CHAPTER 6: GRAPHS OF FUNCTIONS

Defining the Graph of a Function

The graph of a function f is the set of all points in the plane of the form (x, f(x)). We could also define the graph of f to be the graph of the equation y = f(x). So, the graph of a function if a special case of the graph of an equation.

Example 1.

Let **$f(x) = x^2 - 3$**.

Recall that when we introduced graphs of equations we noted that if we can solve the equation for y, then it is easy to find points that are on the graph. We simply choose a number for x, then compute the corresponding value of y. Graphs of functions are graphs of equations that have been solved for y!

The graph of f(x) in this example is the graph of $y = x^2 - 3$. It is easy to generate points on the graph. Choose a value for the first coordinate, then evaluate f at that number to find the second coordinate. The following table shows several values for x and the function f evaluated at those numbers.

x	-2	-1	0	1	2
f(x)	1	-2	-3	-2	1

Each column of numbers in the table holds the coordinates of a point on the graph of f.

Exercise 1:

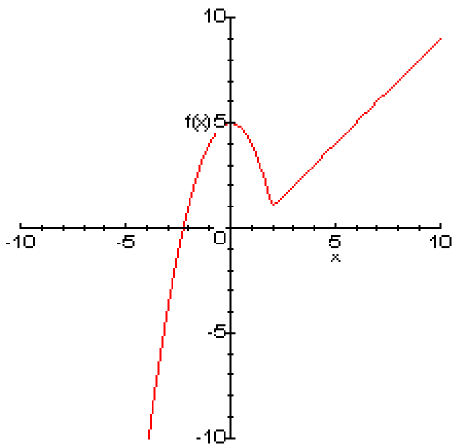
(a) Plot the five points on the graph of f from the table above, and based on these points, sketch the graph of f.

Example 2.

Let f be the piecewise-defined function

$$f(x) = \begin{cases} 5 - x^2, & x \leq 2 \\ x - 1, & x > 2 \end{cases}$$

The graph of f is shown below.



Exercise 2:

Graph the piecewise-defined function

$$f(x) = \begin{cases} 1 - x, & x \leq 4 \\ 2x - 11, & x > 4 \end{cases}$$

We have seen that some equations in x and y do *not* describe y as a function of x. The algebraic way see if an equation determines y as a function of x is to solve for y. If there is not a unique solution, then y is not a function of x.

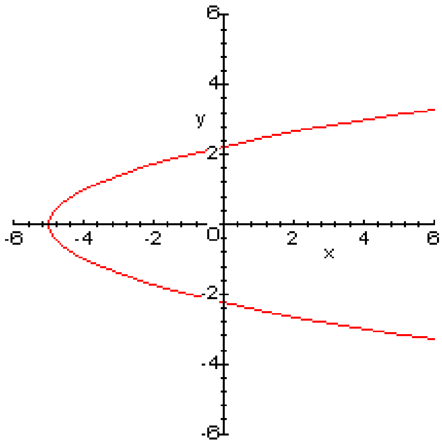
Suppose that we are given the graph of the equation. There is an easy way to see if this equation describes y as a function of x.

Vertical Line Test

A set of points in the plane is the graph of a function if and only if no vertical line intersects the graph in more than one point.

Example 3.

The graph of the equation $\mathbf{y^2 = x + 5}$ is shown below.



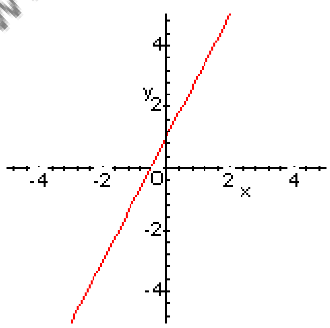
By the vertical line test, this graph is not the graph of a function, because there are many vertical lines that hit it more than once.

Think of the vertical line test this way. The points on the graph of a function f have the form (x, f(x)), so once you know the first coordinate, the second is determined. Therefore, there cannot be two points on the graph of a function with the same first coordinate.

All the points on a vertical line have the same first coordinate, so if a vertical line hits a graph twice, then there are two points on the graph with the same first coordinate. If that happens, the graph is not the graph of a function

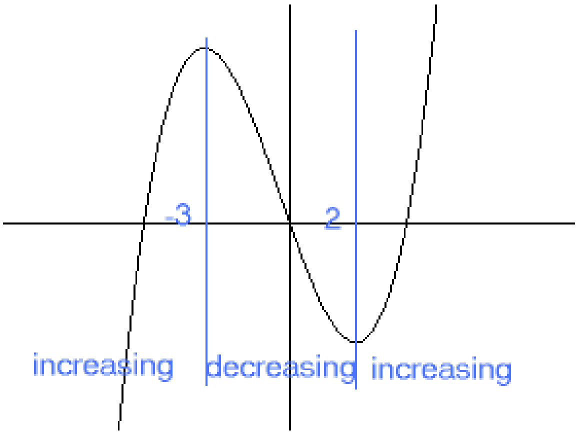
Characteristics of Graphs

Consider the function f(x) = 2 x + 1. We recognize the equation y = 2 x + 1 as the Slope-Intercept form of the equation of a line with slope 2 and y-intercept (0,1).



Think of a point moving on the graph of f. As the point moves toward the right it rises. This is what it means for a function to be *increasing.* Your text has a more precise definition, but this is the basic idea.

The function f above is increasing everywhere. In general, there are intervals where a function is increasing and intervals where it is decreasing.



The function graphed above is decreasing for x between -3 and 2. It is increasing for x less than -3 and for x greater than 2.

Using interval notation, we say that the function is decreasing on the interval (-3, 2) increasing on (-infinity, -3) and (2, infinity)

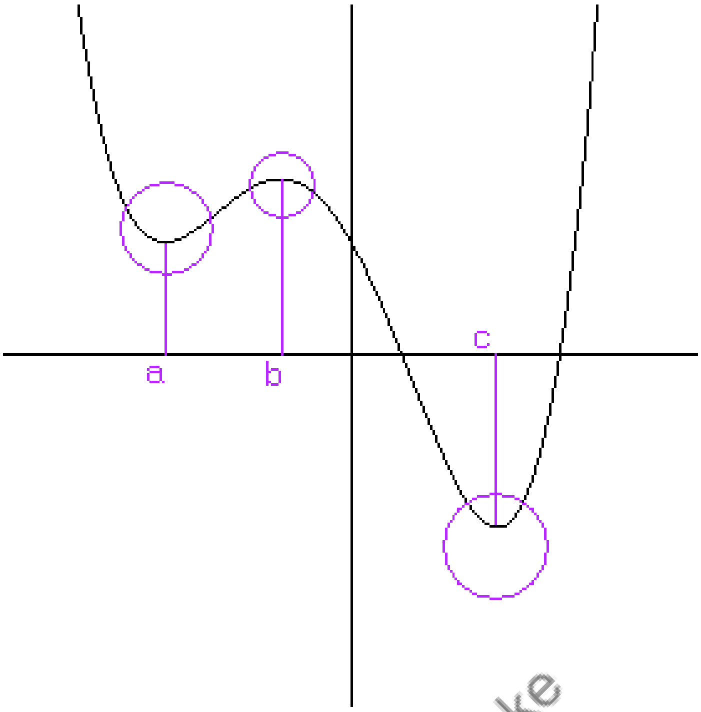
Exercise 3:

Graph the function $f(x) = x^2 - 6x + 7$ and find the intervals where it is increasing and where it is decreasing.

Answer

- decreasing on (-infinity, 3)
- increasing on (3, infinity)

Some of the most characteristics of a function are its **Relative Extreme Values**. Points on the functions graph corresponding to relative extreme values are turning points, or points where the function changes from decreasing to increasing or vice versa. Let f be the function whose graph is drawn below.



f is decreasing on (-infinity, a) and increasing on (a, b), so the point (a, f(a)) is a turning point of the graph. f(a) is called a **relative minimum** of f. Note that f(a) is *not* the smallest function value, f(c) is. However, if we consider only the portion of the graph in the circle above a, then f(a) is the smallest second coordinate. Look at the circle on the graph above b. While f(b) is not the largest function value (this function does not have a largest value), if we look only at the portion of the graph in the circle, then the point (b, f(b)) is above all the other points. So, f(b) is a **relative maximum** of f. f(c) is another relative minimum of f. Indeed, f(c) is the **absolute minimum** of f, but it is also one of the relative minima.

Here again we are giving definitions that appeal to your geometric intuition. The precise definitions are given in your text.

Approximating Relative Extrema

Suppose a is a number such that f(a) is a relative minimum. In applications, it is often more important to know *where* the function attains its relative minimum than it is to know what the relative minimum is.

For example, $f(x) = x^3 - 4x^2 + 4x$ has a relative minimum of 0. It attains this relative minimum at x = 2, so (2,0) is a turning point of the graph of f. We will call the point (2,0) a relative minimum point. In general, a **relative extreme point** is a point on the graph of f whose second coordinate is a relative extreme value of f.

Even and Odd Functions

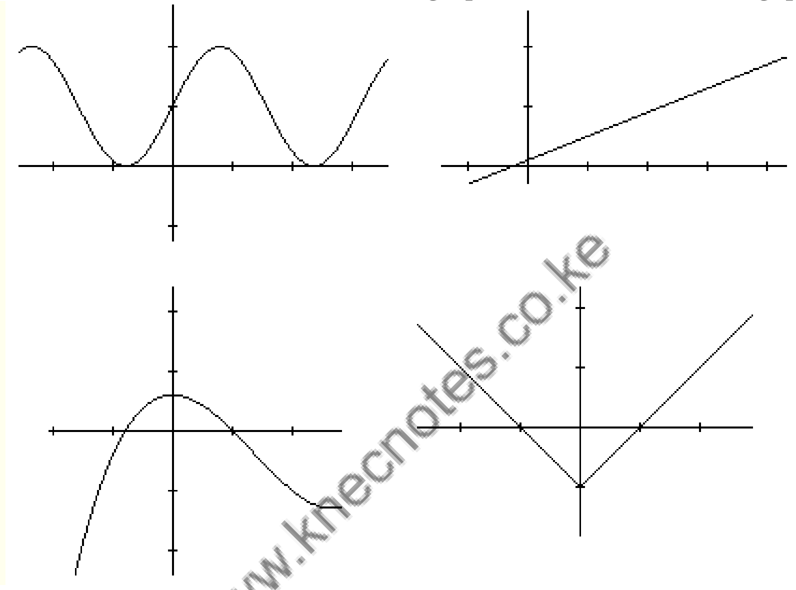
A function f is **even** if its graph is symmetric with respect to the y-axis. This criterion can be stated algebraically as follows: f is even if f(-x) = f(x) for all x in the domain of f. For example, if you evaluate f at 3 and at -3, then you will get the same value if f is even.

A function f is **odd** if its graph is symmetric with respect to the origin. This criterion can be stated algebraically as follows: f is odd if f(-x) = -f(x) for all x in the domain of f. For example, if you evaluate f at 3, you get the negative of f(-3) when f is odd.

Continuous Functions

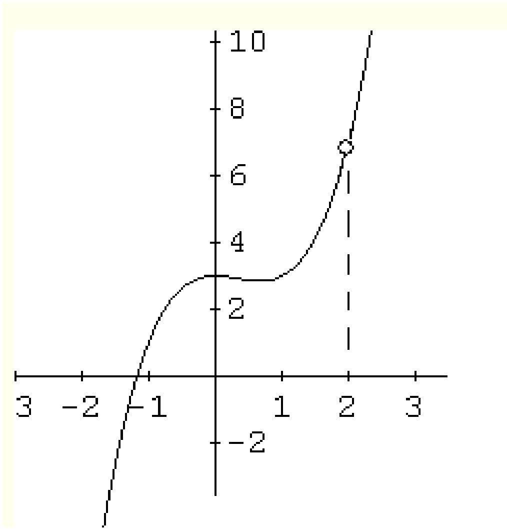
Introduction and Definition of Continuous Functions

We first start with graphs of several continuous functions. The functions whose graphs are shown below are said to be continuous since these graphs have no "breaks", "gaps" or "holes".

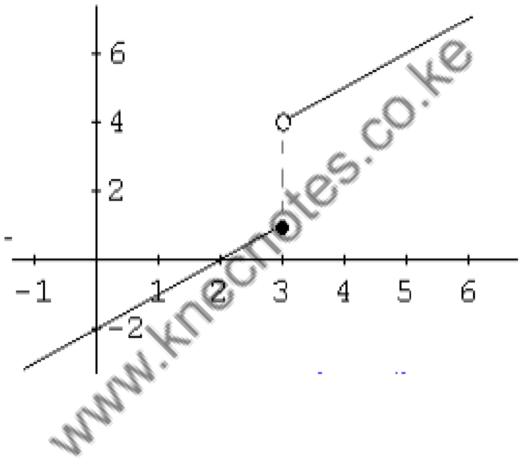


We now present examples of dicontinuous functions. These graphs have: breaks, gaps or points at which they are undefined.

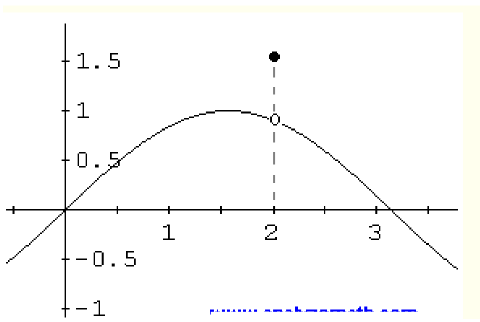
In the graphs below, the function is undefined at x = 2. The graph has a hole at x = 2 and the function is said to be discontinuous.



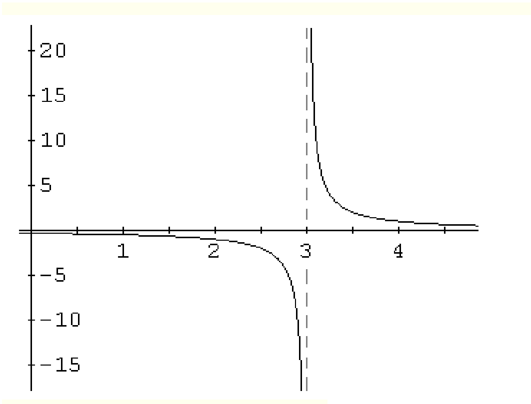
In the graphs below, the limits of the function to the left and to the right are not equal and therefore the limit at x = 3 does not exist. The function is said to be discontinuous.



The limits of the function at x = 2 exists but it is not equal to the value of the function at x = 2. This function is also discontinuous.



The limits of the function at x = 3 does does not exist since to the left and to the right of 3 the function either increases or decreases undefinitely. This function is also discontinuous.



Taking into consideration all the information gathered from the examples of continuous and discontinuous functions shown above, we define a continuous functions as follows:

Function f is continuous at a point **a** if the following conditions are satisfied.

1. f(a) is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

Interpolation and Extrapolation

Definitions

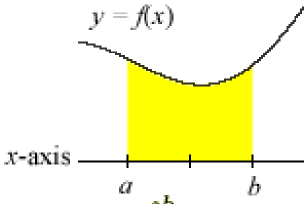
- **Interpolation** is the process of obtaining a value from a graph or table that is located between major points given, or between data points plotted. A ratio process is usually used to obtain the value. *Interpolation allows you to add new data points between pairs of existing data points*
- **Extrapolation** is the process of obtaining a value from a chart or graph that extends beyond the given data. The "trend" of the data is extended past the last point given and an estimate made of the value. *Extrapolation allows you to add data points that extend beyond the beginning or ending values of your data range.*

Area under a Curve

The area between the graph of $y = f(x)$ and the x-axis is given by the definite integral below. This formula gives a positive result for a graph above the x-axis, and a negative result for a graph below the x-axis.

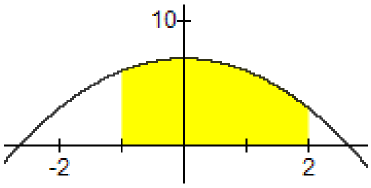
Note: If the graph of $y = f(x)$ is partly above and partly below the x-axis, the formula given below generates the net area. That is, the area above the axis minus the area below the axis.

Formula:



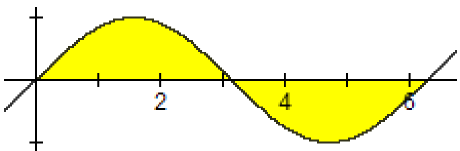
$$\text{Area} = \int_a^b f(x)dx$$

Example 1: Find the area between $y = 7 - x^2$ and the x-axis between the values $x = -1$ and $x = 2$.



$$\begin{aligned}\text{Area} &= \int_{-1}^{2} \left(7 - x^2\right) dx \\ &= \left(7x - \frac{1}{3}x^3\right)\Big|_{-1}^{2} \\ &= \left[7 \cdot 2 - \frac{1}{3}(8)\right] - \left[7(-1) - \frac{1}{3}(-1)\right] \\ &= 18\end{aligned}$$

Example 2: Find the net area between $y = \sin x$ and the x-axis between the values $x = 0$ and $x = 2\pi$.



$$\begin{aligned}\text{Net area} &= \int_0^{2\pi} \sin x dx \\ &= (-\cos x)\big|_0^{2\pi} \\ &= (-1) - (-1) \\ &= 0\end{aligned}$$

Area between Curves

The area between curves is given by the formulas below.

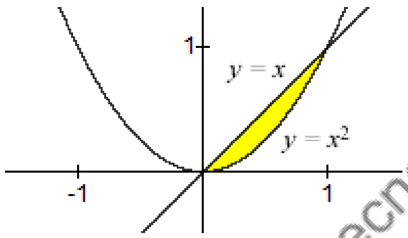
Formula 1: $\text{Area} = \int_a^b |f(x) - g(x)|\,dx$

for a region bounded above and below by $y = f(x)$ and $y = g(x)$, and on the left and right by $x = a$ and $x = b$.

Formula 2: $\text{Area} = \int_c^d |f(y) - g(y)|\,dy$

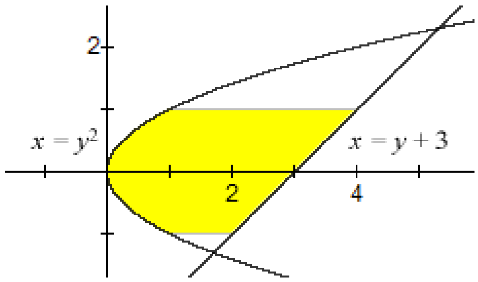
for a region bounded left and right by $x = f(y)$ and $x = g(y)$, and above and below by $y = c$ and $y = d$.

Example 1: Find the area between $y = x$ and $y = x^2$ from $x = 1$ to $x = 2$.



$$\begin{aligned}\text{Area} &= \int_0^1 |x - x^2|\,dx \\ &= \int_0^1 (x - x^2)\,dx \\ &= \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Bigg|_0^1 \\ &= \frac{1}{6}\end{aligned}$$

Example 2: Find the area between $x = y + 3$ and $x = y^2$ from $y = -1$ to $y = 1$.



$$\text{Area} = \int_{-1}^{1} \left| y + 3 - y^2 \right| dy$$
$$= \int_{-1}^{1} \left(y + 3 - y^2 \right) dy$$
$$= \left(\frac{1}{2}y^2 + 3y - \frac{1}{3}y^3 \right) \Big|_{-1}^{1}$$
$$= \left(\frac{1}{2} + 3 - \frac{1}{3} \right) - \left(\frac{1}{2} - 3 + \frac{1}{3} \right)$$
$$= \frac{16}{3}$$

Inequality

In mathematics, an **inequality** is a relation that holds between two values when they are different (see also: equality).

- The notation $a \neq b$ means that a is **not equal to** b.

It does not say that one is greater than the other, or even that they can be compared in size.

If the values in question are elements of an ordered set, such as the integers or the real numbers, they can be compared in size.

- The notation $a < b$ means that a is **less than** b.
- The notation $a > b$ means that a is **greater than** b.

In either case, a is not equal to b. These relations are known as **strict inequalities**. The notation $a < b$ may also be read as "a is strictly less than b".

In contrast to strict inequalities, there are two types of inequality relations that are not strict:

- The notation $a \leq b$ means that a is **less than or equal to** b (or, equivalently, **not greater than** b, or **at most** b).
- The notation $a \geq b$ means that a is **greater than or equal to** b (or, equivalently, **not less than** b, or **at least** b)

An additional use of the notation is to show that one quantity is much greater than another, normally by several orders of magnitude.

- The notation $a \ll b$ means that a is **much less than** b. (In measure theory, however, this notation is used for absolute continuity, an unrelated concept.)
- The notation $a \gg b$ means that a is **much greater than** b.

Solving Inequalities

Sometimes we need to solve Inequalities like these:

Symbol	Words	Example
>	greater than	x + 3 > 2
<	less than	7x < 28
≥	greater than or equal to	5 ≥ x - 1
≤	less than or equal to	2y + 1 ≤ 7

Solving

Our aim is to have x (or whatever the variable is) **on its own** on the left of the inequality sign:

Something like: x < 5

or: y ≥ 11

We call that "solved".

These are things you can do **without affecting** the direction of the inequality:

- Add (or subtract) a number from both sides
- Multiply (or divide) both sides by a **positive** number
- Simplify a side

Example: 3x < 7+3

You can simplify 7+3 without affecting the inequality:

3x < 10

But these things will change the direction of the inequality ("<" becomes ">" for example):

- Multiply (or divide) both sides by a **negative** number
- Swapping left and right hand sides

Example: 2y+7 < 12

When you swap the left and right hand sides, you must also **change the direction of the inequality**:

12 **>** 2y+7

Common Graph Construction Errors

Confusing independent and dependent variables. The Independent variable is what I change." The size of the Dependent variable Depends on the value of the independent variable.

Putting wrong variables on axes. For a matter of convenience, it is common practice to put the independent variable on the horizontal x-axis (bottom) rather than the vertical y-axis (side) when seeking a relationship to define the dependent variable.

For instance, if one wants to arrive at a relationship describing F as a function of a, then F (the dependent variable) should be plotted on the y-axis. The slope becomes the proportionality constant, F = ma.

Failure to understand the significance of "linearizing" data. When data are non-linear (not in a straight line when graphed), it is best to "linearize" the data. This does not mean to fit the curved data points with a straight line. Rather, it means to modify one of the variables in some manner such that when the data are graphed using this new data set, the resulting data points will appear to lie in a straight line. For instance, say the data appear to be an inverse function – as x is doubled, y is halved. To linearize the data for such a function plot x versus 1/y. If this is indeed an inverse function, then the plot of x versus 1/y date will be linear.

Failure to properly relate y = mx + b to the linearized data. When plotting, say, distance (on the y-axis) versus time (on the x-axis), the correct relationship between distance and time can be found by relating y to distance, x to time, m to the slope, and b to the y-intercept. For instance, data have been linearized for the function resulting in a straight-line graph when distance is plotted versus time-squared. The slope is 2m/s2 and the y-intercept is 1 m. The correct form of the relationship between all the variables will be distance = (2m/s2)*time2 + 1 m.

Failure to apply appropriate labeling. Each graph should be appropriately labeled; each axis should be similarly labeled with its variable and units (in parentheses). For instance, time (seconds) or distance (meters).