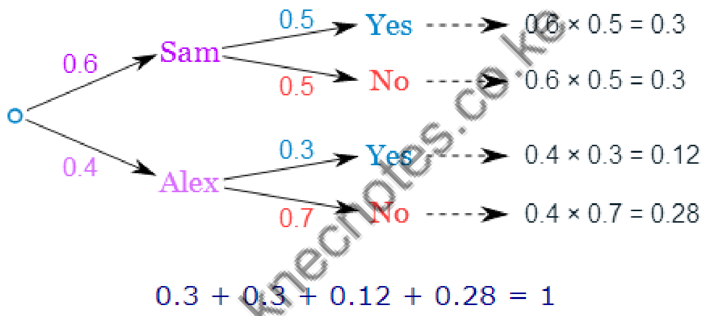
Now we add the column:

0.3 + 0.12 = **0.42 probability** of being a Goalkeeper today

(That is a 42% chance)

Check

One final step: complete the calculations and make sure they add to 1:



Yes, it all adds up.

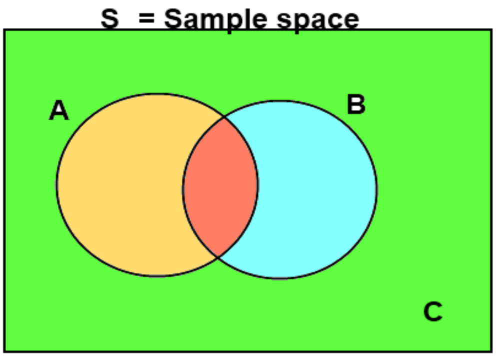
<u>Conclusion</u>

So there you go, when in doubt draw a tree diagram, multiply along the branches and add the columns. Make sure all probabilities add to 1 and you are good to go.

Set Theory in Probability

A sample space is defined as a universal set of all possible outcomes from a given experiment.

Given two events **A** and **B** and given that these events are part of a sample space **S**. This sample space is represented as a set as in the diagram below.



The entire sample space of **S** is given by:

$$S = \{A, B, C\}$$

Remember the following from set theory:

$$C = (A \cup B)'$$

$$A \cup B = A + B - (A \cap B)$$

The different regions of the set **S** can be explained as using the rules of probability.

Rules of Probability

When dealing with more than one event, there are certain rules that we must follow when studying probability of these events. These rules depend greatly on whether the events we are looking at are Independent or dependent on each other.

First acknowledge that

$$P(S) = P(A \cup B \cup C)$$

Multiplication Rule (A∩B)

This region is referred to as 'A intersection B' and in probability; this region refers to the event that both **A** and **B** happen. When we use the word **and** we are referring to multiplication, thus **A and B** can be thought of as **AxB** or (using dot notation which is more popular in probability) **A•B**

If **A** and **B** are dependent events, the probability of this event happening can be calculated as shown below:

$$P(A \cap B) = P(A \cup B) - (P(A \text{ only}) + P(B \text{ only}))$$

If **A** and **B** are independent events, the probability of this event happening can be calculated as shown below:

$$P(A \cap B) = P(A) \times P(B)$$

Conditional probability for two independent events can be redefined using the relationship above to become:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(B|A) = \frac{P(A) \times P(B)}{P(A)}$$

$$P(B|A) = P(B)$$

The above is consistent with the definition of independent events, the occurrence of event **A** in no way influences the occurrence of event **B**, and so the probability that event **B** occurs given that event **A** has occurred is the same as the probability of event **B**.

Additive Rule (A∪ B)

In probability we refer to the addition operator (+) as **or**. Thus when we want to we want to define some event such that the event can be A or B, to find the probability of that event:

$$P(A + B) = P(A \cup B)$$

$$P(A \cup B) = A + B - P(A \cap B)$$

Thus it follows that:

$$P(A + B) = P(A) + P(B) - P(A \cap B)$$

But remember from set theory that and from the way we defined our sample space above:

$$(A \cup B) = 1 - (A \cup B)'$$

and that:

$$(A \cup B)' = C$$

So we can now redefine out event as

$$P(A + B) = 1 - P(A \cup B)'$$

$$P(A + B) = 1 - P(C)$$

The above is sometimes referred to as the subtraction rule.

Mutual Exclusivity

Certain special pairs of events have a unique relationship referred to as mutual exclusivity. Two events are said to be mutually exclusive if they can't occur at the same time. For a given sample space, its either one or the other but not both. As a consequence, mutually exclusive events have their probability defined as follows:

$$P(A) + P(B) = 1$$

An example of mutually exclusive events are the outcomes of a fair coin flip. When you flip a fair coin, you either get a head or a tail but not both, we can prove that these events are mutually exclusive by adding their probabilities:

$$P(\text{head}) + P(\text{tail}) = \frac{1}{2} + \frac{1}{2} = 1$$

For any given pair of events, if the sum of their probabilities is equal to one, then those two events are mutually exclusive.

Rules of Probability for Mutually Exclusive Events

- Multiplication Rule: From the definition of mutually exclusive events, we should quickly conclude the following:

$$P(A \cap B) = 0$$

- Addition Rule: As we defined above, the addition rule applies to mutually exclusive events as follows:

$$P(A + B) = 1$$

- Subtraction Rule: From the addition rule above, we can conclude that the subtraction rule for mutually exclusive events takes the form;

$$P(A \cup B)' = 0$$

Conditional Probability for Mutually Exclusive Events

We have defined conditional probability with the following equation:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

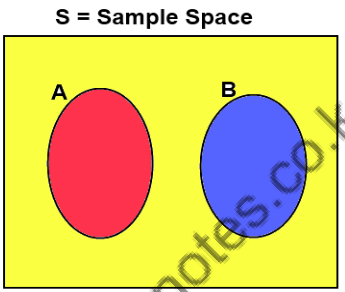
We can redefine the above using the multiplication rule

$$P(A \cap B) = 0$$

hence

$$P(B|A) = \frac{0}{P(A)} = 0$$

Below is a venn diagram of a set containing two mutually exclusive events **A** and **B**.



Probability Distribution

The Poisson random variable satisfies the following conditions:

1. The number of successes in two disjoint time intervals is independent.
2. The probability of a success during a small time interval is proportional to the entire length of the time interval.

Apart from disjoint time intervals, the Poisson random variable also applies to **disjoint regions of space**.

Applications

- the number of deaths by horse kicking in the Prussian army (first application)
- birth defects and genetic mutations
- rare diseases (like Leukemia, but not AIDS because it is infectious and so not independent) - especially in legal cases
- car accidents
- traffic flow and ideal gap distance
- number of typing errors on a page