CHAPTER 1: INTRODUCTION TO COMPUTATIONAL MATHEMATICS

Equations

In mathematics an **equation** is an expression of the shape $A = B$, where A and B are expressions containing one or several variables called *unknowns*.

An algebraic equation or polynomial equation is an equation in which both sides are polynomials

A **system of polynomial equations** is a set of simultaneous equations $f_1 = 0, ..., f_h = 0$ where the f_i are polynomials in several variables, say $x_1, ..., x_n$, over some field k.

Algebraic equation

Mathematical statement of equality between algebraic expressions. An expression is algebraic if it involves a finite combination of numbers and variables and algebraic operations (addition, subtraction, multiplication, division, raising to a power, and extracting a root). Two important types of such equations are linear equations, in the form $y = ax + b$, and quadratic equations, in the form $y = ax^2 + bx + c$. A solution is a numerical value that makes the equation a true statement when substituted for a variable. In some cases it may be found using a formula; in others the equation may be rewritten in simpler form.

1. **A linear equation** is an algebraic equation in which each term is either a constant or the product of a constant and (the first power of) a single variable.

   A common form of a linear equation in the two variables x and y is

$$y = mx + b,$$

where m and b designate constants (parameters). The origin of the name "linear" comes from the fact that the set of solutions of such an equation forms a straight line in the plane. In this particular equation, the constant m determines the slope or gradient of that line, and the constant term b determines the point at which the line crosses the y-axis, otherwise known as the y-intercept.

General (or standard) form

In the general (or standard) form the linear equation is written as:

$$Ax + By = C,$$

where A and B are not both equal to zero. The equation is usually written so that $A \geq 0$, by convention. The graph of the equation is a straight line, and every straight line can be represented by an equation in the above form. If A is nonzero, then the x-intercept, that is, the x-coordinate of the point where the graph crosses the x-axis (where, y is zero), is C/A. If B is nonzero, then the y-intercept, that is the y-

coordinate of the point where the graph crosses the *y*-axis (where x is zero), is *C*/*B*, and the slope of the line is −*A*/*B*. The general form is sometimes written as:

$$ax + by + c = 0,$$

where *a* and *b* are not both equal to zero. The two versions can be converted from one to the other by moving the constant term to the other side of the equal sign.

Slope–intercept form

$$y = mx + b,$$

where *m* is the slope of the line and *b* is the *y*-intercept, which is the *y*-coordinate of the location where line crosses the *y* axis. This can be seen by letting $x = 0$, which immediately gives $y = b$. It may be helpful to think about this in terms of $y = b + mx$; where the line passes through the point (0, *b*) and extends to the left and right at a slope of *m*. Vertical lines, having undefined slope, cannot be represented by this form.

Point–slope form

$$y - y_1 = m(x - x_1),$$

where *m* is the slope of the line and (x_1,y_1) is any point on the line.

The point-slope form expresses the fact that the difference in the *y* coordinate between two points on a line (that is, $y - y_1$) is proportional to the difference in the *x* coordinate (that is, $x - x_1$). The proportionality constant is *m* (the slope of the line).

Two-point form

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

where (x_1, y_1) and (x_2, y_2) are two points on the line with $x_2 \neq x_1$. This is equivalent to the point-slope form above, where the slope is explicitly given as $(y_2 - y_1)/(x_2 - x_1)$.

Multiplying both sides of this equation by $(x_2 - x_1)$ yields a form of the line generally referred to as the **symmetric form**:

$$(x_2 - x_1)(y - y_1) = (y_2 - y_1)(x - x_1).$$

Intercept form

$$\frac{x}{a} + \frac{y}{b} = 1,$$

Where *a* and *b* must be nonzero. The graph of the equation has *x*-intercept *a* and *y*-intercept *b*. The intercept form is in standard form with $A/C = 1/a$ and $B/C = 1/b$. Lines that pass through the origin or which are horizontal or vertical violate the nonzero condition on *a* or *b* and cannot be represented in this form.

<u>Matrix form</u>

Using the order of the standard form

$$Ax + By = C,$$

one can rewrite the equation in matrix form:

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C \end{pmatrix}.$$

Further, this representation extends to systems of linear equations.

$$A_1x + B_1y = C_1,$$
$$A_2x + B_2y = C_2,$$

becomes

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

Since this extends easily to higher dimensions, it is a common representation in linear algebra, and in computer programming. There are named methods for solving system of linear equations, like Gauss-Jordan which can be expressed as matrix elementary row operations.

2. **A quadratic equation** is a univariate polynomial equation of the second degree. A general quadratic equation can be written in the form

$$ax^2 + bx + c = 0,$$

where *x* represents a variable or an unknown, and *a*, *b*, and *c* are constants with $a \neq 0$. (If $a = 0$, the equation is a linear equation.). The constants *a*, *b*, and *c* are called respectively, the quadratic coefficient, the linear coefficient and the constant term

A quadratic equation with real or complex coefficients has two solutions, called *roots*. These two solutions may or may not be distinct, and they may or may not be real.

Methods to Solve Quadratic Equations

Below are the four most commonly used methods to solve quadratic equations. Click on any link to learn more about any of the methods.

- The Quadratic Formula (Quadratic formula in depth)
- Factoring
- Completing the Square
- Graph

Method 1 – Formula

The solution of a quadratic equation is the value of x when you set the equation equal to zero i.e. When you solve the following general equation: $0 = ax^2 + bx + c$

Given a quadratic equation: $ax^2 + bx + c$

The quadratic formula below will solve the equation for zero

The quadratic formula is:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Examples of the quadratic formula to solve an equation

- **Example 1**

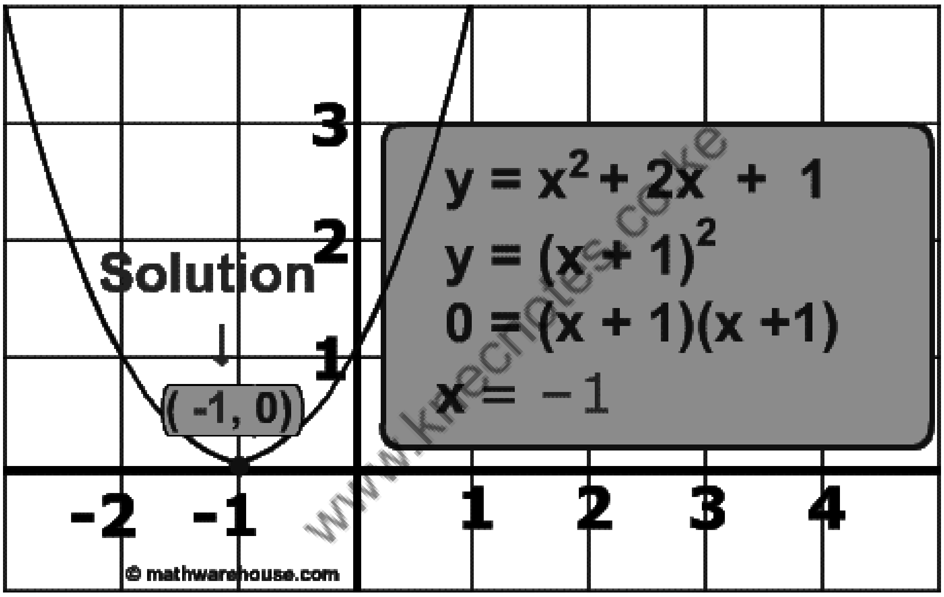
  Quadratic Equation: $\mathbf{y = x^2 + 2x + 1}$

  - a = 1
  - b = 2
  - c = 1

Using the quadratic formula to solve this equation just substitutes a, b, and c into the general formula:

$$\frac{-2 \pm \sqrt{2^2-4(1)(1)}}{2(1)} \rightarrow \frac{-2 \pm \sqrt{4 - 4}}{2} \rightarrow \frac{-2 \pm \sqrt{0}}{2} \rightarrow \frac{-2}{2} \rightarrow -1$$

Method 2 – Graph - Below is a picture representing the graph of $y = x^2 + 2x + 1$ and its solution



Method 3 – Factoring

The solution of a quadratic equation is the value of x when you set the equation equal to zero i.e. When you solve the following general equation: $0 = ax^2 + bx + c$

Given a quadratic equation: $ax^2 + bx + c$

One method to solve the equation for zero is to factor the equations.

General Steps to solve by factoring

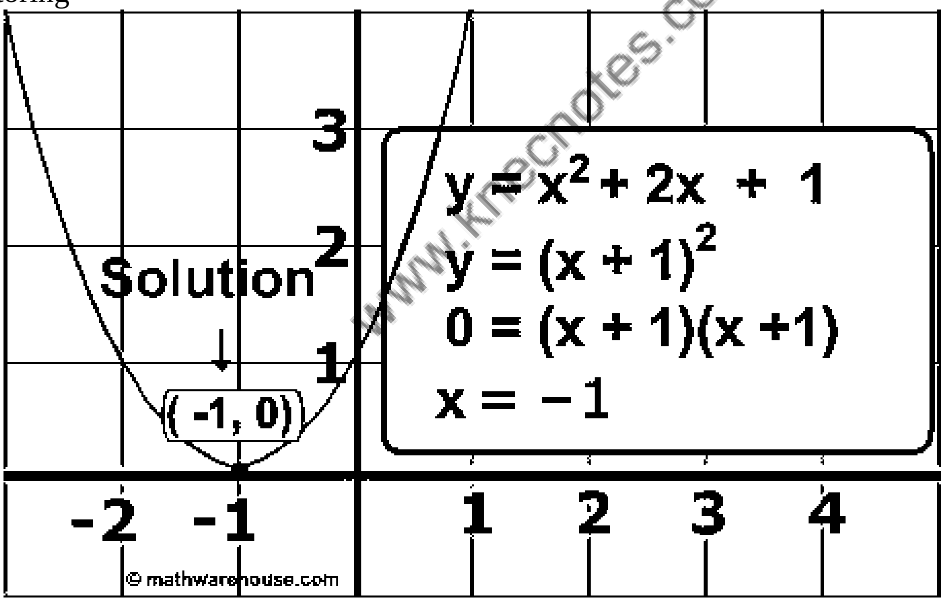
- Step 1) Create a factor chart for all factor pairs of **c**
  - A factor pair is just two numbers that multiply and give you **'c'**
- Step 2) Out of all of the factor pairs from step 1, look for the pair (if it exists) that add up to **b**

- **Note:** if the pair does not exist, you must either complete the square or use the quadratic formula .

- Step 3) Insert the pair you found in step 2 into two binomials
- Step 4) Solve each binomial for zero to get the solutions of the quadratic equation.
- Example of how to solve a quadratic equation by factoring
- Quadratic Equation: $\mathbf{y = x^2 + 2x + 1}$

$y=ax^2+bx+c$ $y=x^2+2x+1$		
1) Create a factor chart	**Factors of c (as pairs)** **1.1**	**Sum of factors (must equal b)** **1+1**
2) Determine which of factor pair of **"c"** has a sum of **"b"**	**1,1**	
3) Insert that pair into binomial factors	$y=(x+1)(x+1)$	
4) Solve each binomial for zero	$0 = x + 1$ $-1 \quad -1$ $-1 = x$	

Below is a picture representing the graph of $y = x^2 + 2x + 1$ as well as the solution we found by factoring



Method 4 – Completing the square

Formula for Completing the Square

First off, a little necessary vocabulary:
A perfect square trinomial is a polynomial that you get by squaring a binomial.(binomials are things like 'x + 3' or 'x − 5')
Examples of perfect square trinomials (the red trinomials)

- $(x+1)^2 = x^2 + 2x + 1$
- $(x+2)^2 = x^2 + 4x + 4$
- $(x+3)^2 = x^2 + 6x + 9$

Examples of trinomials that are **NOT** perfect square trinomials

- $(x+1)(x+2) = x^2 + 3x + 2$
- $(x+2)(x+5) = x^2 + 7x + 10$
- $(x+3)(x-3) = x^2 - 9$

To best understand the formula and logic behind completing the square, look at each example below and you should see the pattern that occurs whenever you square a binomial to produce a perfect square trinomial.

$(x+3)^2$	= =	x^2+6x+9 $x^2+2(3)x+3^2$
$(x+4)^2$	= =	$x^2+8x+16$ $x^2+2(4)x+4^2$
$(x+5)^2$	= =	$x^2+10x+25$ $x^2+2(5)x+5^2$
$(x+6)^2$	= =	$x^2+12x+36$ $x^2+2(6)x+6^2$
$(x+7)^2$	= =	$x^2+14x+49$ $x^2+2(7)x+7^2$
$(x+k)^2$	=	$x^2+2(k)x+k^2$

General Formula

$$(x + \frac{t}{2})^2 = x^2 + 2(\frac{t}{2}) + (\frac{t}{2})^2$$

As the examples above show, the square of a binomial always follows the same pattern and formula.

Given a quadratic equation $\mathbf{x^2 + bx + c}$ that is a SQUARE OF A BINOMIAL

c is always the square of **½(b)**

i.e. $c = ½(b)^2$

3. **Simultaneous equations**

The terms ***simultaneous equations*** and *systems of equations* refer to conditions where two or more unknown variables are related to each other through an equal number of equations.

A set of two or more equations, each containing two or more variables whose values can simultaneously satisfy both or all the equations in the set, the number of variables being equal to or less than the number of equations in the set

There are Four methods to solve a system of equations:

1. Graphing
2. Substitution
3. Elimination or addition method
4. Matrices

For example

6x + 3y = 12

5x + y = 7

Method 1 – Graphing

To solve by graphing, the two lines must be graphed on the same rectangular co-ordinate system. To graph each line, you need to find at least two points on each line. Set up an x and y charts on each. The points on which the two graphs intersect is the solution.

For 6x + 3y = 12	
x	y
0	4
2	0

For 5x + y = 7	
x	y
0	7
1.4	0

Select a value for one variable and solve for the other

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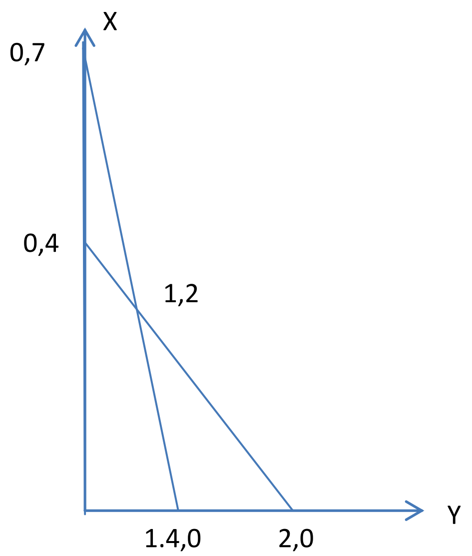


Method 2 - Substitution

To solve by substitution, select one equation (it does not matter which of the equation) and solve for one of the variables (it does not matter which of the variables). Then substitute that expression into the other equation.

For 6x + 3y = 12 I am going to solve for Y (why?, because I want to)

6x + 3y = 12
(6x + 3y) -6x = 12 -6x therefore 3y=-6x+12
3y/3=-(6x+12)/3 hence y=-2x+4

Since I solved for y, I am going to use this to substitute into the y variable of the second equation

For 5x + y = 7
5x + (-2x+4) = 7 now solve for X
5x-2x+4=7 **=>**3x+4=7 **=>**3x=3 =>x=1

Now that we know what x is, use this to solve for y by using one of the original equations (it does no matter which of the equations)

I'll use 6x + 3y = 12(why?, because I want to)

6x + 3y = 12 **=>** 6(1) + 3y = 12 **=>** 6 + 3y = 12 **=>**3y = 12-6 **=>** 3y = 6
therefore y = 6/3=2

Therefore, our answer is (1, 2) which is our point of intersection.

Method 3 – Elimination or Addition method

The objective of elimination method is to add the equations such that like terms add up to zero. If no like terms add up to zero, the multiplication is used to alter the system of equations such that like terms do add up to zero.

If the equation

6x + 3y = 12

5x + y = 7

11x+4y=19

were added now, no like terms world add up to zero

Therefore multiplication is required to alter the equation. To do this legally, remember that whatever you do on one side of the equation you must do the same on the other side of the equation.

To eliminate a variable, do the following

1. Decide which variable to eliminate(it does not matter which of the equation)

   6x + 3y = 12

   5x + y = 7

   I'll pick the X term (why?, because I want to)

2. Looking at the variable from each equation, find the least common multiple of the coefficients (X and Y)

   6x + 3y = 12

   5x + y = 7

   The least common multiple of 5 and 6 is 30

3. Now multiply each equation by the number which will make the co-efficient on the x variable 30

5[6x + 3y = 12]	→	30x + 15y = 60
6[5x + y = 7]		30x + 6y = 42

Notice that if this equations were added, the X terms would still not add up to zero, therefore one of the factors, 5 or 6, which was used to multiply with the equations must be negative, I'll chose 6 to be negative (why?, because I want to)

5[6x + 3y = 12]	→	30x + 15y = 60
-6[5x + y = 7]		-30x - 6y = -42
		9y=18
		Y=2

4. Now substitute this value into one of the original equations(it does not matter which of the equation) to solve for X
6x + 3y = 12(why? because I want to)

6x+3(2) =12 **=>** 6x+6=12 **=>** 6x=6 **=>** x=1

Therefore our answer is (1,2) which happens to be the point of intersection.

Method 4: Matrices

Example 1

EQUATIONS

- x + y + z = 6
- 2y + 5z = -4
- 2x + 5y - z = 27

A Matrix is an array of numbers, right?

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

A Matrix

Well, think about the equations:

$$\begin{array}{ccccccc} x & + & y & + & z & = & 6 \\ & & 2y & + & 5z & = & -4 \\ 2x & + & 5y & - & z & = & 27 \end{array}$$

They could be turned into a table of numbers like this:

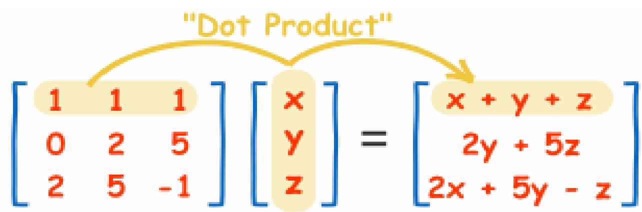
$$\begin{array}{ccccc} 1 & 1 & 1 & = & 6 \\ 0 & 2 & 5 & = & -4 \\ 2 & 5 & -1 & = & 27 \end{array}$$

We could even separate the numbers before and after the "=" into:

$$\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{array} \quad \text{and} \quad \begin{array}{c} 6 \\ -4 \\ 27 \end{array}$$

Now it looks like we have 2 Matrices.

In fact we have a third one, which is [x y z], and the way that matrices are ,we need to set it up like this:



And we know that $x + y + z = 6$, etc, so we can write the system of equations like this:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix}$$

Pretty neat, hey?

The Matrix Solution

We can call the matrices "A", "X" and "B" and the equation becomes:

$$AX = B$$

Where A is the 3x3 matrix of x,y and z coefficients X is x, y and z, and B is 6, -4 and 27

Then (as shown on the **Inverse of a Matrix** page) the solution is this:

$$X = A^{-1}B$$

(Assuming we can calculate the Inverse Matrix A^{-1})

Then multiply A^{-1} by B (you can use the Matrix Calculator again):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-21} \begin{bmatrix} -27 & 6 & 3 \\ 10 & -3 & -5 \\ -4 & -3 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix} = \frac{1}{-21} \begin{bmatrix} -105 \\ -63 \\ 42 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

And we are done! The solution is:

$$x = 5,\ y = 3 \text{ and } z = -2$$

An introduction to MATRICES

Definition of Matrix

A matrix is a collection of numbers arranged into a fixed number of rows and columns. Usually the numbers are real numbers. In general, matrices can contain complex numbers but we won't see those here. Here is an example of a matrix with three rows and three columns:

Cols 1........

Rows 1........

$$\begin{pmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{3} \\ \mathbf{0} & \mathbf{8} & \mathbf{4.6} \\ \mathbf{4} & \mathbf{-1} & \mathbf{0} \end{pmatrix}$$

The top row is row 1. The leftmost column is column 1. This matrix is a 3x3 matrix because it has three rows and three columns. In describing matrices, the format is:

Rows X columns

Each number that makes up a matrix is called an **element** of the matrix. The elements in a matrix have specific locations.

The upper left corner of the matrix is row 1 column 1. In the above matrix the element at row 1 col 1 is the value 1. The element at row 2 column 3 is the value 4.6.

Matrix Dimensions

The numbers of rows and columns of a matrix are called its **dimensions**. Here is a matrix with three rows and two columns:

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}_{3 \times 2}$$

Sometimes the dimensions are written off to the side of the matrix, as in the above matrix. But this is just a little reminder and not actually part of the matrix. Here is a matrix with different dimensions. It has two rows and three columns. This is a different "data type" than the previous matrix

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 8 & 4.6 \\ 4 & -1 & 0 \end{pmatrix} \quad 2\times3$$

Types of Matrices

Square matrix

If a matrix A has n rows and n columns then we say it's a square matrix.

In a square matrix the elements $a_{i,i}$, with i = 1,2,3,... , are called diagonal elements.
Remark. There is no difference between a 1 x 1 matrix and an ordinary number.

Diagonal matrix

A diagonal matrix is a square matrix with all de non-diagonal elements 0.

The diagonal matrix is completely defined by the diagonal elements.

Example.

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

The matrix is denoted by diag(7 , 5 , 6)

Row matrix

A matrix with one row is called a row matrix.

[2 5 -1 5]

Column matrix

A matrix with one column is called a column matrix.

[2]
[4]
[3]
[0]

Matrices of the same kind

Matrix A and B are of the same kind if and only if
A has as many rows as B and A has as many columns as B

```
[7  1  2]        [4  0  3]
[0  5  6]  and  [1  1  4]
[3  4  6]        [8  6  2]
```

The transposed matrix of a matrix

The n x m matrix B is the transposed matrix of the m x n matrix A if and only if
The ith row of A = the ith column of B for (i = 1,2,3,..m)
So $a_{i,j} = b_{j,i}$
The transposed matrix of A is denoted T(A) or A^T

```
          T
[7  1 ]       [7  0  3]
[0  5 ]   =   [1  5  4]
[3  4 ]
```

0-matrix

When all the elements of a matrix A are 0, we call A a 0-matrix.
We write shortly 0 for a 0-matrix.

An identity matrix I

An identity matrix I is a diagonal matrix with all the diagonal elements = 1.

```
[1]

[1 0]
[0 1]

[1 0 0]
[0 1 0]
[0 0 1]

. . .
```

A scalar matrix S

A scalar matrix S is a diagonal matrix whose diagonal elements all contain the same scalar value.
$a_{1,1} = a_{i,i}$ for (i = 1,2,3,..n)

```
[7  0  0]
[0  7  0]
[0  0  7]
```

The opposite matrix of a matrix

If we change the sign of all the elements of a matrix A, we have the opposite matrix -A.
If A' is the opposite of A then $a_{i,j}' = -a_{i,j}$, for all i and j.

A symmetric matrix

A square matrix is called symmetric if it is equal to its transpose.
Then $a_{i,j} = a_{j,i}$, for all i and j.

```
[7  1  5]
[1  3  0]
[5  0  7]
```

A skew-symmetric matrix

A square matrix is called skew-symmetric if it is equal to the opposite of its transpose.
Then $a_{i,j} = -a_{j,i}$, for all i and j.

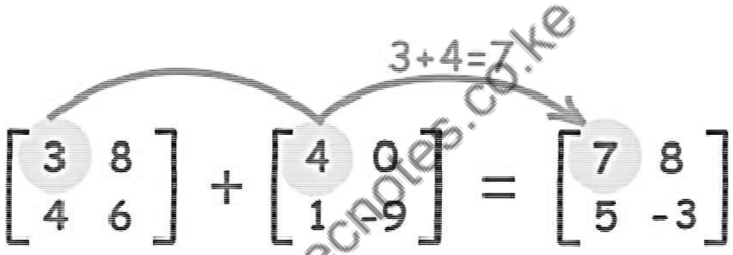
```
[ 0  1 -5]
[-1  0  0]
[5 0 0]
```

Operation of matrices

Addition of Matrices

Adding

To add two matrices: add the numbers in the matching positions:



These are the calculations:

3+4=7 8+0=8

4+1=5 6-9=-3

The two matrices must be the same size, i.e. the rows must match in size, and the columns must match in size.

Example: a matrix with **3 rows** and **5 columns** can be added to another matrix of **3 rows** and **5 columns**.

But it could not be added to a matrix with **3 rows** and **4 columns** (the columns don't match in size)

Denote the sum of two matrices A and B (of the same dimensions) by C = A + B.. The sum is defined by adding entries with the same indices

$c_{ij} \equiv a_{ij} + b_{ij}$

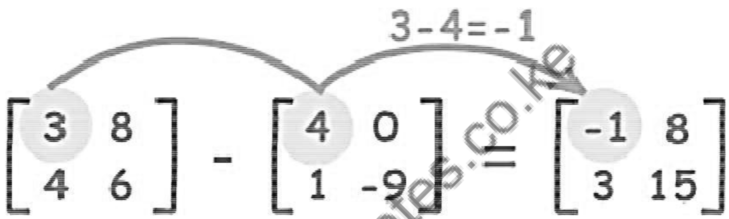
over all i and j.

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Subtraction of Matrices

To subtract two matrices: subtract the numbers in the matching positions:



These are the calculations:

3-4=-1 8-0=8

4-1=3 6-(-9)=15

Note: subtracting is actually defined as the ***addition*** *of a negative matrix: A + (-B)*

Subtraction is performed in analogous way.

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

How to Multiply Matrices

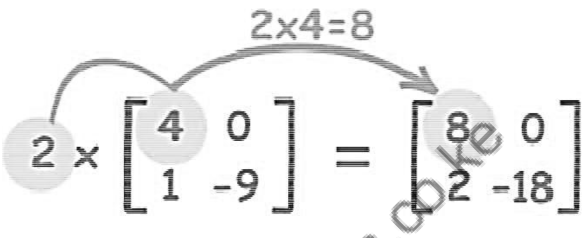
A Matrix is an array of numbers:

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

A Matrix

(This one has 2 Rows and 3 Columns)

To multiply a matrix by a single number is easy:



These are the calculations:

2×4=8 2×0=0 2×1=2

2×-9=-18

We call the number ("2" in this case) a **scalar**, so this is called "scalar multiplication".

Example: The local shop sells 3 types of pies.

- Beef pies cost **$3** each
- Chicken pies cost **$4** each
- Vegetable pies cost **$2** each

And this is how many they sold in 4 days:

	Mon	Tue	Wed	Thu
Beef	**13**	**9**	**7**	**15**
Chicken	**8**	**7**	**4**	**6**
Vegetable	**6**	**4**	**0**	**3**

Now think about this ... the **value of sales** for Monday is calculated this

way: Beef pie value + Chicken pie value + Vegetable pie value

$3×13 + $4×8 + $2×6 = $83

So it is, in fact, the "dot product" of prices and how many were sold:

($3, $4, $2) • (13, 8, 6) = $3×13 + $4×8 + $2×6 = $83

We **match** the price to how many sold, **multiply** each, then **sum** the result.

In other words:

- The sales for Monday were: Beef pies: **$3×13=$39**, Chicken pies: **$4×8=$32**, and Vegetable pies: **$2×6=$12**. Together that is $39 + $32 + $12 = **$83**

- And for Tuesday: **$3×9 + $4×7 + $2×4 = $63**
- And for Wednesday: **$3×7 + $4×4 + $2×0 = $37**
- And for Thursday: **$3×15 + $4×6 + $2×3 = $75**

So it is important to match each price to each quantity.

Now you know why we use the "dot product".

And here is the full result in Matrix form:

$$\begin{bmatrix} \$3 & \$4 & \$2 \end{bmatrix} \times \begin{bmatrix} 13 & 9 & 7 & 15 \\ 8 & 7 & 4 & 6 \\ 6 & 4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \$83 & \$63 & \$37 & \$75 \end{bmatrix}$$

$$\$3 \times 13 + \$4 \times 8 + \$2 \times 6$$

They sold **$83** worth of pies on Monday, **$63** on Tuesday, etc.

(You can put those values into the Matrix Calculator to see if they work.)

Rows and Columns

To show how many rows and columns a matrix has we often write

rows×columns. Example: This matrix is 2×3 (2 rows by 3 columns):

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

When we do multiplication:

- The number of **columns of the 1st matrix** must equal the number of **rows of the 2nd matrix**.
- And the result will have the same number of **rows as the 1st matrix**, and the same number of **columns as the 2nd matrix**.

Example:

$$\begin{bmatrix} \$3 & \$4 & \$2 \end{bmatrix} \times \begin{bmatrix} 13 & 9 & 7 & 15 \\ 8 & 7 & 4 & 6 \\ 6 & 4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \$83 & \$63 & \$37 & \$75 \end{bmatrix}$$

$$\$3 \times 13 + \$4 \times 8 + \$2 \times 6$$

In that example we multiplied a 1×3 matrix by a 3×4 matrix (note the 3s are the same), and the result was a 1×4 matrix.

In General:

To multiply an **m×n** matrix by an **n×p** matrix, the **n**s must be the same, and the result is an **m×p** matrix.

$$m \times n \; \times \; n \times p \rightarrow m \times p$$

Order of Multiplication

In arithmetic we are used to:

$$3 \times 5 = 5 \times 3$$

(The Commutative Law of Multiplication)

But this is **not** generally true for matrices (matrix multiplication is **not commutative**):

$$AB \neq BA$$

When we change the order of multiplication, the answer is (usually) **different**.

Example:

See how changing the order affects this multiplication:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 10 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 7 & 10 \end{bmatrix}$$

Identity Matrix

The "Identity Matrix" is the matrix equivalent of the number "1":

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A 3x3 Identity Matrix

- It is "square" (has same number of rows as columns),
- It has **1**s on the diagonal and **0**s everywhere else.
- Its symbol is the capital letter **I**.

It is a **special matrix**, because when we multiply by it, the original is unchanged:

$$A \times I = A$$

$$I \times A = A$$

Determinant of a Matrix

The determinant of a matrix is a **special number** that can be calculated from a square matrix.

A Matrix is an array of numbers:

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix}$$

A Matrix

(This one has 2 Rows and 2 Columns)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant of that matrix is (calculations are explained later):

$3 \times 6 - 8 \times 4 = 18 - 32 = \mathbf{-14}$

What is it for?

The determinant tells us things about the matrix that are useful in systems of linear equations, helps us find the inverse of a matrix, is useful in calculus and more.

Symbol

The **symbol** for determinant is two vertical lines either side.

Example:

|A| means the determinant of the matrix **A**

(Exactly the same symbol as absolute value.)

Calculating the Determinant

First of all the matrix must be **square** (i.e. have the same number of rows as columns). Then it is just basic arithmetic. Here is how:

For a 2×2 Matrix

For a 2×2 matrix (2 rows and 2 columns):

The determinant is:

$$|A| = ad - bc$$

"The determinant of A equals a times d minus b times c"

It is easy to remember when you think of a cross:

- Blue means positive (+ad),
- Red means negative (-bc)

Example:

$$B = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

$|B| = 4 \times 8 - 6 \times 3$

$= 32 - 18$
$= 14$

Cramer's Rule

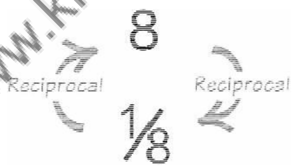
Cramer's Rule, is used to solve systems with matrices. Cramer's Rule was named after the Swiss mathematician Gabriel Cramer, who also did a lot of other neat stuff with math.

Cramer's rule is all about getting determinants of the square matrices that are used to solve systems.

Inverse of a Matrix

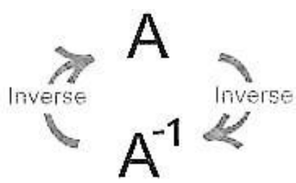
What is the Inverse of a Matrix?

This is the reciprocal of a **Number**:



Reciprocal of a Number

The **Inverse of a Matrix** is the **same idea** but we write it A^{-1}



Why not $^1/_A$? Because we don't divide by a Matrix! And anyway $^1/_8$ can also be written 8^{-1}

And there are other similarities:

When you **multiply a number** by its **reciprocal** you get **1**

$$8 \times (^1/_8) = \mathbf{1}$$

When you **multiply a Matrix** by its **Inverse** you get the **Identity Matrix** (which is like "1" for Matrices):

$$A \times A^{-1} = \mathbf{I}$$

Same thing when the inverse comes first:

$$(^1/_8) \times 8 = \mathbf{1}$$

$$A^{-1} \times A = \mathbf{I}$$

Identity Matrix

We just mentioned the "Identity Matrix". It is the matrix equivalent of the number "1":

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A 3x3 Identity Matrix

- It is "square" (has same number of rows as columns),
- It has **1**s on the diagonal and **0**s everywhere else.
- It's symbol is the capital letter **I**.

The Identity Matrix can be 2×2 in size, or 3×3, 4×4, etc ...

Definition

Here is the definition:

The Inverse of A is A^{-1} only when:

$$A \times A^{-1} = A^{-1} \times A = \mathbf{I}$$

Sometimes there is no Inverse at all.

2x2 Matrix

OK, how do we calculate the Inverse?

Well, for a 2x2 Matrix the Inverse is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

determinant

In other words: **swap** the positions of a and d, put **negatives** in front of b and c, and **divide** everything by the determinant (ad-bc).

Let us try an example:

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4 \times 6 - 7 \times 2} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

How do we know this is the right answer?

Remember it must be true that: $A \times A^{-1} = \mathbf{I}$

So, let us check to see what happens when we multiply the matrix by its inverse:

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 4\times0.6 + 7\times-0.2 & 4\times-0.7 + 7\times0.4 \\ 2\times0.6 + 6\times-0.2 & 2\times-0.7 + 6\times0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 2.4 - 1.4 & -2.8 + 2.8 \\ 1.2 - 1.2 & -1.4 + 2.4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

And, hey!, we end up with the Identity Matrix! So it must be right.

It should **also** be true that: $A^{-1} \times A = \mathbf{I}$

Why don't you have a go at multiplying these? See if you also get the Identity Matrix:

$$\begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Why Would We Want an Inverse?

Because with Matrices we **don't divide**! Seriously, there is no concept of dividing by a Matrix.

But we can **multiply by an Inverse**, which achieves the same thing.

Imagine you couldn't divide by numbers, and someone asked "How do I share 10 apples with 2 people?"

But you could take the **reciprocal** of 2 (which is 0.5), so you could answer:

10 × 0.5 = 5

They get 5 apples each

The same thing can be done with Matrices:

Say that you know Matrix A and B, and want to find Matrix X:

$$XA = B$$

It would be nice to divide both sides by A (to get X=B/A), but remember **we can't divide**.

But what if we multiply both sides by A^{-1} ?

$$XAA^{-1} = BA^{-1}$$

And we know that $AA^{-1} = I$, so:

$$XI = BA^{-1}$$

We can remove I (for the same reason we could remove "1" from 1x = ab for numbers):

$$X = BA^{-1}$$

And we have our answer (assuming we can calculate A^{-1})

In that example we were very careful to get the multiplications correct, because with Matrices the order of multiplication matters. AB is almost never equal to BA.

A Real Life Example

A group took a trip on a bus, at \$3 per child and \$3.20 per adult for a total of \$118.40.

They took the train back at \$3.50 per child and \$3.60 per adult for a total of \$135.20.

How many children, and how many adults?

First, let us set up the matrices (be careful to get the rows and columns correct!):

$$\underset{\text{Child} \quad \text{Adult}}{\begin{bmatrix} x_1 & x_2 \end{bmatrix}} \underset{\text{Bus} \quad \text{Train}}{\begin{bmatrix} 3 & 3.5 \\ 3.2 & 3.6 \end{bmatrix}} = \underset{\text{Bus} \quad \text{Train}}{\begin{bmatrix} 118.4 & 135.2 \end{bmatrix}}$$

This is just like the example above:

$$XA = B$$

So to solve it we need the inverse of "A":

$$\begin{bmatrix} 3 & 3.5 \\ 3.2 & 3.6 \end{bmatrix}^{-1} = \frac{1}{3\times3.6-3.5\times3.2} \begin{bmatrix} 3.6 & -3.5 \\ -3.2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 8.75 \\ 8 & -7.5 \end{bmatrix}$$

Now we have the inverse we can solve using:

$$X = BA^{-1}$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} 118.4 & 135.2 \end{bmatrix} \begin{bmatrix} -9 & 8.75 \\ 8 & -7.5 \end{bmatrix}$$

$$= \begin{bmatrix} 118.4\times-9 + 135.2\times8 & 118.4\times8.75 + 135.2\times-7.5 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 22 \end{bmatrix}$$

There were 16 children and 22 adults!

The answer almost appears like magic. But it is based on good mathematics.

Calculations like that (but using much larger matrices) help Engineers design buildings, are used in video games and computer animations to make things look 3-dimensional, and many other places.

It is also a way to solve Systems of Linear Equations.

The calculations are done by computer, but the people must understand the formulas.

Order is Important

Say that you are trying to find "X" in this case:

$$AX = B$$

This is different to the example above! X is now **after** A.

With Matrices the order of multiplication usually changes the answer. Do not assume that AB = BA, it is almost never true.

So how do we solve this one? Using the same method, but put A^{-1} in front:

$$A^{-1}AX = A^{-1}B$$

And we know that $A^{-1}A = I$, so:

$$IX = A^{-1}B$$

We can remove I:

$$X = A^{-1}B$$

And we have our answer (assuming we can calculate A^{-1})

Why don't we try our example from above, but with the data set up this way around. (Yes, you can do this, just be careful how you set it up.)

This is what it looks like as AX = B:

$$\begin{bmatrix} 3 & 3.2 \\ 3.5 & 3.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 118.4 \\ 135.2 \end{bmatrix}$$

$$\frac{1}{3 \times 3.6 - 3.2 \times 3.5} \begin{bmatrix} 3.6 & -3.2 \\ -3.5 & 3 \end{bmatrix} = \begin{bmatrix} -9 & 8 \\ 8.75 & -7.5 \end{bmatrix}$$

It looks so neat! I think I prefer it like this.

Also note how the rows and columns are swapped over ("Transposed")
compared to the previous example.

To solve it we need the inverse of "A":

It is like the Inverse we got before, but

Transposed (rows and columns swapped over).

Now we can solve using:

$$X = A^{-1}B$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -9 & 8 \\ 8.75 & -7.5 \end{bmatrix} \begin{bmatrix} 118.4 \\ 135.2 \end{bmatrix} = \begin{bmatrix} -9 \times 118.4 + 8 \times 135.2 \\ 8.75 \times 118.4 - 7.5 \times 135.2 \end{bmatrix} = \begin{bmatrix} 16 \\ 22 \end{bmatrix}$$

Same answer: 16 children and 22 adults.

So, Matrices are powerful things, but they do need to be set up correctly!

The Inverse May Not Exist

First of all, to have an Inverse the Matrix must be "Square" (same number of rows and columns).

But also the **determinant cannot be zero** (or you would end up dividing by zero). How about this:

$$\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}^{-1} = \frac{1}{3 \times 8 - 4 \times 6} \begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix}$$

$$= \frac{1}{24-24} \begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix}$$

24-24? That equals 0, and **1/0 is undefined**.

We cannot go any further! This Matrix has no Inverse.

Such a Matrix is called "Singular", which only happens when the determinant is zero.

And it makes sense ... look at the numbers: the second row is just double the first row, and does **not add any new information**.

Imagine in our example above that the prices on the train were exactly, say, 50% higher ... we wouldn't be any closer to figuring out how many adults and children ... we need something different.

And the determinant neatly works this out.

Conclusion

- The Inverse of A is A^{-1} only when $A \times A^{-1} = A^{-1} \times A = \mathbf{I}$
- To find the Inverse of a 2x2 Matrix: **swap** the positions of a and d, put **negatives** in front of b and c, and **divide** everything by the determinant (ad-bc).
- Sometimes there is no Inverse at all

Algebraic Properties of Matrix Operations

In this page, we give some general results about the three operations: addition, multiplication, and multiplication with numbers, called **scalar multiplication**.

From now on, we will not write (mxn) but mxn.

Properties involving Addition. Let *A*, *B*, and *C* be mxn matrices. We have

1. $A+B = B+A$

2. $(A+B)+C = A + (B+C)$

$$A + \mathcal{O} = A$$

Where $\mathcal{O}$ is the mxn zero-matrix (all its entries are equal to 0);

4. $A + B = \mathcal{O}$ if and only if $B = -A$.

Properties involving Multiplication.

1. Let *A*, *B*, and *C* be three matrices. If you can perform the products *AB*, (*AB*)*C*, *BC*, and *A*(*BC*), then we have

$$(AB)C = A\,(BC)$$

Note, for example, that if *A* is 2x3, *B* is 3x3, and *C* is 3x1, then the above products are possible (in this case, (*AB*)*C* is 2x1 matrix).

2. If α and β are numbers, and *A* is a matrix, then we have

$$\alpha(\beta A) = (\alpha\beta)A$$

3. If α is a number, and *A* and *B* are two matrices such that the product $A \cdot B$ is possible, then we have

$$\alpha(AB) = (\alpha A)B = A(\alpha B)$$

4. If *A* is an nxm matrix and $\mathcal{O}$ the mxk zero-matrix, then

$$A\mathcal{O} = \mathcal{O}$$

Note that $A\mathcal{O}$ is the nxk zero-matrix. So if n is different from m, the two zero-matrices are different.

Properties involving Addition and Multiplication.

1. Let *A*, *B*, and *C* be three matrices. If you can perform the appropriate products, then we have
   (*A*+*B*)*C* = *AC* + *BC* and *A*(*B*+*C*) = *AB* + *AC*

2. If α and β are numbers, *A* and *B* are matrices, then we have

$\alpha(A+B) = \alpha A + \alpha B$	And	$(\alpha+\beta)A = \alpha A + \beta A$

Example. Consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 0 & 1 & 5 \end{pmatrix}.$$

Evaluate (*AB*)*C* and *A*(*BC*). Check that you get the same matrix.

Answer. We have

$$AB = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

So

$$(AB)C = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -5 \\ 0 & -2 & -10 \end{pmatrix}.$$

On the other hand, we have

$$BC = \begin{pmatrix} 0 & 2 & 10 \\ 0 & -1 & -5 \end{pmatrix}$$

so

$$A(BC) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 10 \\ 0 & -1 & -5 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -5 \\ 0 & -2 & -10 \end{pmatrix}.$$

Example. Consider the matrices

$$X = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \text{ and } Y = \begin{pmatrix} \alpha & \beta & \nu & \gamma \end{pmatrix}.$$

It is easy to check that

$$X = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and

$$Y = \alpha \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} + \nu \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}.$$

These two formulas are called **linear combinations**. More on linear combinations will be discussed on a different page.

We have seen that matrix multiplication is different from normal multiplication (between numbers). Are there some similarities? For example, is there a matrix which plays a similar role as the number 1? The answer is yes. Indeed, consider the nxn matrix

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

In particular, we have

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix I_n has similar behavior as the number 1. Indeed, for any nxn matrix A, we have

$$A\, I_n = I_n\, A = A$$

The matrix I_n is called the **Identity Matrix** of order n.

Example. Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}.$$

Then it is easy to check that

$$AB = I_2 \text{ and } BA = I_2.$$

The identity matrix behaves like the number 1 not only among the matrices of the form nxn. Indeed, for any nxm matrix A, we have

$$I_n A = A \text{ and } AI_m = A.$$

In particular, we have

$$I_4 \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Applications of Matrix Mathematics

Matrix mathematics applies to several branches of science, as well as different mathematical disciplines. Let's start with computer graphics, then touch on science, and return to mathematics.

We see the results of matrix mathematics in every computer-generated image that has a reflection, or distortion effects such as light passing through rippling water.

Before computer graphics, the science of optics used matrix mathematics to account for reflection and for refraction.

Matrix arithmetic helps us calculate the electrical properties of a circuit, with voltage, amperage, resistance, etc.

In mathematics, one application of matrix notation supports graph theory. In an adjacency matrix, the integer values of each element indicates how many connections a particular node has.

The field of probability and statistics may use matrix representations. A probability vector lists the probabilities of different outcomes of one trial. A stochastic matrix is a square matrix whose rows are probability vectors. Computers run Markov simulations based on stochastic matrices in order to model events ranging from gambling through weather forecasting to quantum mechanics.

Matrix mathematics simplifies linear algebra, at least in providing a more compact way to deal with groups of equations in linear algebra.

Daily Matrix Applications

Matrix mathematics has many applications. Mathematicians, scientists and engineers represent groups of equations as matrices; then they have a systematic way of doing the math. Computers have embedded matrix arithmetic in graphic processing algorithms, especially to render reflection and refraction. Some properties of matrix mathematics are important in math theory.