CHAPTER 4: DISCRETE COUNTING

The Basic Rules/Principles of Counting

The Inclusion-Exclusion and the Pigeonhole Principles are the most fundamental combinatorial techniques. There are two additional rules which are basic to most elementary counting. One is known as the *Sum Rule* (or *Disjunctive Rule*), the other is called *Product Rule* (or *Sequential Rule*.)

The Rules of Sum and Product

The **Rule of Sum** and **Rule of Product** are used to decompose difficult counting problems into simple problems.

- **The Rule of Sum** − If a sequence of tasks $T_1, T_2, \ldots, T_m$ can be done in $w_1, w_2, \ldots w_m$ ways respectively (the condition is that no tasks can be performed simultaneously), then the number of ways to do one of these tasks is $w_1 + w_2 + \cdots + w_m$. If we consider two tasks A and B which are disjoint (i.e. $A \cap B = \emptyset$), then mathematically $|A \cup B| = |A| + |B|$
- **The Rule of Product** − If a sequence of tasks $T_1, T_2, \ldots, T_m$ can be done in $w_1, w_2, \ldots w_m$ ways respectively and every task arrives after the occurrence of the previous task, then there are $w_1 \times w_2 \times \cdots \times w_m$ ways to perform the tasks. Mathematically, if a task B arrives after a task A, then $|A \times B| = |A| \times |B|$

Example

Question − A boy lives at X and wants to go to School at Z. From his home X he has to first reach Y and then Y to Z. He may go X to Y by either 3 bus routes or 2 train routes. From there, he can either choose 4 bus routes or 5 train routes to reach Z. How many ways are there to go from X to Z?

Solution – From X to Y, he can go in $3+2=5$ ways (Rule of Sum). Thereafter, he can go Y to Z in $4+5=9$ ways (Rule of Sum). Hence from X to Z he can go in $5 \times 9 = 45$ ways (Rule of Product).

Pigeonhole Principle

In 1834, German mathematician, Peter Gustav Lejeune Dirichlet, stated a principle which he called the drawer principle. Now, it is known as the pigeonhole principle.

Pigeonhole Principle states that if there are fewer pigeon holes than total number of pigeons and each pigeon is put in a pigeon hole, then there must be at least one pigeon hole with more than one pigeon. If n pigeons are put into m pigeonholes where n > m, there's a hole with more than one pigeon.

Examples

- Ten men are in a room and they are taking part in handshakes. If each person shakes hands at least once and no man shakes the same man's hand more than once then two men took part in the same number of handshakes.
- There must be at least two people in a class of 30 whose names start with the same alphabet.

The Inclusion-Exclusion principle

The **Inclusion-exclusion principle** computes the cardinal number of the union of multiple non-disjoint sets. For two sets A and B, the principle states −

$$|A\cup B|=|A|+|B|-|A\cap B|$$

For three sets A, B and C, the principle states −

$$|A\cup B\cup C|=|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$$

The generalized formula -

$$|\bigcup_{i=1}^{n} A_i|=\sum_{1\le i<j<k\le n}|A_i\cap A_j|+\sum_{1\le i<j<k\le n}|A_i\cap A_j\cap A_k|-\cdots+(-1)^{\pi-1}|A_1\cap\cdots\cap A_2|$$

Problem 1
How many integers from 1 to 50 are multiples of 2 or 3 but not both?

Solution
From 1 to 100, there are $50/2=25$ numbers which are multiples of 2.

There are $50/3=16$ numbers which are multiples of 3.

There are $50/6=8$ numbers which are multiples of both 2 and 3.

So, $|A|=25$, $|B|=16$ and $|A\cap B|=8$.

$$|A\cup B|=|A|+|B|-|A\cap B|=25+16-8=33$$

Problem 2
In a group of 50 students 24 like cold drinks and 36 like hot drinks and each student likes at least one of the two drinks. How many like both coffee and tea?
Solution
Let X be the set of students who like cold drinks and Y be the set of people who like hot drinks.

So, $|X\cup Y|=50$, $|X|=24$, $|Y|=36$

$|X \cap Y| = |X| + |Y| - |X \cup Y| = 24 + 36 - 50 = 60 - 50 = 10$

Counting Techniques

Permutations

A **permutation** is an arrangement of some elements in which order matters. In other words a Permutation is an ordered Combination of elements.

Examples

- From a set S ={x, y, z} by taking two at a time, all permutations are − xy, yx, xz, zx, yz, zy
- We have to form a permutation of three digit numbers from a set of numbers $S = \{1,2,3\}$. Different three digit numbers will be formed when we arrange the digits. The permutation will be = 123, 132, 213, 231, 312, 321

Number of Permutations

The number of permutations of 'n' different things taken 'r' at a time is denoted by nP_r

$$nP_r = n!(n-r)!$$

where $n! = 1.2.3.\ldots(n-1).n$

Proof − Let there be 'n' different elements.

There are n numbers of ways to fill up the first place. After filling the first place (n-1) number of elements is left. Hence, there are (n-1) ways to fill up the second place. After filling the first and second place, (n-2) number of elements is left. Hence, there are (n-2) ways to fill up the third place. We can now generalize the number of ways to fill up r-th place as [n – (r–1)] = n–r+1

So, the total no. of ways to fill up from first place up to r-th-place −

$$nP_r = n(n-1)(n-2).....(n-r+1)$$

$$= [n(n-1)(n-2)...(n-r+1)][(n-r)(n-r-1)\ldots 3.2.1]/[(n-r)(n-r-1)\ldots 3.2.1]$$

Hence,

$$nP_r = n!/(n-r)!$$

Some important formulas of permutation

- If there are *n* elements of which a_1 are alike of some kind, a_2 are alike of another kind; a_3 are alike of third kind and so on and a_r are of r_{th} kind, where $(a_1+a_2+...a_r)=n$.

  Then, number of permutations of these n objects is = $n!/[(a_1!(a_2!)\dots(a_r!)]$.

- Number of permutations of n distinct elements taking n elements at a time = $nP_n=n!$
- The number of permutations of n dissimilar elements taking r elements at a time, when x particular things always occupy definite places = $n-x_{p_{r-x}}$
  - The number of permutations of n dissimilar elements when r specified things always come together is − $r!(n−r+1)!$
  - The number of permutations of n dissimilar elements when r specified things never come together is − $n!–[r!(n−r+1)!]$
  - The number of circular permutations of n different elements taken x elements at time = ${}_np_x/x$
  - The number of circular permutations of n different things = ${}_np_n/n$

Some Problems

Problem 1 − From a bunch of 6 different cards, how many ways we can permute it?

Solution − As we are taking 6 cards at a time from a deck of 6 cards, the permutation will be ${}_6P_6=6!=720$

Problem 2 − In how many ways can the letters of the word 'READER' be arranged?

Solution − There are 6 letters word (2 E, 1 A, 1D and 2R.) in the word 'READER'.

The permutation will be $=6!/[(2!)(1!)(1!)(2!)]=180.$

Problem 3 − In how ways can the letters of the word 'ORANGE' be arranged so that the consonants occupy only the even positions?

Solution − There are 3 vowels and 3 consonants in the word 'ORANGE'. Number of ways of arranging the consonants among themselves $={}_3P_3=3!=6$. The remaining 3 vacant places will be filled up by 3 vowels in ${}_3P_3=3!=6$ ways. Hence, the total number of permutation is $6\times6=36$

Combinations

A **combination** is selection of some given elements in which order does not matter.

The number of all combinations of n things, taken r at a time is −

$$_{n}C_{r} = n!r!(n-r)!$$

Problem 1

Find the number of subsets of the set $\{1,2,3,4,5,6\}$having 3 elements.

Solution

The cardinality of the set is 6 and we have to choose 3 elements from the set. Here, the ordering does not matter. Hence, the number of subsets will be $_{6}C_{3}=20$.

Problem 2

There are 6 men and 5 women in a room. In how many ways we can choose 3 men and 2 women from the room?

Solution

The number of ways to choose 3 men from 6 men is $_{6}C_{3}$ and the number of ways to choose 2 women from 5 women is $_{5}C_{2}$

Hence, the total number of ways is − $_{6}C_{3} \times {}_{5}C_{2} = 20 \times 10 = 200$

Problem 3

How many ways can you choose 3 distinct groups of 3 students from total 9 students?

Solution

Let us number the groups as 1, 2 and 3

For choosing 3 students for 1st group, the number of ways − $_{9}C_{3}$

The number of ways for choosing 3 students for 2nd group after choosing 1st group − $_{6}C_{3}$

The number of ways for choosing 3 students for 3rd group after choosing 1st and 2nd group − $_{3}C_{3}$

Hence, the total number of ways $= {}^{9}C_{3} \times {}^{6}C_{3} \times {}^{3}C_{3} = 84 \times 20 \times 1 = 1680$

Pascal's Identity

Pascal's identity, first derived by Blaise Pascal in 19th century, states that the number of ways to choose k elements from n elements is equal to the summation of number of ways to choose (k-1) elements from (n-1) elements and the number of ways to choose elements from n-1 elements.

Mathematically, for any positive integers k and n: ${}^{n}C_{k} = {}^{n-1}C_{k-1} + {}^{n-1}C_{k}$

Proof −

$${}^{n-1}C_{k-1} + {}^{n-1}C_{k}$$

$= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}$

$= (n-1)!\left(\frac{k}{k!(n-k)!} + \frac{n-k}{k!(n-k)!}\right)$

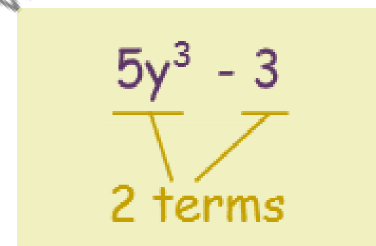
$= (n-1)!\frac{n}{k!(n-k)!}$

$= \frac{n!}{k!(n-k)!}$

$= {}^{n}C_{k}$

Binomial Theorem

A **binomial** is a polynomial with two terms



example of a binomial

What happens when we multiply a binomial by itself ... many times?

Example: a+b

a+b is a binomial (the two terms are **a** and **b**)

Let us multiply **a+b** by itself using Polynomial Multiplication :

$(a+b)(a+b) = \mathbf{a^2 + 2ab + b^2}$

Now take that result and multiply by **a+b** again:

$(a^2 + 2ab + b^2)(a+b) = \mathbf{a^3 + 3a^2b + 3ab^2 + b^3}$

And again:

$(a^3 + 3a^2b + 3ab^2 + b^3)(a+b) = \mathbf{a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4}$

The calculations get longer and longer as we go, but there is some kind of **pattern** developing.

That pattern is summed up by the **Binomial Theorem**:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

The Binomial Theorem

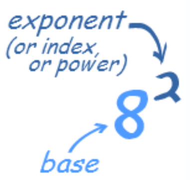
Don't worry ... it will all be explained!

And you will learn lots of cool math symbols along the way.

Exponents

First, a quick summary of Exponents.

An exponent says **how many times** to use something in a multiplication.



Example: $8^2 = 8 \times 8 = 64$

An exponent of **1** means just to have it appear once, so we get the original value:

Example: $8^1 = 8$

An exponent of **0** means not to use it at all, and we have only 1:

Example: $8^0 = 1$

<u>**Exponents of (a+b)**</u>

Now on to the binomial.

We will use the simple binomial a+b, but it could be any binomial.

Let us start with an exponent of **0** and build upwards.

Exponent of 0: When an exponent is 0, we get **1**:

$$(a+b)^0 = 1$$

Exponent of 1: When the exponent is 1, we get the original value, unchanged:

$$(a+b)^1 = a+b$$

Exponent of 2: An exponent of 2 means to multiply by itself.

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

Exponent of 3: For an exponent of 3 just multiply again:

$$(a+b)^3 = (a^2 + 2ab + b^2)(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$$

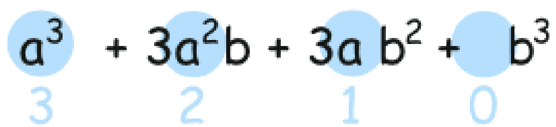
We have enough now to start talking about the pattern.

The Pattern

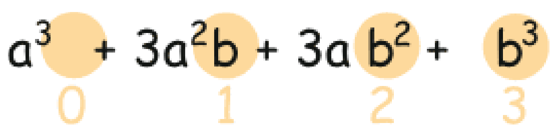
In the last result we got:

$$a^3 + 3a^2b + 3ab^2 + b^3$$

Now, notice the exponents of a. They start at 3 and go down: 3, 2, 1, 0:



Likewise the exponents of b go upwards: 0, 1, 2, 3:



If we number the terms 0 to *n*, we get this:

k=0	k=1	k=2	k=3
a^3	a^2	a	1
1	b	b^2	b^3

Which can be brought together into this:

$$a^{n-k}b^k$$

How about an example to see how it works:

Example: When the exponent, *n*, is 3.

The terms are:

k=0:	k=1:	k=2:	k=3:
$a^{n-k}b^k$ $= a^{3-0}b^0$ $= \mathbf{a^3}$	$a^{n-k}b^k$ $= a^{3-1}b^1$ $= \mathbf{a^2b}$	$a^{n-k}b^k$ $= a^{3-2}b^2$ $= \mathbf{ab^2}$	$a^{n-k}b^k$ $= a^{3-3}b^3$ $= \mathbf{b^3}$

It works like magic!

Coefficients

So far we have:	$a^3 + a^2b + ab^2 + b^3$
But we **really** need:	$a^3 + 3a^2b + 3ab^2 + b^3$

We are **missing the numbers** (which are called *coefficients*).

Let's look at **all the results** we got before, from $(a+b)^0$ up to $(a+b)^3$:

$$1$$
$$a + b$$
$$a^2 + 2ab + b^2$$
$$a^3 + 3a^2b + 3ab^2 + b^3$$

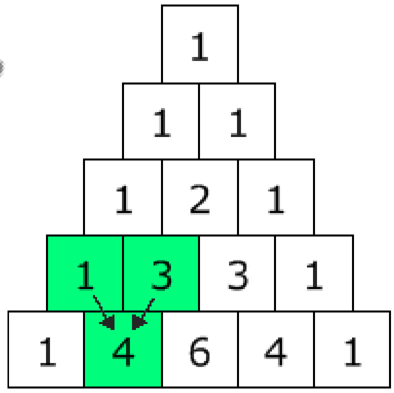
And now look at **just the coefficients** (with a "1" where a coefficient wasn't shown):

$$1$$
$$1a + 1b$$
$$1a^2 + 2ab + 1b^2$$
$$1a^3 + 3a^2b + 3ab^2 + 1b^3$$

They actually make Pascal's Triangle!

Each number is just the two numbers above it added together (except for the edges, which are all "1")

(Here I have highlighted that **1+3 = 4)**



Armed with this information let us try something new ... an **exponent of 4**:

a exponents go 4,3,2,1,0:		a^4	+	a^3	+	a^2	+	a	+	1	
b exponents go 0,1,2,3,4:		a^4	+	a^3b	+	a^2b^2	+	ab^3	+	b^4	
coefficients go 1,4,6,4,1:		a^4	+	$4a^3b$	+	$6a^2b^2$	+	$4ab^3$	+	b^4	

And that is the correct answer (compare to the top of the page).

We have success!

We can now use that pattern for exponents of 5, 6, 7, ... 50, ... 112, ... you name it!

That pattern is the essence of the Binomial Theorem.

Now you can take a break.

When you come back see if you can work out $\mathbf{(a+b)^5}$ yourself.

Answer (hover over): $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Formula

Our last step is to write it all as a formula.

But hang on, how do we write a formula for **"find the coefficient from Pascal's Triangle"** ... ?

Well, there **is** such a formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

It is commonly called "n choose k" because it is how many ways to choose k elements from a set of n.

You can read more at Combinations and Permutations

The "!" means "factorial", for example 4! = 4×3×2×1 = 24

And it matches to Pascal's Triangle like this:

(Note how the top row is row zero and also the leftmost column is zero!)

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Example: Row 4, term 2 in Pascal's Triangle is "6".

Let's see if the formula works:

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1} = 6$$

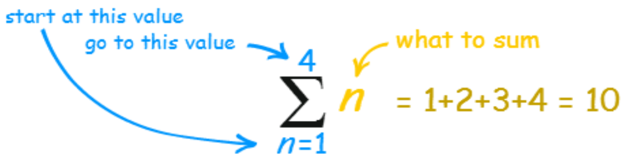
Yes, it works! Try another value for yourself.

Putting It All Together

The last step is to put all the terms together into **one formula**.

But we are adding lots of terms together ... can that be done using one formula?

Yes! The handy Sigma Notation allows us to sum up as many terms as we want:



Sigma Notation

Now it can all go into one formula:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

The Binomial Theorem

Use It

OK ... it won't make much sense without an example.

So let's try using it for $n = 3$:

$$\begin{aligned}(a+b)^3 &= \sum_{k=0}^{3} \binom{3}{k} a^{3-k} b^k \\ &= \binom{3}{0} a^{3-0} b^0 + \binom{3}{1} a^{3-1} b^1 + \binom{3}{2} a^{3-2} b^2 + \binom{3}{3} a^{3-3} b^3 \\ &= 1 \cdot a^3 b^0 + 3 \cdot a^2 b^1 + 3 \cdot a^1 b^2 + 1 \cdot a^0 b^3 \\ &= a^3 + 3a^2 b + 3ab^2 + b^3\end{aligned}$$

BUT ... it is usually **much easier** just to remember the **patterns**:

- The first term's exponents start at **n and go down**
- The second term's exponents start at **0 and go up**
- Coefficients are from Pascal's Triangle, or by calculation using **n!/(k!(n-k)!)**

Like this:

Example: What is $(y+5)^4$

Start with exponents:	$y^4 5^0$	$y^3 5^1$	$y^2 5^2$	$y^1 5^3$	$y^0 5^4$
Include Coefficients:	$1y^4 5^0$	$4y^3 5^1$	$6y^2 5^2$	$4y^1 5^3$	$1y^0 5^4$

Then write down the answer (including all calculations, such as 4×5, 6×5^2, etc):

$$(y+5)^4 = y^4 + 20y^3 + 150y^2 + 500y + 625$$

We may also want to calculate just one term:

Example: What is the coefficient for x^3 in $(2x+4)^8$

Example: What is the coefficient for x^3 in $(2x+4)^8$

The **exponents** for x^3 are **8-5** (=3) and **5**:

$$(2x)^3 4^5$$

The **coefficient** is "8 choose 5". We can use Pascal's Triangle, or calculate directly:

$$\frac{n!}{k!(n-k)!} = \frac{8!}{5!(8-5)!} = \frac{8!}{5!3!} = \frac{8\times7\times6}{3\times2\times1} = 56$$

And we get:

$$56(2x)^3 4^5$$

Which simplifies to:

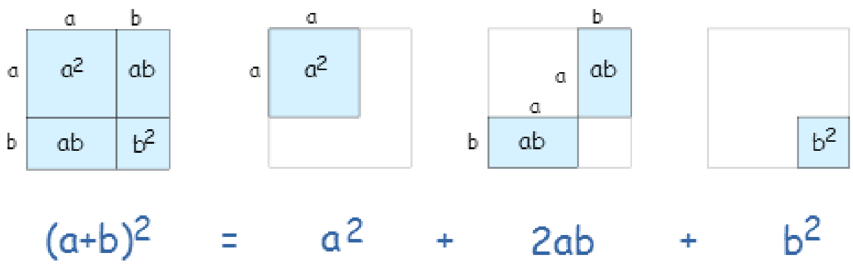
$$\mathbf{458752}\ x^3$$

A large coefficient, isn't it?

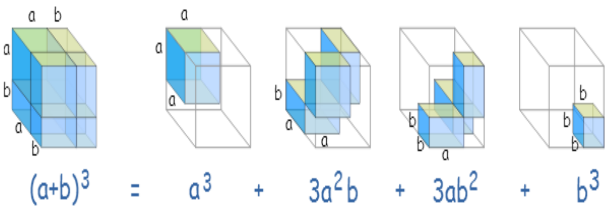
Geometry

Want to see the Binomial Theorem using Geometry?

In 2 dimensions, $(\mathbf{a+b})^2 = \mathbf{a}^2 + \mathbf{2ab} + \mathbf{b}^2$



In 3 dimensions, $(\mathbf{a+b})^3 = \mathbf{a}^3 + \mathbf{3a^2b} + \mathbf{3ab^2} + \mathbf{b}^3$



In 4 dimensions, $(\mathbf{a+b})^4 = \mathbf{a}^4 + \mathbf{4a^3b} + \mathbf{6a^2b^2} + \mathbf{4ab^3} + \mathbf{b}^4$

(Sorry, I am not good at drawing in 4 dimensions!)

Advanced Example

And one last, most amazing example:

Example: A formula for e (Euler's Number)

We can use the Binomial Theorem to calculate e (Euler's number).

e = **2.718281828459045...** (the digits go on forever without repeating)

It can be calculated using:

$$(1 + 1/n)^n$$

(It gets more accurate the higher the value of **n**)

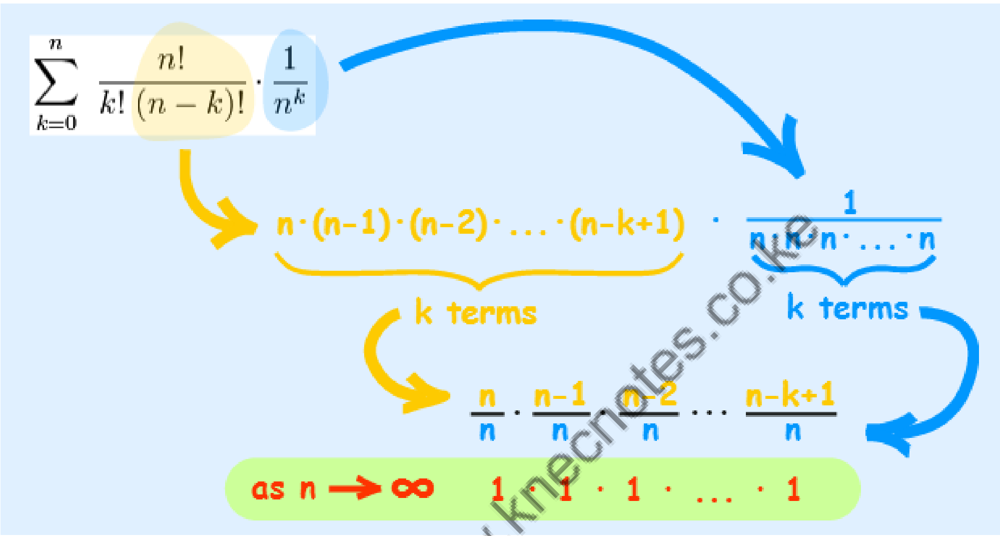
That formula is a **binomial**, right? So let's use the Binomial Theorem:

$$(1+\tfrac{1}{n})^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (\tfrac{1}{n})^k$$

First, we can drop 1^{n-k} as it is always equal to 1:

$$= \sum_{k=0}^{n} \binom{n}{k} (\tfrac{1}{n})^k$$

And, quite magically, most of what is left goes to **1** as n goes to infinity:



Which just leaves:

$$\sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ldots$$

With just those first few terms we get e ≈ 2.7083...

Try calculating more terms for a better approximation!

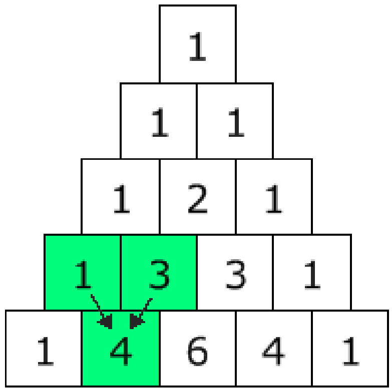
Pascal's Triangle

One of the most interesting Number Patterns is Pascal's Triangle (named after *Blaise Pascal*, a famous French Mathematician and Philosopher).

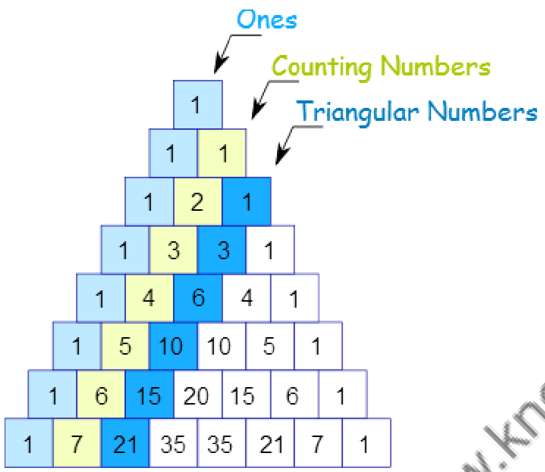
To build the triangle, start with "1" at the top, then continue placing numbers below it in a triangular pattern.

Each number is the numbers directly above it added together.

(Here I have highlighted that **1+3 = 4)**



Patterns Within the Triangle



Diagonals

The first diagonal is, of course, just "1"s

The next diagonal has the Counting Numbers (1,2,3, etc).

The third diagonal has the triangular numbers

(The fourth diagonal, not highlighted, has the tetrahedral numbers.)

Symmetrical

The triangle is also symmetrical. The numbers on the left side have identical matching numbers on the right side, like a mirror image.