
**STANDARD LECTURE NOTES
FOR
STATISTICS FOR DIPLOMA**

**QUANTITATIVE
TECHNIQUES**

QUANTITATIVE TECHNIQUES

Course Outline

1. Introduction to Quantitative Techniques
2. Fundamentals of Mathematics and Statistics.
3. Data Collection
4. Measures Of Central Tendency
5. Measures Of Variation /Depression
6. Correlation Analysis
7. Regression Analysis
8. Index Number
9. Time Series Analysis
10. Probability
11. Sampling
12. Test of Hypothesis
13. Linear Programming

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QUANTITATIVE TECHNIQUES

GENERAL OBJECTIVES

At the end of this course unit, the trainee should be able to;-

- ❖ broaden his/her knowledge in mathematical application;
- ❖ understand and appreciate the role of quantitative methods in decision making;
- ❖ collect and organize statistical data for management;
- ❖ analyze quantitative data for management decision making;
- ❖ Apply quantitative methods in solving business problems.

Introduction

Def; quantitative techniques are those techniques which provides the decision maker with a systematic and powerful means of analysis and help, based on quantitative data in exploring policies for achieving pre-determined goals Involves the use of numbers, symbols and other mathematical expressions.

They are essentially helpful in supplementing to judgment and intuition. These techniques evaluate planning factors of alternatives as and when they arise rather than prescribe courses of action. They are particularly relevant to problems of complex business enterprises.

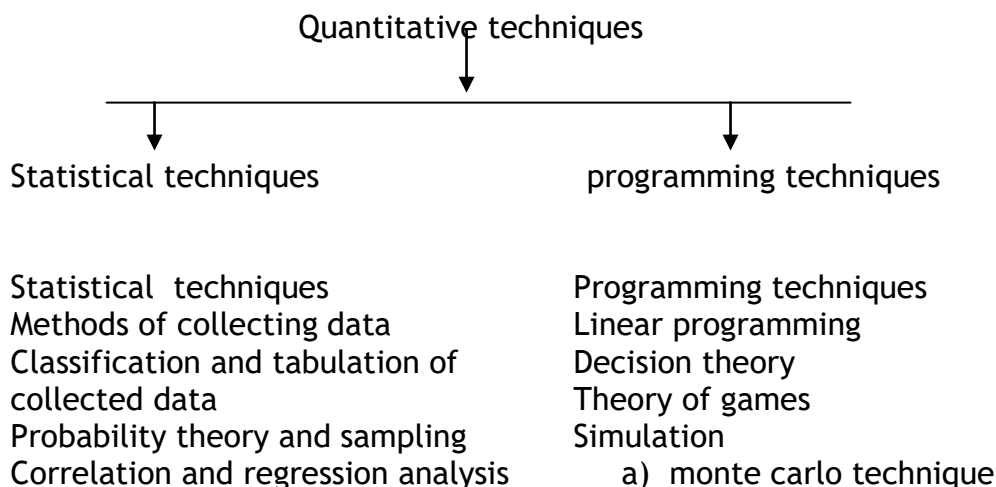
Classification of Q.T

a) Statistical Techniques

Are those techniques which are used in conducting the statistical inquiry concerning a certain phenomenon? They include statistical methods beginning from the collection of data till the task of interpretation of the data collected.

b) Programming Techniques

Are the model building techniques used by decision maker?



Index numbers
Time series analysis
Interpretation and extrapolation
Survey techniques and methodology
Ratio analysis
Statistical quality control
Analysis of variance
Statistical inference and interpretation
Theory of attributes.

b) system simulation
queuing (waiting line) theory
inventory planning
network analysis/ PERT
integrated production model
others;
non-linear programming
the theory of replacement
quadratic programming
Parametric programming etc.

QT and Business management

Production Management

- Selecting building site fro a plant, scheduling and controlling its development and designing its layout
- Locating within the plant and controlling movement required materials and finished goods inventories.
- Scheduling and sequencing production by adequate preventive maintenance with optimum number of operatives by proper allocation of machines
- Calculating optimum product mix.

Personnel Management

- Optimum manpower planning
- No of persons to be maintained on permanent or full time roll
- The no. of persons to be kept in work pool intended fro meeting the absenteeism.
- Optimum manner of sequencing and routing of personnel to a variety of jobs
- Studying personnel recruiting procedures, accidents rates and labor turnover

Market Management

- Where distribution and warehousing should be located the size , quantity to be stocked and choice of customers.
- Optimum allocation of sales budget to direct selling of promotional expenses.
- Choice of different media of advertising and bidding strategies.
- Financial management
- Finding long range capital requirement as well as how to generate theses requirements
- Determining optimum replacement policies
- Working out a profit plan for the firm
- Developing capital investment plans
- Estimating credit and investment risks.

Limitation of Q.T s

1. the inherent limitation concerning mathematical expressions
2. high costs involved in the use of QTs
3. They do not take into consideration the intangible factors i.e. non-measurable human factors.
4. Quantitative techniques are just the tools of analysis and not the complete decision making process.

Role of QT in business and industry

1. they provide a tool for scientific analysis
these techniques provides executives with a more precise description of the cause and effect relationship and risks underlying the business operations in measurable terms and this eliminates the conventional intuitive and subjective basis on which management used to formulate their decisions.

2. they provide solution for various business problems
Are used in the field of production, procurement, marketing, finance and other allied fields. Problems like how best can managers and executive4s allocate available resources to various products so that in a given time the profits are maximized or costs are minimized. Is it possible for an enterprise to arrange the time and quantity of orders of its stocks such that the overall profit with given resources is maximized?

3. they enable proper development of resources
E.g. programmed evaluation and review techniques (PERT) enables us to determine earliest and the latest time fro each of he events and activities and thereby helps in the identification of the critical path
All these helps in deployment of resources from one activity to another to enable the project completion on time.

4. They help in minimizing waiting and servicing time.
The queuing theory helps management in minimizing the total waiting of servicing costs. It also analyses the feasibility of adding facilities and thereby helping to take correct and profitable decision.

5. They enable management to decide when to buy and how much to buy.
The main objective of inventory planning is to achieve balance between the costs of holding stock and benefits of holding stock. Helps in determining when to buy and how much to buy.

6. They assist in choosing an optimum strategy.
In a competitive situation game theory helps to determine optimum strategy which maximizes profits or minimizes loses but adopting optimum strategy.

7. they render great help in optimum resource allocation
8. they facilitate the process of decision making
9. Through various QTs management can know the reactions of the integrated business systems.

CHAPTER ONE

FUNDAMENTALS OF MATHEMATICS AND STATISTICS

Specific Objectives

At the end of this topic, the trainee should be able to:

- ❖ Form and solve algebraic equations.
- ❖ Apply the various techniques of counting to solving management decision problems;
- ❖ Applying set theory to business decision problems;
- ❖ Derive and apply the binomial theorem to business problems;
- ❖ Evaluate mathematical series.

ALGEBRAIC EQUATIONS

Algebra

Algebra is a branch of mathematics in which, instead of using numbers, we use letters to represent numbers.

We all know that $2+3=5$.

Suppose, though, that we substitute letters for the first two numbers, so that:

$$2 = a$$

$$3 = b$$

We can then write:

$$a + b = 5$$

All that has happened is that we have replaced the numbers with letters. However, a number is a specific quantity - e.g., 5 is more than 4, but less than 6 - whereas a letter can be used to represent any number. Thus in the above expression, 'a' could be 4 and 'b' could be 1. We only know that they are 2 and 3 respectively because we defined them as such before.

The main consequence of this is that algebra uses general expression and gives general results, whereas arithmetic (using numbers) uses definite numbers and gives definite results. Arithmetic is specific whereas algebra is general.

Equations

An equation is an expression with an equal sign (=)
Equations are classified into two main groups' linear equations and non linear equations. Examples of linear equations are

$$x + 13 = 15$$

$$7x + 6 = 0$$

Non linear equations in the variable x are equations in which x appears in the second or higher degrees. They include quadratic and cubic equations amongst others. For example

$$5x^2 + 3x + 7 = 0 \text{ (quadratic equation)}$$

$$2x^3 + 4x^2 + 3x + 8 = 0 \text{ (cubic equation)}$$

The solution of equations or the values of the variables for which the equations hold is called the roots of the equation or the solution set.

Solution of linear equations.

Supposing M, N, and P are expressions that may or may not involve variables, then the following constitute some rules which will be useful in the solution of linear equations

Rule 1: Additional rule

$$\text{If } M = N \text{ then } M + P = N + P$$

Rule 2: Subtraction rule

$$\text{If } M = N, \text{ Then } M - P = N - P$$

Rule 3: multiplication rule

$$\text{If } M = N \text{ and } P \neq 0 \text{ then } M \times P = N \times P$$

Rule 4: Division rule

$$\text{If } P \times M = N \text{ and } P \neq 0$$

$$\text{And } N/P = Q \text{ Q being a rational number then}$$

$$M = N/P$$

Example

i. Solve $3x + 4 = - 8$

ii. Solve $\frac{y}{3} = - 4$

Solutions

$$\begin{aligned}
 \text{i.} \quad & 3x + 4 = -8 \\
 & 3x + 4 - 4 = -8 - 4 \quad (\text{by subtraction rule}) \\
 & 3x = -12 \quad (\text{simplifying}) \\
 & \frac{3x}{3} = \frac{-12}{3} \quad (\text{by division rule}) \\
 & x = -4 \quad (\text{simplifying}) \\
 \\
 \text{ii.} \quad & 3 \times \frac{y}{3} = -4 \times 3 \\
 & y = -12 \quad (\text{simplifying})
 \end{aligned}$$

Solutions of inequalities

The solutions sets of inequalities frequently contain many elements. In a number of cases they contain infinite elements.

Example

Solve and graph the following inequalities

$$x - 2 > 2; \quad x \subset w \quad (\text{where } x \text{ is a subset of } w)$$

Solution

$$x - 2 > 2 \text{ so } x - 2 + 2 > 2 + 2$$

Thus, $x > 4$

The solution set is infinite, being all the elements in w greater than 4



Example

Solve and graph

$$3x - 7 < -13;$$

Solution

$$3x - 7 < -13$$

$$\Rightarrow 3x - 7 + 7 < -13 + 7$$

$$\Rightarrow 3x < -6$$

$$\frac{3x}{3} < \frac{-6}{3}$$

$$x < -2$$



Rules for solving linear inequalities

Suppose M , M_1 , N , N_1 and P are expressions that may or may not involve variables, then the corresponding rules for solving inequalities will be:

Rule 1: Addition rule

$$\text{If } M > N \text{ and } M_1 > N_1$$

Then $M + P > N + P$ and

$$M_1 + P > N_1 + P$$

Rule 2: Subtraction Rule

$$\text{If } M < N \text{ and } M_1 \geq N_1$$

Then $M - P < N - P$ and

$$M_1 - P \geq N_1 - P$$

Rule 3: Multiplication rule

$$\text{If } M \geq N \text{ and } M_1 > N_1 \text{ and } P \neq 0$$

$$\text{Then } MP \geq NP; M_1P > N_1P$$

$$M(-P) \leq N(-P) \text{ and } M_1(-P) < N_1(-P)$$

Rule 4: Division

$$\text{If } M > N \text{ and } M_1 < N_1 \text{ and } P \neq 0$$

Then $M/P > N/P$: $M_1/P < N_1/P$

$M/(-P) < N/(-P)$: and $M_1/(-P) > N_1/(-P)$

Rule 5: Inversion Rule

If $M/P \leq N/Q$ where $P, Q \neq 0$

$M_1/P > N_1/Q$

Then $P/M \geq Q/N$ and $P/M_1 < Q/N_1$

Note: The rules for solving equations are the same as those for solving equations with one exception; when both sides of an equation is multiplied or divided by a negative number, the inequality symbol must be reversed (see rule 3 & Rule 4 above).

Example

Solve and graph the following:

i. $7 - 2x > -11$;

ii. $-5x + 4 \leq 2x - 10$;

iii. $-3 \leq 2x + 1 < 7$;

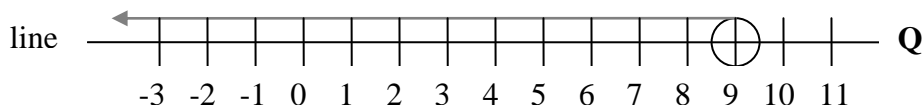
Solutions

i. $7 - 2x > -11$

$-2x > -18$ (subtraction rule)

$\frac{-2x}{-2} < \frac{-18}{-2}$ (by division rule)

$x < 9$

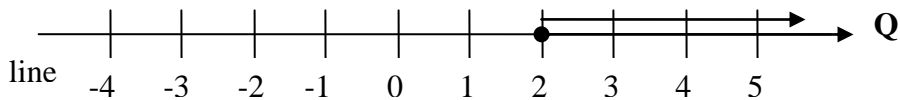


ii. $-5x + 4 \leq 2x - 10$

$$-7x + 4 \leq -10 \quad (\text{by subtraction rule})$$

$$-7x \leq -14 \quad (\text{by subtraction rule})$$

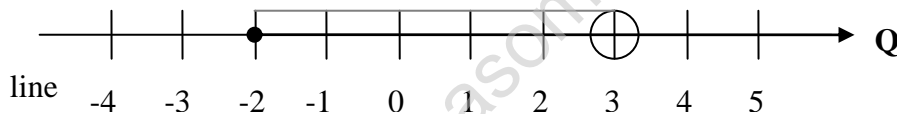
$$x \geq 2 \quad (\text{by division rule})$$



iii. $-3 \leq 2x + 1 < 7$

$$-4 \leq 2x < 6 \quad (\text{by subtraction rule})$$

$$-2 \leq x < 3 \quad (\text{by division rule})$$



Linear inequalities in two variables: relations

An expression of the form

$$Y \geq 2x - 1$$

Is technically called a relation. It corresponds to a function, but different from it in that, corresponding to each value of the independent variable x , there is more than one value of the dependent variable y

Relations can be successfully presented graphically and are of major importance in linear programming.

Solutions of linear simultaneous equations.

Two or more equations will form a system of linear simultaneous equations if such equations be linear in the same two or more variables.

For instance, the following systems of the two equations is simultaneous in the two variables x and y .

$$2x + 6y = 23$$

$$4x + 7y = 10$$

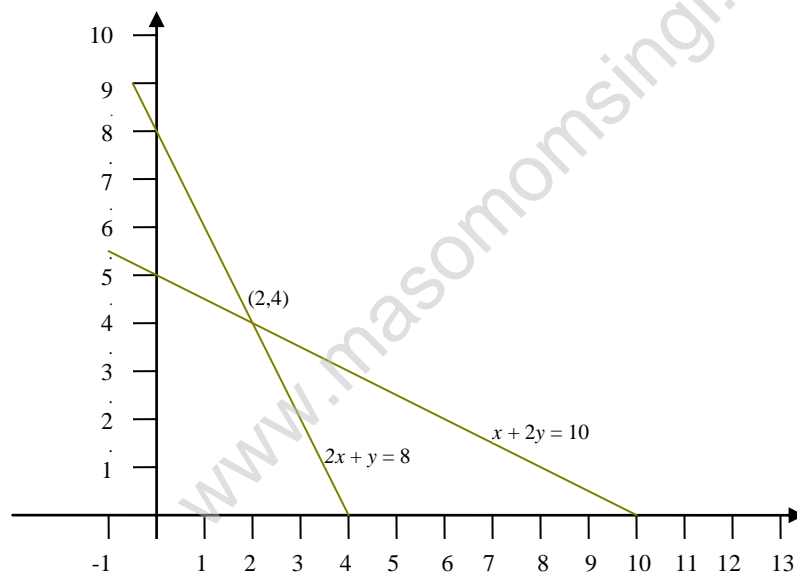
The solution of a system of linear simultaneous equations is a set of values of the variables which simultaneously satisfy all the equations of the system.

Solution techniques

a) The graphical technique

The graphical technique of solving a system of linear equations consists of drawing the graphs of the equations of the system on the same rectangular coordinate system. The coordinates of the point of intersection of the equations of the system would then be the solution.

Example



The above figure illustrates:

Solution by graphical method of two equations

$$2x + y = 8$$

$$x + 2y = 10$$

The system has a unique solution $(2, 4)$ represented by the point of intersection of the two equations.

b) The elimination technique

This method requires that each variable be eliminated in turn by making the absolute value of its coefficients equal in the equations of the system and then adding or subtracting the equations. Making the absolute values of the coefficients equal necessitates the multiplication of each equation by an appropriate numerical factor.

Consider the system of two equations (i) and (ii) below

$$2x - 3y = 8 \dots\dots\dots (i).$$

$$3x + 4y = -5 \dots\dots\dots (ii).$$

Step 1

Multiply (i) by 3

$$6x - 9y = 24 \dots\dots\dots (iii).$$

Multiply (ii) By 2

$$6x + 8y = -10 \dots\dots\dots (iv).$$

Subtract (iii) from (iv).

$$17y = -34 \dots\dots\dots (v).$$

$$\therefore y = -2$$

Step 2

Multiply (i) by 4

$$8x - 12y = 32 \dots\dots\dots (vi)$$

Multiply (ii) by 3

$$9x + 12y = -15 \dots\dots\dots (vii)$$

Add (vi) to (vii)

$$17x = 17 \dots\dots\dots (viii)$$

$$\therefore x = 1$$

Thus $x = 1, y = -2$ i.e. $\{1, -2\}$

c) The substitution technique

To illustrate this technique, consider the system of two equations (i) and (ii) reproduced below

$$\dots 2x - 3y = 8 \dots\dots\dots (i).$$

$$\dots 3x + 4y = -5 \dots\dots\dots (ii).$$

The solution of this system can be obtained by

- a) Solving one of the equations for one variable in terms of the other variable;
- b) Substituting this value into the other equation(s) thereby obtaining an equation with one unknown only
- c) Solving this equation for its single variable finally
- d) Substituting this value into any one of the two original equations so as to obtain the value of the second variable

Step 1

Solve equation (i) for variable x in terms of y

$$2x - 3y = 8$$

$$x = 4 + \frac{3}{2}y \quad \text{(iii)}$$

Step 2

Substitute this value of x into equation (ii). And obtain an equation in y only

$$3x + 4y = -5$$

$$3(4 + \frac{3}{2}y) + 4y = -5$$

$$8\frac{1}{2}y = -17 \dots\dots \quad \text{(iv)}$$

Step 3

Solve the equation (iv). For y

$$8\frac{1}{2}y = -17$$

$$y = -2$$

Step 4

Substitute this value of y into equation (i) or (iii) and obtain the value of x

$$2x - 3y = 8$$

$$2x - 3(-2) = 8$$

$$x = 1$$

Example

Solve the following by substitution method

$$2x + y = 8$$

$$3x - 2y = -2$$

Solution

Solve the first equation for y

$$y = 8 - 2x$$

Substitute this value of y into the second equation and solve for x

$$3x - 2y = -2$$

$$3x - 2(8 - 2x) = -2$$

$$x = 2$$

Substitute this value of x into either the first or the second original equation and solve for y

$$2x + y = 8$$

$$(2)(2) + y = 8$$

$$y = 4$$

TECHNIQUES OF COUNTING

Permutations

This is an order arrangement of items in which the order must be strictly observed

Example

Let x , y and z be any three items. Arrange these in all possible permutations

1 st	2 nd	3 rd	} Six different permutations
X	Y	Z	
X	Z	Y	
Y	X	Z	
Y	Z	X	
Z	Y	X	
Z	X	Y	

NB: The above 6 permutations are the maximum one can ever obtain in a situation where there are only 3 items but if the number of items exceeds 3 then determining the no. of permutations by outlining as done above may be cumbersome. Therefore we use a special formula to determine such permutations. The formula is given below

The number of permutations of ' r ' items taken from a sample of ' n ' items

may be provided as ${}^n P_r = \frac{n!}{(n-r)!}$ where $! = \text{factorial}$

e.g.

i. ${}^3P_3 = \frac{3!}{(3-3)!}$
 $= \frac{3 \times 2 \times 1}{0!}$ **note; 0! = 1**
 $= \frac{6}{1} = 6$

ii. ${}^5P_3 = \frac{5!}{(5-3)!}$
 $= \frac{5 \times 4 \times 3 \times 2 \times 1}{1 \times 2}$
 $= 60$

iii. ${}^7P_5 = \frac{7!}{(7-5)!}$
 $= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1}$
 $= \frac{5040}{2}$
 $= 2520$

Example

There are 6 contestants for the post of chairman secretary and treasurer. These positions can be filled by any of the 6. Find the possible no. of ways in which the 3 positions may be filled.

Solution

Chairman	Secretary	Treasurer
6	5	4

Therefore the no of ways of filing the three positions is $6 \times 5 \times 4 = 120$

$${}^6P_3 = \frac{6!}{(6-3)!}$$

$$= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}$$

$$= \frac{720}{6}$$

$$= 120$$

Combinations

A combination is a group of times in which order is not important. For a combination to hold at any given time it must comprise of the same items but if a new item is added to the group or removed from the group then we have a new combination

Example

3 items x, y and z will have 6 different permutations but only one combination.

The following formula is usually used to determine the no. of combinations in a given situation.

$${}^n C_r = \frac{n!}{r!(n-r)!}$$

Example

i.
$${}^8 C_7 = \frac{8!}{7!(8-7)!}$$
$$= \frac{8!}{7!1!} = \frac{8 \times 7!}{1 \times 7!}$$
$$= 8$$

ii.
$${}^6 C_4 = \frac{6!}{4!(6-4)!}$$
$$= \frac{6!}{4!2!} = \frac{6 \times 5 \times 4!}{4! \times 2 \times 1}$$
$$= 15$$

iii.
$${}^8 C_3 = \frac{8!}{3!5!}$$
$$= \frac{8 \times 7 \times 6 \times 5!}{3 \times 2 \times 1 \times 5!}$$
$$= 56$$

Application

Example

There is a committee to be selected comprising of 5 people from a group of 5 men and 6 women. If the selection is randomly done. Find the possibility of having the following possibilities (combinations)

- i. Three men and two women
- ii. At least one man and at least one woman must be in the committee
- iii. One particular man and one particular woman must not be in the committee (one man four women)

Solution

- i. The committee size = 5 people
The group size = 5m + 6w

∴ assuming no restrictions the committee can be selected in ${}_{11}C_5$
 The committee has to consist of 3m & 2w

∴ these may be selected as follows.

$${}^5C_3 \times {}^6C_2$$

P(committee 3m and 2w)

$$= \frac{{}^5C_3 \times {}^6C_2}{{}_{11}C_5} \quad \text{note that this formula can be fed directly to}$$

your scientific calculator and attain a solution.

$$= \frac{\frac{5!}{3!2!} \times \frac{6!}{4!2!}}{\frac{11!}{5!6!}}$$

$$= \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} \times \frac{6 \times 5 \times 4!}{2 \times 1 \times 4!} \times \frac{5 \times 4 \times 3 \times 2 \times 1 \times 6!}{11 \times 10 \times 9 \times 8 \times 7 \times 6!}$$

$$= \frac{27}{77}$$

- ii. P(at least one man and at least one woman must be in the committee)

The no. of possible combinations of selecting the committee without any woman = 5C_5

The probability of having a committee of five men only

$$= \frac{{}^5C_5}{{}_{11}C_5} = \frac{1}{462}$$

The probability of having a committee of five women only

$$= \frac{{}^6C_5}{{}_{11}C_5} = \frac{\frac{6!}{5!1!}}{\frac{11!}{5!6!}}$$

$$= \frac{6 \times 5!}{5!1!} \times \frac{5!6!}{11 \times 10 \times 9 \times 8 \times 7 \times 6!}$$

$$= \frac{1}{77}$$

∴ P(at least one man and at least one woman)

$$= 1 - \{P(\text{no man}) + P(\text{no woman})\}$$

$$= 1 - \left\{ \frac{1}{77} + \frac{1}{462} \right\}$$

$$\begin{aligned}
&= 1 - \frac{(6+1)}{462} \\
&= 1 - \frac{7}{462} \\
&= \frac{455}{462}
\end{aligned}$$

- iii. P(one particular man and one particular woman must not be in the committee would be determined as follows

$$\begin{aligned}
\text{The group size} &= 5m + 6w \\
\text{Committee size} &= 5 \text{ people}
\end{aligned}$$

$$\begin{aligned}
\text{Actual groups size from which to} & \\
\text{Select the committee} &= 4m + 5w \\
\text{Committee} &= 1m + 4w
\end{aligned}$$

The committee may be selected in 9C_5
The one man may be selected in 4C_1 ways
The four women may be selected in 5C_4 ways

\therefore P (committee of 4w1man).

$$\begin{aligned}
&= \frac{{}^5C_4 \times {}^4C_1}{{}^9C_5} \\
&= \frac{\frac{5!}{4!1!} \times \frac{4!}{1!3!}}{\frac{9!}{4!5!}} \\
&= \frac{5 \times 4!}{1!4!} \times \frac{4 \times 3!}{1 \times 3!} \times \frac{4!5!}{9 \times 8 \times 7 \times 6 \times 5!} \\
&= \frac{10}{63}
\end{aligned}$$

SETS THEORY

Introduction

Sets and set theory

A set is a collection of distinct objects. We may consider all the ocean in the world to be a set with the objects being whales, sea plants, sharks, octopus etc, similarly all the fresh water lakes in Africa can form a set. Supposing A to be a set

$$A = \{4, 6, 8, 13\}$$

The objects in the set, that is, the integers 4, 6, 8 and 13 are referred to as the members or elements of the set. The elements of a set can be listed in any order. For example,

$$A = \{4, 6, 8, 13\} = \{8, 4, 13, 6\}$$

Sets are always precisely defined. Each element occurs once and only once in a set.

The notion \in is used to indicate membership of a set. \notin represents non membership. However, in order to represent the fact that one set is a subject of another set, we use the notion \subset . A set "S" is a subject of another set "T" if every element in "S" is a member of "T"

Example

If $A = \{4, 6, 8, 13\}$ then

- i) $4 \in \{4, 6, 8, 13\}$ or $4 \in A$; $16 \notin A$
- ii) $\{4, 8\} \subset A$; $\{5, 7\} \not\subset A$; $A \subset A$

Methods of set representation

Capital letters are normally used to represent sets. However, there are two different methods for representing members of a set:

- i. The descriptive method and
- ii. The enumerative method

The descriptive method involves the description of members of the set in such a way that one can determine the elements of the set without difficulty.

The enumerative method requires that one writes out all the members of the set within the curly brackets.

For example, the set of numbers 0, 1, 2, 3, 4, 5, 6 and 7 can be represented as follows

$$P = \{0, 1, 2, 3, 4, 5, 6, 7\}, \quad \text{enumerative method}$$

$$P = \{x/x = 0, 1, 2, \dots, 7\} \quad \text{descriptive method}$$

Or

$$P = \{x/0 \leq x \leq 7\} \text{ where } x \text{ is an integer.}$$

Types of sets

- a) Finite and infinite sets

A set can be classified as a finite or infinite set, depending upon the number of elements it has. A finite set has a finite number of elements whereas an infinite set has an infinite number of elements.

For example, set P below has ten elements and is therefore a finite set. Set S, on the other hand, is an infinite set since it has an infinite number of elements.

$$P = \{2, 4, 6 \dots 20\}$$

$$S = \{1, 3, 5 \dots\}$$

b) Universal set

The term refers to the set that contains all the elements that an analyst wishes to study.

The notation U or ξ is generally used to denote universal sets

c) The null set or empty set

This is a set which contains no elements. It is normally designated by a Greek letter \emptyset , or $\{\}$.

The sets \emptyset and $\{\emptyset\}$ are not the same thing since the former has no elements in it, while the latter has one element in it, namely zero

d) Equal or equivalent sets

Two sets C and D are said to be equal if every member of set C belongs to D and every member of set D also belongs to C

e) Complement of a set

The complement of set A is written as A' . This set contains all those elements of universal set which are not in A

f) Intersection and union

$B \cap C$ Denotes the intersection of B and C. it is the set containing all those elements, which belong to both B and C

$$\text{If } B = \{5, 8, 11, 20, 25\} \text{ and } C = \{1, 3, 5, 7, 9, 11, 13\}$$

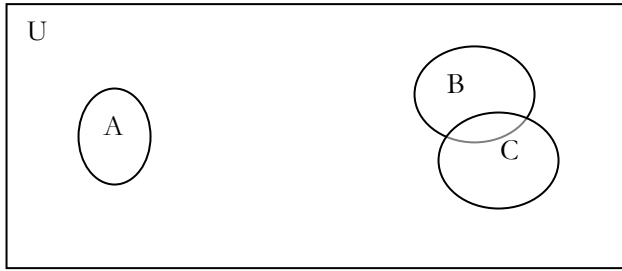
$$\text{Then } B \cap C = \{5, 11\}$$

$$B \cup C = \{1, 3, 5, 7, 8, 9, 11, 13, 20, 25\}$$

Set Operations and Laws

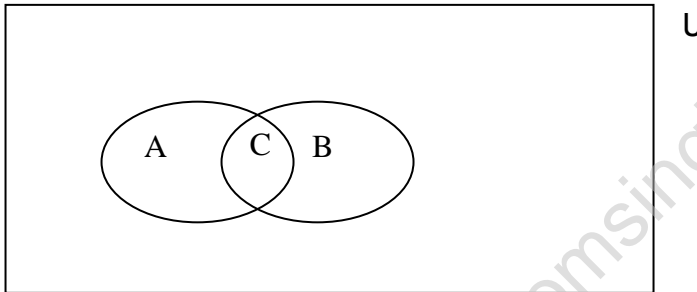
A simple way of representing sets and relations between sets is by means of the Venn diagram. Venn diagram consists of a rectangle that represents the universal set. Subjects of the universal set are represented by circles drawn within the rectangle, or the universe.

Suppose that the universal set is designated by U and the sets A, B and C are subject of U. The Venn diagram below can be used to illustrate the sets as follows



Venn diagram below representing the intersection of set A and B or $A \cap B = C$ is illustrated as follows

Intersection of sets



Example:

You are given the universal set

$$T = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

And the following subjects of the universal set:

$$A = \{3, 4, 5, 6\}$$

$$B = \{1, 3, 4, 7, 8\}$$

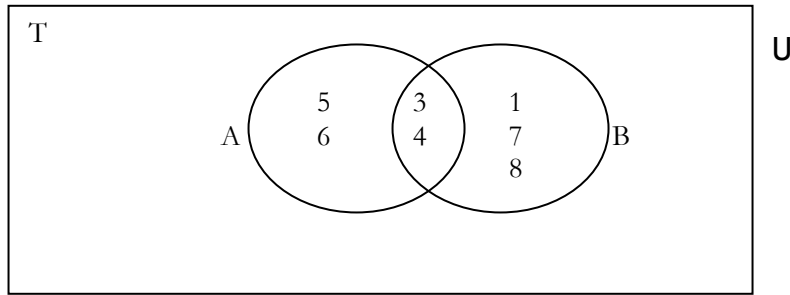
Determine the intersection of A and B

Solution

The intersection of A and B is the subject of T, containing elements that belong to both A and B

$$\begin{aligned} A \cap B &= \{3, 4, 5, 6\} \cap \{1, 3, 4, 7, \text{ and } 8\} \\ &= \{3, 4\} \end{aligned}$$

Or



Example

Consider the following universal set T and its subjects C , D and E

$$T = \{0, 2, 4, 6, 8, 10, 12\}$$

$$C = \{4, 8\}$$

$$D = \{10, 2, 0\}$$

$$E = \{0\}$$

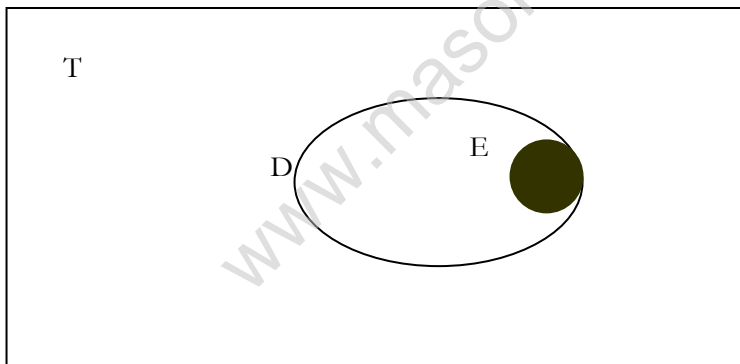
Find

i) $D \cap E$

ii) $C \cap D \cap E$

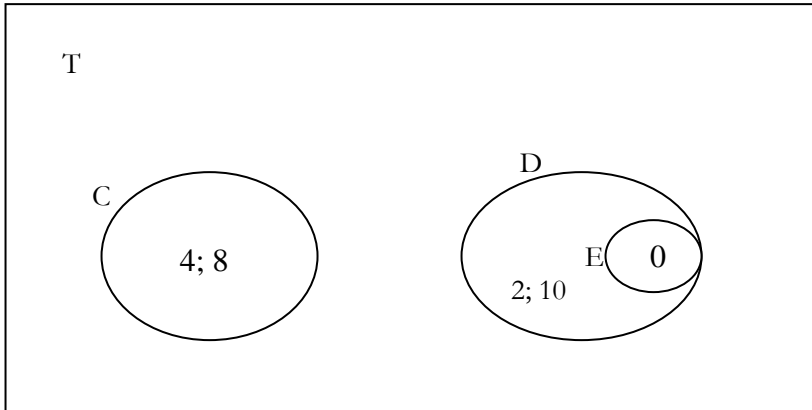
Solution

ii) $D \cap E = \{10, 2, 0\} \cap \{0\} = \{0\}$



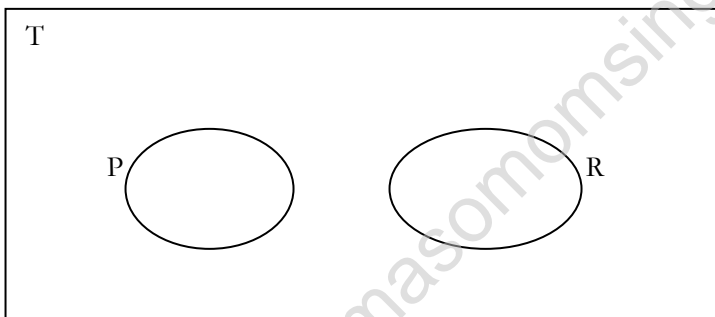
$$D \cap E = \text{Shaded area}$$

ii) $C \cap D \cap E = \{4, 8\} \cap \{10, 2, 0\} \cap \{0\} = \{\} = \emptyset$



Mutually exclusive or disjoint sets

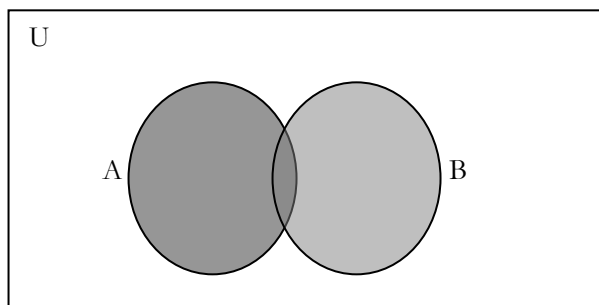
Two sets are said to be disjoint or mutually exclusive if they have no elements in common. Sets P and R below are disjoint



Disjoint sets are represented by a null set in this case $P \cap R = \emptyset$

The union of sets

Venn diagram representing the union of sets A and B or $A \cup B =$ Shaded area is illustrated below;-



Example

Consider the universal set T and its subsets A, B and C below:

$$T = \{a, b, c, d, e, f\}$$

$$A = \{a, d\}$$

$$B = \{b, c, f\}$$

$$C = \{a, c, e, f\}$$

Find

i) $A \cup B$

ii) $A \cup C$

iii) $B \cup C$

iv) $A \cup B \cup C$

Solution

i) $A \cup B = \{a, d\} \cup \{b, c, f\} = \{a, b, c, d, f\}$

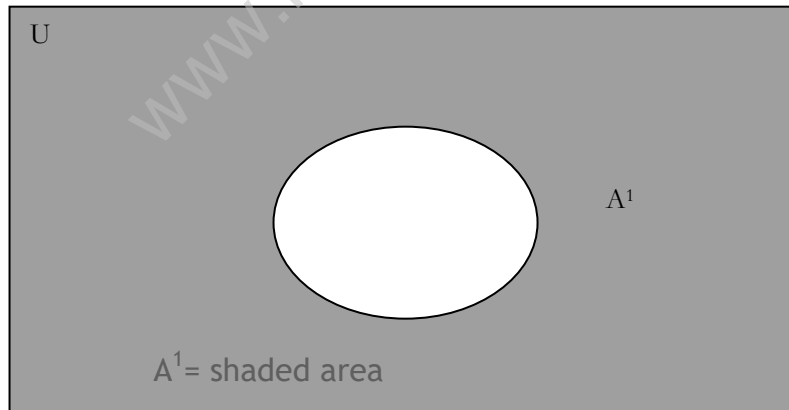
ii) $A \cup C = \{a, d\} \cup \{a, c, e, f\} = \{a, c, d, e, f\}$

iii) $B \cup C = \{b, c, f\} \cup \{a, c, e, f\} = \{a, b, c, e, f\}$

iv) $A \cup B \cup C = \{a, d\} \cup \{b, c, f\} \cup \{a, c, e, f\} = \{a, b, c, d, e, f\} = T$

Complement of a set

Venn diagram representing the complement of a set say A represented by A^1 is illustrated below.



Example

For the universal set $T = \{1, 2, 3, 4, 5\}$ and its subset $A = \{2, 3\}$ and $B = \{5, \}$

Find

ii) A^1

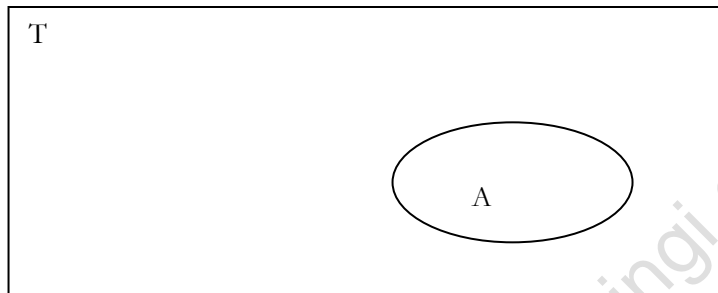
- iii) $(A^1)^1$
- iv) $(B^1)^1$

Solution

- i) $A^1 = \{2, 3\}^1 = \{1, 4, 5\}$
- ii) $(A^1)^1 = (\{2, 3\}^1)^1 = \{1, 4, 5\}^1 = \{2, 3\} = A$
- iii) $(B^1)^1 = (\{5\}^1)^1 = \{1, 2, 3, 4\}^1 = \{5\} = B$

Laws of Set Algebra

From the following Venn diagram where T is the universal set and A its subset, we can deduce a number of laws.



- i) $A \cup \emptyset = A$
- ii) $A \cup T = T$
- iii) $A \cup A = A$
- iv) $A \cap A = A$
- v) $A \cap T = A$
- vi) $A \cup A^1 = T$
- vii) $A \cap A^1 = \emptyset$
- viii) $(A^1)^1 = A$

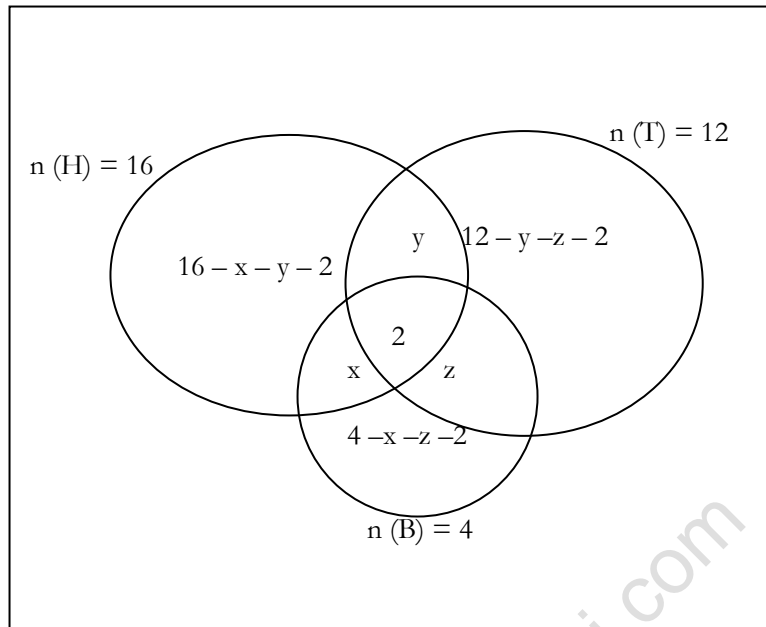
Applications

Example 1

Of the 20 girls in a form, 16 play hockey 12 play tennis and 4 play basketball. Every girl plays at least one game and two plays all the three. How many play two and only two games.

Solution

$$N(\xi) = 20$$



H = Hockey

T = Tennis

B = Basketball

Those who play two and only two games = $x + y + z$

Using the diagram

$$(14 - x - y) + (10 - y - z) + (2 - x - z) + 2 + x + y + z = 20$$

$$28 - x - y - z = 20$$

$$x + y + z = 8$$

Example 2

250 members of a certain society have voted to elect a new chairman. Each member may vote for either one or two candidates. The candidate elected is the one who polls most votes

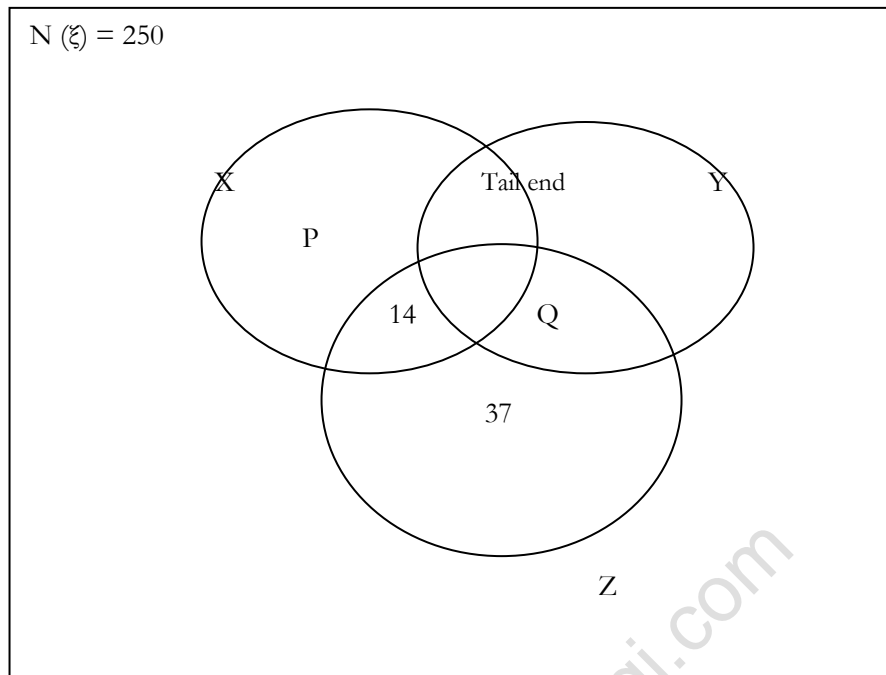
Three candidates x, y, z stood for election and when the votes were counted, it was found that

- 59 voted for y only, 37 voted for z only
- 12 voted for x and y, 14 voted for x and z
- 147 voted for either x or y or both x and y but not for z
- 102 voted for y or z or both but not for x

Required

- i) How many voters did not vote
- ii) How many voters voted for x only
- iii) Who won the elections

Solution



$$P + 12 + 59 = 147 \text{ giving } P = 76$$

$$Q + 59 + 37 = 102 \text{ giving } Q = 6$$

- i) Those who did not vote
 $= 250 - (76 + 12 + 14 + 59 + 6 + 37)$
 $= 250 - 204 = 46$
- ii) $x = 76 + 12 + 14 = 102$
 $y = 12 + 59 + 6 = 77$
 $z = 37 + 14 + 6 = 57$
- iii) X won the election

BINOMIAL THEOREM

PROGRESSION

CHAPTER TWO

DATA COLLECTION AND PRESENTATION

Specific objectives

At the end of this topic the trainee should be able to:

- ❖ Discuss the basic consideration for data collection.
- ❖ Classify collected data into various categories.
- ❖ Tabulate collected data.
- ❖ Diagrammatically and graphically present data.

Introduction

a) Statistics

Statistics viewed as a subject is a process of collecting, tabulating and analyzing numerical data upon which significant conclusions are drawn. Statistics may also be defined as numerical data, which has been, collected from a given source and for a particular purpose e.g. population statistics from the ministry of planning, Agricultural statistics from the ministry of Agriculture

Statistics may also refer to the values, which have been obtained from statistical calculations e.g. the mean, mode, range e.t.c.

b) Application of statistics

1. Quality Control

Usually there is a quality control departments in every industry which is charged with the responsibility of ensuring that the products made do meet the customers standards e.g. the Kenya bureau of standards (KeBS) is one of the national institutions which on behalf of the government inspects the various products to ensure that they do meet the customers specification. The KeBS together with other control department have developed quality control charts. They use these charts to check whether the products are up to standards or not.

2. Statistics may be used in making or ordering economic order quantities (EOQ).

It is important for a business manager to realize that it is an economic cost if one orders a large quantity of items which have to be stored for too long before they are sold. This is because the large stock holds a lot of capital which could otherwise be used in buying other items for sale.

It is also important to realize that the longer the items are stored in the stores the more will be the storage costs

On the other hand if one orders a few items for sale he will incur relatively low storage expenses but may not be able to satisfy all the clients. These

may lose their customers if the goods are out of stock. Therefore it is advisable to work out the EOQ which will be sufficient for the clients in a certain period before delivery.

The EOQ will also ensure that minimal costs are incurred in terms of storage

3. Forecasting

Statistics is very important for business managers when predicting the future of a business for example if a given business situation involves a dependent and independent variables one can develop an equation which can be used to predict the output under certain given conditions.

4. Human resource management

Statistics may be used in efficient use of human resources for example we may give questionnaires to workers to find out where the management is weak

By compiling the statistics of those who were signing it may be found useful to analyze such data to establish the causes of resignation thus whether it is due to frustration or by choice.

BASICS FOR DATA COLLECTION

Introduction

A statistical investigation involves a number of stages:

- ❖ Definition of the problem or issue;
- ❖ Collection of relevant data;
- ❖ Classification and analysis of the collected data;
- ❖ Presentation of the results.

Even before the collection of data starts, then, there are some important points to consider when planning a statistical investigation.

Preliminary considerations.

It is important to be aware of these issues as they impact on the data which is to be collected and analyzed.

- ❖ Exact definition of the problem

This is necessary in order to ensure that nothing important is omitted from the enquiry, and effort is not wasted by collecting irrelevant data. The problem as originally put to the statistician is often of a very general type and it needs to be specified precisely before work can begin.

- ❖ Definition of the units

The results must appear in comparable units for any analysis to be valid. If the analysis is going to involve comparisons, then the data must all be in the same units. It is no use just asking for 'output' from several factories - some may give their answers in numbers of

items, some in weight of items, some in number of inspected batches and so on.

❖ Scope of the enquiry.

No investigation should be got under way without defining the field to be covered. Are we interested in all departments of our business, or only some? Are we to concern ourselves with our own business only, or with others of the same kind?

❖ Accuracy of the data

To what degree of accuracy is data to be recorded? For example, are ages of individuals to be given to the nearest year or to the nearest month or as the number of completed years? If some of the data is to come from measurements, then the accuracy of the measuring instrument will determine the accuracy of the results. The degree of precision required in an estimate might affect the amount of data we need to collect. In general, the more precisely we wish to estimate a value, the more readings we need to take.

Data sources and types

Data constitute the foundation of statistical analysis and interpretation.

Data can be obtained from three sources namely;

- Primary source
- Secondary source
- Internal records

Primary data

Primary data are measurement, observed and recorded as part of an original study. There are two basic methods of obtaining primary data namely;

- Questionnaires
- Observation
- Interviews
- Sampling

Primary data is data which is both original and has been obtained in order to solve the specific problem in hand. Primary data is, therefore, raw data and has to be classified and processed using appropriate statistical methods in order to reach a solution to the problem.

Secondary sources

Secondary data can be obtained from journals, reports, and government publications, publications of research organization, trade and professional bodies.

Secondary data must be used with utmost care, before using secondary data the investigator should examine the following;

1. Whether the data are suitable for the purpose of investigation.
2. whether the data are adequate for the purpose of investigation

3. Whether the data are reliable.

Secondary data is any data other than primary data. Thus, it includes any data which has been subject to the processes of classification or tabulation or which has resulted from the application of statistical methods to primary data, and all published statistics.

Internal data

Internal data refers to the measurement that are the by products of routine businesses records keeping like accounting , finance, production personnel, quality control , sales , R&D.

Since internal data originate within the business collecting the desired information does not usually offer much difficulty. The particular procedure depends largely upon the nature of the facts being collected and the form in which they exist.

Data collection methods and limitations

The methods usually available are as follows:

- ❖ Questionnaires
- ❖ Observation
- ❖ Interview
- ❖ Use of published statistics

a. Questionnaire

As the name suggest is distinguished by the fact that data are collected by asking questions from people who are thought to have the desired information.

A formal list of such questions is called a questionnaire

Questionnaire refers to a device for securing answers to questions by using a form which the respondent fills in it.

b. Observation

The investigator observes the object or action in which he is interested.

Sometime individual makes the observation on other occasions mechanical device observe and record the desired information.

Observation method does not automatically produce accurate data. Physical difficulties in the observation situation on the part of the observer may result in errors.

Classification of Data

Classification is grouping of related facts into different classes. Facts in one class differ from another class with respect to some characteristics called a basis of classification.

Sorting facts on one basis of classification and then another basis is called cross-classification

Rules of Classification

-
1. the number of classes should preferably be between 5 and 15
 2. as far as possible one should avoid odd values of class intervals
 3. the starting point i.e. the lower limit of the first class should either be zero or 5 or multiple of 5
 4. to ensure continuity and to get correct interval we should adopt exclusive method of classification
 5. Whenever possible all classes should be of the same size.

Types of classification

- ❖ Geographical
- ❖ Chronological
- ❖ Qualitative
- ❖ Quantitative

Geographical classification; in geographical classification data are classified on the basis of geographical or location difference between the various items.

Chronological classification; when data are observed over a period of time, the type of classification is known as chronological classification.

Qualitative classification; in qualitative classification data re classified on the basis of some attributes or quality such as sex, color of hair, literacy, religion.

QUALITATIVE ANALYSIS DATA COLLECTION LIMITATION OF STATISTICS.

Despite the usefulness of statistics in many fields, impression should not be carried that statistics are like magical devices which always provide the correct solution of problems . unless the data are properly collected and critically interpreted there is every likelihood of drawing wrong conclusion. Therefore it is necessary to know the limitation and the possible misuse of statistics.

The following are the important limitation of the science of statistics:

- Statistics does not deal with isolated measurement.
- Statistics deals only with quantitative characteristic i.e quantitative characteristic such as honesty , efficiency , intelligence, blindness and deafness cannot be studied directly
- Statistics result are true only an average.
- Statistic is only a means not a end.
- Statistics can be misused.

Classification functions.

Classification of data is a function very similar to text of sorting letters in a post office. It is well known that the letter collected in a post office are sorted into different geographical basis i.e in accordance with their

destination as Nairobi , Mombasa , kampala etc .they are then put in separate bags each containing letters with a common characteristic , or having the same destination.

Classification of statistical data is comparable to the sorting operation, the process of classification gives prominence to important information gathered while dropping unnecessary details facilitates comparison and enables a statistical treatment of the material collected.

Formation of frequency distribution.

Here we just count the number of times a particular value is repeated which is called the frequency. In order to facilitate counting prepare a column of "tally" in another column all possible values of variables from the lowest to the highest. Then put a bar(vertical line) opposite the particular value to which it relates.

To facilitate counting blocks of five bars are prepared and some space is left in between each block.

We finally count the number of blocks and bars corresponding to each value to each value of the variable and place it in the column and frequency.

Example .

Construct a frequency distribution from the following data.

23, 30,20,26,30,30,20,23,40,40,26,20,23,40,28,26,30,40,28,28,30.

class	tally	frequency
20	111	3
23	1111	4
26	111	3
28	111	3
30	1111	5
40	1111	4
	total	22

Classification according to class interval.

The following technical terms are important when data are classified according to class intervals:

- (a) Class limit. these are the lowest and the highest values that can be included in the class e.g take the class 20-40. the lowest value of this class is 20 and the highest is 40.

The two boundaries of a class is the value below which there can be no value in that class, while the upper limit of a class is the value above which no value can belong to that class.

The class 70- 89 .70 is the lower limit and 80 is the upper limit i.e in this class there can be no value which is less than 70 or more than 89.

- (b) Class interval .it is the span of a class i.e the difference between the upper limit and the lower limit is known as class interval , for example in the class 20-40 , the class interval is 20 (i.e 40 - 20) the size of the class interval is determined by the number of classes and the total range in the data.
- (c) Class frequency . this is the number of observation corresponding to the particular class.it is also known as the frequency of that class or the class frequency .if we add together the frequency of all individual classes , we obtain the total frequency.
- (d) Class mid - point.it is the value lying half way between the lower and the upper class limit of a class interval.

Mid point of a class is ascertained as follows:

Mid point of a class = $\frac{\text{upper limit of the class} + \text{lower limit of the class}}{2}$

2

There are two methods of classifying the data according to class interval namely

- Exclusive method.
- Inclusive method.

(a) Exclusive method.

When the class interval are so fixed that the upper limit of one class is the lower limit of the next class, it is known as the exclusive method of classification .this can be illustrated as follow.

Class interval	No of items
10-20	5
20-30	3
30-40	4
40-50	6
50-60	2
60-70	1

It is clear that the exclusive method ensures continuity of data in as much as the upper limit of one class is the lower limit of the next class.

Whenever this method is used it is always assumed that the upper limit is exclusive i.e the observation exactly equals to the upper limit is not included in that class.

(b) Inclusive method.

Under this method the upper limit of one class is included in that class itself. this can be illustrated as below:

Class interval	frequency
1-10	2
11-20	5
21-30	4
31-40	10
41-50	15
51-60	30
61-70	12
71-80	3
81-90	2

Whenever inclusive method is used for equal class interval is obtained by taking the difference between the two upper limit or lower limit.

DATA TABULATION.

- a table is a systematic arrangements of statistical data in column and rows.
- Rows are horizontal arrangements whereas columns are vertical ones.
- The purpose of a table is to simplify the presentation and to facilitate comparisons.
- The simplification result from the clear cut and systematic arrangements which enables the reader to quickly locate desired information.

Parts of a table.

- The various parts of a table may vary from case to case depending upon the given data, but a good table must contain atleast te following parts:
 - (a) Table number
 - (b) Title of the table
 - (c) Caption
 - (d) Stub
 - (e) Body of the table
 - (f) Headnote
 - (g) Footnote

-
- (a) table number. each table should be numbered.
 - (b) Title of the table. every table must have a suitable title.
 - (c) Caption. it refers to the column heading .it represent what the column represent.
 - (d) Stub .are the designation of the rows or row heading .they are at the extreme left and perform the same function for the horizontal rows or numbers I the table as the Column heading do for the horizontal rows or numbers in the table as the column heading do for the vertical column or numbers.
 - (e) Body of the table. It contains the numerical information .data presented in the body arranged according to description are classification of the captions and stubs.
 - (f) Head note. It is a brief explanatory statement applying to all or a major part of the material in the table and is placed below the tittle entered and closed in brackets.
 - (g) Footnote .they are placed directly below the body of the table.

DIAGRAMATIC AND GRAPHICAL PRESENTATION

Diagrams.

Rules for construction of diagrams.

- There must be the title of the diagram .
- Proper proportion between width and height should be maintained.
- The scale selected should be appropriate
- If necessary footnotes should be given at the bottom of the diagram.
- Diagram should be absolutely neat and clean
- An index illustration different types of lines or different shades , column , should be give so that the reader can easily make out the meaning of the diagram.
- Diagram should be as simple as possible.

Types of diagrams.

- One dimensional diagram e.g bar diagram.
- Two dimensional diagram e.g squares .
- Picto grams and cartograms.

One dimensional diagrams.

- Bar diagrams are the most common type of diagram used in practice , a bar is a thick line whose width is shown merely for attention.
- They are called one dimensional because it is only the lengh of the bar that matters and not the width.

Merits of bar diagrams.

- They are readily understood even by those unaccustomed to reading charts or those who are not chart minded.
- They possess the outstanding advantage that they are the simplest and the easiest to make.
- When a large number of observations are to be compared they are the only form that can be used effectively..

Points to mind when constructing bar diagrams.

- The width of the bars should be uniform throughout the diagram.
- The gap between one bar and another should be uniform throughout.
- Bars may either be horizontal or vertical. The vertical bars should be preferred because they give a better look and also facilitate comparison.
- While constructing the bar diagram it is desirable to write the respective figure at the end of each bar so that the reader can know the precise value without looking at the scale.

Types of bar diagrams.

- (a) Simple bar charts.
- (b) Sub divided or component bar charts.
- (c) Multiple bar charts
- (d) Percentage component bar charts.

(a) Simple bar charts.

It is used to represent only one variable i.e. shows only totals. However an important limitation of such diagrams is that they can prefer only one classification or one category of data e.g. for example while presenting the population for the last five decades, one can only depict the total population in the simple bar diagram and not its sex-wise distribution.

(b) Component bar charts.

These diagrams are used to represent various parts of the total, while constructing such charts, the various components in each bar should be kept in the same order.

A common and helpful arrangement is that of presenting each bar in the order of magnitude from the largest component at the base of the bar to the smallest at the end.

To distinguish between the different components, it is useful to use different shades or colours.

They usually show component totals and the totals.

(c) multiple bar charts
in multiple bar chart two or more set of interrelated data are represented .the technigue of drawing such a diagram is the same as that of simple bar charts.

The only difference is that since more than one phenomenon is represented ,different shades , colours .,dots or crossing are used to distinguish between the bars. they normally show the component totals only.

(d) Percentage component bar charts.
They are particularly useful in statistical work which require the portrayal of relative changes in data.
When such diagrams are prepared , the length of the bars is kept equal to 100 and segment are cut in these bars to represent the component of an aggregate.

Illustration.

Using the data below , construct the following charts.

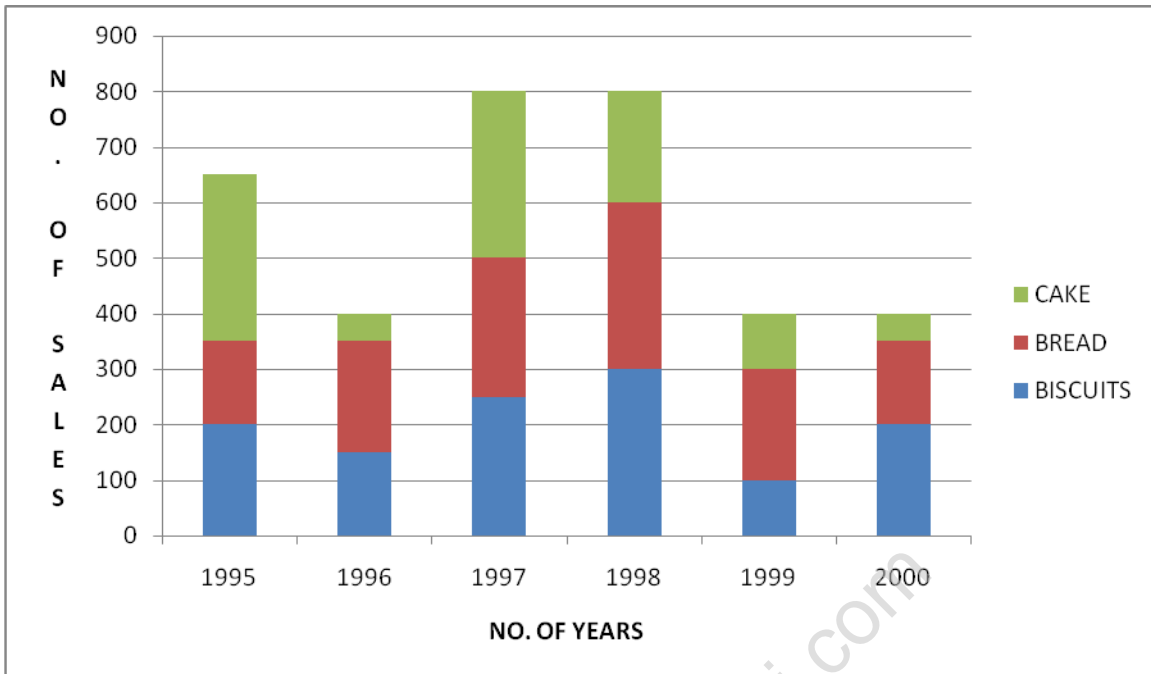
- (a) Simple bar chart.
- (b) Component bar chart.
- (c) Multiple bar charts.
- (d) Percentage component bar chart.

Shows the sales of Xyz Ltd.

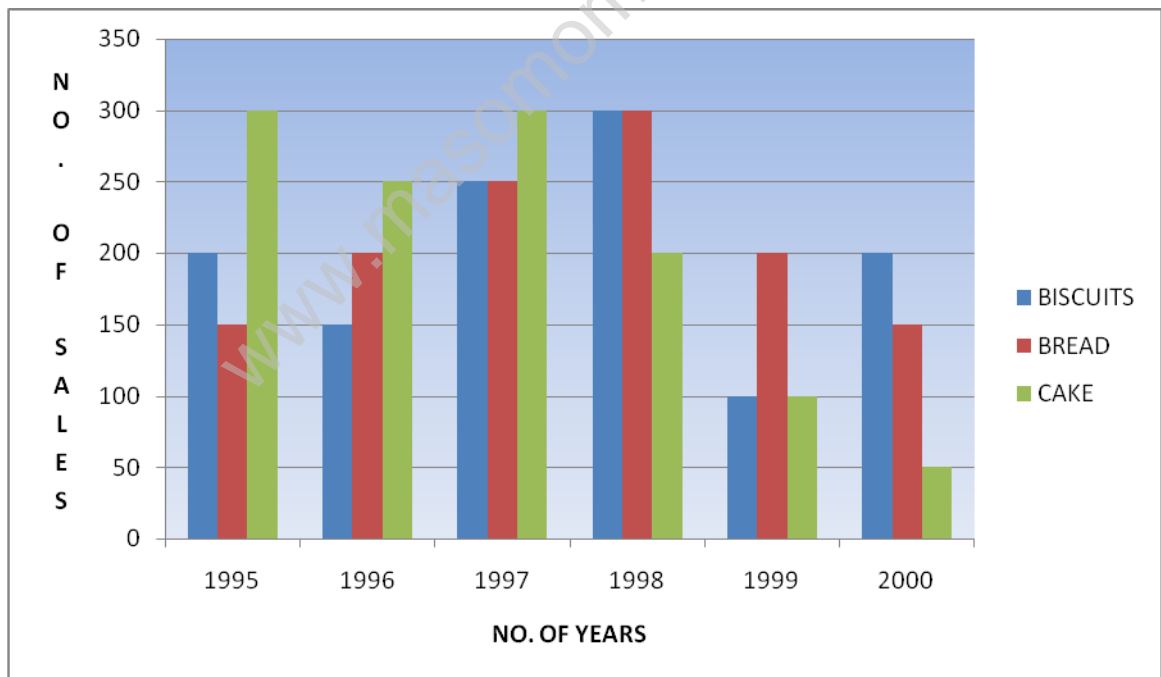
year	biscuits	bread	cake	total
1995	200	150	300	650
1996	150	200	250	600
1997	250	250	300	700
1998	300	300	200	750
1999	100	200	100	400
2000	200	150	50	400

Solution

- (a) Component chart



b) simple chart



TWO DIMENSIONAL DIAGRAMS.

As distinguished from one dimensional diagram in which only the length of the bars is taken into account in two dimension diagram, the length as well as the width of the bars is considered.

Thus the area of the bar represents the given data. Two dimensional diagrams are also known as surface diagrams example of this are-

- Rectangles
- Squares
- Circles

Pie chart or diagram

This type of diagram enables us to show the partitioning of a total into component parts.

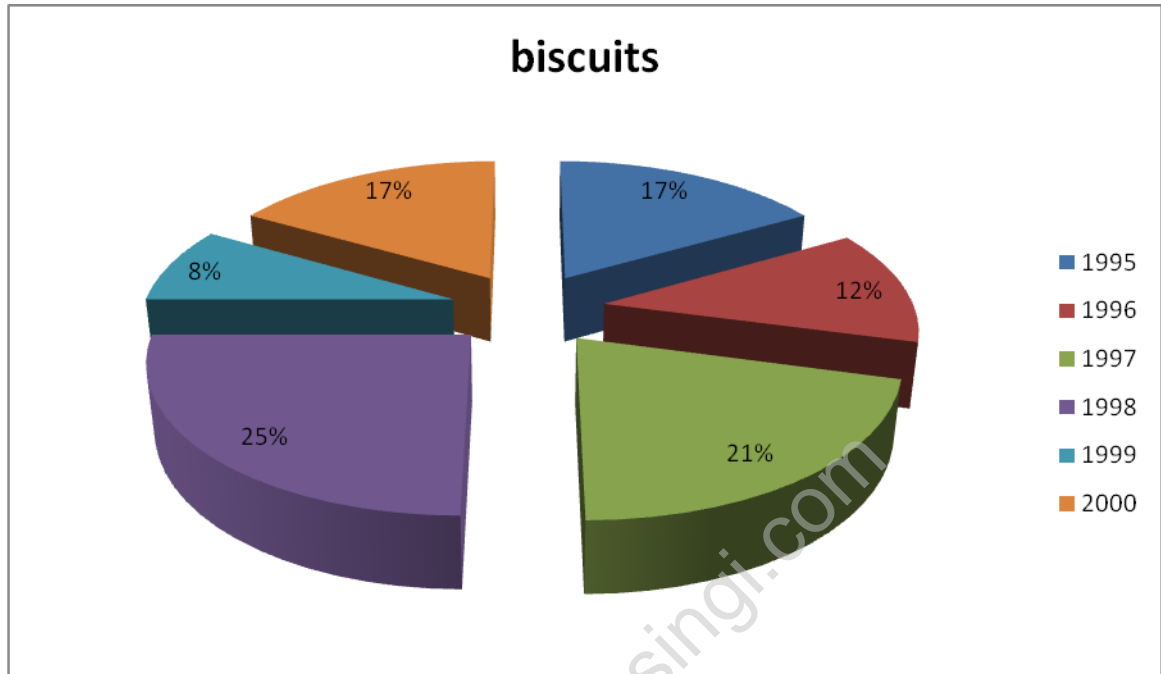
In constructing a pie chart , the steps involved are:

- Prepare the data so that the various components values are in percentage by applying

$$\% \text{ component} = \frac{x}{\text{Total}} \times 360$$

- Draw a circle of appropriate size with a compass.
- Measure points on the circle representing the size of each component with the help of a protractor.

Pie chart of the above.



Quantitative classification; it refers to the classification of data according to some characteristics that can be measured such as height, weight, income, sales.

Tabulation of Data

One of the simplest and most revealing devices of summarizing data and presenting them in meaningful fashion is the statistical tables.

Types of tabulation

Tabulation may be classified as:

1. simple tabulation
2. complex or matrix tabulation

Simple tabulation only one characteristic is shown hence this type of table is also known as one -path table. This can be illustrated as follows:

Table1: Distribution of workers by workshop

Workshop	Number Of Employees
A	600
B	360
C	660
D	840
E	540
Total	3,000

N/B: simple tabulation does not tell us very much - although it may be enough for the question of the moment.

Complex tabulation has two or more characteristics are shown i.e. two or more aspects of a problem are dealt with at the same time. It is some times called two path tables. Such tables show two characteristics and are formed when either the stub or the caption is divided into two coordinate parts.

Age (in years)	Employees		total
	Males	females	
Bellow 25	32	18	50
25-35	40	27	67
35-45	25	18	43
45-55	10	5	15
55 and above	5	-	5
Total	112	68	180

Presentation of data

After the data has been collected the next step is to present them in a suitable form.

Charting data

One of the most convincing and appealing ways in which data may be presented is through charts

A chart can take the shape of either a diagram or a graph.

Types of diagrams

- ❖ One dimensional diagram e.g. bar graphs
- ❖ Two dimensional diagram e.g. rectangles squares
- ❖ Pictograms and cartograms (circles)

One dimensional diagram

Bar diagrams is the most common type of diagram used in practice.

Merits of bar diagrams

- ❖ They're readily understood
- ❖ They are simplest and easiest to make
- ❖ They are effective if the large number of observation is to be compared.

Example

Two dimensional diagrams

As distinguished from one dimensional diagrams in which only the length of the bar is taken into account in two dimensional diagram the length as well as the width of the bars is considered. Thus the area of the bar represents the given data.

Pictograms

Also known as picture grams are very popularly used in presenting situation data. They are no abstract presentation such as lines or bars

When pictograms are used data are represented through a pictorial symbol that is carefully selected.

Illustration

The following table gives the production of tea in India by a leading company.

Year	Production (million kgs)
2004	421
2005	561
2006	587
2007	645
2008	660

Solution for representing the above data by a pictogram we will use the symbol of a star

Year	Production of tea
2004	◇ ◇ ◇ ◇
2005	◇ ◇ ◇ ◇ ◇ ◇
2006	◇ ◇ ◇ ◇ ◇ ◇
2007	◇ ◇ ◇ ◇ ◇ ◇ ◇
2008	◇ ◇ ◇ ◇ ◇ ◇ ◇

Merits of diagrams

- ❖ Pictograms have a greater attraction thus stimulate interest in the information being presented
- ❖ Facts portrayed in pictorial form are generally remembered longer.

Limitations

- ❖ They are difficult to construct

Graphs

A large variety of graphs are used in practice. Graphs can be divided under the following headings

- ❖ Time series.
- ❖ Z- Charts.
- ❖ Scatter graphs
- ❖ Semi- logarithmic graphs
- ❖ Graphs of frequency distribution

Graphs of time series

When we observe the values of a variable or different point of time the series forms is known as time series. The technique of graphic presentation is extremely helpful in analyzing change at different point of time. Graphs of time series can be constructed either on a natural scale or on a ratio scale.

Graphs of one variable

When one variable is to be represented on x-axis measure time and on y-axis in that various points are plotted and joined by one straight line, the fluctuation of this line show the variation in the variable

Illustration

Represent the following data by suitable graph

Year	Imports(in million tones)
2003-04	1.5
2004-05	2.5
2005-06	2.0
2006-07	2.7

Solution

Draw a graph

Graphs of frequency distribution

A frequency distribution can be presented graphically in any of the following ways;

Histogram

Frequency polygon

Smoothed frequency curve

Cumulative frequency curve

Histogram

Histograms are a set of vertical bars whose areas are proportional to the frequencies represented. Histogram is most widely used for graphical presentation of a frequency distribution.

Illustration

Represent the following data by a histogram

Marks	number of students
0-10	8
10-20	12
20-30	22
30-40	35
40-50	40

Solution

Draw a histogram

Frequency polygon

A frequency polygon is a graph of frequency distribution it has more than four sides

It is particularly effective in comparing two or more frequency distribution.

Illustration

The daily profits (in thousand rupees) of 100 shops are distributed as follows

Daily profit	No of shops
0-50	12
50-100	18
100-150	21
150-200	20
200-250	17

Solution

The frequency polygon of the above data are shown below

Draw graph

Smoothed frequency curve

The following points should be kept in mind while smoothing a frequency graph.

Only frequency distribution based on sample should be smoothed

Only continuous series should be smoothed

The total area under the curve should be equal to the area under the original histogram or polygon.

Illustration

Draw a histogram frequency polygon and frequency curve representing the following information

Length of service (in years)

Number of employees

5-10	5
10-15	12
15-20	25
20-25	48
25-30	32
30-35	6
	1

Histogram frequency polygon and curve

Draw

Cumulative frequency curve / ogives

Sometimes one needs to know the answers to questions like how many workers of a factory earn more than Rs 1500 per month

There are two methods of constructing a cumulative frequency curve namely;

The less than method

The more than method

Less than method; in the less than method we start with the upper limit of the classes and go on adding the frequencies. When these frequencies are plotted we get a rising curve.

More than method; in the more than method, we start with the upper limits of the classes and from the total frequencies we subtract the frequency of each class. When the frequencies are plotted we get a declining curve.

given by less than method

Yearly profit(Rs)	Frequency
Less than 850	21
Less than 900	50
Less than 950	69
Less than 1000	108
Less than 1050	151
Less than 1100	245
Less than 1150	318
Less than 1200	422
Less than 1250	467
Less than 1300	494
Less than 1350	542
Less than 1400	563
Less than 1450	575
Less than 1500	580

Ogive by more than method

Yearly Rs

Frequency

More than 800	580
More than 850	559
More than 900	530
More than 950	511
More than 1000	472
More than 1050	429
More than 1100	335
More than 1150	262
More than 1200	194
More than 1250	158
More than 1300	113
More than 1350	86
More than 1400	38
More than 1450	17
More than 1500	5

Less than and more than ogive and median

Draw

Limitation of charts

- They can present only approximate values
- They can appropriately represent only limited amount of information
- They are intended mostly to explain quantitative facts to the general public.
- They can be easily misinterpreted

CHAPTER THREE

MEASURES OF CENTRAL TENDENCY

Specific Objectives

At the end of this topic the trainee should be able to:

- Define the measures of central tendency;
- State the properties of the measure of central tendency;
- Determine the measures of central tendency.

Introduction

Measures of central tendency show the tendency of some central value around which data tends to cluster. These are statistical values which tend to occur at the centre of any well ordered set of data. Whenever these measures occur they do not indicate the centre of that data.

Objectives of averaging

- To get one single value that describes the characteristics of the entire data.
- To facilitate comparison.

Properties of Good Measure

- It should be easy to understand i.e. since statistical methods are designed to simplify complexity, it is described that an average be such that can be readily understood, its use is bound to be very limited.
- Should be simple to compute.
- Should be based on all observation.
- Should be rigidly defined, i.e. an average should be properly defined so that it has one and only one interpretation.
- Should be capable of further algebraic treatment.
- Should have sampling stability.
- Should not be unduly affected by the presence of extreme values.

N/B. The following statistical terms are commonly used in statistical calculations. They must therefore be clearly understood.

i) Class limits

These are numerical values which limits the extend of a given class i.e. all the observations in a given class are expected to fall within the interval which is bounded by the class limits e.g. 15 & 19 are class limits as in the table of the example above.

ii) Class boundaries

These are statistical boundaries, which separate one class from the other. They are usually determined by adding the lower class limit to the next upper class limit and dividing by 2 e.g. in the above table the class

boundary between 19 and 20 is 19.5 which is = $\frac{19+20}{2}$.

iii) Class mid points

These are very important values which mark the center of a given class. They are obtained by adding together the two limits of a given class and dividing the result by 2.

iv) Class interval/width

This is the difference between an upper class boundary and lower class boundary. The value usually measures the length of a given class.

The following are important measures of central tendency

- Arithmetic mean
- Median
- Mode
- Geometric mean
- Harmonic

Arithmetic Mean

The most popular and widely used measure for representing the entire data by one value is an average. Its value is contained by adding together all the observation and dividing this total by the number of observation

These is commonly known as average or mean_it is obtained by first of all summing up the values given and by dividing the total value by the total no. of observations.

$$\text{l.e. mean} = \frac{\sum X}{n}$$

Where x = no. of values

Σ = summation

n = no of observations

Example

The mean of 60, 80, 90, 120

$$\frac{60 + 80 + 90 + 120}{4}$$

$$= \frac{350}{4}$$

$$= 87.5$$

The arithmetic mean is very useful because it represents the values of most observations in the population.

The mean therefore describes the population quite well in terms of the magnitudes attained by most of the members of the population

Computation of the mean from grouped Data i.e. in classes.

The following data was obtained from the manufacturers of electronic cells. A sample of electronic cells was taken and the life spans were recorded as shown in the following table.

Life span hrs	No. of cells (f)	Class MP(x)	X - A = d	fd
1600 - 1799	25	1699.5	-600	-15000
1800 - 1999	32	1899.5	-400	-12800
2000-2199	46	2099.5	-200	-9200
2200 - 2399	58	2299.5(A)	0	0
2400 - 2599	40	2499.5	200	8000
2600 - 2799	30	2699.5	400	12000
2800 - 2999	7	2899.5	600	4200

A = Assumed mean, this is an arbitrary number selected from the data, MP = mid point

$$\begin{aligned} \text{Arithmetic mean} &= \text{assumed mean} + \frac{\sum fd}{\sum f} = 2299.5 + -\frac{12800}{238} \\ &= 2299.5 + -53.78 \\ &= 2245.72 \text{ hours} \end{aligned}$$

Example 2 - (use of the coded method)

The following data was obtained from students who were registered in a certain college.

The table shows the age distribution

Age (yrs)	No. of Students (f)	mid points (x)	x-a = d	D/c = u	fu
15 - 19	21	17	-15	-3	-63
20 - 24	35	22	-10	-2	-70
25 - 29	38	27	-5	-1	-38
30 - 34	49	32(A)	0	0	0
35 - 39	31	37	+5	+	31
40 - 44	19	42	+10	+2	38
	193				-102

Required calculate the mean age of the students using the coded method

$$\text{Actual mean} = A(\text{assumed mean}) + \frac{\sum fu}{\sum f} \times c$$

$$= 32 + \frac{-102}{193} \times 5$$

$$= 29.36 \text{ years}$$

Merits of arithmetic mean

- It possess first six our seven characteristics of good average and no other average possess such large number of characteristics
- It is unduly affected by the presence of extreme value.
- It utilizes all the observations given.
- It is a very useful statistic in terms of applications. It has several applications in business management e.g. hypothesis testing, quality control e.t.c.
- It is the best representative of a given set of data if such data was obtained from a normal population.
- The a.m. can be determined accurately using mathematical formulas.

Limitation

- It is difficult to compute mean without making assumption regarding the size of the class- interval of the open end classer.

- If the data is not drawn from a 'normal' population, then the a.m. may give a wrong impression about the population.
- In some situations, the a.m. may give unrealistic values especially when dealing with discrete variables e.g. when working out the average no. of children in a no. of families. It may be found that the average is 4.4 which is unrealistic in human beings.

The Median

This is a statistical value which is normally located at the center of a given set of data which has been organized in the order of magnitude or size e.g. consider the set 14, 17, 9, 8, 20, 32, 18, 14.5, 13. When the data is ordered it will be 8, 9, 13, 14, 14.5, 17, 18, 20, and 32. The middle number/median is 14.5. The importance of the median lies in the fact that it divides the data into 2 equal halves. The no. of observations below and above the median is equal. In order to determine the value of the median from grouped data, the data is grouped the median may be determined by using the following methods

- Graphical method using the cumulative frequency curve (ogive).
- The formula

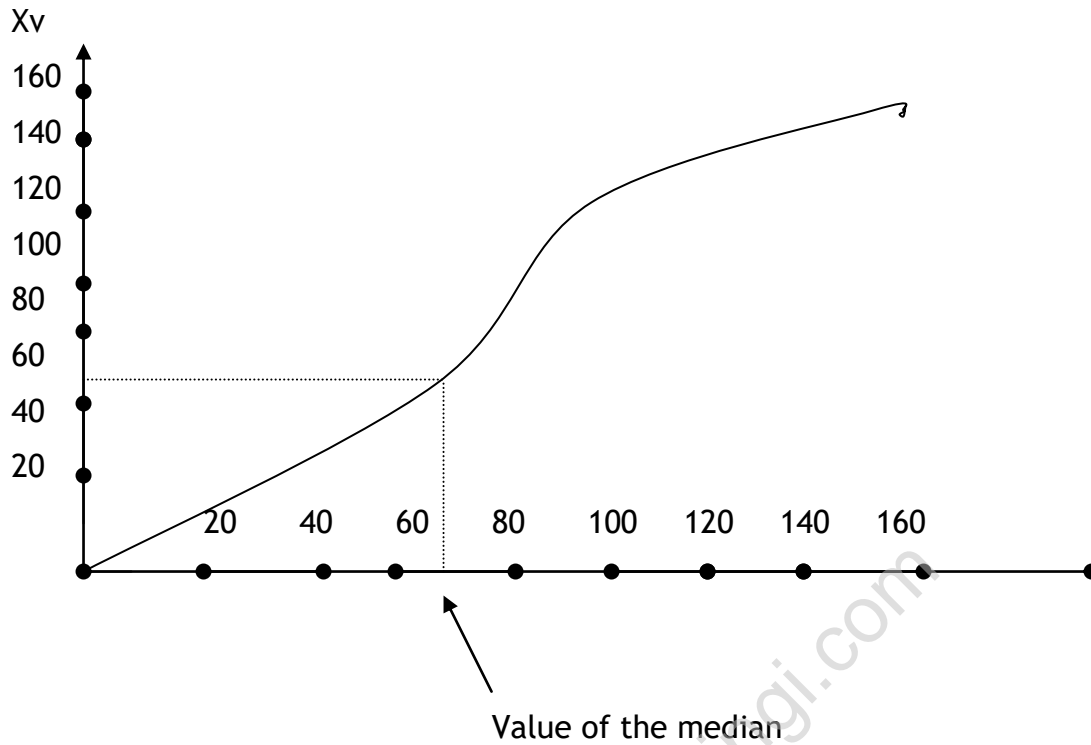
Example

Referring to the table in 105, determine the median using the methods above

The graphical method

IQ	No of resid	UCB	Cumulative Frequency
0 - 20	6	20	6
20 - 40	18	40	24
40 - 60	32	60	56
60 - 80	48	80	104
80 - 100	27	100	131
100 - 120	13	120	144
120 - 140	2	140	146

146



The position of the median = $\frac{n+1}{2} = \frac{146+1}{2}$

Computation

The formula used is

$$\text{Media} = L + i/f\{m-c\}$$

Where L = Lower class boundary of the class containing the median

$$m = (n/2)\text{th or } (n+1/2)\text{th}$$

c = Cumulative frequency of the class before that containing the median.

f = Frequency of the class containing the median

$$\text{Media} = L + i/f\{m-c\}$$

$$= 60 + \frac{73.5 - 56}{48} \times 20$$

$$= 60 + 7.29$$

$$= 67.29$$

Merits of median

- It is not influenced by the presence extreme values
- The sum of the deviation of observation from median is minimum
- It shows the centre of a given set of data
- Knowledge of the determination of the median may be extended to determine the quartiles
- The median can easily be defined
- It can be obtained easily from the cumulative frequency curve
- It can be used in determining the degrees of skewness

Limitation

- Its value is not determined by each and every observation.
- Median is not capable of algebraic treatment.
- Median is less reliable.
- It is affected by sampling fluctuation.
- In some situations where the no. of observations is even, the value of the median obtained is usually imaginary.
- The computation of the median using the formulas is not well understood by most businessmen.
- In business environment the median has got very few applications.

Quartiles, Deciles, Percentiles

The procedure for computing quartiles deciles etc is the same as for median For grouped data the following formula are used for quartiles, deciles and percentiles:

$$Q1 = L + i/f (m-c)$$

Where, Q1= First or lower quartile.

L= Lower class limit of the lower quartile group.

i = class interval of the lower quartile group.

f = frequency of the lower quartile group.

$$m = n/4$$

c = cumulative frequency of the group preceding the lower quartile group.

Decile:

$$D2 = L + i/f (m-c)$$

Where, D2 = second decile

$$m = 2n/10$$

Percentile:

$$P40 = L + i/f (m-c)$$

Where, P40 = fortieth percentile.
 $m = 40n/100$

Example

The profit earned by 100 companies during 2003-04 are given bellow

Profits (Kshs.)	Number of companies
20-30	4
30-40	8
40-50	18
50-60	30
60-70	15
70-80	10
80-90	8
90-100	7

Calculation Q median, D4 and P80 and interpret the values.

Solution

Calculation of Q1, Q2, Q4, and P80

Profits (Kshs.)	f	cf
20-30	4	4
30-40	8	12
40-50	18	30
50-60	30	60
60-70	15	75
70-80	10	85
80-90	8	93
90-100	7	100

Q1 = size of $N/4^{\text{th}}$ observation = $100/4 = 25^{\text{th}}$ observation

Hence q1 lies in the class 40-50

$$Q1 = L + i/f (m-c)$$

$$= 40 + 10/18(25-12)$$

$$= 40 + 7.22 = 47.22$$

$$= 47.22$$

$$D4 = L + i/f (m-c)$$

$$= 50 + 10/30 (40-30)$$

$$= 50 + 3.33$$

$$= 53.33$$

Thus 40 percent of the companies earn an annual profit of kshs.53.33 or less

P80 = size of 80/100 the observation = 80 the observation

P80 lies in the class 70-80

$$\begin{aligned} P80 &= L + i/f (m-c) \\ &= 70 + 10/10(80-75) \\ &= 70 + 5 \\ &= 75 \end{aligned}$$

This means that 80 percent of the companies earn an annual profit of kshs.75 or less

20 percent of the companies earn an annual profit of more than kshs. 75

Mode

This is one of the measures of central tendency. The mode is defined as a value within a frequency distribution which has the highest frequency. Sometimes a single value may not exist as such in which case we may refer to the class with the highest frequency. Such a class is known as a **modal class**

The mode is a very important statistical value in business activities quite often business firms tend to stock specific items which are heavily on demand e.g. footwear, clothes, construction materials (beams, wires, iron sheets e.t.c.

The mode can easily be determined from ungrouped data by arranging the figures given and determining the one with the highest frequency.

When determining the values of the mode from the grouped data we may use the following methods;-

- i. The graphical method which involves use of the histogram
- ii. The computation method which involves use of formula

Example

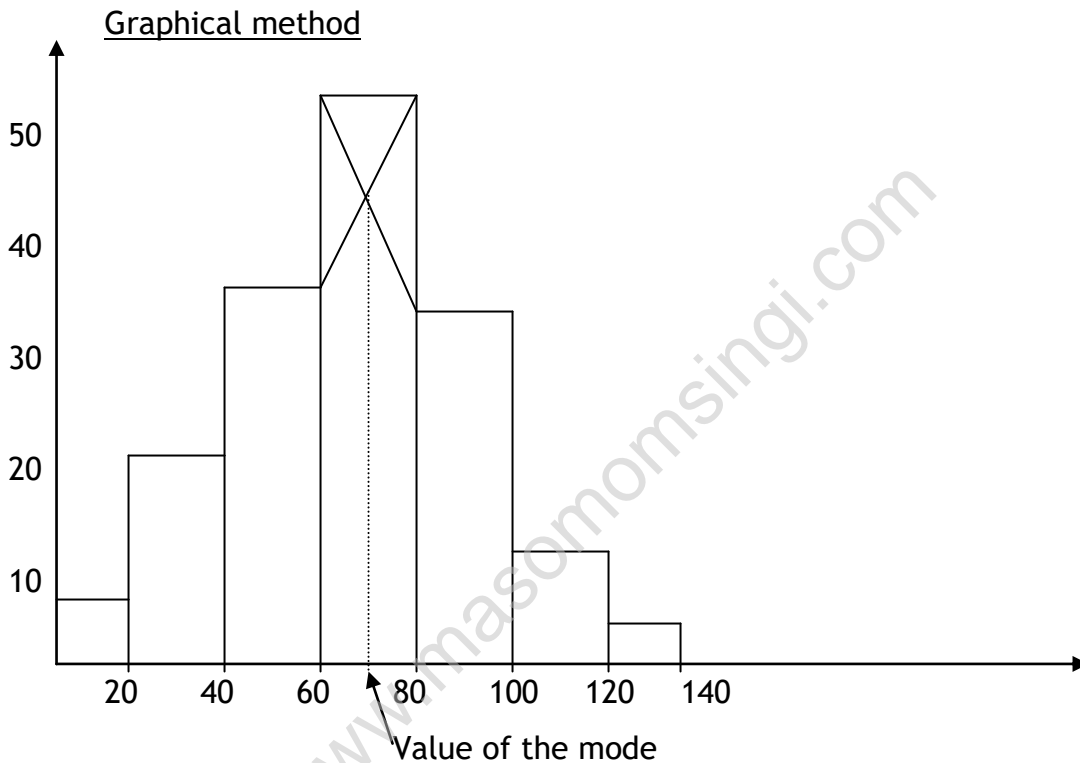
In a social survey in which the main purpose was to establish the intelligence quotient (IQ) of resident in a given area, the following results were obtained as tabulated below:

IQ	No. of residents	Upper class bound	CF
1 - 20	6	20	6
21 - 40	18	40	24
41 - 60	32 fo	60	56
61 - 80	48 f1	80	104
81 - 100	27 f2	100	131
101 - 120	13	120	144
121 - 140	2	140	146

Required

Calculate the modal value of the IQ's tabulated above using

- i. The graphical method and
- ii. Formula



Computation method

$$\text{Mode} = L + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2} \right) \times c$$

Where L = Lower class boundary of the class containing the mode

f_0 = Frequency of the class below the modal class

f_1 = Frequency of the class containing the mode

f_2 = frequency of the class above the modal class

c = Class interval

$$\text{Therefore Mode} = 60.50 + \left(\frac{(48 - 32)}{2(48) - 32 - 27} \right) \times 20$$

$$= 69.14$$

Merits of mode

- Mode is not affected by extreme values.
- It can be easily used to decide qualitative phenomenon.
- Mode happens to be meaningful as an average.
- It can be determined from incomplete data provided the observations with the highest frequency are already known.
- The mode has several applications in business.
- The mode can be easily defined.
- It can be determined easily from a graph.

Limitation of mode

- Mode cannot always be computed
- If the data is quite large and ungrouped, determination of the mode can be quite cumbersome
- Use of the formula to calculate the mode is unfamiliar to most business people
- The mode may sometimes be non-existent or there may be two modes for a given set of data. In such a case therefore a single mode may not exist

Geometric mean

This is a measure of central tendency normally used to measure industrial growth rates. It is defined as the n th root of the product of 'n' observations or values

- i.e. $GM = \sqrt[n]{x_1 \times x_2 \times \dots \times x_n}$

Example

In 1995 five firms registered the following economic growth rates; 26%. 32% 41% 18% and 36%.

Required

Calculate the GM for the above values

$$GM = \sqrt[5]{26 \times 32 \times 41 \times 18 \times 36}$$

$$= \frac{1}{5} [\text{Log } 26 + \text{Log } 32 + \text{Log } 41 + \text{Log } 18 + \text{Log } 36]$$

No.	Log
26	1.4150
32	1.5052

41	1.6128
18	1.2533
36	1.5563
	7.3446

Therefore Log of GM = $1/5 \times 7.3446 = 1.46892$

So GM = Antilog of 1.46892

= 29.43

Merits

- i. It makes use of all the values given (except when $x = 0$ or negative)
- ii. It is the best measure for industrial growth rates

Demerits

- i. The determination of the GM by using logarithms is not familiar process to all those expected to use it e.g. managers
- ii. If the data contains **zeros** or **-ve** values, the GM ceases to exist

Harmonic mean

This is a measure of central tendency which is used to determine the average growth rates for natural economies. It is defined as the reciprocal of the average of the reciprocals of all the values given by HM.

$$HM = \frac{1}{\frac{1}{n}(\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n})}$$

Example

The economic growth rates of five countries were given as 20%, 15%, 25%, 18% and 5%

Calculate the harmonic mean

$$\text{The HM} = \frac{1}{\frac{1}{5} \left(\frac{1}{20} + \frac{1}{15} + \frac{1}{25} + \frac{1}{10} + \frac{1}{5} \right)}$$

$$= \frac{1}{0.2(0.05 + 0.07 + 0.04 + 0.10 + 0.2)}$$

$$= \frac{1}{0.092}$$

10.86%

Merits - same as the arithmetic mean

Demerits - same as the arithmetic mean

Weighted mean

- This is the mean which uses arbitrarily given weights
- It is a useful measure especially where assessment is being done yet the conditions prevailing are not the same. This is particularly true when assessment of students is being done given that the subjects being taken have different levels of difficulties.

Examples

The following table shows that marks scored by a student doing section 3 and 4 of CPA

Subject	Scores (x)	Weight (w)	wx
STAD	65	50	3250
BF	63	40	2520
FA2	62	45	2340
LAW	80	35	2800
QT	69	55	3795
FA3	55	60	3300
		w = 285	wx = 18005

Weighted mean

$$\frac{E_{wx}}{E_w}$$

$$= \frac{18005}{285}$$

$$= 63.17\%$$

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CHAPTER FOUR

MEASUREMENT OF VARIATION /DISPERSION

Specific objectives

At the end of this topic, the trainee should be able to:-

- State the characteristics of a good measure of dispersion.
- Differentiate between the absolute and relative measures.
- Calculate and interpret the measures of dispersion.

Introduction

Measures of variation help us in studying the important characteristics of a distribution. The measures of dispersion are very useful in statistical work because they indicate whether the rest of the data are scattered around the mean or away from the mean. If the data is approximately dispersed around the mean then the measure of dispersion obtained will be small therefore indicating that the mean is a good representative of the sample data. But on the other hand, if the figures are not closely located to the mean then the measures of dispersion obtained will be relatively big indicating that the mean does not represent the data sufficiently.

Significance of measuring variation or dispersion

- i) to determine the reliability of an average
- ii) to serve as a basis for the control of the variability
- iii) to compare two or more series with regard to their variability
- iv) to facilitate the use of other statistical measures

Property of a good measure of variation/dispersion.

A good measure of variation should possess as far as possible

- i) It should be simple to understand
- ii) It should be easy to compute
- iii) It should be rigidly defined
- iv) It should be based on every observation of the distribution
- v) It should be amenable to further algebraic treatment
- vi) It should have sampling stability
- vii) It should not be unduly affected by extreme observation

Methods of studying variation

The following are the most important methods of studying variation

- The range
- The Mean deviation
- The interquartile range or quartile deviation
- The standard deviation

-
- The Lorenz curve

Absolute and relative measures of variation

Measures of variation may be either absolute or relative. Absolute measures of variation are expressed in the same statistical unit in which the original data are given. Such shillings, kilograms, tones, etc

These values may be used to compare the variation in two or more than two distributions provided the variables are expressed in the same units and have almost the same average value.

In case the two sets of data are expressed in different units such as manager's salary versus workers salary, the absolute measures of variation are not comparable. In such cases measures of relative variation should be used.

A measure of relative variation is the ratio of a measure of absolute variation to an average. It is sometimes called a coefficient of variation, because coefficient means a pure number that is independent of the unit of measurement.

a) The Range

The range is defined as the difference between the highest and the smallest values in a frequency distribution. This measure is not very efficient because it utilizes only 2 values in a given frequency distribution. However the smaller the value of the range, the less dispersed the observations are from the arithmetic mean and vice versa

The range is not commonly used in business management because 2 sets of data may yield the same range but end up having different interpretations regarding the degree of dispersion.

Range is the simplest method of studying variation. It is defined as the difference between the value of the smallest observation and the value of the largest observation

Range = largest value- smallest value

R= L-S

Coefficient of Range = $\frac{L-S}{L+S}$

Example

The following are the prices of shares of a company from Monday to Saturday

days	Prices
Monday	200
Tuesday	210
Wednesday	208
Thursday	160
Friday	220

Saturday

250

Calculate the range and coefficient of range.

Solution

Range = L-S

L=250

S= 160

Range =250-160
= 90

Coefficient of range = $\frac{L-S}{L+S} = \frac{250-160}{250+160} = \frac{90}{410} = 0.219$

Merits of range

- It is the simplest to understand and easiest to compute
- It takes minimum time to calculate the value of range

Limitation

- Range is not based on each and every observation of the distribution
- It is subject to fluctuation of considerable magnitude from sample to sample
- Range cannot be computed in case of open -end distribution
- Range cannot tell us anything about the character of the distribution within two extreme observations

Uses of range

1. Quality control; the objective of quality control is to keep a check on the quality of the products without 100% inspection control charts are prepared.
2. fluctuation in the shares price; range is useful in studying the variation in the prices in stocks and shares and other commodities
3. Weather forecast; the meteorological department does make use of the range in determining the difference between the minimum temperature and maximum temperature.

(b) Mean Deviation

This is a useful measure of dispersion because it makes use of all the values given. The average deviation is obtained by calculating the absolute deviation of each observation from median (or mean) and then averaging this deviation by taking their arithmetic mean. The formula for average deviation may be written as;

$$A.D = \frac{\sum |x - med|}{N}$$

Incase deviation are taken from mean the formula shall be written as;

$$A.D x = \frac{\sum |x - \bar{x}|}{N}$$

Computation of average deviation

The formula for computing average deviation is

$$A.D (med) = \frac{\sum |x - med|}{N}$$

N

If average deviation is small the distribution is highly compact or uniform since more than half of the cases are concentrated within a small range around the mean.

The relative measure corresponding to the average deviation called the coefficient of average deviation is obtained by dividing average by the measure of central tendency used, i.e. mean or median.

Example 1

In a given exam the scores for 10 students were as follows

Student	Mark (x)	$ x - \bar{x} $
A	60	1.8
B	45	16.8
C	75	13.2
D	70	8.2
E	65	3.2
F	40	21.8
G	69	7.2
H	64	2.2
I	50	11.8
J	80	18.2
Total	618	104.4

Required

Determine the absolute mean deviation

$$\text{Mean, } \bar{x} = \frac{618}{10} = 61.8$$

$$\text{Therefore AMD} = \frac{\sum |X - \bar{X}|}{N} = \frac{104.4}{10} = 10.44$$

Example 2

The following data was obtained from a given financial institution. The data refers to the loans given out in 1996 to several firms

Firms (f)	Amount of loan per firm (x)	fx	$ x - \bar{x} $	$ x - \bar{x} \cdot f$
3	20000	60000	4157.9	12473.70
4	60000	240000	35842.1	143368.40
1	15000	15000	9157.9	9157.9
5	12000	60000	12157.9	60789.50
6	14000	84000	10157.9	60947.40
$\Sigma f = 19$		$\Sigma fx = 459000$		286736.90

Required

Calculate the mean deviation for the amount of items given

$$\bar{X} = \frac{\sum fx}{\sum f} = \frac{459,000}{19} = 24157.9$$

$$\therefore AMD = \frac{\sum |X - \bar{X}|}{\sum f} = \frac{286736.90}{19}$$

$$= \text{Shs } 15,091.40$$

NB if the absolute mean deviation is relatively small it implies that the data is more compact and therefore the arithmetic mean is a fair sample representative.

(b) The interquartile range or quartile deviation

This is a measure of dispersion which involves the use of quartile. A quartile is a mark or a value which lies at the boundary of a division when any given set of data is divided into four equal divisions

Each of such divisions normally carries 25% of all the observations

The semi interquartile range is a good measure of dispersion because it shows how the rest of the data are generally spread around the mean

The quartiles normally used are three namely;

- i. The lower quartile (first quartile Q1) this usually binds the lower 25% of the data
- ii. The median (second quartile Q2)
- iii. The upper quartile (third quartile Q3)

Interquartile range = Q3-Q1

Very often the interquartile range is reduced to the form of the semi-interquartile range or quartile deviation by dividing it by 2

Q.D = quartile deviation

$$Q.D = \frac{Q_3 - Q_1}{2}$$

Quartile deviation gives the average amount by which the quartile differs from the median. Quartile deviation is an absolute measure of variation. The relative measure corresponding to the measure called the coefficient of quartile deviation.

$$\text{Coefficient of Q.D} = \frac{Q_3 - Q_1}{Q_3 + Q_1}$$

The semi-interquartile range,

$$SIR = \frac{Q_3 - Q_1}{2}$$

Example 1

The weights of 15 parcels recorded at the GPO were as follows:
16.2, 17, 20, 25(Q1) 29, 32.2, 35.8, 36.8(Q2) 40, 41, 42, 44(Q3) 49, 52, 55
(in kgs)

Required

Determine the semi interquartile range for the above data

$$SIR = \frac{Q_3 - Q_1}{2} = \frac{44 - 25}{2} = \frac{19}{2} = 8.5$$

Example 2 (Grouped Data)

The following table shows the levels of retirement benefits given to a group of workers in a given establishment.

Retirement benefits £ '000	No of retirees (f)	UCB	cf
20 - 29	50	29.5	50
30 - 39	69	39.5	119
40 - 49	70	49.5	189
50 - 59	90	59.5	279
60 - 69	52	69.5	331
70 - 79	40	79.5	371
80 - 89	11	89.5	382

Required

- Determine the semi interquartile range for the above data
- Determine the minimum value for the top ten per cent.(10%)
- Determine the maximum value for the lower 40% of the retirees

Solution

The lower quartile (Q1) lies on position

$$\frac{N+1}{4} = \frac{382+1}{4}$$

$$= 95.75$$

$$\therefore \text{the value of Q1} = 29.5 + \frac{(95.75 - 50)}{69} \times 10$$

$$= 29.5 + 6.63$$

$$= \text{£}36.13$$

The upper quartile (Q3) lies on position

$$= 3 \left(\frac{N+1}{4} \right)$$

$$= 3 \left(\frac{382+1}{4} \right)$$

$$= 287.25$$

$$\therefore \text{the value of Q3} = 59.5 + \frac{(287.25 - 279)}{52} \times 10$$

$$= 61.08$$

$$\text{The semi interquartile range} = \frac{Q3 - Q1}{2}$$

$$= \frac{61.08 - 36.13}{2}$$

$$= 12.475$$

$$= \text{£}12,475$$

ii. The top 10% is equivalent to the lower 90% of the retirees

The position corresponding to the lower 90%

$$= \frac{90}{100} (n + 1) = 0.9 (382 + 1)$$

$$= 0.9 \times 383$$

$$= 344.7$$

∴ the benefits (value) corresponding to the minimum value for top 10%

$$= 69.5 + \frac{(344.7 - 331)}{40} \times 10$$

$$= 72.925$$

$$= \text{£ } 72925$$

iii. The lower 40% corresponds to position

$$= \frac{40}{100} (382 + 1)$$

$$= 153.20$$

∴ Retirement benefits corresponding to its position

$$= 39.5 + \frac{(153.2 - 119)}{70} \times 10$$

$$= 39.5 + 4.88$$

$$= 44.38$$

$$= \text{£ } 44380$$

The 10th - 90th percentile range

This is a measure of dispersion which uses percentile. A percentile is a value which separates one division from the other when a given data is divided into 100 equal divisions.

This measure of dispersion is very important when calculating the coefficient of skewness.

Example

Using the above data for retirees calculate the 10th - 90th percentile. The tenth percentile 10th percentile lies on position

$$\frac{10}{100} (382 + 1) = 0.1 \times 383$$

$$= 38.3$$

∴ the value corresponding to the tenth percentile

$$= 19.5 + \frac{(38.3 \times 10)}{50}$$

$$= 19.5 + 7.66$$

$$= 27.16$$

The 90th percentile lies on position

$$\frac{90}{100} (382 + 1) = 0.9 \times 383$$

$$= 344.7$$

∴ the value corresponding to the 90th percentile

$$= 69.5 + \frac{(344.7 - 331)}{40} \times 10$$

$$= 69.5 + 3.425$$

$$= 72.925$$

∴ the required value of the 10th - 90th percentile = 72.925 - 27.16 = 45.765

Merits of quartile deviation

- i) In certain respect it is superior to range as a measure of variation
- ii) It has a special utility in measuring variation in cases of open-ended distribution
- iii) It has also useful in erratic or highly skewed distribution where the other measure of variation would be warped by extreme values. It is not affected by the presence of extreme values.

Limitation

- i) Quartile deviation ignores 50% item as the value of quartile deviation does not depend upon every observation it cannot be regarded as a good method of measuring variation.
- ii) It is not capable of mathematical manipulation

- iii) Its value is very much affected by sampling fluctuation
- iv) It is in fact not a measure of variation as it really does not show the scatter around an average but rather a distance on scale.

(d) Standard deviation

The standard deviation measures the absolute dispersion (or variability of distribution, the greater amount of dispersion or variability). The greater the standard deviation the greater will be the magnitude of the deviations of the value from their mean.

A small standard deviation means a high degree of uniformity of the observation as well as homogeneity of a series.

A large standard deviation means just the opposite.

Difference between deviation and standard deviation

Both these measure of dispersion are based on each and every item of the distribution. But they differ in the following respects.

- Algebraic signs are ignored while calculating mean deviation whereas in the calculation of standard deviation signs are taken into account.
- Mean deviation can be computed either from median or mean. The standard deviation on the other hand is always computed from the arithmetic mean because the sum of the squares of the deviation of items from arithmetic mean is least.

This is one of the most accurate measures of dispersion. It has the following advantages;

- i. It utilizes all the values given
- ii. It makes use of both negative and positive values if they occur
- iii. The standard deviation reflects an accurate impression of how much the sample data varies from the mean. This is because its suitability can also be tested using other statistical methods

Example

A sample comprises of the following observations; 14, 18, 17, 16, 25, 31

Determine the standard deviation of this sample

Observation.

	x	$(x - \bar{x})$	$(x - \bar{x})^2$
	14	-6.1	37.21
	18	-2.1	4.41
	17	-3.1	9.61
	16	-4.1	16.81
	25	4.9	24.01
	31	10.9	118.81
Total	121		210.56

$$\bar{X} = \frac{121}{6} = 20.1$$

$$\therefore \text{Standard deviation, } \sigma = \sqrt{\frac{\sum (x - \bar{x})^2}{n}} = \sqrt{\frac{210.56}{6}}$$

$$= 5.93$$

Alternative method

	x	X ²
	14	196
	18	324
	17	289
	16	256
	25	625
	31	961
Total	121	2651

$$\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2} = \sqrt{\frac{2651}{6} - \left(\frac{121}{6}\right)^2}$$

$$= 5.93$$

Example 2

The following table shows the part-time rate per hour of a given no. of laborers in the month of June 1997.

Rate per hr (x) Shs	No. of laborers (f)	fx	fx ²
230	7	1610	370300
400	6	2400	960000
350	2	700	245000
450	1	450	202500
200	8	1600	320000
150	11	1650	247500
Total	35	8410	2345300

Calculate the standard deviation from the above table showing how the hourly payment were varying from the respective mean

$$\begin{aligned}
 \therefore \text{Standard deviation, } \sigma &= \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} \\
 &= \sqrt{\frac{2345300}{35} - \left(\frac{8410}{35}\right)^2} \\
 &= \sqrt{67008.6 - 577372} \\
 &= \sqrt{9271.4} \\
 &= 96.29
 \end{aligned}$$

Example 3 - Grouped data

In business statistical work we usually encounter a set of grouped data. In order to determine the standard deviation from such data, we use any of the three following methods

- i. The long method
- ii. The shorter method
- iii. The coded method

The above methods are used in the following examples

Example 3.1

The quality controller in a given firm had an accurate record of all the iron bars produced in May 1997. The following data shows those records

i. Using long method

Bar lengths (cm)	No. of bars(f)	Class mid point (x)	fx	fx ²
201 - 250	25	225.5	5637.5	1271256.25
251 - 300	36	275.5	9918	2732409
301 - 350	49	325.5	15949.5	5191562.25
351 - 400	80	375.5	30040	11280020
401 - 450	51	425.5	21700.5	9233562.75
451 - 500	42	475.5	19971	9496210.50
501 - 550	30	525.5	15765	8284507.50
	313		118981.50	47489526

Calculate the standard deviation of the lengths of the bars

$$\begin{aligned} \therefore \text{Standard deviation, } \sigma &= \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} \\ &= \sqrt{\frac{47489526}{313} - \left(\frac{118981.50}{313}\right)^2} \\ &= 84.99 \text{ cm} \end{aligned}$$

ii. Using the shorter method

Bar lengths (cm)	No. of bars(f)	mid point (x)	x-A = d	fd	Fd ²
201 - 250	25	225.5	-150	-3750	562500
251 - 300	36	275.5	-100	-3600	360000
301 - 350	49	325.5	-50	-2450	122500
351 - 400	80	375.5 (A)	0	0	0
401 - 450	51	425.5	50	2550	127500
451 - 500	42	475.5	100	4200	420000
501 - 550	30	525.5	150	4500	675000
Total	313			1450	2267500

Calculate the standard deviation using the shorter method

$$\begin{aligned} \therefore \text{Standard deviation, } \sigma &= \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2} \\ &= \sqrt{\frac{2267500}{313} - \left(\frac{1450}{313}\right)^2} \\ &= \sqrt{7244.40 - 21.50} \\ &= \sqrt{7222.90} \\ &= 84.99 \text{ cm} \end{aligned}$$

iii. Using coded method

Bar lengths (cm)	(f)	mid point (x)	x-A = d	d/c = u	fu	fu ²
201 - 250	25	225.5	-150	-3	-75	225
251 - 300	36	275.5	-100	-2	-72	144

301 - 350	49	325.5	-50	-1	-49	49
351 - 400	80	375.5 (A)	0	0	0	0
401 - 450	51	425.5	50	1	51	51
451 - 500	42	475.5	100	2	84	168
501 - 550	30	525.5	150	3	90	270
	313				29	907

$C = 50$ where c is an arbitrary number, try picking a different figure say 45 the answer should be the same.

Standard deviation using the coded method. This is the most preferable method among the three methods

$$\begin{aligned}\sigma &= c \times \sqrt{\frac{\sum fu^2}{\sum f} - \left(\frac{\sum fu}{\sum f}\right)^2} \\ &= 50 \times \sqrt{\frac{907}{313} - \left(\frac{29}{313}\right)^2} \\ &= 50 \times 1.6997 \\ &= 84.99\end{aligned}$$

Variance

Square of the standard deviation is called variance.

(e) Lorenz curve

The Lorenz curve derived by Max O. Lorenz, famous economic statistician, is a graphic method of studying dispersion. This curve was used by him for the first time to measure the distribution of wealth and income. Now the curve is also used to study the distribution of profits, wages, turnover, etc.

Illustration

In the following table is given the number of companies belonging to two areas A and B according to the amount of profits earned by them. Draw in the same diagram their Lorenz curve and interpret them.

Profits earned shs (000s)	Number of companies	
	Area A	Area B
6	6	2
25	11	38
60	13	52
84	14	28

105	15	38
150	17	26
170	10	12
400	14	4

Solution

Relative measures of dispersion

Def: A relative measure of dispersion is a statistical value which may be used to compare variations in 2 or more samples.

The measures of dispersion are usually expressed as decimals or percentages and usually they do not have any other units

Example

The average distance covered by vehicles in a motor rally may be given as 2000 km with a standard deviation of 5 km.

In another competition set of vehicles covered 3000 km with a standard deviation of 10 kms

NB: The 2 standard deviations given above are referred to as absolute measures of dispersion. These are actual deviations of the measurements from their respective mean

However, these are not very useful when comparing dispersions among samples.

Therefore the following measures of dispersion are usually employed in order to assess the degree of dispersion.

- i. Coefficient of mean deviation

$$= \frac{\text{Mean deviation}}{\text{mean}}$$

ii. Coefficient of quartile deviation

$$= \frac{\frac{1}{2}(Q_3 - Q_1)}{Q_2}$$

Where Q_1 = first quartile

Q_3 = third quartile

iii. Coefficient of standard deviation

$$= \frac{\text{Standard deviation}}{\text{mean}}$$

iv. Coefficient of variation

$$= \frac{\text{standard deviation}}{\text{mean}} \times 100$$

Example (see information above)

First group of cars: mean = 2000 kms

Standard deviation = 5 kms

$$\therefore \text{C.O.V} = \frac{5}{2000} \times 100$$

$$= 0.25\%$$

Second group of cars: mean = 3000 kms

Standard deviation = 10kms

$$\therefore \text{C.O.V} = \frac{10}{3000} \times 100$$

$$= 0.33\%$$

Conclusion

Since the coefficient of variation is greater in the 2nd group, than in the first group we may conclude that the distances covered in the 1st group are much closer to the mean than in the 2nd group.

Example 2

In a given firm located in the UK the average salary of the employees is £ 3500 with a standard deviation of £150

The same firm has a local branch in Kenya in which the average salaries are Kshs 8500 with a standard deviation of Kshs.800

Determine the coefficient of variation in the 2 firms and briefly comment on the degree of dispersion of the salaries in the 2 firms.

First firm in the UK

$$\begin{aligned} \text{C.O.V} &= \frac{150}{3500} \times 100 \\ &= 4.29\% \end{aligned}$$

Second firm in Kenya

$$\begin{aligned} \text{C.O.V} &= \frac{800}{8500} \times 100 \\ &= 9.4\% \end{aligned}$$

Conclusively, since $4.29\% < 9.4\%$ then the salaries offered by the firm in UK are much closer to the mean given them in the case to the local branch in Kenya

COMBINED MEAN AND STANDARD DEVIATION

Sometimes we may need to combine 2 or more samples say A and B. It is therefore essential to know the new mean and the new standard deviation of the combination of the samples.

Combined mean

Let m be the combined mean

Let x_1 be the mean of first sample

Let x_2 be the mean of the second sample

Let n_1 be the size of the 1st sample

Let n_2 be the size of the 2nd sample

Let s_1 be the standard deviation of the 1st sample

Let s_2 be the standard deviation of the 2nd sample

$$\therefore \text{combined mean} = \frac{n_1x_1 + n_2x_2}{n_1 + n_2}$$

$$\text{combined standard deviation} = \sqrt{\frac{n_1 s_1^2 + n_1 (m - x_1)^2 + n_2 s_2^2 + n_2 (m - x_2)^2}{n_1 + n_2}}$$

Example

A sample of 40 electric batteries gives a mean life span of 600 hrs with a standard deviation of 20 hours.

Another sample of 50 electric batteries gives a mean lifespan of 520 hours with a standard deviation of 30 hours.

If these two samples were combined and used in a given project simultaneously, determine the combined new mean for the larger sample and hence determine the combined or pulled standard deviation.

Size	x	s
40(n ₁)	600 hrs(x ₁)	20hrs (s ₁)
50 (n ₁)	520 hrs (x ₂)	30 hrs (s ₂)

$$\text{Combined mean} = \frac{40(600) + 50(520)}{40 + 50} = \frac{50,000}{90} = 555.56$$

Combined standard deviation

$$= \sqrt{\frac{40(20^2) + 40(555.56 - 600)^2 + 50(30)^2 + 50(555.56 - 520)^2}{40 + 50}}$$

$$= \sqrt{\frac{1600 + 78996.54 + 45000 + 63225.68}{90}}$$

$$= 47.52 \text{ hrs}$$

SKEWNESS

- This is a concept which is commonly used in statistical decision making. It refers to the degree in which a given frequency curve is deviating away from the normal distribution
- There are 2 types of skew ness namely
 - i. Positive skew ness
 - ii. Negative skew ness

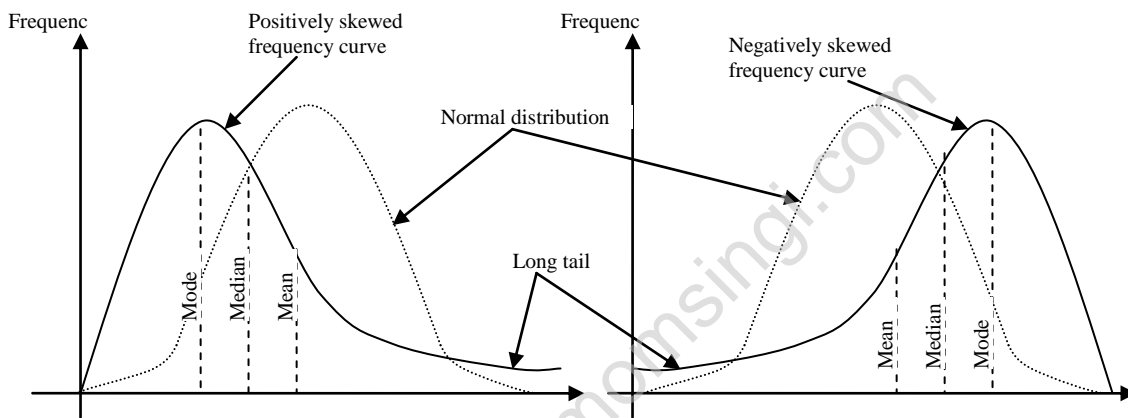
1. Positive Skewness

- This is the tendency of a given frequency curve leaning towards the left. In a positively skewed distribution, the long tail extended to the right.

In this distribution one should note the following

- The mean is usually bigger than the mode and median
- The median always occurs between the mode and mean
- There are more observations below the mean than above the mean

This frequency distribution as represented in the skewed distribution curve is characteristic of the age distributions in the developing countries



2. Negative Skewness

This is an asymmetrical curve in which the long tail extends to the left

NB: This frequency curve for the age distribution is characteristic of the age distribution in developed countries

- The mode is usually bigger than the mean and median
- The median usually occurs in between the mean and mode
- The no. of observations above the mean are usually more than those below the mean (see the shaded region)

MEASURES OF SKEWNESS

- These are numerical values which assist in evaluating the degree of deviation of a frequency distribution from the normal distribution.

- Following are the commonly used measures of skewness.

1. Coefficient Skewness

$$= 3 \times \frac{(\text{mean} - \text{median})}{\text{Standard deviation}}$$

2. Coefficient of skewness

$$= \frac{\text{mean} - \text{mode}}{\text{Standard deviation}}$$

NB: These 2 coefficients above are also known as Pearsonian measures of skewness.

3. Quartile Coefficient of skewness

$$= \frac{Q_3 + Q_1 - 2Q_2}{Q_3 + Q_1}$$

Where $Q_1 = 1^{\text{st}}$ quartile

$Q_2 = 2^{\text{nd}}$ quartile

$Q_3 = 3^{\text{rd}}$ quartile

NB: The Pearsonian coefficients of skewness usually range between -ve 3 and +ve 3. These are extreme value i.e. +ve 3 and -ve 3 which therefore indicate that a given frequency is negatively skewed and the amount of skewness is quite high.

Similarly if the coefficient of skewness is +ve it can be concluded that the amount of skewness of deviation from the normal distribution is quite high and also the degree of frequency distribution is positively skewed.

Example

The following information was obtained from an NGO which was giving small loans to some small scale business enterprises in 1996. the loans are in the form of thousands of Kshs.

Loans	Units (f)	Midpoints(x)	$x-a=d$	$d/c=u$	fu	Fu ²	UCB	cf
46 - 50	32	48	-15	-3	-96	288	50.5	32
51 - 55	62	53	-10	-2	-124	248	55.5	94
56 - 60	97	58	-5	-1	-97	97	60.5	191
61 -65	120	63 (A)	0	0	0	0	0	0
66 -70	92	68	5	+1	92	92	70.5	403
71 -75	83	73	10	+2	166	332	75.5	486
76 - 80	52	78	15	+3	156	468	80.5	538
81 - 85	40	83	20	+4	160	640	85.5	57.8
86 - 90	21	88	25	+5	105	525	90.5	599
91 - 95	11	93	30	+6	66	396	95.5	610
Total	610				428	3086		

Required

Using the Pearsonian measure of skewness, calculate the coefficients of skewness and hence comment briefly on the nature of the distribution of the loans.

$$\text{Arithmetic mean} = \text{Assumed mean} + \frac{c(\sum fu)}{\sum f}$$

$$= 63 + \frac{(428 \times 5)}{610}$$

$$= 66.51$$

It is very important to note that the method of obtaining arithmetic mean (or any other statistic) by misusing assumed mean (A) from X and then dividing by c can be abit confusing, if this is the case then just use the straight forward method of:

$$\text{Arithmetic mean} = \frac{\sum f.x}{\sum f} \quad \text{where } x \text{ is the midpoint, the answers are the same.}$$

$$\text{The standard deviation} = c \times \sqrt{\frac{\sum fu^2}{\sum f} - \left(\frac{\sum fu}{\sum f}\right)^2}$$

$$= 5 \times \sqrt{\frac{3086}{610} - \left(\frac{428}{610}\right)^2}$$

$$= 10.68$$

$$\begin{aligned} \text{The Position of the median lies } m &= \frac{n+1}{2} \\ &= \frac{610+1}{2} = 305.5 \end{aligned}$$

$$= 60.5 + \frac{(305.5 - 191)}{120} \times 5$$

$$= 60.5 + \frac{(114.4)}{120} \times 5$$

$$\text{Median} = 65.27$$

Therefore the Pearsonian coefficient

$$= 3 \frac{(66.51 - 65.27)}{10.68}$$

$$= 0.348$$

Comment

The coefficient of skewness obtained suggests that the frequency distribution of the loans given was positively skewed. This is because the coefficient itself is positive. But the skewness is not very high implying the degree of deviation of the frequency distribution from the normal distribution is small.

Example 2

Using the above data calculate the quartile coefficient of skewness

$$\text{Quartile coefficient of skewness} = \frac{Q_3 + Q_1 - 2Q_2}{Q_3 + Q_1}$$

$$\text{The position of } Q_1 \text{ lies on} = \frac{610+1}{4} = 152.75$$

$$\therefore \text{actual value } Q_1 = 55.5 + \frac{(152.75 - 94)}{97} \times 5 = 58.53$$

$$\text{The position of } Q_3 \text{ lies on} = 3 \frac{(610+1)}{4} = 458.25$$

$$\therefore \text{actual value } Q_3 = 70.55 + \frac{(458.25 - 403)}{83} \times 5 = 73.83 \times 5$$

$$Q_2 \text{ position: i.e. } 2 \frac{(610+1)}{4} = 305.5$$

$$\text{Actual } Q_2 \text{ value} = 60.5 + \frac{(305.5 - 191)}{120} \times 5 = 65.27$$

The required coefficient of skewness

$$= \frac{73.83 + 58.53 - 2(65.27)}{73.83 + 58.53} = 0.013$$

Conclusion

Same as above when the Pearsonian coefficient was used

KURTOSIS

- This is a concept, which refers to the degree of peakedness of a given frequency distribution. The degree is normally measured with reference to normal distribution.
- The concept of kurtosis is very useful in decision making processes i.e. if a frequency distribution happens to have either a higher peak or a lower peak, then it should not be used to make statistical inferences.
- Generally there are 3 types of kurtosis namely;
 - i. Leptokurtic
 - ii. Mesokurtic
 - iii. Platykurtic

Leptokurtic

- a) A frequency distribution which is leptokurtic has generally a higher peak than that of the normal distribution. The coefficient of kurtosis when determined will be found to be more than 3. Thus frequency distributions with a value of more than 3 are definitely leptokurtic.
 - b) Some frequency distributions when plotted may produce a curve similar to that of the normal distribution. Such frequency distributions are referred to as mesokurtic. The degree of kurtosis is usually equal to 3.
 - c) When the frequency curve contacted produces a peak which is lower than that of a normal distribution when such a curve is said to be platykurtic. The coefficient of such is usually less than 3.
- It is necessary to calculate the numerical measure of kurtosis. The commonly used measure of kurtosis is the percentile coefficient of kurtosis. This coefficient is normally determined using the following equation

$$\text{Percentile measure of kurtosis, } K (\text{Kappa}) = \frac{1}{2} \frac{(Q_3 - Q_1)}{P_{90} - P_{10}}$$

Example

Refer to the table above for loans to small business firms/units

Required

Calculate the percentile coefficient of Kurtosis

$$\begin{aligned} P_{90} &= \frac{90}{100}(n+1) = 0.9(610+1) \\ &= 0.9(611) \\ &= 549.9 \end{aligned}$$

The actual loan for a firm in this position

$$(549.9) = 80.5 + \frac{(549.9 - 538)}{40} \times 5 = 81.99$$

$$P_{10} = \frac{10}{100}(n+1) = 0.1(611) = 61.1$$

The actual loan value given to the firm on this position is

$$50.5 + \frac{(61.1-32)}{62} \times 5 = 52.85$$

$$= 0.9 (611)$$

$$= 549.9$$

∴ Percentile measure of kurtosis

$$K \text{ (Kappa)} = \frac{1}{2} \frac{(Q3 - Q1)}{P90 - P10}$$

$$= \frac{1}{2} \frac{(73.83 - 58.53)}{81.99 - 52.85}$$

$$= 0.26$$

Since $0.26 < 3$, it can be concluded that the frequency distribution exhibited by the distribution of loans is platykurtic. Kurtosis is also measured by moment statistics, which utilize the exact value of each observation.

i. M1 the first moment = $M1 = \frac{\sum X}{n} = \text{Mean } M1 \text{ or } M_1$

$$M2 = \frac{\sum X^2}{n}$$

$$M3 = \frac{\sum X^3}{n}$$

$$M4 = \frac{\sum X^4}{n}$$

3. M2 second moment about the mean $M2$ or f^2

$$M2 = M_2 - M1^2$$

4. M3 third moment about the mean $M3$ (a measure of the absolute skewness)

$$M3 = M_3 - 3M_2M_1 + 2M1^3$$

5. M4 fourth moment about the mean $M4$ (a measure of the absolute Kurtosis)

$$M4 = M_4 - 4M_3M_1 + 6M_2M_1^2 + 3M1^4$$

An alternative formula

$$M4 = \frac{\sum (x-m)^4 f}{\sum f} \text{ Where } m \text{ is mean}$$

Moment coefficient of Kurtosis $\frac{M_4}{S_4}$

Example

Find the moment coefficient of the following distribution

X	f
12	1
14	4
16	6
18	10
20	7
22	2

X	F	xf	(x-m)	(x-m) ²	(x-m) ² f	(x-m) ⁴ f
12	1	12	-5.6	31.36	31.36	983.45
14	4	56	-3.6	12.96	51.84	671.85
16	6	96	-1.6	2.56	15.36	39.32
18	10	180	.4	0.16	1.60	0.256
20	7	140	2.4	5.76	40.32	232.24
22	2	44	4.4	19.36	38.72	749.62
	30	528			179.20	2,676.74

$$M = \frac{528}{30} = 17.6$$

$$\sigma^2 = \frac{179.20}{30} = 5.973$$

$$\sigma^4 = 35.677$$

$$M_4 = \frac{\sum (x-m)^4 f}{\sum f} = \frac{2,676.74}{30} = 89.22$$

$$\text{Moment coefficient of Kurtosis} = \frac{89.22}{35.677} = 2.5$$

Note Coefficient of kurtosis can also be found using the method of assumed mean.

CHAPTER FIVE

CORRELATION AND REGRESSION

Specific Objectives

At the end of the topic the trainee should be able to:

- Draw the scatter diagram;
- Differentiate between the various forms of correlation;
- Determine the correlation coefficient and interpret;
- Determine the coefficient of determination and interpret;
- Apply the linear regression models.

Introduction

When the relationship is of a quantitative nature, the appropriate statistical tool for discovering and measuring the relationship and expressing it in a brief formula is known as a correlation. This is an important statistical concept which refers to interrelationship or association between variables. The purpose of studying correlation is for one to be able to establish a relationship, plan and control the inputs (independent variables) and the output (dependent variables).

In business one may be interested to establish whether there exists a relationship between the

- i. Amount of fertilizer applied on a given farm and the resulting harvest
- ii. Amount of experience one has and the corresponding performance
- iii. Amount of money spent on advertisement and the expected incomes after sale of the goods/service

There are two methods that measure the degree of correlation between two variables these are denoted by R and r .

- (a) **Coefficient of correlation** denoted by r , this provides a measure of the strength of association between two variables one the dependent variable the other the independent variable r can range between $+1$ and -1 for perfect positive correlation and perfect negative correlation respectively with zero indicating no relation i.e. for perfect positive correlation y increase linearly with x increment.
- (b) **Rank correlation coefficient** denoted by R is used to measure association between two sets of ranked or ordered data. R can also vary from $+1$, perfect positive rank correlation and -1 perfect negative rank correlation where 0 or any number near zero representing no correlation.

Significance of the study of correlation

- Most of the variation shows some kind of relationship between price and supply income and expenditure. With the help of correlation analysis the degree of relationship existing between the variable can be measured.
- The value of one variable can be estimated once it has been known that they closely related. It can be done with the help of regression analysis.
- It contributes to the economic behavior, aids in locating the critically important variable on which other depend.
- Nature has been found to be multiplying of interrelated force.
- It helps in determining the degree of relationship between two or more variable.

Types of correlation

- Positive and negative correlation
- Simple, partial and multiple
- Linear and non linear.

Positive and negative correlation

If both the variable is varying in the same direction i.e. if one variable is increasing the other on an average is also increasing or if one variable is decreasing the other on an average is also decreasing. Correlation is said to be positive. On the other hand if they are varying in the opposite directions i.e. as one variable is increasing the other is decreasing or vice versa, correlation is said to be negative.

Positive correlation

X	T
10	15
12	20
15	22
18	25
20	37

negative correlation

X	T
20	40
30	30
40	22
60	15
80	16

Simple partial and multiple correlations

The distribution between simple, partial and multiple correlations are based upon the number of variable studied.

When only two variables are studied it is a problem of simple correlation.

When three or more variables are studied it is a problem of either

Multiple or partial correlation.

In multiple correlations three or more variables are studied simultaneously.

Linear and non linear correlation

The distinction between linear and non linear correlation is based upon the constancy of the ratio of change between the variables. If the amount of change in one variable tends to bear constant ratio to the amount of change in the other variable then the correlation is said to be linear. If such variable are plotted on a graph paper, all the plotted points would fall on a straight line. Correlation would be non linear or curvilinear, If the amount of change in one variable does not bear constant ratio to the amount of change in the other variable.

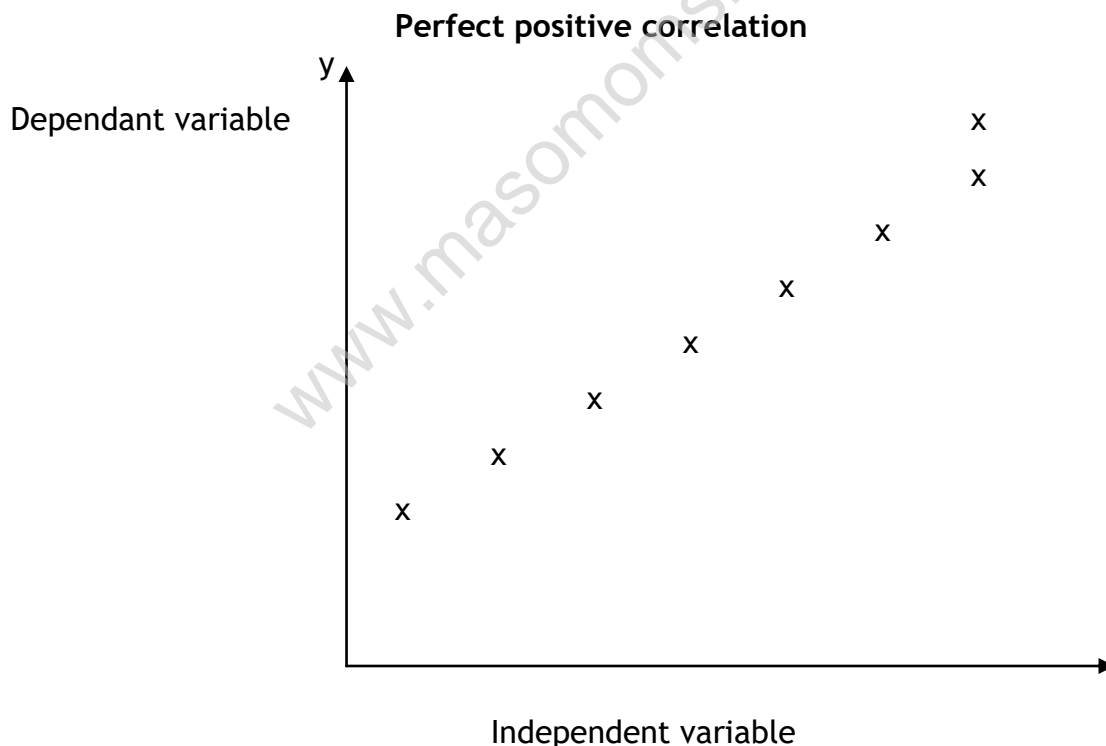
Scatter diagram

The simplest device for studying correlation in two variables is a special type of dot chart called dotogram or scatter diagram.

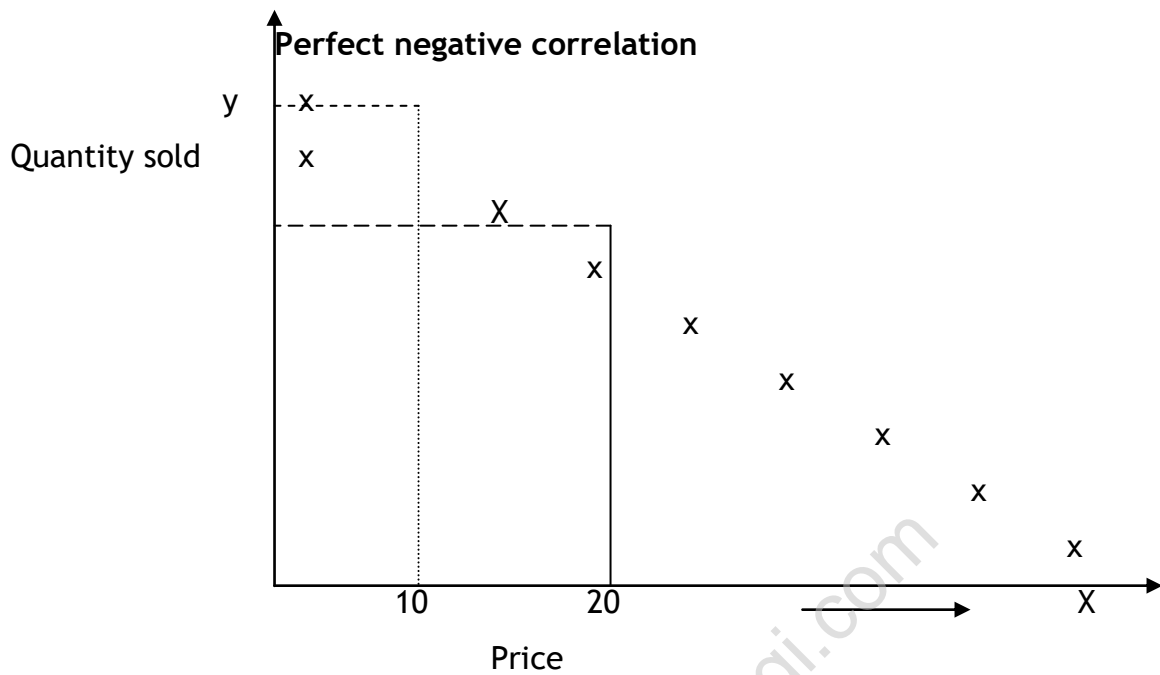
A scatter graph is a graph which comprises of points which have been plotted but are not joined by line segments

The pattern of the points will definitely reveal the types of relationship existing between variables

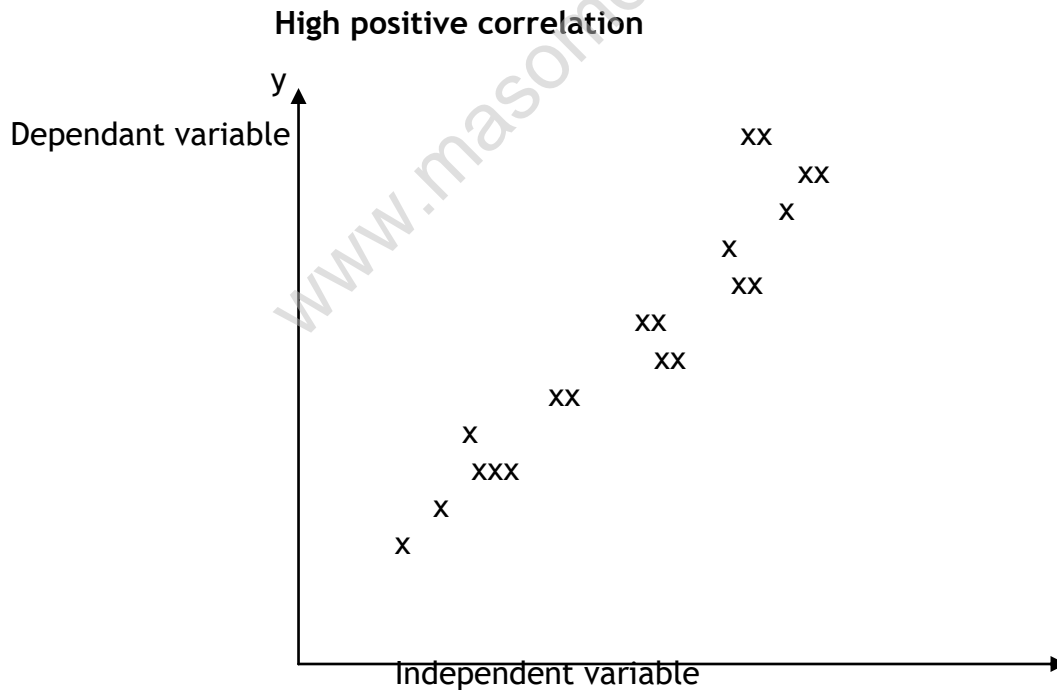
The following sketch graphs will greatly assist in the interpretation of scatter graphs.



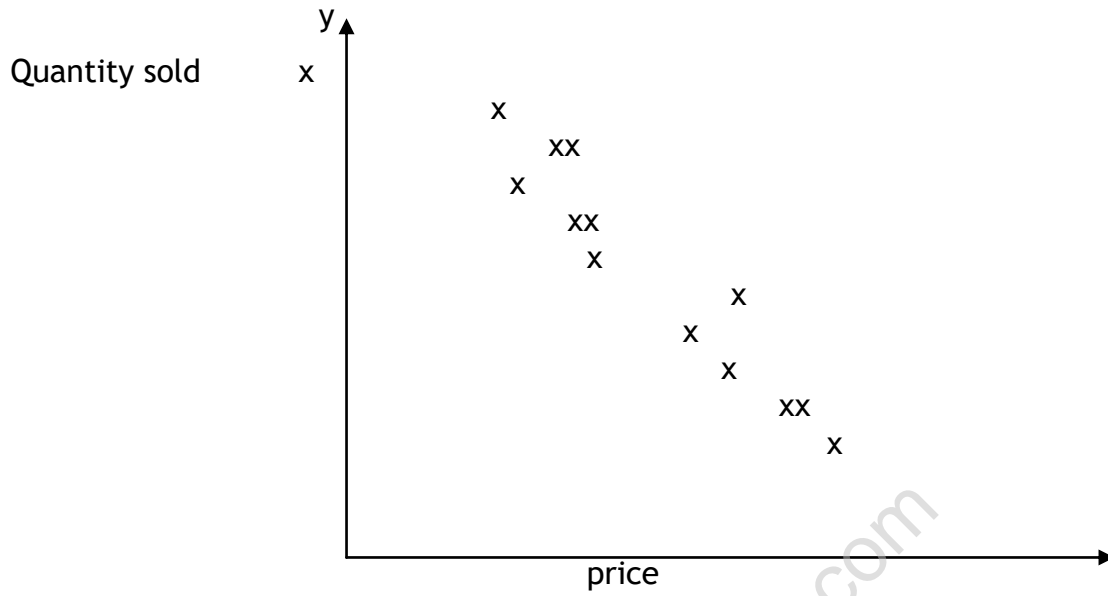
NB: For the above pattern, it is referred to as perfect because the points may easily be represented by a single line graph e.g. when measuring relationship between volumes of sales and profits in a company, the more the company sales the higher the profits.



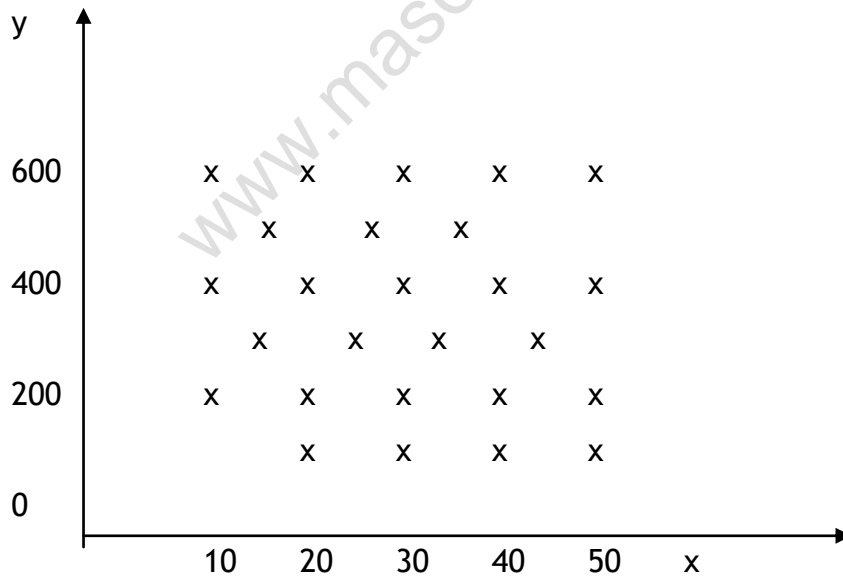
This example considers volume of sale in relation to the price, the cheaper the goods the bigger the sale.



High negative correlation



No correlation



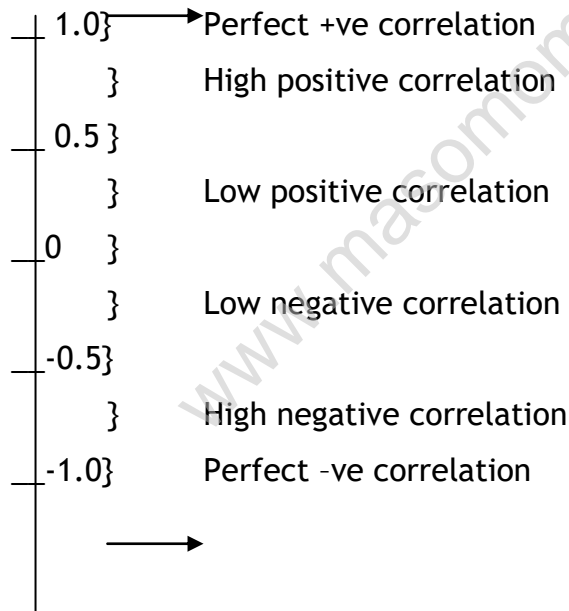
h) Spurious Correlations

in some rare situations when plotting the data for x and y we may have a group showing either positive correlation or -ve correlation but when you analyze the data for x and y in normal life there may be no convincing evidence that there is such a relationship. This implies therefore that the relationship only exists in theory and hence it is referred to as spurious or non sense e.g. when high pass rates of student show high relation with increased accidents.

Correlation coefficient

These are numerical measures of the correlations existing between the dependent and the independent variables. These are better measures of correlation than scatter graphs (diagrams). The range for correlation coefficients lies between +ve 1 and -ve 1. A correlation coefficient of +1 implies that there is perfect positive correlation. A value of -ve shows that there is perfect negative correlation. A value of 0 implies no correlation at all.

The following chart will be found useful in interpreting correlation coefficients.



There are usually two types of correlation coefficients normally used namely;-

Merits of scatter diagram

- It is simple and non mathematical method of studying correlation between the variables.

- It is not influenced by the size of extreme values whereas most of the mathematical methods of findings correlation are influenced by extreme values.

Limitation

- Exact degree of correlation can not be established between the variable

Product Moment Coefficient (r)

It gives an indication of the strength of the linear relationship between two variables.

$$r = \frac{n\sum xy - \sum x \sum y}{\sqrt{n\sum x^2 - (\sum x)^2} \times \sqrt{n\sum y^2 - (\sum y)^2}}$$

Note that this formula can be rearranged to have different outlooks but the resultant is always the same.

Example

The following data was observed and it is required to establish if there exists a relationship between the two.

X	15	24	25	30	35	40	45	65	70	75
Y	60	45	50	35	42	46	28	20	22	15

Solution

Compute the product moment coefficient of correlation (r)

X	Y	X ²	Y ²	XY
15	60	225	3,600	900
24	45	576	2,025	1,080
25	50	625	2,500	1,250
30	35	900	1,225	1,050
35	42	1,225	1,764	1,470
40	46	1,600	2,116	1,840
45	28	2,025	784	1,260
65	20	4,225	400	1,300
70	22	4,900	484	1,540
75	15	5,625	225	1,125
$\sum X = 424$	$\sum Y = 363$	$\sum X^2 = 21,926$	$\sum Y^2 = 15,123$	$\sum XY = 12,815$

$$r = \frac{n\sum xy - \sum x \sum y}{\sqrt{n\sum x^2 - (\sum x)^2} \times \sqrt{n\sum y^2 - (\sum y)^2}}$$

$$r = \frac{10 \times 12,815 - 424 \times 363}{\sqrt{(10 \times 21,926 - 424^2)} \times \sqrt{(10 \times 15,123 - 363^2)}} \\ = \frac{-25,762}{\sqrt{(39,484)} \times \sqrt{(19,461)}} = -0.93$$

The correlation coefficient thus indicates a strong negative linear association between the two variables.

Interpretation of r - Problems in interpreting r values

NOTE:

- A high value of r (+0.9 or - 0.9) only shows a strong association between the two variables but doesn't imply that there is a causal relationship i.e. change in one variable causes change in the other it is possible to find two variables which produce a high calculated r yet they don't have a causal relationship. This is known as spurious or nonsense correlation e.g. high pass rates in QT in Kenya and increased inflation in Asian countries.
- Also note that a low correlation coefficient doesn't imply lack of relation between variables but lack of linear relationship between the variables i.e. there could exist a curvilinear relation.
- A further problem in interpretation arises from the fact that the r value here measures the relationship between a single independent variable and dependent variable, where as a particular variable may be dependent on several independent variables (e.g. crop yield may be dependent on fertilizer used, soil exhaustion, soil acidity level, season of the year, type of seed etc.) in which case multiple correlation should be used instead.

The Rank Correlation Coefficient (R)

Also known as the spearman rank correlation coefficient, its purpose is to establish whether there is any form of association between two variables where the variables arranged in a ranked form.

$$R = 1 - \frac{6 \sum d^2}{n(n^2 - 1)}$$

Where d = difference between the pairs of ranked values.

n = numbers of pairs of rankings

Example

A group of 8 accountancy students are tested in Quantitative Techniques and Law II. Their rankings in the two tests were.

Student	Q. T. ranking	Law II ranking	d	d ²
A	2	3	-1	1
B	7	6	1	1
C	6	4	2	4
D	1	2	-1	1
E	4	5	-1	1
F	3	1	2	4
G	5	8	-3	9
H	8	7	1	1

$\sum d^2 = 22$

$d = \text{Q. T. ranking} - \text{Law II ranking}$

$$R = 1 - \frac{6 \sum d^2}{n(n^2 - 1)} = 1 - \frac{6 \times 22}{8(8^2 - 1)}$$

$$= 0.74$$

Thus we conclude that there is a reasonable agreement between student's performances in the two types of tests.

NOTE: in this example, if we are given the actual marks then we find r. R varies between +1 and -1.

Tied Rankings

A slight adjustment to the formula is made if some students tie and have the same ranking the adjustment is

$$\frac{t^3 - t}{12} \text{ where } t = \text{number of tied rankings the adjusted formula}$$

becomes

$$R = 1 - \frac{6 \left(\sum d^2 + \frac{t^3 - t}{12} \right)}{n(n^2 - 1)}$$

Example

Assume that in our previous example student E & F achieved equal marks in Q. T. and were given joint 3rd place.

Solution

Student	Q. T. ranking	Law II ranking	d	d ²
A	2	3	-1	1
B	7	6	1	1
C	6	4	2	4
D	1	2	-1	1
E	3 ½	5	-1 ½	2 ¼
F	3 ½	1	2 ½	6 ¼
G	5	8	-3	9
H	8	7	1	1

$\sum d^2 = 26\frac{1}{2}$

$$R = 1 - \frac{6\left(\sum d^2 + \frac{t^3-t}{12}\right)}{n(n^2-1)} = 1 - \frac{6\left(26\frac{1}{2} + \frac{2^3-2}{12}\right)}{8(8^2-1)} \quad \text{since } t = 2$$

$$= 0.68$$

NOTE: It is conventional to show the shared rankings as above, i.e. E, & F take up the 3rd and 4th rank which are shared between the two as 3½ each.

Coefficient of Determination

This refers to the ratio of the explained variation to the total variation and is used to measure the strength of the linear relationship. The stronger the linear relationship the closer the ratio will be to one.

$$\text{Coefficient determination} = \frac{\text{Explained variation}}{\text{Total variation}}$$

Example (Rank Correlation Coefficient)

In a beauty competition 2 assessors were asked to rank the 10 contestants using the professional assessment skills. The results obtained were given as shown in the table below

Contestants	1 st assessor	2 nd assessor
A	6	5
B	1	3
C	3	4
D	7	6
E	8	7
F	2	1
G	4	8
H	5	2

J	10	9
K	9	10

REQUIRED

Calculate the rank correlation coefficient and hence comment briefly on the value obtained

			d	d ²
A	6	5	1	1
B	1	3	-2	4
C	3	4	-1	1
D	7	6	1	1
E	8	7	1	1
F	2	1	1	1
G	4	8	-4	16
H	5	2	3	9
J	10	9	+1	1
K	9	10	-1	1

$$\Sigma d^2 = 36$$

∴ The rank correlation coefficient R

$$\begin{aligned}
 R &= 1 - \frac{6 \sum d^2}{n(n^2 - 1)} \\
 &= 1 - \frac{6 \times 36}{10(10^2 - 1)} \\
 &= 1 - \frac{216}{990} \\
 &= 1 - 0
 \end{aligned}$$

Comment: since the correlation is 0.78 it implies that there is high positive correlation between the ranks awarded to the contestants. $0.78 > 0$ and $0.78 > 0.5$

Example

Contestant	1 st assessor	2 nd assessor	d	d ²
A	1	2	-1	1
B	5 (5.5)	3	2.5	6.25
C	3	4	-1	1
D	2	1	1	1
E	4	5	-1	1
F	5 (5.5)	6.5	-1	1
G	7	6.5	-0.5	0.25
H	8	8	0	0

$$\Sigma d^2 = 11.25$$

Required: Complete the rank correlation coefficient

$$\begin{aligned} \therefore R &= 1 - \frac{6 \Sigma d^2}{n(n^2 - 1)} = 1 - \frac{6 \times 11.25}{8(63)} \\ &= 1 - \frac{67.5}{504} \\ &= 1 - 0.13 \\ &= 0.87 \end{aligned}$$

This implies high positive correlation

Example (Rank Correlation Coefficient)

Sometimes numerical data which refers to the quantifiable variables may be given after which a rank correlation coefficient may be worked out. In such a situation, the rank correlation coefficient will be determined after the given variables have been converted into ranks. See the following example;

Candidates	Math	r	Accounts	r	d	d ²
P	92	1	67	5	-4	16
Q	82	3	88	1	2	4
R	60	5(5.5)	58	7(7.5)	-2	4
S	87	2	80	2	0	0
T	72	4	69	4	0	0
U	60	5(5.5)	77	3	-2.50	6.25
V	52	8	58	7(7.5)	0.5	0.25
W	50	9	60	6	3	9
X	47	10	32	10	0	0
Y	59	7	54	9	-2	4

$$\Sigma d^2 = 43.5$$

$$\therefore \text{Rank correlation } r = 1 - \frac{6 \Sigma d^2}{n(n^2 - 1)}$$

$$= 1 - \frac{6 \times 43.5}{10(10^2 - 1)} = 1 - \frac{261}{990}$$

$$= 0.74 \text{ (High positive correlation between mathematics marks and accounts)}$$

Example

(Product moment correlation)

The following data was obtained during a social survey conducted in a given urban area regarding the annual income of given families and the corresponding expenditures.

Family	(x)Annual income £ 000	(y)Annual expenditure £ 000	xy	x ²	y ²
A	420	360	151200	176400	129600
B	380	390	148200	144400	152100
C	520	510	265200	270400	260100
D	610	500	305000	372100	250000
E	400	360	144000	160000	129600
F	320	290	92800	102400	84100
G	280	250	70000	78400	62500
H	410	380	155800	168100	144400
J	380	240	91200	144400	57600
K	300	270	81000	90000	72900
Total	4020	3550	1504400	1706600	1342900

Required

Calculate the product moment correlation coefficient briefly comment on the value obtained

The produce moment correlation

$$r = \frac{n \sum xy - \sum x \sum y}{\sqrt{n \sum x^2 - (\sum x)^2} \times \sqrt{n \sum y^2 - (\sum y)^2}}$$

Workings:

$$\bar{X} = \frac{4020}{10} = 402 \qquad \bar{Y} = \frac{3550}{10} = 355$$

$$r = \frac{10(1,504,400) - (4020)(3550)}{\sqrt{10(1,706,600) - 4020^2} \times \sqrt{10(1,342,900) - (3550)^2}}$$

$$= 0.89$$

Comment: The value obtained 0.89 suggests that the correlation between annual income and annual expenditure is high and positive. This implies that the more one earns the more one spends.

REGRESSION

BASIC CONCEPTS

This is a concept, which refers to the changes which occur in the dependent variable as a result of changes occurring on the independent variable. Knowledge of regression is particularly very useful in business statistics where it is necessary to consider the corresponding changes on dependant variables whenever independent variables change. It should be noted that most business activities involve a dependent variable and either one or more independent variable. Therefore knowledge of regression will enable a business statistician to predict or estimate the expenditure value of a dependant variable when given an independent variable e.g. consider the above example for annual incomes and annual expenditures. Using the regression techniques one can be able to determine the estimated expenditure of a given family if the annual income is known and vice versa.

Difference between correlation and regression analysis

There are two important points of difference between correlation and regression analysis

1. Correlation coefficient as a measure of degree of relationship between x and y. while regression analysis is to study the nature of relationship between the variables.
2. The cause and effect relation is clearly indicated through regression analysis while correlation is merely a tool of ascertaining the degree of relationship between two variables.

The general equation used in simple regression analysis is as follows

$$y = a + bx$$

Where y = Dependant variable

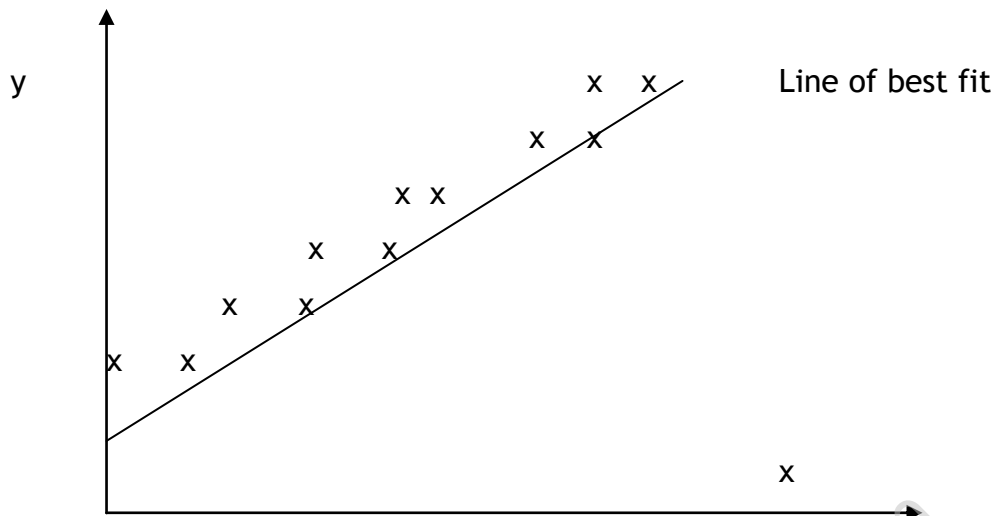
a= Interception y axis (constant)

b = Slope on the y axis

x = Independent variable

- i. The determination of the regression equation such as given above is normally done by using a technique known as "the method of least squares".

Regression equation of y on x i.e. $y = a + bx$



The following sets of equations normally known as normal equation are used to determine the equation of the above regression line when given a set of data.

$$\Sigma y = an + b\Sigma x$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

Where Σy = Sum of y values

Σxy = sum of the product of x and y

Σx = sum of x values

Σx^2 = sum of the squares of the x values

a = The intercept on the y axis

b = Slope gradient line of y on x

NB: The above regression line is normally used in one way only i.e. it is used to estimate the y values when the x values are given.

Regression line of x on y i.e. $x = a + by$

- The fact that regression lines can only be used in one way leads to what is known as a regression paradox
- This means that the regression lines are not ordinary mathematical line graphs which may be used to estimate the x and y simultaneously
- Therefore one has to be careful when using regression lines as it becomes necessary to develop an equation for x and y before doing the estimation.

The following example will illustrate how regression lines are used

Example

An investment company advertised the sale of pieces of land at different prices. The following table shows the pieces of land their acreage and costs

Piece of land	(x) Acreage Hectares	(y) Cost £ 000	xy	x ²
A	2.3	230	529	5.29
B	1.7	150	255	2.89
C	4.2	450	1890	17.64
D	3.3	310	1023	10.89
E	5.2	550	2860	27.04
F	6.0	590	3540	36
G	7.3	740	5402	53.29
H	8.4	850	7140	70.56
J	5.6	530	2969	31.36
	$\Sigma x = 44.0$	$\Sigma y = 4400$	$\Sigma xy = 25607$	$\Sigma x^2 = 254.96$

Required

Determine the regression equations of

i. y on x and hence estimate the cost of a piece of land with 4.5 hectares

ii. Estimate the expected average if the piece of land costs £ 900,000

$$\Sigma y = an + b\Sigma xy$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

By substituting of the appropriate values in the above equations we have

$$4400 = 9a + 44b \dots\dots (i)$$

$$25607 = 44a + 254.96b \dots\dots (ii)$$

By multiplying equation (i) by 44 and equation (ii) by 9 we have

$$193600 = 396a + 1936b \dots\dots (iii)$$

$$230463 = 396a + 2294.64b \dots\dots (iv)$$

By subtraction of equation (iii) from equation (iv) we have

$$36863 = 358.64b$$

$$102.78 = b$$

by substituting for b in (i)

$$4400 = 9a + 44(102.78)$$

$$4400 - 4522.32 = 9a$$

$$-122.32 = 9a$$

$$-13.59 = a$$

Therefore the equation of the regression line of y on x is

$$Y = 13.59 + 102.78x$$

When the acreage (hectares) is 4.5 then the cost

$$(y) = -13.59 + (102.78 \times 4.5)$$

$$= 448.92$$

$$= \text{£ } 448, 920$$

Note that

Where the regression equation is given by

$$y = a + bx$$

Where a is the intercept on the y axis and

b is the slope of the line or regression coefficient

n is the sample size

then,

$$\text{intercept } a = \frac{\sum y - b \sum x}{n}$$

$$\text{Slope } b = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2}$$

Example

The calculations for our sample size $n = 10$ are given below. The linear regression model is

$$y = a + bx$$

Table

Distance x miles	Time y mins	xy	x^2	y^2
3.5	16	56.0	12.25	256
2.4	13	31.0	5.76	169
4.9	19	93.1	24.01	361
4.2	18	75.6	17.64	324
3.0	12	36.0	9.0	144
1.3	11	14.3	1.69	121
1.0	8	8.0	1.0	64
3.0	14	42.0	9.0	196
1.5	9	13.5	2.25	81
4.1	16	65.6	16.81	256
$\Sigma x = 28.9$	$\Sigma y = 136$	$\Sigma xy = 435.3$	$\Sigma x^2 = 99.41$	$\Sigma y^2 = 1972$

$$\text{The Slope } b = \frac{10 \times 435.3 - 28.9 \times 136}{10 \times 99.41 - 28.9^2} = \frac{422.6}{158.9}$$

$$= 2.66$$

$$\text{and the intercept } a = \frac{136 - (2.66 \times 28.9)}{10}$$

$$= 5.91$$

We now insert these values in the linear model giving

$$y = 5.91 + 2.66x$$

or

Delivery time (mins) = 5.91 + 2.66 (delivery distance in miles)

The slope of the regression line is the estimated number of minutes per mile needed for a delivery. The intercept is the estimated time to prepare for the journey and to deliver the goods which is the time needed for each journey other than the actual traveling time.

PREDICTION WITHIN THE RANGE OF SAMPLE DATA

We can use the linear regression model to predict the mean of dependant variable for any given value of independent variable

For example if the sample model is given by

$$\text{Time (min)} = 5.91 + 2.66 (\text{distance in miles})$$

Then if the distance is 4.0 miles then our estimated mean time is

$$\hat{Y} = 5.91 + 2.66 \times 4.0 = 16.6 \text{ minutes}$$

Multiple Linear Regression Models

There are situations in which there is more than one factor which influence the dependent variable

Example

Cost of production per week in a large department depends on several factors;

- i. Total numbers of hours worked
- ii. Raw material used during the week
- iii. Total number of items produced during the week
- iv. Number of hours spent on repair and maintenance

It is sensible to use all the identified factors to predict department costs

Scatter diagram will not give the relationship between the various factors and total costs

The linear model for multiple linear regression is of the type; (which is the line of best fit).

$$y = \alpha + b_1x_1 + b_2x_2 + \dots + b_nx_n$$

We assume that errors or residuals are negligible.

In order to choose between the models we examine the values of the multiple correlation coefficient r and the standard deviation of the residuals α .

A model which describes well the relationship between y and x 's has multiple correlation coefficient r close to ± 1 and the value of α which is small.

Example

Odino chemicals limited are aware that its power costs are semi variable cost and over the last six months these costs have shown the following relationship with a standard measure of output.

Month	Output (standard units)	Total power costs £ 000
1	12	6.2
2	18	8.0
3	19	8.6
4	20	10.4
5	24	10.2
6	30	12.4

Required

- i. Using the method of least squares, determine an appropriate linear relationship between total power costs and output
- ii. If total power costs are related to both output and time (as measured by the number of the month) the following least squares regression equation is obtained
 Power costs = 4.42 + (0.82) output + (0.10) month
 Where the regression coefficients (i.e. 0.82 and 0.10) have t values 2.64 and 0.60 respectively and coefficient of multiple correlation amounts to 0.976
 Compare the relative merits of this fitted relationship with one you determine in (a). Explain (without doing any further analysis) how you might use the data to forecast total power costs in seven months.

Solution

a)

Output (x)	Power costs (y)	x^2	y^2	xy
12	6.2	144	38.44	74.40
18	8.0	324	64.00	144.00
19	8.6	361	73.96	163.40
20	10.4	400	108.16	208.00
24	10.2	576	104.04	244.80
30	12.4	900	153.76	372.00
$\Sigma x = 123$	$\Sigma y = 55.8$	$\Sigma x^2 = 2705$	$\Sigma y^2 = 542.36$	$\Sigma xy = 1,206.60$

$$b = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2}$$

$$= \frac{6 \times 1206.6 - 123 \times 55.8}{6 \times 2705 - (123)^2}$$

$$= \frac{376.2}{1101} = 0.342$$

$$\begin{aligned}
 \text{a} &= \frac{1}{n} (\Sigma y - b \Sigma x) \\
 &= \frac{1}{6} \times (55.8 - 0.342) \times 123 \\
 &= 2.29
 \end{aligned}$$

\therefore (Power costs) = 2.29 + 0.342 (output)

b. For linear regression calculated above, the coefficient of correlation r is

$$\begin{aligned}
 r &= \frac{(6 \times 1206.6) - (123 \times 55.8)}{\sqrt{6 \times 2705 - 123 \times 123} \sqrt{6 \times 542.36 - 55.8 \times 55.8}} \\
 &= \frac{376.2}{\sqrt{1101 \times 140.52}} \\
 &= 0.96
 \end{aligned}$$

This shows a strong correlation between power cost and output. The multiple correlations when both output and time are considered at the same time are 0.976.

We observe that there has been very little increase in r which means that inclusion of time variable does not improve the correlation significantly.

The value for time variable is only 0.60 which is insignificant as compared with a t value of 2.64 for the output variable.

In fact, if we work out correlation between output and time, there will be a high correlation. Hence there is no necessity of taking both the variables. Inclusion of time does improve the correlation coefficient but by a very small amount.

If we use the linear regression analysis and attempt to find the linear relationship between output and time i.e.

Month	Output
1	12
2	18
3	19
4	20
5	24
6	30

The value of b and a will turn out to be 3.11 and 9.6 i.e. relationship will be of the form

Output = $9.6 + 3.11 \times \text{month}$
For this equation forecast for 7th month will be

$$\begin{aligned}\text{Output} &= 9.6 + 3.11 \times 7 \\ &= 9.6 + 21.77 \\ &= 31.37 \text{ units}\end{aligned}$$

Using the equation, Power costs = $2.29 + 0.34 \times \text{output}$
 $= 2.29 + 0.34 \times 31.37$
 $= 2.29 + 10.67$
 $= 12.96$ i.e. £ 12,960

Non Linear Relationships

If the scatter diagram and the correlation coefficient do not indicate linear relationship, then the relationship may be non - linear

Two such relationships are of peculiar interest

$$y = ab^x \quad \text{and} \quad y = ax^b$$

Both of these can be reduced to linear model. Simple or multiple linear regression methods are then used to determine the values of the coefficients

i. Exponential model

$$y = ab^x$$

Take log of both sides

$$\log y = \log a + \log b^x$$

$$\log y = \log a + x \log b$$

Let $\log y = Y$ and $\log a = A$ and $\log b = B$

Thus we get $Y = A + Bx$. This is a linear regression model

ii. Geometric model

$$y = ax^b$$

Using the same technique as above

$$\log y = \log a + b \log x$$

$$Y = A + bX$$

Where $Y = \log y$

$$A = \log a$$

$$X = \log x$$

Using linear regression technique (the method of least squares), it is possible to calculate the value of a and b

CHAPTER SIX ELEMENTS OF PROBABILITY

SPECIFIC OBJECTIVES

At the end of the topic the trainee should be able to:

- Discuss the basic concepts of probability;
- Apply the techniques of counting and set theory;
- Apply the laws of probability;
- Apply probability distribution concepts to decision problems.

Introduction

Probability is the ratio of the number of favorable cases to the total number of equal likely cases. Probability is a very popular concept in business management. This is because it covers the risks which may be involved in certain business situations. It is a fact that when a business investment is being arranged, the outcome is usually uncertain. Therefore the concept of probability may be used to describe the **degree of uncertainty of a particular business outcome**. Probability may therefore be defined as the chances of a given event occurring. Numerically, probability values range between 0 and 1. a probability of 0 implies that the event cannot occur at all. A probability of 1 implies that the event will certainly occur.

Therefore other events have their probabilities with values lying between 0 and 1

The formula used to determine probability is as follow

$$\text{Probability (x)} = \frac{r}{n} = \frac{\text{Favourable outcomes}}{\text{Total outcomes}}$$

Uses of Probability in Business

1. Business games of chance e.g. Raffles Lotteries e.t.c.
2. Insurance firms: this is usually done when a new client or property is being insured. The company has to be certain about the chances of the insured risks occurring.
3. Business decision making regarding viability of projects thus the projects with a greater probability has greater chances.

Example

A bag contains 80 balls of which 20 are red, 25 are blue and 35 are white. A ball is picked at random what is the probability that the ball picked is:

- (i) Red ball
- (ii) Black ball
- (iii) Red or Blue ball

Solution

(i) Probability of a red ball = $\frac{\text{Number of red balls in the bag}}{\text{Total number of balls in the bag}} = P(R)$
 $= \frac{20}{80} = \frac{1}{4}$

(ii) Probability of black ball = $\frac{\text{Number of black balls in the bag}}{\text{Total number of balls}} = P(B)$
 $= \frac{0}{80} = 0$

(iii) $P(R \text{ or } B) = \frac{20}{80} \text{ or } \frac{25}{80} = \frac{20}{80} + \frac{25}{80}$
 $= \frac{9}{16}$

Note: in probability or is replaced by a plus (+) sign. See addition rule.

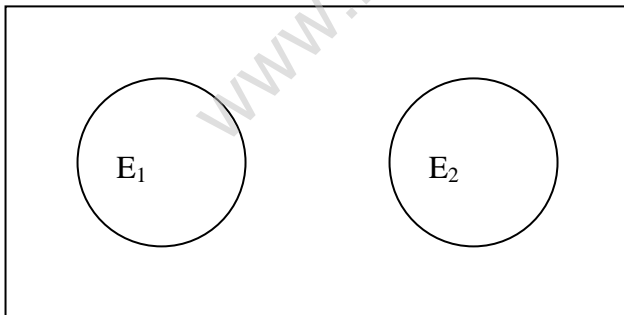
Common terms

Events: an event is a possible outcome of an experiment or a result of a trial or an observation.

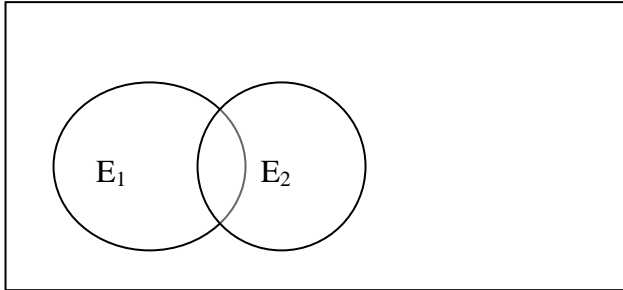
Mutually exclusive events

A set of events is said to be mutually exclusive if the occurrence of any one of the events precludes the occurrence of any of the other events e.g. when tossing a coin, the events are a head or a tail these are said to be mutually exclusive since the occurrence of heads for instance implies that tails cannot and has not occurred.

It can be represented in Venn diagram as.



$$E_1 \cap E_2 = \emptyset$$



Non-mutually exclusive events
(independent events)

$$E_1 \cap E_2 \neq \emptyset$$

Consider a survey in which a random sample of registered voters is selected. For each voter selected their sex and political party affiliation are noted. The events “KANU” and “woman” are not mutually exclusive because the selection of KANU does not preclude the possibility that the voter is also a woman.

Independent Events

Events are said to be independent when the occurrence of any of the events does not affect the occurrence of the other(s).

E.g. the outcome of tossing a coin is independent of the outcome of the preceding or succeeding toss.

Example

From a pack of playing cards what is the probability of;

- (i) Picking either a ‘Diamond’ or a ‘Heart’ → mutually exclusive
- (ii) Picking either a ‘Flower’ or an ‘Ace’ → independent events

Solution.

$$\begin{aligned} \text{(i)} \quad & P(\text{Diamond or Heart}) \\ &= P(\text{Diamond}) + P(\text{Heart}) \\ &= \frac{13}{52} + \frac{13}{52} = \frac{26}{52} \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad & P(\text{Flower or Ace}) \\ &= P(\text{Flower}) + P(\text{Ace}) - P(\text{Flower and Ace}) \\ &= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} \\ &= \frac{4}{52} = 0.31 \end{aligned}$$

Note: that the formula used in case of independent events is different to the one of mutually exclusive.

There are mainly four schools of thought on probability

- i) the classical to priors approach
- ii) the relative frequency of empirical approach
- iii) the axiomatic approach
- iv) the personalistic approach

The classical approach

This school of thought assumes that all the possible outcomes of an experiment are mutually exclusive and equally likely.

$$P(E) = \frac{\text{number of outcomes favorable to the occurrence of event E}}{\text{Total number of outcome}} = \frac{a}{a+b}$$

Relative axiomatic approach

The axiomatic approach theory of probability is an honest attempt at constructing a theory probability large tree from the inadequacies of both the classical and empirical approaches.

Personality approach

According to the personality or subjective concept the probability of an event is the degree of confidence placed in the occurrence of an event by a particular individual based on the evidence available to him.

Relative frequency approach

The relative frequency theoreticians agree that the only valid procedure for determining event probabilities is through repetitive experiments

$$P(E) = a/n$$

Rules of Probability

- (a) Additional Rule - This rule is used to calculate the probability of two or more mutually exclusive events. In such circumstances the probability of the separate events must be added.

Example

What is the probability of throwing a 3 or a 6 with a throw of a die?

Solution

$$P(\text{throwing a 3 or a 6}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

- (b) Multiplicative rule

This is used when there is a string of independent events for which individual probability is known and it is required to know the overall probability.

Example

What is the probability of a 3 and a 6 with two throws of a die?

Solution

P (throwing a 3) and P (6)

$$= P (3) \text{ and } P (6) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

Note: 1) In probability 'and' is replaced by 'x' - multiplication.

- 2) $P(x) \text{ and } P(y) \neq P(x \text{ and } y)$ note that these two are different. The first implies P(x) happening and P(y), but if the order of which happened first is unimportant then we have p(x and y).

In the example above:

$$P (3) \text{ and } P (6) = \frac{1}{36}$$

But

$$\begin{aligned} P (3 \text{ and } 6) &= P (3 \text{ followed by } 6) \text{ or } P (6 \text{ followed by } 3) \\ &= [P (3) P (6)] \text{ or } [P (6) P (3)] \\ &= \frac{1}{36} + \frac{1}{36} = \frac{1}{18} \end{aligned}$$

Conditional Probability

This is the probability associated with combinations of events but given that some prior result has already been achieved with one of them.

It's expressed in the form of

$P (x|y)$ = Probability of x given that y has already occurred.

$$P (x|y) = \frac{P(xy)}{P(y)} \rightarrow \text{conditional probability formula.}$$

Example:

In a competitive examination, 30 candidates are to be selected. In all 600 candidates appear in a written test, and 100 will be called for the interview.

- (i) What is the probability that a person will be called for the interview?
- (ii) Determine the probability of a person getting selected if he has been called for the interview?
- (iii) Probability that person is called for the interview and is selected?

Solution:

Let event A be that the person is called for the interview and event B that he is selected.

- (i) $\therefore P(A) = \frac{100}{600} = \frac{1}{6}$
- (ii) $P(B|A) = \frac{30}{100} = \frac{3}{10}$
- (iii) $P(AB) = P(A) \times P(B|A)$
 $= \frac{1}{6} \times \frac{3}{10} = \frac{3}{60} = \frac{1}{20}$

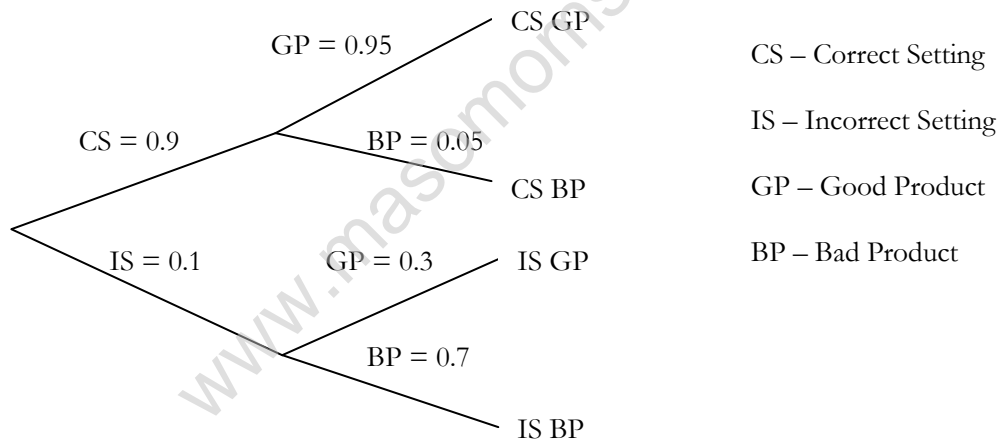
Example:

From past experience a machine is known to be set up correctly on 90% of occasions. If the machine is set up correctly then 95% of good parts are expected but if the machine is not set up correctly then the probability of a good part is only 30%.

On a particular day the machine is set up and the first component produced and found to be good. What is the probability that the machine is set up correctly.

Solution:

This is displayed in the form of a probability tree or diagram as follows:



$$P(\text{CSGP}) = 0.9 \times 0.95 = 0.855$$

$$P(\text{CSBP}) = 0.9 \times 0.05 = 0.045$$

$$P(\text{ISGP}) = 0.1 \times 0.3 = 0.03$$

$$P(\text{ISBP}) = 0.1 \times 0.7 = \underline{0.07}$$

$$1.00$$

$$\begin{aligned}
 \text{Probability of getting a good part (GP)} &= \text{CSGP or ISGP} \\
 &= \text{CSGP} + \text{ISGP} \\
 &= 0.855 + 0.03 = 0.885
 \end{aligned}$$

Note: Good parts may be produced when the machine is correctly set up and also when it's incorrectly setup. In 1000 trials, 855 occasions when its correctly setup and good parts produced (CSGP) and 30 occasions when it's incorrectly setup and good parts produced (ISGP).

Probability that the machine is correctly set up after getting a good part.

$$= \frac{\text{Number of favourable outcomes}}{\text{Total possible outcomes}} = \frac{P(\text{CSGP})}{P(\text{GP})} = \frac{0.855}{0.885} = 0.966$$

Or

$$= P(\text{CS}|\text{GP}) = \frac{P(\text{CSGP})}{P(\text{GP})} = \frac{0.855}{0.885} = 0.966$$

Example

In a class of 100 students, 36 are male and studying accounting, 9 are male but not studying accounting, 42 are female and studying accounting, 13 are female and are not studying accounting.

Use these data to deduce probabilities concerning a student drawn at random.

Solution:

	Accounting A	Not accounting \bar{A}	Total
Male M	36	9	45
Female F	42	13	55
Total	78	22	100

$$P(M) = \frac{45}{100} = 0.45$$

$$P(F) = \frac{55}{100} = 0.55$$

$$P(A) = \frac{78}{100} = 0.78$$

$$P(\bar{A}) = \frac{22}{100} = 0.22$$

$$P(M \text{ and } A) = P(A \text{ and } M) = \frac{36}{100} = 0.36$$

$$P(M \text{ and } \bar{A}) = 0.09$$

$$P(F \text{ and } \bar{A}) = 0.13$$

These probabilities can be express differently as;

$$P(M) = P(M \text{ and } A) \text{ or } P(M \text{ and } \bar{A})$$

$$= 0.36 + 0.09 = 0.45$$

$$P(F) = P(F \text{ and } A) \text{ or } P(F \text{ and } \bar{A}) \\ = 0.42 + 0.13 = 0.55$$

$$P(A) = P(A \text{ and } M) + P(A \text{ and } F) = 0.36 + 0.42 = 0.78$$

$$P(\bar{A}) = P(\bar{A} \text{ and } M) + P(\bar{A} \text{ and } F) = 0.09 + 0.13 = 0.22$$

Now calculate the probability that a student is studying accounting given that he is male.

This is a conditional probability given as $P(A|M)$

$$P(A|M) = \frac{P(A \text{ and } M)}{P(M)} = \frac{0.36}{0.45} = 0.80$$

From the formula above we get that,
 $P(A \text{ and } M) = P(M) P(A|M)$ (i)
 Note that $P(A|M) \neq P(M|A)$

Since $P(M|A) = \frac{P(A \text{ and } M)}{P(A)}$ this is known as the Bayes' rule.

Bayes' rule/Theorem

This rule or theorem is given by

$$P(A|B) = \frac{P(A) \times P(B|A)}{P(B)}$$

It's used frequently in decision making where information is given the in form of conditional probabilities and the reverse of these probabilities must be found.

Example

Analysis of questionnaire completed by holiday makers showed that 0.75 classified their holiday as good at Malindi. The probability of hot weather in the resort is 0.6. If the probability of regarding holiday as good given hot

weather is 0.9, what is the probability that there was hot weather if a holiday maker considers his holiday good?

Solution

$$P(A|B) = \frac{P(A) \times P(B|A)}{P(B)}$$

Let H = hot weather

G = Good

P(G) = 0.75 P(H) = 0.6 and P(G|H) = 0.9 (Probability of regard holiday as good given hot weather)

Now the question requires us to get

P(H|G) = Probability of (there was) hot weather given that the holiday has been rated as good).

$$= \frac{P(H)P(G|H)}{P(G)} = \frac{(0.6)(0.9)}{0.75} = 0.72.$$

Application

1. A machine comprises of 3 transformers A, B and C. The machine may operate if at least 2 transformers are working. The probabilities of each transformer working are given as shown below;

P(A) = 0.6, P(B) = 0.5, P(C) = 0.7

A mechanical engineer went to inspect the working conditions of those transformers. Find the probabilities of having the following outcomes

- i. Only one transformer operating
- ii. Two transformers are operating
- iii. All three transformers are operating
- iv. None is operating
- v. At least 2 are operating
- vi. At most 2 are operating

Solution

P(A) = 0.6 P(\bar{A}) = 0.4 P(B) = 0.5 P(\sim B) = 0.5
P(C) = 0.7 P(\bar{C}) = 0.3

i. P(only one transformer is operating) is given by the following possibilities

	1 st	2 nd	3 rd	
P	(A	\bar{B}	\bar{C})	= 0.6 x 0.5 x 0.3 = 0.09
P	(\bar{A}	B	\bar{C})	= 0.4 x 0.5 x 0.3 = 0.06
P	(\bar{A}	\bar{B}	C)	= 0.4 x 0.5 x 0.7 = 0.14

∴ P(Only one transformer working)
= 0.09 + 0.06 + 0.14 = 0.29

ii. P(only two transformers are operating) is given by the following possibilities.

	1 st	2 nd	3 rd	
P	(A	B	\bar{C})	= 0.6 x 0.5 x 0.3 = 0.09
P	(A	\bar{B}	C)	= 0.6 x 0.5 x 0.7 = 0.21
P	(\bar{A}	B	C)	= 0.4 x 0.5 x 0.7 = 0.14

∴ P(Only two transformers are operating)

$$= 0.09 + 0.21 + 0.14 = 0.44$$

iii. P(all the three transformers are operating).

$$= P(A) \times P(B) \times P(C)$$

$$= 0.6 \times 0.5 \times 0.7$$

$$= 0.21$$

iv. P(none of the transformers is operating).

$$= P(\bar{A}) \times P(\bar{B}) \times P(\bar{C})$$

$$= 0.4 \times 0.5 \times 0.3$$

$$= 0.06$$

v. P(at least 2 working).

$$= P(\text{exactly 2 working}) + P(\text{all three working})$$

$$= 0.44 + 0.21$$

$$= 0.65$$

vi. P(at most 2 working).

$$= P(\text{Zero working}) + P(\text{one working}) + P(\text{two working})$$

$$= 0.06 + 0.29 + 0.44$$

$$= 0.79$$

POISSON PROBABILITY DISTRIBUTION

This is a set of probabilities which is obtained for discrete events which are described as being rare. Occasions similar to binominal distribution but have very low probabilities and large sample size.

Examples of such events in business are as follows:

- i. Telephone congestion at midnight
- ii. Traffic jams at certain roads at 9 o'clock at night
- iii. Sales boom
- iv. Attaining an age of 100 years (Centureon)

Poisson probabilities are frequently applied in business situations in order to determine the numerical probabilities of such events occurring.

The formula used to determine such probabilities is as follows

$$P(x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

Where x = No. of successes

λ = mean no. of the successes in the sample ($\lambda = np$)

$e = 2.718$

Example 1

A manufacturer assures his customers that the probability of having defective item is 0.005. A sample of 1000 items was inspected. Find the probabilities of having the following possible outcomes

- i. Only one is defective
- ii. At most 2 defective
- iii. More than 3 defective

$$P(x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$(\lambda = np = 1000 \times 0.005) = 5$$

- i. $P(\text{only one is defective}) = P(1) = P(x = 1)$

$$= \frac{2.718^{-5} \times 5^1}{1!} \quad \text{Note that } 2.718^{-5} = \frac{1}{2.718^5}$$

$$= \frac{5}{2.718^5}$$

$$= \frac{5}{148.33}$$

$$= 0.0333$$

- ii. $P(\text{at most 2 defective}) = P(x \leq 2)$

$$= P(0) + P(1) + P(2)$$

$$\begin{aligned}
 P(x = 0) &= \frac{e^{-5} 5^0}{0!} \\
 &= 2.718^{-5} \\
 &= \frac{1}{2.718^{+5}} \\
 &= \frac{1}{148.336} \\
 &= 0.00674
 \end{aligned}$$

$$P(1) = 0.0337$$

$$\begin{aligned}
 P(2) &= \frac{2.718^5 5^2}{2!} \\
 &= \frac{25}{2 \times 148.336} \\
 &= 0.08427
 \end{aligned}$$

$$\begin{aligned}
 P(x \leq 2) &= 0.00674 + 0.0337 + \\
 &0.08427 \\
 &= 0.12471
 \end{aligned}$$

iii. $P(\text{more than 3 defective}) = P(x > 3)$

$$= 1 - [P(0) + P(1) + P(2) + P(3)]$$

POISSON MATHEMATICAL PROPERTIES

1. The mean or expected value = $np = \lambda$
 Where; n = Sample Size
 p = Probability of success
2. The variance = $np = \lambda$
3. Standard deviation = $\sqrt{np} = \sqrt{\lambda}$

Example

The probability of a rare disease striking a given population is 0.003. A sample of 10000 was examined. Find the expected no. suffering from the disease and hence determine the variance and the standard deviation for the above problem

Solution

Sample size n = 10000

$P(\text{a person suffering from the disease}) = 0.003 = p$

\therefore expected number of people suffering from the disease

$$\begin{aligned}
 \text{Mean} &= \lambda = 10000 \times 0.003 \\
 &= 30
 \end{aligned}$$

$$= np = \lambda$$

Variance = $np = 30$

$$\begin{aligned}
 \text{Standard deviation} &= \sqrt{np} = \sqrt{\lambda} \\
 &= \sqrt{30} \\
 &= 5.477
 \end{aligned}$$

NORMAL PROBABILITY DISTRIBUTION

In a continuous distribution, the variable can take any value within a specified range, e.g. 2.21 or 1.64 compared to the specific values taken by a discrete variable e.g. 1 or 3. The probability is represented by the area under the probability density curve between the given values.

The uniform distribution, the normal probability distribution and the exponential distribution are examples of a continuous distribution

The normal distribution is a probability distribution which is used to determine probabilities of continuous variables

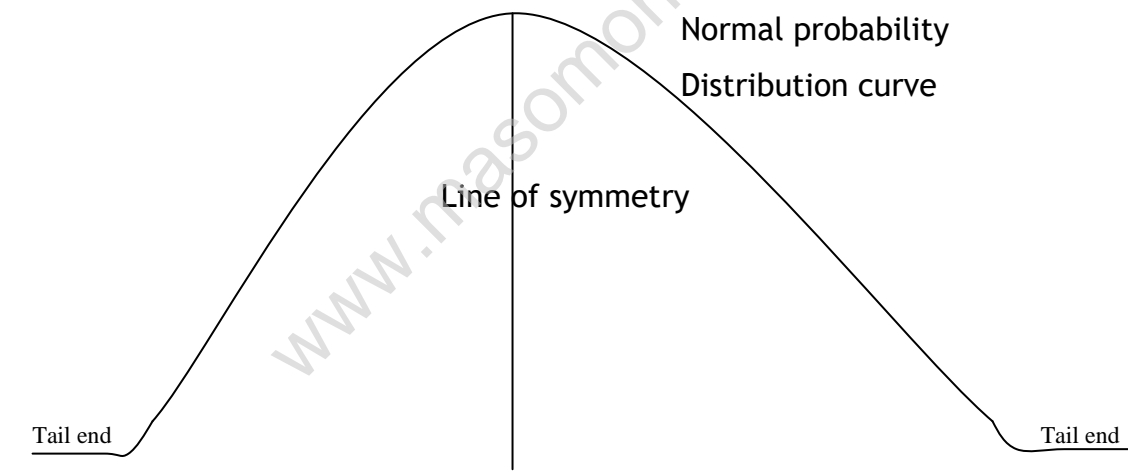
Examples of continuous variables are

- Distances
- Times
- Weights
- Heights
- Capacity e.t.c

Usually continuous variables are those, which can be measured by using the appropriate units of measurement.

Following are the properties of the normal distribution:

1. The total area under the curve is = 1 which is equivalent to the maximum value of probability



2. The line of symmetry divides the curve into two equal halves
3. The two ends of the normal distribution curve continuously approach the horizontal axis but they never cross it
4. The values of the mean, mode and median are all equal

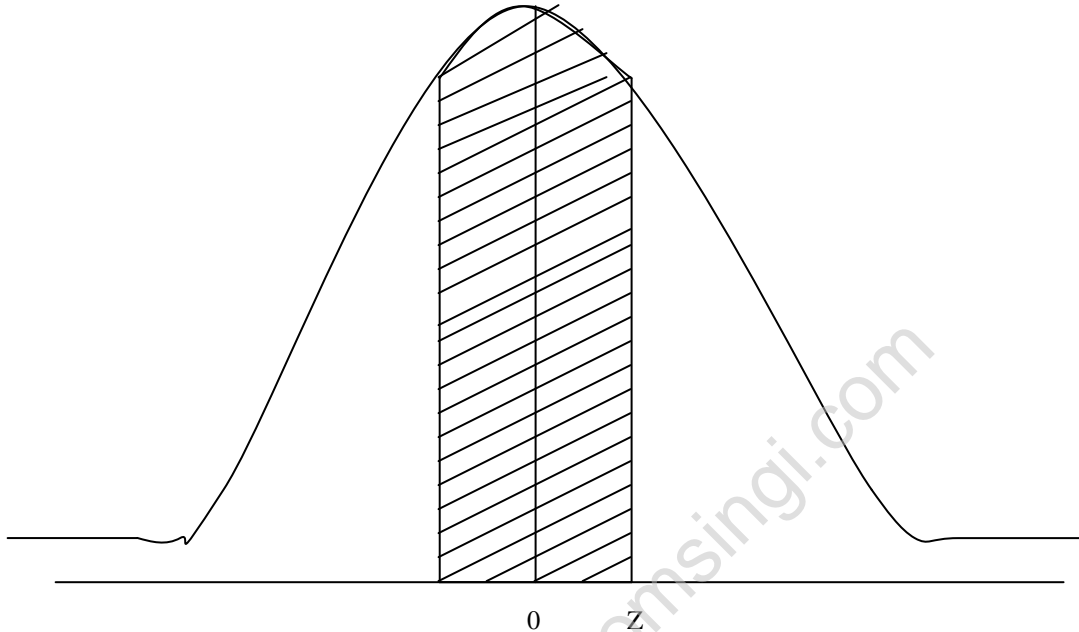
NB: The above distribution curve is referred to as normal probability distribution curve because if a frequency distribution curve is plotted from measurements of a given sample drawn from a normal population then a graph similar to the normal curve must be obtained.

It should be noted that 68% of any population lies within one standard deviation, $\pm 1\sigma$

95% lies within two standard deviations $\pm 2\sigma$

99% lies within three standard deviations $\pm 3\sigma$

Where σ = standard deviation



STANDARDIZATION OF VARIABLES

Before we use the normal distribution curve to determine probabilities of the continuous variables, we need to standardize the original units of measurement, by using the following formula.

$$Z = \frac{x - \mu}{\sigma}$$

Where x = Value to be standardized

Z = Standardization of x

μ = population mean

σ = Standard deviation

Example

A sample of students had a mean age of 35 years with a standard deviation of 5 years. A student was randomly picked from a group of 200 students. Find the probability that the age of the student turned out to be as follows

- i. Lying between 35 and 40
- ii. Lying between 30 and 40
- iii. Lying between 25 and 30
- iv. Lying beyond 45 yrs
- v. Lying beyond 30 yrs
- vi. Lying below 25 years

Solution

(i). The standardized value for 35 years

$$Z = \frac{\chi - \mu}{\sigma} = \frac{35 - 35}{5} = 0$$

The standardized value for 40 years

$$Z = \frac{\chi - \mu}{\sigma} = \frac{40 - 35}{5} = 1$$

∴ the area between $Z = 0$ and $Z = 1$ is 0.3413 (These values are checked from the normal tables see appendix)

The value from standard normal curve tables.

When $z = 0$, $p = 0$

And when $z = 1$, $p = 0.3413$

Now the area under this curve is the area between $z = 1$ and $z = 0$
 $= 0.3413 - 0 = 0.3413$

∴ the probability age lying between 35 and 40 yrs is 0.3413

(ii). 30 and 40 years

$$Z = \frac{\chi - \mu}{\sigma} = \frac{30 - 35}{5} = \frac{-5}{5} = -1$$

$$Z = \frac{\chi - \mu}{\sigma} = \frac{40 - 35}{5} = 1$$

∴ the area between $Z = -1$ and $Z = 1$ is

$= 0.3413$ (lying on the positive side of zero) + 0.3413 (lying on the negative side of zero)

$P = 0.6826$

∴ the probability age lying between 30 and 40 yrs is 0.6826

(iii). 25 and 30 years

$$Z = \frac{\chi - \mu}{\sigma} = \frac{25 - 35}{5} = \frac{-10}{5} = -2$$

$$Z = \frac{\chi - \mu}{\sigma} = \frac{30 - 35}{5} = -1$$

∴ the area between $Z = -2$ and $Z = -1$

Probability area corresponding to $Z = -2$

$= 0.4772$ (the z value to check from the tables is 2)

Probability area corresponding to $Z = -1$

$= 0.3413$ (the z value for this case is 1)

∴ the probability that the age lies between 25 and 30 yrs

$= 0.4772 - 0.3413$ (The area under this curve)

$P = 0.1359$

iv). P(beyond 45 years) is determined as follow = $P(x > 45)$

$$Z = \frac{x - \mu}{\sigma} = \frac{45 - 35}{5} = \frac{+10}{5} = + 2$$

Probability corresponding to $Z = 2 = 0.4772 =$ probability of between 35 and 45

$$\therefore P(\text{Age} > 45\text{yrs}) = 0.5000 - 0.4772 \\ = 0.0228$$

PRACTICE QUESTION

Question One

The quality controller, Mr. Brooks, at Queensville Engineers has become aware of the need for an acceptance sampling programme to check the quality of bought-in components. This is of particular importance for a problem the company is currently having with batches of pump shafts bought in from a local supplier. Mr. Brooks proposes the following criteria to assess whether or not to accept a large batch of pump shafts from this supplier.

From each batch received take a random sample of 50 shafts, and accept that batch if no more than two defectives are found in the sample.

Mr. Brooks needed to calculate the probability of accepting a batch P_a , when the proportion of defectives in the batch, p , is small (under 10%, say)

Required:

- Explain why the Poisson distribution is appropriate to investigate this situation.
- Using the Poisson distribution, determine the probability of accepting a batch P_a , containing $p=2\%$ defectives if the method is used. Determine P_a , for $p = 0\%, 2\%, 5\%, 10\%, 15\%$

Question Two

A woven cloth is liable to contain faults and is subjected to an inspection procedure. Any fault has a probability of 0.7 that it will be detected by the procedure, independent of whether any other fault is detected or not.

Required:

- If a piece of cloth contains three faults, A, B and C,

-
- i) Calculate the probability that A and C are detected, but that B is undetected;
 - ii) Calculate the probability that any two of A, B and C be detected, the other fault being undetected;
 - iii) State the relationship between your answers to parts (i) and (ii) and give reasons for this.
- b) Suppose now that, in addition to the inspection procedure given above, there is a secondary check which has a probability of 0.6 of detecting each fault missed by the first inspection procedure. This probability of 0.6 applies independently to each and every fault undetected by the first procedure.
- i) Calculate the probability that a piece of cloth with one fault has this fault undetected by both the inspection procedure and the secondary check;
 - ii) Calculate the probability that a piece of cloth with two faults has one of these faults detected by either the inspection procedure or the secondary check, and one fault undetected by both;
 - iii) Of the faults detected, what proportions are detected by the inspection procedure and what proportion by the secondary check?

Question Three

A company has three production sections S_1 , S_2 and S_3 which contribute 40%, 35% and 25%, respectively, to total output. The following percentages of faulty units have been observed:

S_1	2%	(0.02)
S_2	3%	(0.03)
S_3	4%	(0.04)

There is a final check before output is dispatched. Calculate the probability that a unit found faulty at this check has come from section 1, S_1

Question Four

Assuming a Binomial Distribution what is the probability of a salesman making 0, 1, 2, 3, 4, 5 or 6 sales in 6 visits if the probability of making a sale on a visit is 0.3?

Do not use tables for this question.

Question Five

Records show that 60% of students pass their examinations at first attempt. Using the normal approximation to the binomial, calculate the probability that at least 65% of a group of 200 students will pass at the first attempt.

Question Six

A batch of 5000 electric lamps has a mean life of 1000 hours and a standard deviation of 75 hours. Assume a normal distribution.

- a) How many lamps will fail before 900 hours?
- b) How many lamps will fail between 950 and 1000 hours?
- c) What proportion of lamps will fail before 925 hours?
- d) Given the same mean life, what would the standard deviation have to be to ensure that no more than 20% of lamps fail before 916 hours?

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CHAPTER SEVEN

SAMPLING

SPECIFIC OBJECTIVE

At the end of the topic the trainee should be able to:

- Discuss the reasons for sampling;
- Differentiate between sampling and census;
- Discuss the various types of sampling.

Introduction

Census method and sampling method can be used in sampling.

Sampling is the only tool which helps to know the characteristics of the universe or population by examining only a small part of it.

Advantages of sampling method

- i) It is cheaper to collect data as only a small part of the whole population is studied.
- ii) The data are collected and analyzed more quickly. Thus sampling saves a lot of time.
- iii) A good quality of labor with better supervision can be provided since only a part of the whole population is to be studied.
- iv) An investigation of small part of the population gives us more detailed information.

Disadvantages of sampling

- Non accurate
- Reliability is low.

Size of the sample

An important decision that has to be taken while adopting a sampling technique is about the size of the sample. However the following two considerations may be kept in mind in determining the appropriate size of the sample;

- a) The size of the sample should increase as the variation in the individual item increases.
- b) The greater the degree of accuracy desired the larger should be the sample size.

Sampling and non- sampling errors

A sample being only a part of a population cannot represent the population no matter how carefully the sample is selected. This gives rise to the difference between the value of sample statistics and the true value of the corresponding population parameters. Such difference is called sampling error for that sample. These errors can be avoided through proper selection

of questionnaires, following up the non-response, proper training of the investigator, and correct manipulation of the collected information.

Sampling errors are of two types.

- I. Biased errors
- II. Unbiased errors

Biased errors

These errors arise from any bias in selecting estimation.

Unbiased errors

These errors arise due to chance difference between the member of population included in the sample and those not included. Thus the total sampling errors is made up of errors due to bias and the random sampling.

Causes of bias

- I. Faulty process of selection
- II. Faulty work during the collection of information

Faulty methods of analysis.

Sampling bias means a systematic component of error which deprives statistical results of its representatives. Bias is introduced by the following methods of selection.

i) Deliberate selection

Bias originates from deliberate selection which is biased on personal judgment of what is representative.

ii) substitution

Sometime sit becomes difficult to make contact with certain member or information is not obtained from certain units then we substitute members or units that are conveniently available such substitution introduce bias.

iii) incomplete coverage

Bias also enters when we fail to cover the whole of he selected sample.

iv) haphazard selection

Haphazard human selection can also introduce bias as every human being has a tendency away from randomness in his choice.

v) Inadequate interviewing

Bias also enters when the interviewing is hasty, incomplete and misleading.

Sampling distribution

A sampling distribution is defined as a probability distribution of the values of a statistics such as mean, standard deviation, proportion computed from all possible samples of the same size. Sampling distribution constitutes the theoretical basis of statistical inference and are considerable important in business decision making.

Methods of Sampling

a). Random or probability sampling methods

They include:

- i. Simple random sampling
- ii. Stratified sampling
- iii. Systematic sampling
- iv. Multi stage sampling

b). Non random probability sampling methods

These consist of:

- i. Judgment sampling
- ii. Quota sampling
- iii. Cluster sampling

Simple Random Sampling

This refers to the sampling technique in which each and every item of the population is given an equal chance of being included in the sample. Since selection of items in the sample depends entirely on chance, this method is also called chance selection or representative sampling.

It is assumed that if the sample is chosen at random and if the size of the sample is sufficiently large, it will represent all groups in the population. Random sampling is of 2 types; sampling with replacement and sampling without replacement.

Sampling is said to be with replacement when from a finite population a sampling unit is drawn observed and then returned to the population before another unit is drawn. The population in this case remains the same and a sampling unit might be selected more than once.

If on the other hand a sampling unit is chosen and not returned to the population after it has been observed the sampling is said to be without replacement.

Random samples may be selected by the help of lottery method or table of random numbers (such as tippet's table of random numbers, Fischer and Yates numbers or Kendall and Babington Smith numbers.)

Stratified sampling

In this case the population is divided into groups in such a way that units within each group are as similar as possible in a process called stratification. The groups are called strata. Simple random samples from each of the strata are collected and combined into a simple. This technique of collecting a sample from a population is called stratified sampling. Stratification may be by age, occupation income group e.t.c.

Systematic Sampling

This sampling is a part of simple random sampling in ascending or descending orders. In systematic sampling a sample is drawn according to some predetermined object. Suppose a population consists of 1000 Units, then every tenth, 20th or 50th item are selected. This method is very easy and economical. It also saves a lot of time

Multistage sampling

This is similar to stratified sampling except division is done on geographical/location basis, e.g. a country can be divided into provinces and then survey is done in 4 towns in each province. This helps to cut traveling costs for a surveyor.

Cluster Sampling

This is where a few geographical regions e.g. a location, town or village are selected at random and say every single household or shop in that area is interviewed. This again cuts on costs.

Judgment Sampling

Here the interviewer selects whom to interview believing that their view is more fundamental since they might be directly affected e.g. to find out effects of public transport one may chose to interview only people who don't own cars and travel frequently to work.

Merits of random or probability sampling

- i) Since the sample is objective and unbiased, it is defensive before the supervisor.
- ii) The size of the sample does not depend upon the expediency or mere tradition but on demonstrable statistical method.
- iii) Te degree of deviation from the parameter i.e. the statistical measure based on population can be estimated.
- iv) It provides a more accurate method of drawing conclusion.

The sample may be combined and evaluated even though accomplished by different individual.

PRACTICE QUESTIONS

Question One

A firm purchases a very large quantity of metal off-cuts and wishes to know the average weight of an off-cut. A random sample of 625 off-cuts is weighed and it is found that the mean sample weight is 150 grams with a sample standard deviation of 30 grams. What is the estimate of the population mean and what is the standard error of the mean? What would be the standard error if the sample size was 1225?

Question Two

A sample of 80 is drawn at random from a population of 800. The sample standard deviation was found to be 6 grams.

- What is the finite population correction factor?
- What is the approximation of the correction factor?
- What is the standard error of the mean?

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CHAPTER EIGHT

ESTIMATION AND TEST OF HYPOTHESIS

Specific Objectives

At the end of this topic the trainee should be able to:

- Define estimation
- Differentiate between the two types of estimation
- Determine the sampling distribution of a statistics
- Determine the confidence interval for a parameter;
- Design a simple hypothesis;
- Define errors in hypothesis testing;
- Test various hypothesis.

Introduction

Statistical inference

It is the process of drawing conclusions about attributes of a population based upon information contained in a sample (taken from the population). It is divided into estimation of parameters and testing of hypothesis. Symbols for statistic of population parameters are as follows.

	Sample Statistic	Population Parameter
Arithmetic mean	\bar{x}	μ
Standard deviation	s	σ
Number of items	n	N

Statistical estimation

Def: It is the procedure of using statistic to estimate a population parameter. It is divided into point estimation (where an estimate of a population parameter is given by a single number) and interval estimation (where an estimate of a population is given by a range in which the parameter may be considered to lie) e.g. a bus meant to take a class of 100 students (population N) for trip has a limit to the maximum weight of 600kg of which it can carry, the teacher realizes he has to find out the weight of the class but without enough time to weigh everyone he picks 25 students selected at random (sample $n = 25$). These students are weighed and their average weight recorded as 64kg (\bar{x} - mean of a sample) with a standard deviation (s), now using this the teacher intends to estimate the average weight of the whole class (μ - population mean) by using the statistical parameters standard deviation (s), and mean of the sample (\bar{x}).

Characteristic of a good estimator

- (i) Unbiased: where the expected value of the statistic is equal to the population parameter e.g. if the expected mean of a sample is equal to the population mean
- (ii) Consistency: where an estimator yields values more closely approaching the population parameter as the sample increases
- (iii) Efficiency: where the estimator has smaller variance on repeated sampling.
- (iv) Sufficiency: where an estimator uses all the information available in the data concerning a parameter

Confidence Interval

The interval estimate or a 'confidence interval' consists of a range (upper confidence limit and lower confidence limit) within which we are confident that a population parameter lies and we assign a probability that this interval contains the true population value

The confidence limits are the outer limits to a confidence interval. Confidence interval is the interval between the confidence limits. The higher the confidence level the greater the confidence interval. For example

A normal distribution has the following characteristic

- i. Sample mean $\pm 1.960 \sigma$ includes 95% of the population
- ii. Sample mean $\pm 2.575 \sigma$ includes 99% of the population

Sampling distribution

LARGE SAMPLES

These are samples that contain a sample size greater than 30 (i.e. $n > 30$)

(a) Estimation of population mean

Here we assume that if we take a large sample from a population then the mean of the population is very close to the mean of the sample

Steps to follow to estimate the population mean includes

- i. Take a random sample of n items where ($n > 30$)
- ii. Compute sample mean (\bar{X}) and standard deviation (S)
- iii. Compute the standard error of the mean by using the following formula

$$S_{\bar{X}} = \frac{S}{\sqrt{n}}$$

Where $S_{\bar{X}}$ = Standard error of mean

S = standard deviation of the sample

n = sample size

- iv. Choose a confidence level e.g. 95% or 99%

- v. Estimate the population mean as under

$$\text{Population mean } \mu = \bar{x} \pm (\text{appropriate number}) \times S_{\bar{x}}$$

'Appropriate number' means confidence level e.g. at 95% confidence level is 1.96 this number is usually denoted by Z and is obtained from the normal tables.

Example

The quality department of a wire manufacturing company periodically selects a sample of wire specimens in order to test for breaking strength. Past experience has shown that the breaking strengths of a certain type of wire are normally distributed with standard deviation of 200 kg. A random sample of 64 specimens gave a mean of 6200 kgs. Find out the population mean at 95% level of confidence

Solution

$$\text{Population mean} = \bar{x} \pm 1.96 S_{\bar{x}}$$

Note that sample size is already $n > 30$ whereas s and \bar{x} are given thus step i), ii) and iv) are provided.

Here: $\bar{x} = 6200$ kgs

$$S_{\bar{x}} = \frac{s}{\sqrt{n}} = \frac{200}{\sqrt{64}} = 25$$

$$\begin{aligned} \text{Population mean} &= 6200 \pm 1.96(25) \\ &= 6200 \pm 49 \\ &= 6151 \text{ to } 6249 \end{aligned}$$

At 95% level of confidence, population mean will be in between 6151 and 6249

Finite Population Correction Factor (FPCF)

If a given population is relatively of small size and sample size is more than 5% of the population then the standard error should be adjusted by multiplying it by the finite population correction factor

$$\text{FPCF is given by} = \sqrt{\frac{N-n}{n-1}}$$

Where N = population size
 n = sample size

Example

A manager wants an estimate of sales of salesmen in his company. A random sample 100 out of 500 salesmen is selected and average sales are found to be Shs. 75,000. If a sample standard deviation is Shs. 15000 then find out the population mean at 99% level of confidence

Solution

Here $N = 500$, $n = 100$, $\bar{X} = 75000$ and $S = 15000$

Now

Standard error of mean

$$\begin{aligned} S_{\bar{x}} &= \frac{s}{\sqrt{n}} \times \sqrt{\frac{N-n}{n-1}} \\ &= \frac{15000}{\sqrt{100}} \times \sqrt{\frac{(500-100)}{(500-1)}} \\ &= \frac{15000}{10} \times \sqrt{\frac{400}{499}} \\ &= \frac{15000}{10} (0.895) \end{aligned}$$

$$S_{\bar{x}} = 1342.50 \text{ at } 99\% \text{ level of confidence}$$

$$\begin{aligned} \text{Population mean} &= \bar{X} \pm 2.58 S_{\bar{x}} \\ &= \text{shs } 75000 \pm 2.58(1342.50) \\ &= \text{shs } 75000 \pm 3464 \\ &= \text{Shs } 71536 \text{ to } 78464 \end{aligned}$$

(b) Estimation of difference between two means

We know that the standard error of a sample is given by the value of the standard deviation (σ) divided by the square root of the number of items in the sample (\sqrt{n}).

But, when given two samples, the standard errors is given by

$$S_{(\bar{x}_A - \bar{x}_B)} = \sqrt{\frac{S_A^2}{n_A} + \frac{S_B^2}{n_B}}$$

Also note that we do estimate the interval not from the mean but from the difference between the two sample means i.e. $(\bar{X}_A - \bar{X}_B)$.

The appropriate number of confidence level does not change

Thus the confidence interval is given by;

$$\begin{aligned} &(\bar{X}_A - \bar{X}_B) \pm \text{Confidence level } S_{(\bar{x}_A - \bar{x}_B)} \\ &= (\bar{X}_A - \bar{X}_B) \pm Z S_{(\bar{x}_A - \bar{x}_B)} \end{aligned}$$

Example

Given two samples A and B of 100 and 400 items respectively, they have the means $\bar{X}_1 = 7$ and $\bar{X}_2 = 10$ and standard deviations of 2 and 3 respectively. Construct confidence interval at 70% confidence level?

Solution

Sample	A	B	
$\bar{X}_1 =$	7	$\bar{X}_2 =$	10
$n_1 =$	100	$n_2 =$	400
$S_1 =$	2	$S_2 =$	3

The standard error of the samples A and B is given by

$$\begin{aligned} S_{(\bar{x}_A - \bar{x}_B)} &= \sqrt{\frac{4}{100} + \frac{9}{400}} \\ &= \sqrt{\frac{25}{400}} = \frac{5}{20} \\ &= \frac{1}{4} = 0.25 \end{aligned}$$

At 70% confidence level, then appropriate number is equal to 1.04 (as read from the normal tables)

$$\bar{X}_1 - \bar{X}_2 = 7 - 10 = -3 = 3$$

We take the absolute value of the difference between the means e.g. the value of $|X|$ = absolute value of X i.e. a positive value of X.

Confidence interval is therefore given by

= $3 \pm 1.04 (0.25)$ From the normal tables a z value of 1.04 gives a value of 0.7.

$$= 3 \pm 0.26$$

$$= 3.26 \text{ and } 2.974$$

$$\text{Thus } 2.974 \leq X \leq 3.26$$

Example 2

A comparison of the wearing out quality of two types of tyres was obtained by road testing. Samples of 100 tyres were collected. The miles traveled until wear out were recorded and the results given were as follows

Tyres	T1	T2
Mean	$\bar{X}_1 = 26400$ miles	$\bar{X}_2 = 25000$ miles
Variance	$S^2_1 = 1440000$ miles	$S^2_2 = 1960000$ miles

Find a confidence interval at the confidence level of 70%

Solution

$$\begin{aligned} \bar{X}_1 &= 26400 \\ \bar{X}_2 &= 25000 \end{aligned}$$

Difference between the two means

$$\begin{aligned}(\bar{X}_1 - \bar{X}_2) &= (26400 - 25000) \\ &= 1,400\end{aligned}$$

Again we take the absolute value of the difference between the two means

We calculate the standard error as follows

$$\begin{aligned}S_{(\bar{x}_A - \bar{x}_B)} &= \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} \\ &= \sqrt{\frac{1,440,000}{100} + \frac{1,960,000}{100}} \\ &= 184.4\end{aligned}$$

Confidence level at 70% is read from the normal tables as 1.04 (Z = 1.04).

Thus the confidence interval is calculated as follows

$$= 1400 \pm (1.04) (184.4)$$

$$= 1400 \pm 191.77$$

$$\text{or } (1400 - 191.77) \text{ to } (1400 + 191.77)$$

$$1,208.23 \leq X \leq 1591.77$$

c) Estimation of population proportions

This type of estimation applies at the times when information cannot be given as a mean or as a measure but only as a fraction or percentage

The sampling theory stipulates that if repeated large random samples are taken from a population, the sample proportion “p” will be normally distributed with mean equal to the population proportion and standard error equal to

$$S_p = \sqrt{\frac{Pq}{n}} = \text{Standard error for sampling of population}$$

proportions

Where n is the sample size and q = 1 - p.

The procedure for estimating a proportion is similar to that for estimating a mean, we only have a different formula for calculating standard.

Example 1

In a sample of 800 candidates, 560 were male. Estimate the population proportion at 95% confidence level.

Solution

Here

$$\text{Sample proportion (P)} = \frac{560}{800} = 0.70$$

$$q = 1 - p = 1 - 0.70 = 0.30$$

$$n = 800$$

$$\sqrt{\frac{pq}{n}} = \sqrt{\frac{(0.70)(0.30)}{800}}$$

$$S_p = 0.016$$

Population proportion

$$= P \pm 1.96 S_p \text{ where } 1.96 = Z.$$

$$= 0.70 \pm 1.96 (0.016)$$

$$= 0.70 \pm 0.03$$

$$= 0.67 \text{ to } 0.73$$

$$= \text{between } 67\% \text{ to } 73\%$$

Example 2

A sample of 600 accounts was taken to test the accuracy of posting and balancing of accounts where in 45 mistakes were found. Find out the population proportion. Use 99% level of confidence

Solution

Here

$$n = 600; p = \frac{45}{600} = 0.075$$

$$q = 1 - 0.075 = 0.925$$

$$S_p = \sqrt{\frac{pq}{n}} = \sqrt{\frac{(0.075)(0.925)}{600}}$$

$$= 0.011$$

Population proportion

$$= P \pm 2.58 (S_p)$$

$$= 0.075 \pm 2.58 (0.011)$$

$$= 0.075 \pm 0.028$$

= 0.047 to 0.10

= between 4.7% to 10%

d) Estimation of difference between population proportions

Let the two proportions be given by P_1 and P_2 , respectively

Then the difference (absolute) between the two proportions is given by $(P_1 - P_2)$

The standard error is given by

$$S_{(P_1 - P_2)} = \sqrt{\frac{pq}{n_1} + \frac{pq}{n_2}} \text{ where } p = \frac{P_1 n_1 + P_2 n_2}{n_1 + n_2} \text{ and } q = 1 - p$$

Then given the confidence level, the confidence interval between the two population proportions is given by

$$(P_1 - P_2) \pm \text{Confidence level } S_{(P_1 - P_2)}$$

$$= (P_1 - P_2) \pm Z \sqrt{\frac{pq}{n_1} + \frac{pq}{n_2}}$$

Where $P = \frac{P_1 n_1 + P_2 n_2}{n_1 + n_2}$ always remember to convert P_1 & P_2 to P .

2. SMALL SAMPLES

(a) Estimation of population mean

If the sample size is small ($n < 30$) the arithmetic mean of small samples are not normally distributed. In such circumstances, students' t-distribution must be used to estimate the population mean.

In this case

$$\text{Population mean } \mu = \bar{X} \pm t S_x$$

\bar{X} = Sample mean

$$S_x = \frac{s}{\sqrt{n}}$$

S = standard deviation of samples = $\sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}$ for small samples.

n = sample size

v = $n - 1$ degrees of freedom.

The value of t is obtained from students t-distribution tables for the required confidence level

Example

A random sample of 12 items is taken and is found to have a mean weight of 50 grams and a standard deviation of 9 grams
What is the mean weight of population?

- a) with 95% confidence
- b) with 99% confidence

Solution

$$\bar{X} = 50; S = 9; v = n - 1 = 12 - 1 = 11; \quad S_x = \frac{s}{\sqrt{n}} = \frac{9}{\sqrt{12}}$$

$$\mu = \bar{x} \pm t_{S_x}$$

At 95% confidence level

$$\begin{aligned} \mu &= 50 \pm 2.262 \left(\frac{9}{\sqrt{12}} \right) \\ &= 50 \pm 5.72 \text{ grams} \end{aligned}$$

Therefore we can state with 95% confidence that the population mean is between 44.28 and 55.72 grams

At 99% confidence level

$$\begin{aligned} \mu &= 50 \pm 3.25 \left(\frac{9}{\sqrt{12}} \right) \\ &= 50 \pm 8.07 \text{ grams} \end{aligned}$$

Therefore we can state with 99% confidence that the population mean is between 41.93 and 58.07 grams

Note: To use the t distribution tables it is important to find the degrees of freedom ($v = n - 1$). In the example above $v = 12 - 1 = 11$

From the tables we find that at 95% confidence level against 11 and under 0.05, the value of $t = 2.201$

HYPOTHESIS TESTING

Definition

- A hypothesis is a claim or an opinion about an item or issue. Therefore it has to be tested statistically in order to establish whether it is correct or not correct
- Whenever testing a hypothesis, one must fully understand the 2 basic hypotheses to be tested namely
 - i. The null hypothesis (H_0)

-
- ii. The alternative hypothesis(H_1)

The null hypothesis

This is the hypothesis being tested, the belief of a certain characteristic e.g. Kenya Bureau of Standards (KBS) may walk to a sugar making company with an intention of confirming that the 2kgs bags of sugar produced are actually 2kgs and not less, they conduct hypothesis testing with the null hypothesis being: H_0 = each bag weighs 2kgs. The testing will set out to confirm this or to refute it.

The alternative hypothesis

While formulating a null hypothesis we also consider the fact that the belief might be found to be untrue hence we will reject it. We therefore formulate an alternative hypothesis which is a contradiction to the null hypothesis, thus when we reject the null hypothesis we accept the alternative hypothesis.

In our example the alternative hypothesis would be

$$H_1 = \text{each bag does not weigh 2kg}$$

Acceptance and rejection regions

All possible values which a test statistic may either assume consistency with the null hypothesis (acceptance region) or lead to the rejection of the null hypothesis (rejection region or **critical region**)

The values which separate the rejection region from the acceptance region are called **critical values**

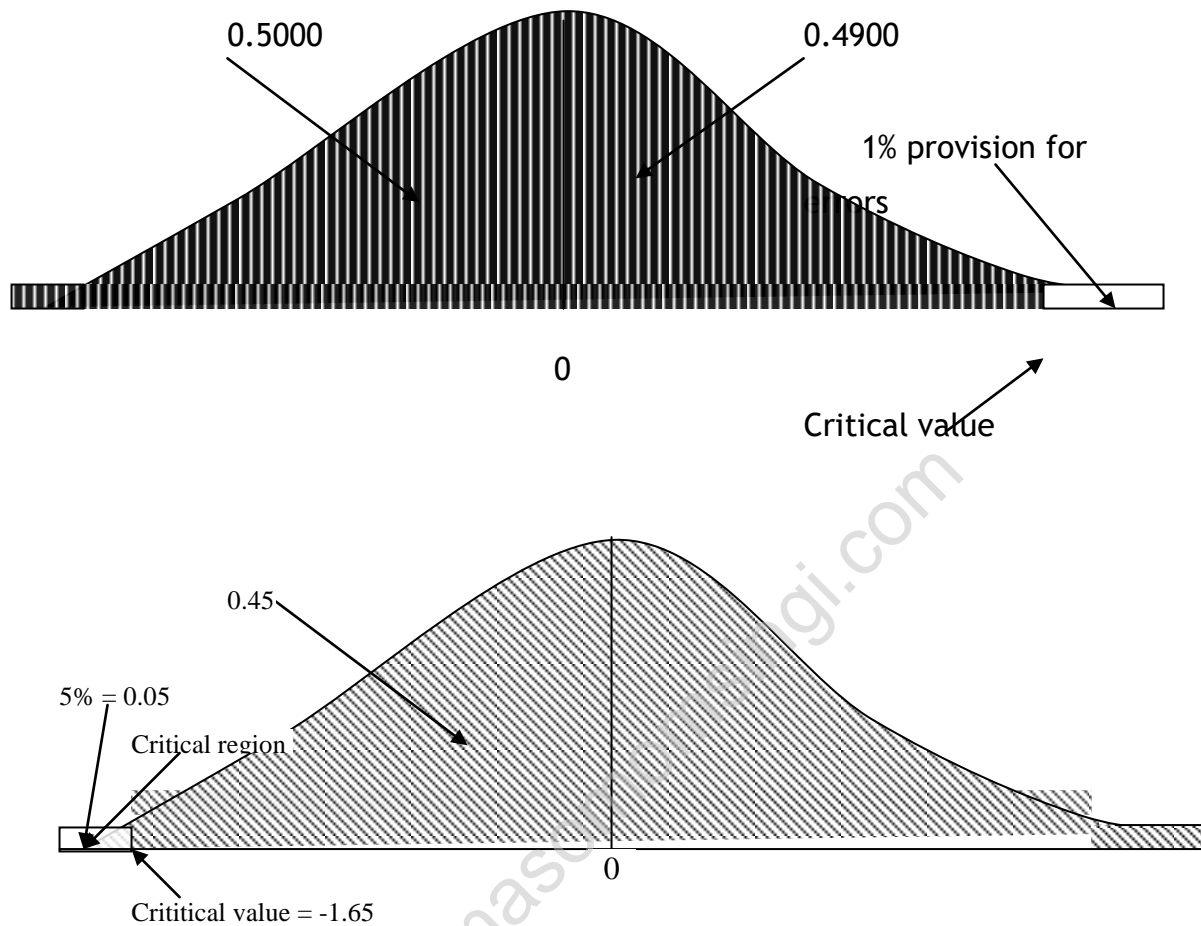
Type I and type II errors

While testing hypothesis (H_0) and deciding to either accept or reject a null hypothesis, there are four possible occurrences.

- Acceptance of a true hypothesis (correct decision) - accepting the null hypothesis and it happens to be the correct decision. Note that statistics does not give absolute information, thus its conclusion could be wrong only that the probability of it being right are high.
- Rejection of a false hypothesis (correct decision).
- Rejection of a true hypothesis - (incorrect decision) - this is called type I error, with probability = α .
- Acceptance of a false hypothesis - (incorrect decision) - this is called type II error, with probability = β .

Levels of significance

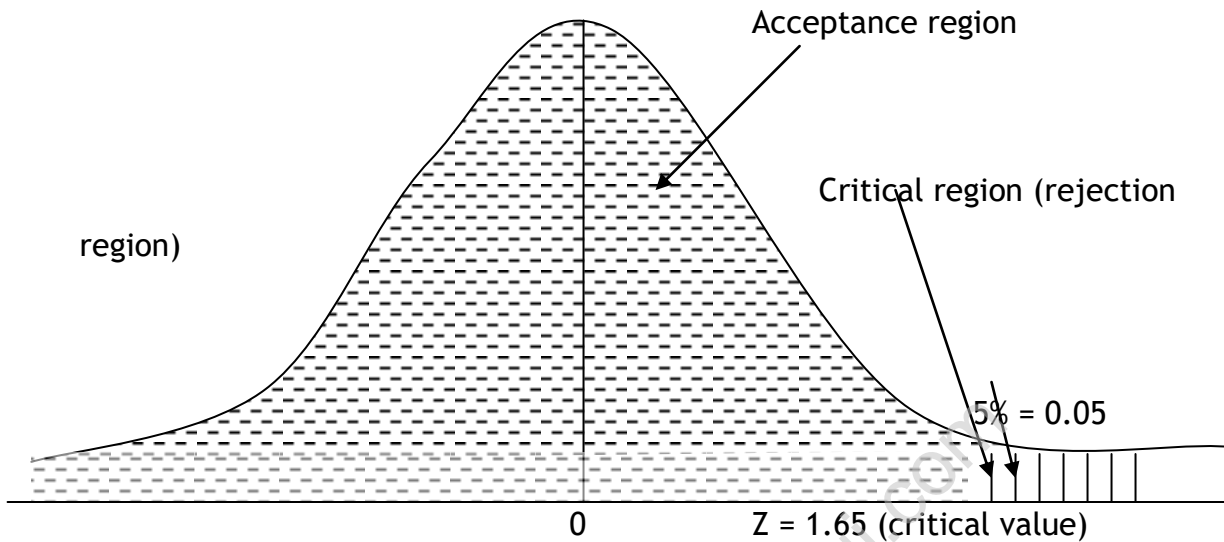
A level of significance is a probability value which is used when conducting tests of hypothesis. A level of significance is basically the probability of one making an incorrect decision after the statistical testing has been done. Usually such probability used are very small e.g. 1% or 5%



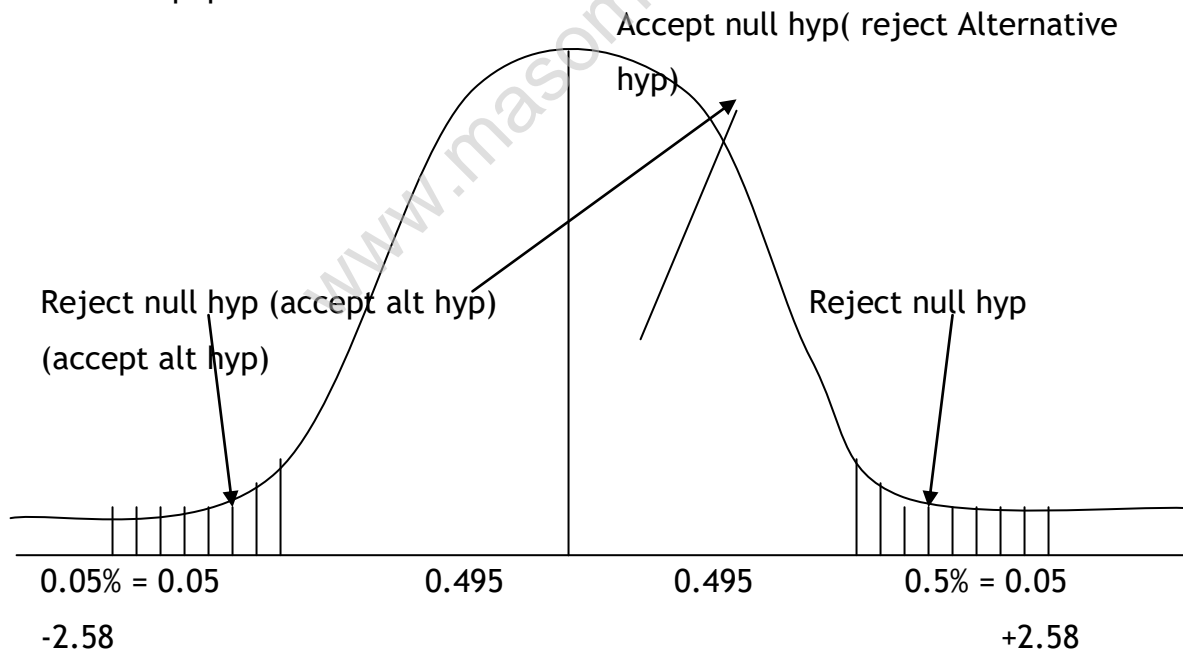
NB: If the standardized value of the mean is less than -1.65 we reject the null hypothesis (H_0) and accept the alternative Hypothesis (H_1) but if the standardized value of the mean is more than -1.65 we accept the null hypothesis and reject the alternative hypothesis

The above sketch graph and level of significance are applicable when the sample mean is $<$ (i.e. less than the population mean)

The following is used when sample mean > population mean



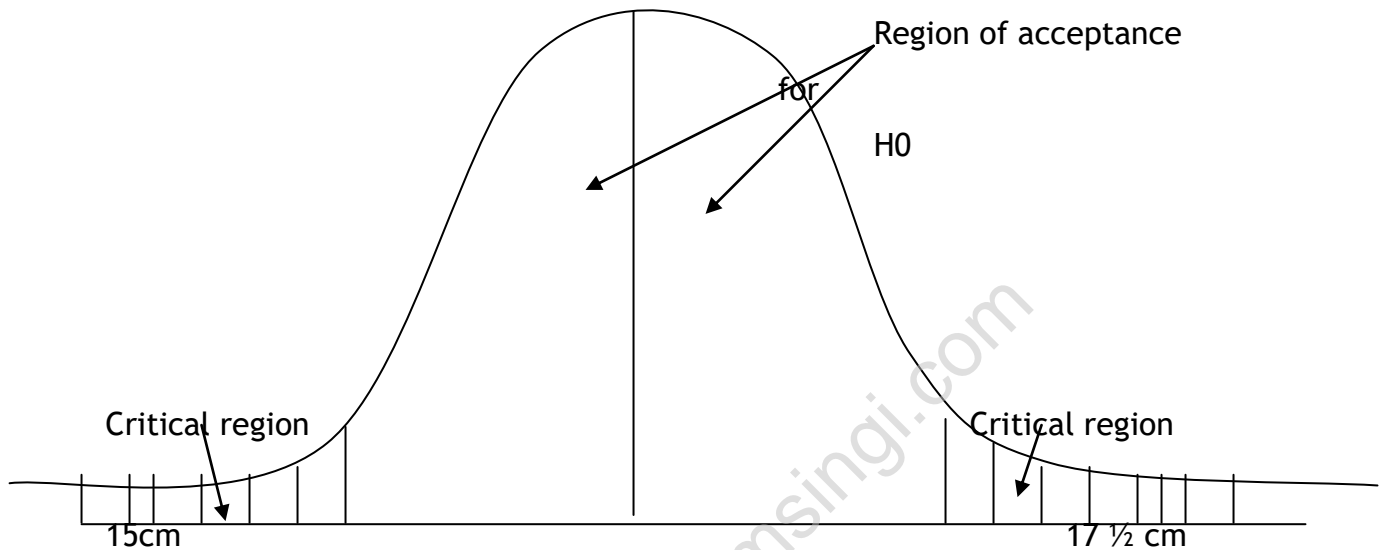
NB: If the sample means standardized value < 1.65 , we accept the null hypothesis but reject the alternative. If the sample means value > 1.65 we reject the null hypothesis and accept the alternative hypothesis
 The above sketch is normally used when the sample mean given is greater than the population mean



NB: if the standardized value of the sample mean is between -2.58 and +2.58 accept the null hypothesis but otherwise reject it and therefore accept the alternative hypothesis

TWO TAILED TESTS

A two tailed test is normally used in statistical work (tests of significance) e.g. if a complaint lodged by the client is about a product not meeting certain specifications i.e. the item will generate a complaint if its measurements are below the **lower tolerance limit** or above the **upper tolerance limit**

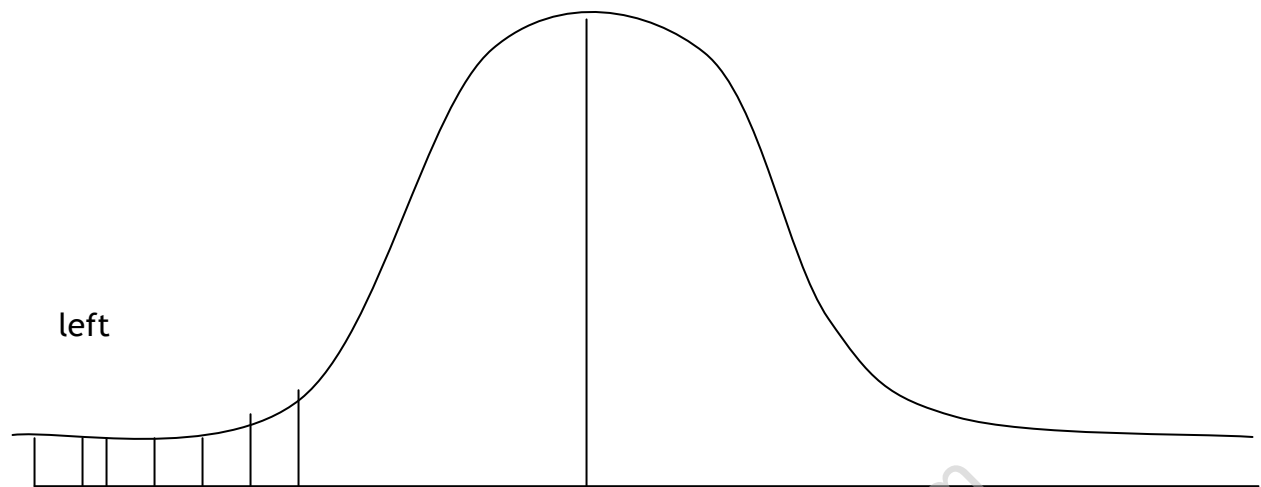


NB: Alternative hypothesis is usually rejected if the standardized value of the sample mean lies beyond the tolerance limits (15cm and 17 1/2 cm).

ONE TAILED TEST

This is a test where the alternative hypothesis (H_1) is only concerned with one of the tails of the distribution e.g. to test a business complaint if the complaint is above the measurements of item being shorter than is required.

E.g. a manufacturer of a given brand of bread may state that the average weight of the bread is 500 gms but if a consumer takes a sample and weighs each of the pieces of bread and happens to have a mean of 450 gms he will definitely complain about the bread which is underweight. The statistical analysis to be done will concentrate on the left tail of the normal distribution in which one will have to establish whether 450 gms being less than 500g is statistically significant. Such a test therefore is referred to as one tailed test.



On the other hand the test may compuliate on the right hand tail of the normal distribution when this happens the major complaint is likely to do with oversize items bought. Therefore the test is known as one tailed as the focus is on one end of the normal distribution.

		Number of standard errors	
		Two tailed test	One tailed test
5% level of significance	of	1.96	1.65
1% level of significance	of	2.58	2.33

HYPOTHESIS TESTING PROCEDURE

Whenever a business complaint comes up there is a recommended procedure for conducting a statistical test. The purpose of such a test is to establish whether the null hypothesis or alternative hypothesis is to be accepted.

The following are steps normally adopted

1. Statement of the null and alternative hypothesis
2. Statement of the level of significance to be used.
3. Statement about the test statistic i.e. what is to be tested e.g. the sample mean, sample proportion, difference between sample means or sample proportions
4. Type of test whether two tailed or one tailed.
5. Statement on critical values using the appropriate level of significance
6. Standardizing the test statistic
7. Conclusion showing whether to accept or reject the null hypothesis

STANDARD HYPOTHESIS TESTS

In principal, we can test the significance of any statistic related to any probability distribution. However we will be interested in a few standard cases. The sample statistics mean, proportion and variance, are related to the normal, t, F, and chi squared distributions

Thus

1. Normal test

Test a sample mean (\bar{X}) against a population mean (μ) (where samples size $n > 30$ and population variance σ^2 is known) and sample proportion, P (where sample size $np > 5$ and $nq > 5$ since in this case the normal distribution can be used to approximate the binomial distribution)

2. t test

Tests a sample mean (\bar{X}) against a population mean and especially where the population variance is unknown and $n < 30$.

3. Variance ratio test or f test

It is used to compare population variances and it is used with samples of any size drawn from normal populations.

4. Chi squared test

It can be used to test the association between attributes or the goodness of fit of an observed frequency distribution to a standard distribution

Example 1

A certain NGO carried out a survey in a certain community in order to establish the average at which the girls are married. The results of the survey indicated that the marriage age for the girls is 19 years

In order to establish the validity of the mean marital age, a sample of 50 women was interviewed and the average age indicated that they got married at the age of 16 years. However the different ages at which they were married differed with the standard deviation of 2.1 years

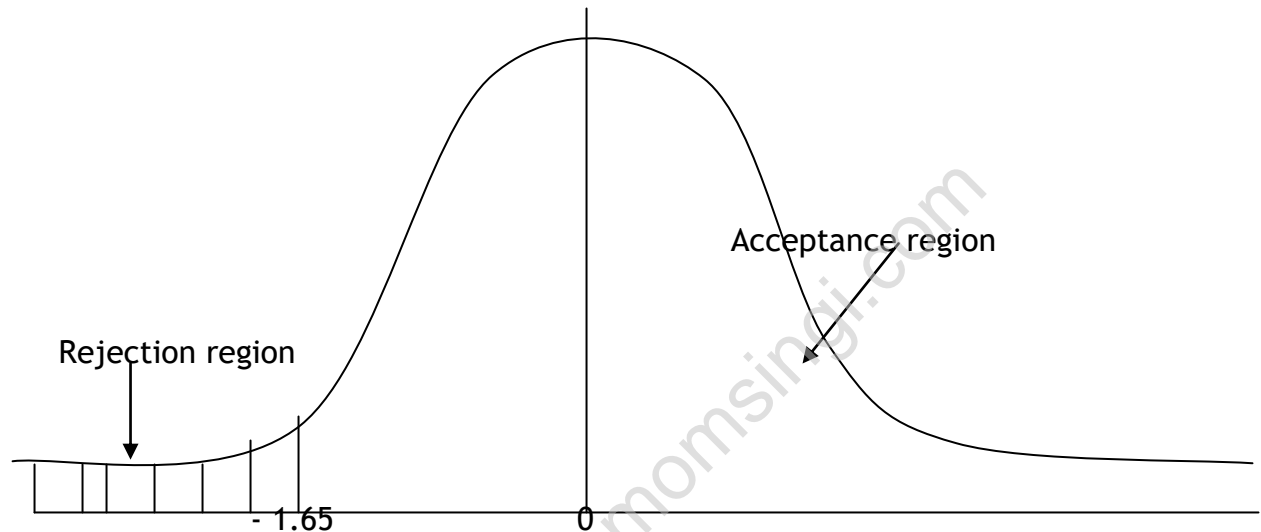
The sample data indicates that the marital age is less 19 years. Is this conclusion true or not?

Required

Conduct a statistical test to either support the above conclusion drawn from the sample statistics i.e. the marriage age is less than 19 years, use a level of significance of 5%

Solution

1. Null hypothesis
 $H_0: \mu$ (mean marital age) = 19 years
Alternative hypothesis $H_1: \mu$ (mean marital age) < 19 years
2. The level of significance is 5%
3. The test statistics is the sample mean age, $\bar{X} = 16$ years
4. The critical value of the one tailed test (one tailed because the alternative hypothesis is an inequality) at 5% level of significance is -1.65



5. The standardized value of the sample mean is

$$Z = \frac{\bar{X} - \mu}{S_{\bar{X}}} \quad \text{where } S_{\bar{x}} = \frac{S}{\sqrt{n}}$$

Where,
 \bar{X} = Sample mean
 μ = Population mean
 S = sample standard deviation
 n = sample size
 z = standard value (as per computation)

The standard value Z must fall within the acceptance region for us to accept the null hypothesis. Thus it must be > -1.65 otherwise we accept the alternative hypothesis.

$$Z = \frac{16-19}{\frac{2.1}{\sqrt{50}}} = -10.1$$

6. Since $-10.1 < -1.65$, we reject the null hypothesis but accept the alternative hypothesis at 5% level of significance i.e. the marriage age in this community is significantly lower than 19 years

Example 2

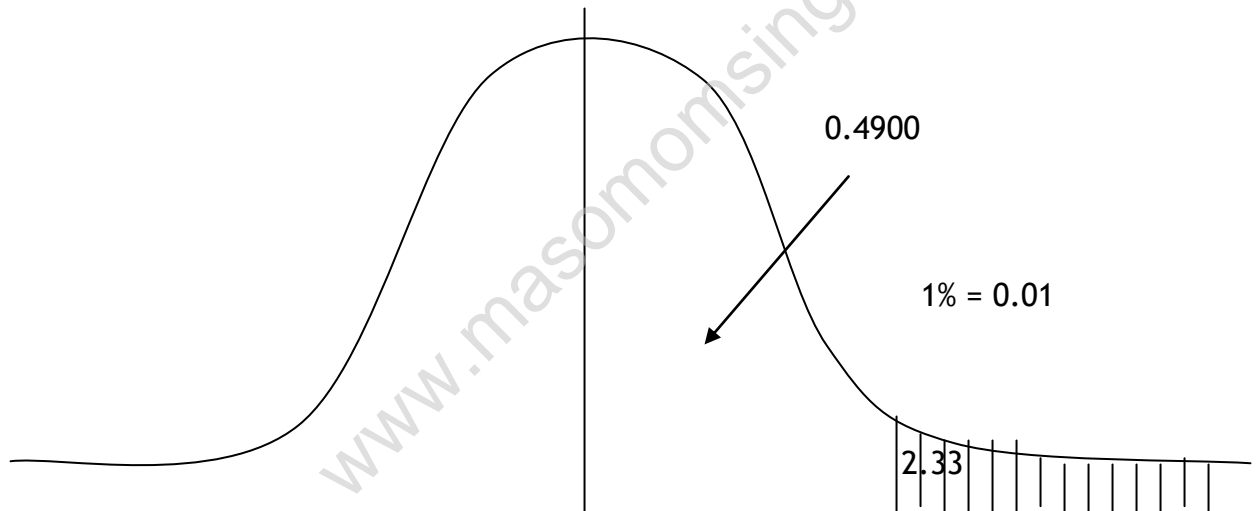
A foreign company which manufactures electric bulbs has assured its customers that the lifespan of the bulbs is 28 month with a standard deviation of 4months

Recently the company embarked on a quality improvement research for their product. After the research using new technology, a sample of 70 bulbs was tested and they gave a mean lifespan of 30.2 months

Does this justify the research undertaken? Use 1% level of significance to conduct a statistical test in order to establish the truth about the above question.

Testing procedure

1. Null hypothesis $H_0: \mu = 28$
Alternative hypothesis $H_1: \mu > 28$
2. The level of significance is 1% (one tailed test)
3. The test statistics is the sample mean age, $\bar{x}' = 30.2$
4. The critical value of the one tailed test at 5% level of significance is + 2.33



5. The standardized value of the sample mean is

$$Z = \frac{\bar{X} - \mu}{S_{\bar{x}}} = \frac{30.2 - 28}{\frac{4}{\sqrt{70}}} = 4.6$$

6. Since $4.6 > 2.33$, we reject the null hypothesis but accept the alternative hypothesis at 1% level of significance i.e. the new sample mean life span is statistically significant higher than the population mean

Therefore the research undertaken was worth while or justified

Example 3

A construction firm has placed an order that they require a consignment of wires which have a mean length of 10.5 meters with a standard deviation of 1.7 m

The company which produces the wires delivered 90 wires, which had a mean length of 9.2 m., The construction company rejected the consignment on the grounds that they were different from the order placed.

Required

Conduct a statistical test to indicate whether you support or not support the action taken by the construction company at 5% level of significance.

Solution

Null hypothesis $\mu = 10.5$ m

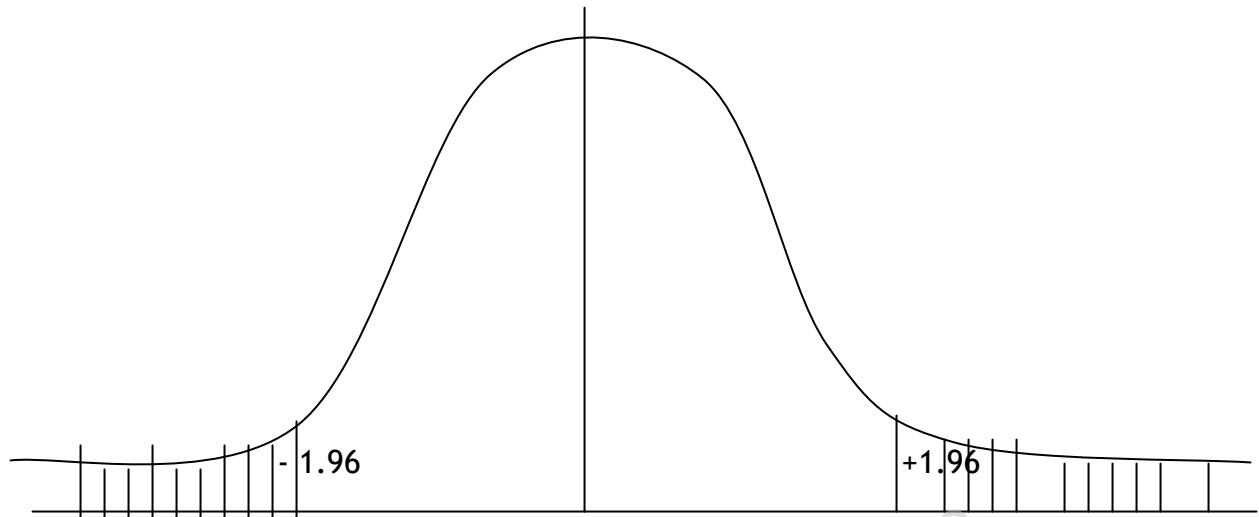
Alternative hypothesis $\mu \neq 10.5$ m

Level of significance be 5%

The test statistics is the sample mean $\bar{X} = 9.2$ m

The critical value of the two tailed test at 5% level of significance is ± 1.96 (two tailed test).

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The standardized value of the test Z =

$$Z = \frac{|\bar{X} - \mu|}{\frac{S_{\bar{X}}}{\sqrt{90}}} = \frac{9.2 - 10.5}{\frac{1.7}{\sqrt{90}}} = -7.25$$

Since $7.25 < 1.96$, reject the null hypothesis but accept the alternative hypothesis at 5% level of significance i.e. the sample mean is statistically different from the consignment ordered by the construction company. Therefore support the action taken by the construction company

TESTING THE DIFFERENCE BETWEEN TWO SAMPLE MEANS (LARGE SAMPLES)

A large sample is defined as one which contains 30 or more items ($n \geq 30$) Where n is the sample size

In a business those involved are constantly observant about the standards or specifications of the item which they sell e.g. a trader may receive a batch of items at one time and another batch at a later time at the end he may have concluded that the two samples are different in certain specifications e.g. mean weight mean lifespan, mean length e.t.c. further it may become necessary to establish whether the observed differences are statistically significant or not. If the differences are statistically significant then it means that such differences must be explained i.e. there are known causes but if they are not statistically significant then it means that the difference observed have no known causes and are mainly due to chance

If the differences are established to be statistically significant then it implies that the complaints, which necessitated that kind of test, are justified

Let X_1 and X_2 be any two samples whose sizes are n_1 and n_2 and mean \bar{X}_1 and \bar{X}_2 . Standard deviation S_1 and S_2 respectively. In order to test the difference between the two sample means, we apply the following formulas

$$Z = \frac{\bar{X}_1 - \bar{X}_2}{S(\bar{X}_1 - \bar{X}_2)} \quad \text{where } S(\bar{X}_1 - \bar{X}_2) = \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$$

Example 1

An agronomist was interested in the particular fertilizer yield output. He planted maize on 50 equal pieces of land and the mean harvest obtained later was 60 bags per plot with a standard deviation of 1.5 bags. The crops grew under natural circumstances and conditions without the soil being treated with any fertilizer. The same agronomist carried out an alternative experiment where he picked 60 plots in the same area and planted the same plant of maize but a fertilizer was applied on these plots. After the harvest it was established that the mean harvest was 63 bags per plot with a standard deviation of 1.3 bags

Required

Conduct a statistical test in order to establish whether there was a significant difference between the mean harvest under the two types of field conditions. Use 5% level of significance.

Solution

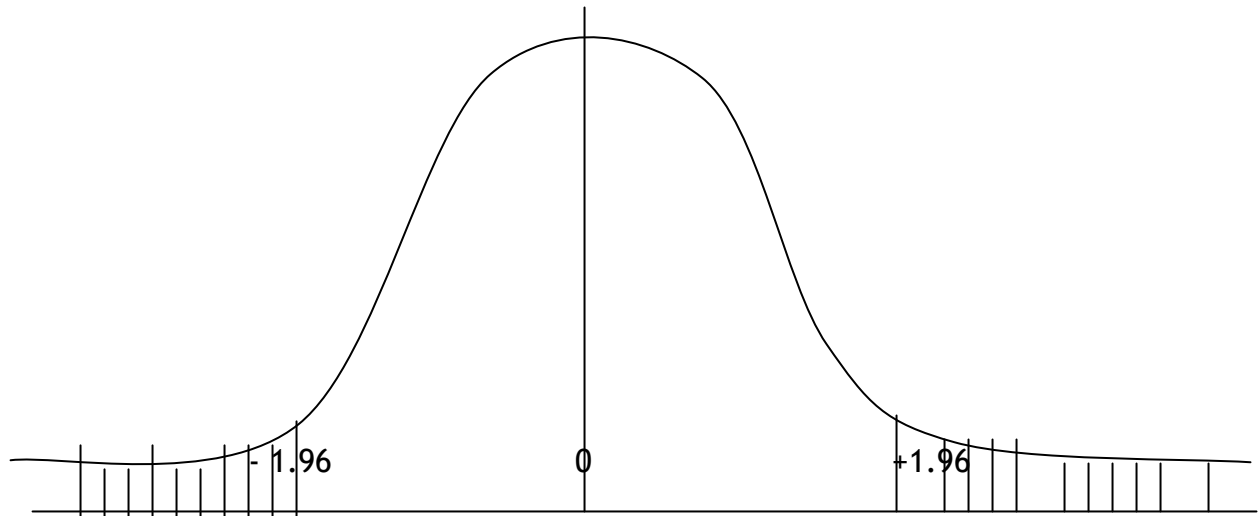
$$H_0 : \mu_1 = \mu_2$$

$$H_1 : \mu_1 \neq \mu_2$$

Critical values of the two tailed test at 5% level of significance are 1.96
The standardized value of the difference between sample means is given by Z where

$$Z = \frac{\bar{X}_1 - \bar{X}_2}{S(\bar{X}_1 - \bar{X}_2)} \quad \text{where } S(\bar{X}_1 - \bar{X}_2) = \sqrt{\frac{1.5^2}{50} + \frac{1.3^2}{60}}$$

$$\begin{aligned} Z &= \frac{(60 - 63)}{\sqrt{0.045 + 0.028}} \\ &= 11.11 \end{aligned}$$



Since $11.11 < -1.96$, we reject the null hypothesis but accept the alternative hypothesis at 5% level of significance i.e. the difference between the sample mean harvest is statistically significant. This implies that the fertilizer had a positive effect on the harvest of maize
Note: You don't have to illustrate your solution with a diagram.

Example 2

An observation was made about reading abilities of males and females. The observation lead to a conclusion that females are faster readers than males. The observation was based on the times taken by both females and males when reading out a list of names during graduation ceremonies.

In order to investigate into the observation and the consequent conclusion a sample of 200 men were given lists to read. On average each man took 63 seconds with a standard deviation of 4 seconds

A sample of 250 women were also taken and asked to read the same list of names. It was found that they on average took 62 seconds with a standard deviation of 1 second.

Required

By conducting a statistical hypothesis testing at 1% level of significance establish whether the sample data obtained does support earlier observation or not

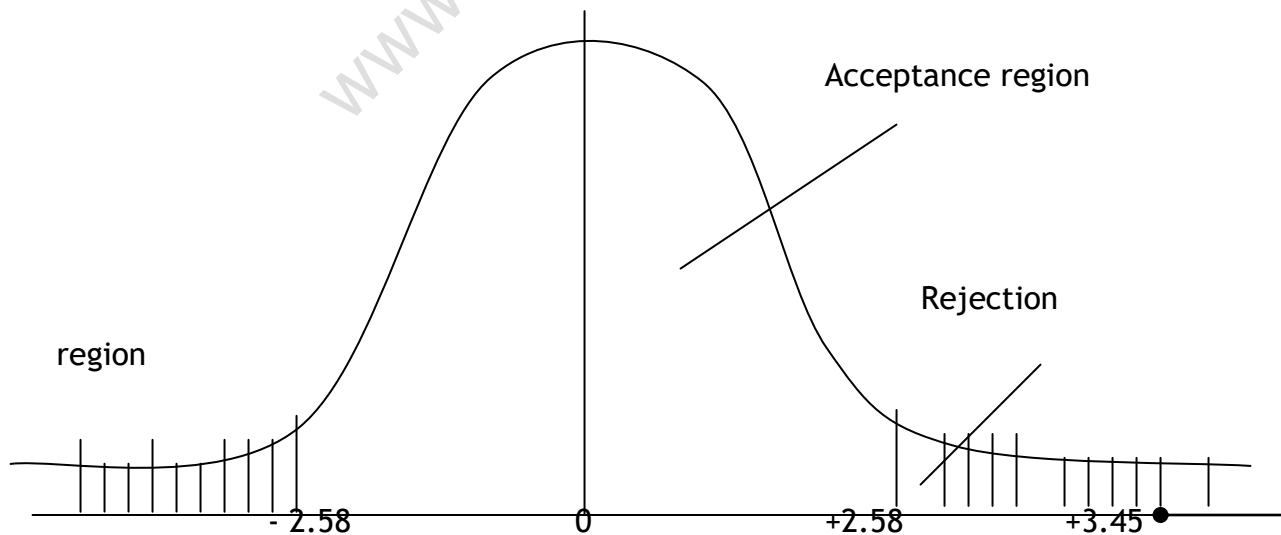
Solution

$$H_0: \mu_1 = \mu_2$$

$$H_1: \mu_1 \neq \mu_2$$

Critical values of the two tailed test is at 1% level of significance is 2.58.

$$Z = \frac{|\bar{X}_1 - \bar{X}_2|}{S(\bar{X}_1 - \bar{X}_2)}$$
$$Z = \frac{|63 - 62|}{\sqrt{\frac{4^2}{200} + \frac{1^2}{250}}} = 3.45$$



Since $3.45 > 2.33$ reject the null hypothesis but accept the alternative hypothesis at 1% level of significance i.e. there is a significant difference between the reading speed of Males and females, thus females are actually faster readers.

TEST OF HYPOTHESIS ON PROPORTIONS

This follows a similar method to the one for means except that the standard error used in this case:

$$S_p = \sqrt{\frac{Pq}{n}}$$

Z score is calculated as, $Z = \frac{P - \Pi}{S_p}$ Where P = Proportion found

in the sample.

Π - the hypothetical

proportion.

Example

A member of parliament (MP) claims that in his constituency only 50% of the total youth population lacks university education. A local media company wanted to ascertain that claim thus they conducted a survey taking a sample of 400 youths, of these 54% lacked university education.

Required:

At 5% level of significance confirm if the MP's claim is wrong.

Solution.

Note: This is a two tailed tests since we wish to test the hypothesis that the hypothesis is different (\neq) and not against a specific alternative hypothesis e.g. $<$ less than or $>$ more than.

H_0 : $\pi = 50\%$ of all youth in the constituency lack university education.

H_1 : $\pi \neq 50\%$ of all youth in the constituency lack university education.

$$S_p = \sqrt{\frac{pq}{n}} = \sqrt{\frac{0.5 \times 0.5}{400}} = 0.025$$

$$Z = \left| \frac{0.54 - 0.50}{0.025} \right| = 1.6$$

at 5% level of significance for a two-tailed test the critical value is 1.96 since calculated Z value $<$ tabulated value (1.96).

i.e. $1.6 < 1.96$ we accept the null hypothesis.

Thus the MP's claim is accurate.

HYPOTHESIS TESTING OF THE DIFFERENCE BETWEEN PROPORTIONS

Example

Ken industrial manufacturers have produced a perfume known as “fianchetto.” In order to test its popularity in the market, the manufacturer carried a random survey in Back rank city where 10,000 consumers were interviewed after which 7,200 showed preference. The manufacturer also moved to area Rook town where he interviewed 12,000 consumers out of which 1,0000 showed preference for the product.

Required

Design a statistical test and hence use it to advise the manufacturer regarding the differences in the proportion, at 5% level of significance.

Solution

$$H_0 : \pi_1 = \pi_2$$

$$H_1 : \pi_1 \neq \pi_2$$

The critical value for this two tailed test at 5% level of significance = 1.96.

$$\text{Now } Z = \frac{(P_1 - P_2) - (\Pi_1 - \Pi_2)}{S(P_1 - P_2)}$$

But since the null hypothesis is $\pi_1 = \pi_2$, the second part of the numerator disappear i.e.

$\pi_1 - \pi_2 = 0$ which will always be the case at this level.

$$\text{Then } Z = \frac{(P_1 - P_2)}{S(P_1 - P_2)}$$

Where;

	Sample 1	Sample 2
Sample size	$n_1 =$ 10,000	$n_2 =$ 12,000
Sample proportion of success	$P_1 = 0.72$	$P_2 = 0.83$
Population proportion of success.	Π_1	Π_2

$$\text{Now } S(p_1 - p_2) = \sqrt{\frac{pq}{n_1} + \frac{pq}{n_2}}$$

$$\text{Where } P = \frac{P_1 n_1 + P_2 n_2}{n_1 + n_2}$$

$$\text{And } q = 1 - p$$

∴ in our case

$$\begin{aligned} P &= \frac{10,000(0.72) + 12,000(0.83)}{10,000 + 12,000} \\ &= \frac{84,000}{22,000} \\ &= 0.78 \end{aligned}$$

$$\therefore q = 0.22$$

$$\begin{aligned} S(P_1 - P_2) &= \sqrt{\frac{0.78(0.22)}{10,000} + \frac{0.78(0.22)}{12,000}} \\ &= 0.00894 \end{aligned}$$

$$Z = \left| \frac{0.72 - 0.83}{0.00894} \right| = 12.3$$

Since $12.3 > 1.96$, we reject the null hypothesis but accept the alternative. The differences between the proportions are statistically significant. This implies that the perfume is much more popular in Rook town than in back rank city.

HYPOTHESIS TESTING ABOUT THE DIFFERENCE BETWEEN TWO PROPORTIONS

Is used to test the difference between the proportions of a given attribute found in two random samples.

The null hypothesis is that there is no difference between the population proportions. It means two samples are from the same population.

Hence

$$H_0 : \pi_1 = \pi_2$$

The best estimate of the standard error of the difference of P_1 and P_2 is given by pooling the samples and finding the pooled sample proportions (P) thus

$$P = \frac{P_1 n_1 + P_2 n_2}{n_1 + n_2}$$

Standard error of difference between proportions

$$S(p_1 - p_2) = \sqrt{\frac{pq}{n_1} + \frac{pq}{n_2}}$$

$$\text{And } Z = \left| \frac{P_1 - P_2}{S(p_1 - p_2)} \right|$$

Example

In a random sample of 100 persons taken from village A, 60 are found to be consuming tea. In another sample of 200 persons taken from a village B, 100 persons are found to be consuming tea. Do the data reveal significant difference between the two villages so far as the habit of taking tea is concerned?

Solution

Let us take the hypothesis that there is no significant difference between the two villages as far as the habit of taking tea is concerned i.e. $\pi_1 = \pi_2$

We are given

$$P_1 = 0.6; \quad n_1 = 100$$
$$P_2 = 0.5; \quad n_2 = 200$$

Appropriate statistic to be used here is given by

$$p = \frac{P_1 n_1 + P_2 n_2}{n_1 + n_2}$$
$$= \frac{(0.6)(100) + (0.5)(200)}{100 + 200} = \frac{60 + 100}{300}$$
$$= 0.53$$
$$q = 1 - 0.53$$
$$= 0.47$$

$$S(P_1 - P_2) = \sqrt{\frac{pq}{n_1} + \frac{pq}{n_2}}$$
$$= \sqrt{\frac{(0.53)(0.47)}{100} + \frac{(0.53)(0.47)}{200}}$$
$$= 0.0608$$

$$Z = \left| \frac{0.6 - 0.5}{0.0608} \right|$$
$$= 1.64$$

Since the computed value of Z is less than the critical value of $Z = 1.96$ at 5% level of significance therefore we accept the hypothesis and conclude that there is no significant difference in the habit of taking tea in the two villages A and B

t distribution (student's t distribution) tests of hypothesis (test for small samples $n < 30$)

For small samples $n < 30$, the method used in hypothesis testing is exactly similar to the one for large samples except that t values are used from t distribution at a given degree of freedom v , instead of z score, the standard error Se statistic used is also different.

Note that $v = n - 1$ for a single sample and $n_1 + n_2 - 2$ where two sample are involved.

a) Test of hypothesis about the population mean

When the population standard deviation (S) is known then the t statistic is defined as

$$t = \frac{|\bar{X} - \mu|}{S_{\bar{X}}} \quad \text{where } S_{\bar{X}} = \frac{S}{\sqrt{n}}$$

Follows the students t distribution with (n-1) d.f. where

\bar{X} = Sample mean

μ = Hypothesis population mean

n = sample size

and S is the standard deviation of the sample calculated by the formula

$$S = \sqrt{\frac{\sum (X - \bar{X})^2}{n-1}} \quad \text{for } n < 30$$

If the calculated value of t exceeds the table value of t at a specified level of significance, the null hypothesis is rejected.

Example

Ten oil tins are taken at random from an automatic filling machine. The mean weight of the tins is 15.8 kg and the standard deviation is 0.5kg. Does the sample mean differ significantly from the intended weight of 16kgs? Use 5% level of significance.

Solution

Given that n = 10; $\bar{x} = 15.8$; S = 0.50; $\mu = 16$; v = 9

$H_0 : \mu = 16$

$H_1 : \mu \neq 16$

$$\begin{aligned} &= S_{\bar{x}} = \frac{0.5}{\sqrt{10}} \\ t &= \frac{|15.8 - 16|}{\frac{0.5}{\sqrt{10}}} \\ &= \frac{|0.2|}{0.16} \\ &= -1.25 \end{aligned}$$

The table value for t for 9 d.f. at 5% level of significance is 2.26. the computed value of t is smaller than the table value of t. therefore, difference is insignificant and the null hypothesis is accepted.

b) Test of hypothesis about the difference between two means

The t test can be used under two assumptions when testing hypothesis concerning the difference between the two means; that the two are normally distributed (or near normally distributed) populations and that the

standard deviation of the two is the same or at any rate not significantly different.

Appropriate test statistic to be used is

$$t = \frac{\bar{X}_1 - \bar{X}_2}{S_{(\bar{X}_1 - \bar{X}_2)}} \quad \text{at } (n_1 + n_2 - 2) \text{ d.f.}$$

The standard deviation is obtained by pooling the two sample standard deviation as shown below.

$$S_p = \sqrt{\frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}}$$

Where S_1 and S_2 are standard deviation for sample 1 & 2 respectively.

$$\text{Now } S_{\bar{X}_1} = \frac{Sp}{\sqrt{n_1}} \text{ and } S_{\bar{X}_2} = \frac{Sp}{\sqrt{n_2}}$$

$$S_{(\bar{X}_1 - \bar{X}_2)} = \sqrt{S_{\bar{X}_1}^2 + S_{\bar{X}_2}^2}$$

$$\text{Alternatively } S_{(\bar{X}_1 - \bar{X}_2)} = Sp \sqrt{\frac{n_1 + n_2}{n_1 n_2}}$$

Example

Two different types of drugs A and B were tried on certain patients for increasing weights, 5 persons were given drug A and 7 persons were given drug B. the increase in weight (in pounds) is given below

Drug A	8	12	16	9	3		
Drug B	10	8	12	15	6	8	11

Do the two drugs differ significantly with regard to their effect in increasing weight? (Given that $v = 10$; $t_{0.05} = 2.23$)

Solution

$$H_0 : \mu_1 = \mu_2$$

$$H_1 : \mu_1 \neq \mu_2$$

$$t = \frac{|\bar{X}_1 - \bar{X}_2|}{S_{(\bar{X}_1 - \bar{X}_2)}}$$

Calculate for \bar{X}_1 , \bar{X}_2 and S

X_1	$X_1 - \bar{X}_1$	$(X_1 - \bar{X}_1)^2$	X_2	$(X_2 - \bar{X}_2)$	$(X_2 - \bar{X}_2)^2$
8	-1	1	10	0	0
12	+3	9	8	-2	4
13	+4	16	12	+2	4
9	0	0	15	+5	25
3	-6	36	6	-4	16
			8	-2	4
			11	+1	1
$\Sigma X_1 = 45$	$\Sigma(X_1 - \bar{X}_1) = 0$	$\Sigma(X_1 - \bar{X}_1)^2 = 62$	$\Sigma X_2 = 70$	$\Sigma(X_2 - \bar{X}_2) = 0$	$\Sigma(X_2 - \bar{X}_2)^2 = 54$

$$\bar{X}_1 = \frac{\sum X_1}{n_1} = \frac{45}{5} = 9 \quad \bar{X}_2 = \frac{\sum X_2}{n_2} = \frac{70}{7} = 10$$

$$S_1 = \sqrt{\frac{62}{4}} = 3.94 \quad S_2 = \sqrt{\frac{54}{6}} = 3$$

$$S_p = \sqrt{\frac{(4)15.4 + (6)9}{10}}$$

$$= 3.406$$

$$S_{(\bar{X}_1 - \bar{X}_2)} = \sqrt{\frac{11.6}{5} + \frac{11.6}{7}} \text{ Or } 3.406 \sqrt{\frac{7+5}{5(7)}}$$

$$= 1.99$$

$$t = \frac{|\bar{X}_1 - \bar{X}_2|}{S_{(\bar{X}_1 - \bar{X}_2)}} = \frac{|9 - 10|}{1.99}$$

$$= 0.50$$

Now $t_{0.05}$ (at $v = 10$) = 2.23 > 0.5

Thus we accept the null hypothesis.

Hence there is no significant difference in the efficacy of the two drugs in the matter of increasing weight

Example

Two salesmen A and B are working in a certain district. From a survey conducted by the head office, the following results were obtained. State whether there is any significant difference in the average sales between the two salesmen at 5% level of significance.

	A	B
No. of sales	20	18
Average sales in shs	170	205
Standard deviation in shs	20	25

Solution

$$H_0 : \mu_1 = \mu_2$$

$$H_1 : \mu_1 \neq \mu_2$$

Where

$$S_p = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

$$S_{(\bar{X}_1 - \bar{X}_2)} = S_p \sqrt{\frac{n_1 + n_2}{n_1 n_2}}$$

Where: $\bar{X}_1 = 170$, $\bar{X}_2 = 205$, $n_1 = 20$, $n_2 = 18$, $S_1 = 20$, $S_2 = 25$, $V = 36$

$$S_p = \sqrt{\frac{(19)(20^2) + (17)(25^2)}{20 + 18 - 2}}$$

$$= 22.5$$

$$S_{(\bar{X}_1 - \bar{X}_2)} = 22.5 \sqrt{\frac{38}{360}}$$

$$= 7.31$$

$$t = \left| \frac{170 - 205}{7.31} \right|$$

$$= 4.79$$

$$t_{0.05}(36) = 1.9 \text{ (Since d.f. > 30 we use the normal tables)}$$

The table value of t at 5% level of significance for 36 d.f. when d.f. > 30, that t distribution is the same as normal distribution is 1.9. Since the value computed value of t is more than the table value, we reject the null hypothesis. Thus, we conclude that there is significant difference in the average sales between the two salesmen

Testing the hypothesis equality of two variances

The test for equality of two population variances is based on the variances in two independently selected random samples drawn from two normal populations

Under the null hypothesis $\sigma_1^2 = \sigma_2^2$

$$F = \frac{\frac{s_1^2}{\sigma_1^2}}{\frac{s_2^2}{\sigma_2^2}} \quad \text{Now under the } H_0 : \sigma_1^2 = \sigma_2^2 \text{ it follows that}$$
$$F = \frac{S_1^2}{S_2^2} \text{ which is the test statistic.}$$

Which follows F - distribution with V_1 and V_2 degrees of freedom. The larger sample variance is placed in the numerator and the smaller one in the denominator

If the computed value of F exceeds the table value of F, we reject the null hypothesis i.e. the alternate hypothesis is accepted

Example

In one sample of observations the sum of the squares of the deviations of the sample values from sample mean was 120 and in the other sample of 12 observations it was 314. test whether the difference is significant at 5% level of significance

Solution

Given that $n_1 = 10$, $n_2 = 12$, $\Sigma(x_1 - \bar{X}_1)^2 = 120$
 $\Sigma(x_2 - \bar{X}_2)^2 = 314$

Let us take the null hypothesis that the two samples are drawn from the same normal population of equal variance

$$H_0 : \sigma_1^2 = \sigma_2^2$$

$$H_1 : \sigma_1^2 \neq \sigma_2^2$$

Applying F test i.e.

$$F = \frac{S_1^2}{S_2^2}$$
$$= \frac{\frac{\Sigma(x_1 - \bar{X}_1)^2}{n_1 - 1}}{\frac{\Sigma(x_2 - \bar{X}_2)^2}{(n_2 - 1)}}$$
$$= \frac{\frac{120}{9}}{\frac{314}{11}}$$
$$= \frac{13.33}{28.55}$$

since the numerator should be greater than denominator

$$F = \frac{28.55}{13.33} = 2.1$$

The table value of F at 5% level of significance for $V_1 = 9$ and $V_2 = 11$. Since the calculated value of F is less than the table value, we accept the hypothesis. The samples may have been drawn from the two populations having the same variances.

Chi square hypothesis tests (Non-parametric test)(X^2)

They include amongst others

- i. Test for goodness of fit
- ii. Test for independence of attributes
- iii. Test of homogeneity
- iv. Test for population variance

The Chi square test (x^2) is used when comparing an actual (observed) distribution with a hypothesized or explained distribution.

$$\text{It is given by; } x^2 = \sum \frac{(O-E)^2}{E} \quad \text{Where } O = \text{Observed frequency}$$

$E = \text{Expected frequency}$

The computed value of x^2 is compared with that of tabulated x^2 for a given significance level and degrees of freedom.

- i. Test for goodness of fit

These tests are used when we want to determine whether an actual sample distribution matches a known theoretical distribution

The null hypothesis usually states that the sample is drawn from the theoretical population distribution and the alternate hypothesis usually states that it is not.

Example

Mr. Nguku carried out a survey of 320 families in Ateka district, each family had 5 children and they revealed the following distribution

No. of boys	5	4	3	2	1	0
No. of girls	0	1	2	3	4	5
No. of families	14	56	110	88	40	12

Is the result consistent with the hypothesis that male and female births are equally probable at 5% level of significance?

Solution

If the distribution of gender is equally probable then the distribution conforms to a binomial distribution with probability $P(X) = \frac{1}{2}$.

Therefore

H_0 = the observed number of boys conforms to a binomial distribution with $P = \frac{1}{2}$

H_1 = the observations do not conform to a binomial distribution.

On the assumption that male and female births are equally probable the probability of a male birth is $P = \frac{1}{2}$. The expected number of families can be calculated by the use of binomial distribution. The probability of male births in a family of 5 is given by

$$P(x) = {}^5C_x P^x q^{5-x} \quad (\text{for } x = 0, 1, 2, 3, 4, 5,)$$

$$= {}^5C_x \left(\frac{1}{2}\right)^5 \quad (\text{Since } P = q = \frac{1}{2})$$

To get the expected frequencies, multiply $P(x)$ by the total number $N = 320$. The calculations are shown below in the tables

x	P(x)	Expected frequency = NP(x)
0	${}^5C_0 \left(\frac{1}{2}\right)^5 = \frac{1}{32}$	$320 \times \frac{1}{32} = 10$
1	${}^5C_1 \left(\frac{1}{2}\right)^5 = \frac{5}{32}$	$320 \times \frac{5}{32} = 50$
2	${}^5C_2 \left(\frac{1}{2}\right)^5 = \frac{10}{32}$	$320 \times \frac{10}{32} = 100$
3	${}^5C_3 \left(\frac{1}{2}\right)^5 = \frac{10}{32}$	$320 \times \frac{10}{32} = 100$
4	${}^5C_4 \left(\frac{1}{2}\right)^5 = \frac{5}{32}$	$320 \times \frac{5}{32} = 50$
5	${}^5C_5 \left(\frac{1}{2}\right)^5 = \frac{1}{32}$	$320 \times \frac{1}{32} = 10$

Arranging observed and expected frequencies in the following table and calculating χ^2

O	E	$(O - E)^2$	$(O - E)^2 / E$
14	10	16	1.60
56	50	16	0.72
110	100	100	1.00
88	100	144	1.44
40	50	100	2.00
12	10	4	0.40
			$\Sigma(O - E)^2 / E = 7.16$

$$\chi^2 = \sum \frac{(O - E)^2}{E}$$

$$= 7.16$$

The table of χ^2 for $V = 6 - 1 = 5$ at 5% level of significance is 11.07. The computed value of $\chi^2 = 7.16$ is less than the table value. Therefore the hypothesis is accepted. Thus it can be concluded that male and female births are equally probable.

ii) Test of independence of attributes

This test disclosed whether there is any association or relationship between two or more attributes or not. The following steps are required to perform the test of hypothesis.

1. The null and alternative hypothesis are set as follows
 H_0 : No association exists between the attributes
 H_1 : an association exists between the attributes
2. Under H_0 an expected frequency E corresponding to each cell in the contingency table is found by using the formula

$$E = \frac{R \times C}{n}$$

Where R = a row total, C = a column total and n = sample size

3. Based upon the observed values and corresponding expected frequencies the χ^2 statistic is obtained using the formula

$$\chi^2 = \sum \frac{(O - E)^2}{E}$$

4. The characteristic of this distribution are defined by the number of degrees of freedom (d.f.) which is given by
 $d.f. = (r-1)(c-1)$,

Where r is the number of rows and c is number of columns corresponding to a chosen level of significance, the critical value is found from the chi squared table

5. The calculated value of χ^2 is compared with the tabulated value χ^2 for $(r-1)(c-1)$ degrees of freedom at a certain level of significance. If the computed value of χ^2 is greater than the tabulated value, the null hypothesis of independence is rejected. Otherwise we accept it.

Example

In a sample of 200 people where a particular disease was selected, 100 were given a drug and the others were not given any drug. The results are as follows

	Drug	No drug	Total
Cured	65	55	120
Not cured	35	45	80
Total	100	100	200

Test whether the drug will be effective or not, at 5% level of significance.

Solution

Let us take the null hypothesis that the drug is not effective in curing the disease.

Applying the χ^2 test

The expected cell frequencies are computed as follows

$$E_{11} = \frac{R_1 C_1}{n} = \frac{120 \times 100}{200} = 60$$

$$E_{12} = \frac{R_1 C_2}{n} = \frac{120 \times 100}{200} = 60$$

$$E_{21} = \frac{R_2 C_1}{n} = \frac{80 \times 100}{200} = 40$$

$$E_{22} = \frac{R_2 C_2}{n} = \frac{80 \times 100}{200} = 40$$

The table of expected frequencies is as follows

60	60	120
40	40	80
100	100	200

O	E	$(O - E)^2$	$(O - E)^2 / E$
65	60	25	0.417
55	60	25	0.625
35	40	25	0.417
45	40	25	0.625
			$\Sigma(O - E)^2 / E = 2.084$

Arranging the observed frequencies with their corresponding frequencies in the following table we get

$$\chi^2 = \sum \frac{(O - E)^2}{E}$$

$$= 2.084$$

$$V = (r - 1)(c - 1) = (2 - 1)(2 - 1) = 1; \chi^2_{tabulated(0.05)} = 3.841$$

The calculated value of χ^2 is less than the table value. The hypothesis is accepted. Hence the drug is not effective in curing the disease.

Test of homogeneity

It is concerned with the proposition that several populations are homogenous with respect to some characteristic of interest e.g. one may be interested in knowing if raw material available from several retailers are homogenous. A random sample is drawn from each of the population and the number in each of sample falling into each category is determined. The sample data is displayed in a contingency table

The analytical procedure is the same as that discussed for the test of independence

Example

A random sample of 400 persons was selected from each of three age groups and each person was asked to specify which types of TV programs be preferred. The results are shown in the following table

Type of program

Age group	A	B	C	Total
Under 30	120	30	50	200
30 - 44	10	75	15	100
45 and above	10	30	60	100
Total	140	135	125	400

Test the hypothesis that the **populations** are homogenous with respect to the types of television program they prefer, at 5% level of significance.

Solution

Let us take hypothesis that the populations are homogenous with respect to different types of television programs they prefer

Applying χ^2 test

O	E	$(O - E)^2$	$(O - E)^2 / E$
120	70.00	2500.00	35.7143
10	35.00	625.00	17.8571
10	35.00	625.00	17.8571
30	67.50	1406.25	20.8333
75	33.75	1701.56	50.4166
30	33.75	14.06	0.4166

50	62.50	156.25	2.500
15	31.25	264.06	8.4499
60	31.25	826.56	26.449
			$\Sigma(O - E)^2 / E = 180.4948$

$$x^2 = \sum \frac{(O - E)^2}{E}$$

The table value of x^2 for 4d.f. at 5% level of significance is 9.488

The calculated value of x^2 is greater than the table value. We reject the hypothesis and conclude that the populations are not homogenous with respect to the type of TV programs preferred, thus the different age groups vary in choice of TV programs.

SUMMARY OF FORMULAE IN HYPOTHESIS

Testing

(a) Hypothesis testing of mean

For $n > 30$

$$Z = \left| \frac{\bar{X} - \mu}{S_{\bar{X}}} \right| \text{ Where } S_{\bar{X}} = \frac{S}{\sqrt{n}} \text{ at } \alpha \text{ level of significance.}$$

For $n < 30$

$$t = \left| \frac{X - \mu}{S_{\bar{X}}} \right| \text{ where } S_{\bar{X}} = \frac{S}{\sqrt{n}}$$

at $n - 1$ d.f

α level of significance

(b) Difference between means (Independent samples)

For $n > 30$

$$Z = \left| \frac{\bar{X}_1 - \bar{X}_2}{S_{(\bar{X}_1 - \bar{X}_2)}} \right|$$

$$\text{Where } S_{(\bar{X}_1 - \bar{X}_2)} = \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$$

At $\alpha =$ level of significance

For $n < 30$

$$t = \frac{\bar{X}_1 - \bar{X}_2}{S_{(\bar{X}_1 - \bar{X}_2)}} \text{ at } n_1 + n_2 - 2 \text{ d.f}$$

$$\text{where } S_{(\bar{X}_1 - \bar{X}_2)} = S_p \sqrt{\frac{n_1 + n_2}{n_1 n_2}}$$

$$\text{and } S_p = \sqrt{\frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}}$$

(c) Hypothesis testing of proportions

$$Z = \frac{p - \pi}{S_p}$$

$$\text{Where: } S_p = \sqrt{\frac{pq}{n}}$$

p = Proportion found in sample

q = 1 - p

π = hypothetical proportion

(d) Difference between proportions

$$Z = \frac{P_1 - P_2}{S_{(P_1 - P_2)}}$$

Where:

$$S_{(P_1 - P_2)} = \sqrt{\frac{pq}{n_1} + \frac{pq}{n_2}}$$

$$p = \frac{p_1 n_1 + p_2 n_2}{n_1 + n_2}$$

$$q = 1 - P$$

(e) Chi-square test

$$\chi^2 = \sum \frac{(O - E)^2}{E}$$

Where O = observed frequency

$$E = \frac{\text{Column total} \times \text{Row total}}{\text{Sample Size}} = \text{expected frequency}$$

(f) F - test (variance test)

$$F = \frac{S_1^2}{S_2^2}$$

Here the bigger value between the standard deviations makes the numerator.

LESSON 6 REINFORCING QUESTIONS

QUESTION ONE

A firm purchases a very large quantity of metal off-cuts and wishes to know the average weight of an off-cut. A random sample of 625 off-cuts is weighed and it is found that the mean sample weight is 150 grams with a sample standard deviation of 30 grams. What is the estimate of the population mean and what is the standard error of the mean? What would be the standard error if the sample size was 1225?

QUESTION TWO

A sample of 80 is drawn at random from a population of 800. The sample standard deviation was found to be 6 grams.

- What is the finite population correction factor?
- What is the approximation of the correction factor?
- What is the standard error of the mean?

QUESTION THREE

State the Central Limit Theorem

QUESTION FOUR

- a) What is statistical inference?
- b) What is the purpose of estimation?
- c) What are the properties of good estimators?
- d) What is the standard error of the mean?
- e) What are confident limits?
- f) When is the Finite Population Correction Factor used? What is the formula?
- g) How are population proportions estimated?
- h) What are the characteristics of the t distribution?

QUESTION FIVE

A market research agency takes a sample of 1000 people and finds that 200 of them know of Brand X. After an advertising campaign a further sample of 1091 people is taken and it's found that 240 know of Brand X.

It is required to know if there has been an increase in the number of people having an awareness of Brand X at the 5% level.

QUESTION SIX

The monthly bonuses of two groups of salesmen are being investigated to see if there is a difference in the average bonus received. Random samples of 12 and 9 are taken from the two groups and it can be assumed that the bonuses in both groups are approximately normally distributed and that the standard deviations are about the same. The same level of significance is to be used.

$n_1=12$	$n_2=9$
$X_1=£1060$	$X_2=£970$
$S_1=£63$	$S_2=£76$

The sample results were

QUESTION SEVEN

Torch bulbs are packed in boxes of 5 and 100 boxes are selected randomly to test for the number of defectives

Number of Defectives	Number of boxes	Total defectives
0	40	0
1	37	37
2	17	34
3	5	15
4	1	4
5	0	0
	<u>100</u>	<u>90</u>

The number of any individual bulb being a reject is

$$\frac{90}{100} \div 5 = 0.18$$

and it is required to test at the 5% level whether the frequency of rejects conforms to a binomial distribution.

QUESTION EIGHT

- Define type I and type II errors.
- What is a two-tail test?
- What is the best estimate of the population standard deviation when the two samples are taken

QUESTION NINE

Express Packets guarantee 95% of their deliveries are on time. In a recent week 80 deliveries were made and 6 were late and the management says that, at the 95% level there has been a significant improvement in deliveries.

Can the MD's statement be supported?

If not, at what level of confidence can it be supported?

A batch of weighing machines has been purchased and one machine is selected at random for testing. Ten weighing tests have been conducted and the errors found are noted as follows:

Test	Errors (gms)
1	4.6
2	8.2
3	2.1
4	6.3
5	5.0
6	3.6
7	1.4
8	4.1
9	7.0
10	4.5

The purchasing manager has previously accepted machines with a mean error of 3.8 gms and asserts that these tests are below standard.

Test the assertions at 5% level.

Compare your answers with those given in lesson 9 of the study pack

CHAPTER NINE

TIME SERIES ANALYSIS

Specific Objectives

At the end of the this topic the trainee should be able to:

- Describe the components of time series;
- Describe the time series models;
- Evaluate trend and seasonal variation in the series;
- Explain the concept of deseasonalization;
- Apply time series analysis to business problems

INTRODUCTION

Businesses and governments use statistical analysis of information collected at regular intervals over extensive periods of time to plan future policies. For example, sales values or unemployment levels recorded at yearly, quarterly or monthly intervals are examined in an attempt to predict their future behavior. Such sets of values observed at regular intervals over a period of time are called time series. The analysis of this data is a complex problem as many variable factors may influence the changes. The first step is to plot the observations on a scattergraph, which differs from those scattergraphs we have considered previously as the points are evenly spaced on the time axis in the order in which they are observed, and the time variable is always the independent Variable. This scattergraph gives us a good visual guide to the actual changes, but is of very little help in showing the component factors causing these changes or in predicting future movements of the dependent variable. Statisticians have constructed a number of mathematical models to describe the behavior of time series, and several of these will be discussed in this study unit.

COMPONENTS OF A TIME SERIES

These mathematical models assume that the changes are caused by the variation of four main factors; they differ in the relationship between these factors. It will be easier to understand the theory in detail if we relate it to a simple time series so that we can see the calculations necessary at each stage. Consider a factory employing a number of people in producing a particular commodity, say thermometers. Naturally, at such a factory during the course of a year some employees will be absent for various reasons. The following table shows the number of days lost through sickness over a five-year period. Each year has been broken down into four quarters of three months. We have assumed that the number of employees at the factory remained constant over the five years.

Table 9.1: Days lost through sickness at a thermometer factory

Year	Quarter	Days Lost
2003	1	30
	2	20
	3	15
	4	35
2004	1	40
	2	25
	3	18
	4	45
2005	1	45
	2	30
	3	22
	4	55
2006	1	50
	2	32
	3	28
	4	60
2007	1	60
	2	35
	3	30
	4	70

We will begin by plotting a time-series graph for the data, as shown in Figure 9.1.

Note the following characteristics of a time-series graph:

- It is usual to join the points by straight lines. The only function of these lines is to help your eyes to see the pattern formed by the points.
- Intermediate values of the variables cannot be read from the graph.
- Every time-series graph will look similar to this, but a careful study of the change of
- Pattern over time will suggest which model should be used for analysis.

There are four factors that influence the changes in a time series - trend, seasonal variations, cyclical fluctuations, and irregular or random fluctuations. Now we will consider each in turn.

Trend

This is the change in general level over the whole time period and is often referred to as the secular trend. You can see in Figure 9.1 that the trend is definitely upwards, in spite of the obvious fluctuations from one quarter to

the next. A trend can thus be defined as a clear tendency for the time series data to travel in a particular direction in spite of other large and small fluctuations.

Seasonal Variations

These are variations which are repeated over relatively short periods of time. Those most frequently observed are associated with the seasons of the year, e.g. ice cream sales tend to rise during the summer months and fall during the winter months. You can see in our example of employees' sickness that more people are sick during the winter than in the summer. If you can establish the variation throughout the year then this seasonal variation is likely to be similar from one year to the next, so that it would be possible to allow for it when estimating values of the variable in other parts of the time series. The usefulness of being able to calculate seasonal variation is obvious as, for example, it allows ice cream manufacturers to alter their production schedules to meet these seasonal changes.

As this type of fluctuation is difficult to determine, it is often considered with the final (fourth) element, and the two together are called the residual variation.

Irregular or Random Fluctuations

Careful examination of Figure 9.1 shows that there are other relatively small irregularities which we have not accounted for and that do not seem to have any easily seen pattern. We call these irregular or random fluctuations; they may be due to errors of observation or to some one-off external influence which is difficult to isolate or predict. In our example (Table1) there may have been a measles epidemic in 2006, but it would be extremely difficult to predict when and if such an epidemic would occur again.

Summary

To sum up, a time series (Y) can be considered as a combination of the following four factors:

- Trend (T)
- Seasonal variation (S)
- Cyclical fluctuation (C)
- Irregular fluctuations (I).

Time series models:

Additive Model

Time series value = $T + S + C + R$

Where S, C and R are expressed in absolute value.

This model is best suited where the component factors are independent e.g. where the seasonal variation is unaffected by trend.

Multiplicative Model:

Time series value = $T \times S \times C \times R$

Where S, C and are expressed as percentage or proportions.

This model is best applied where characteristics interact e.g. where high trends increase seasonal variations. Multiplicative model is more commonly used in practice.

Of the four elements of time series the most important are **trend** and **seasonal variation**. The following illustration shows how the trend (T) and seasonal variation (S) are separated out from a time series and how the calculated T and S values are used to prepare forecast. The process of separating out the trend and seasonal variation is known as deseasonalising the data.

There are two approaches to this process: one is based on regression through the actual data points and the other calculates the regression line through moving average trend points. The method using the actual data is demonstrated first followed by the moving average method.

CHAPTER TEN

INDEX NUMBERS

SPECIFIC OBJECTIVES

At the end of this topic the trainee should be able to:

- Define index numbers;
- Differentiate between the types of index numbers
- Explain the uses of index numbers
- List the limitations of index numbers.

INTRODUCTION

An index number is a device which shows by its variation the changes in a magnitude which is not capable of accurate measurement in itself or of direct valuation in practice. An index number may be described as a specialized average designed to measure the level of a phenomenon with respect to time geographic location or other characteristics such as income.

Uses of index numbers

- They help in framing suitable policies
- They reveal trends and tendencies
- Index numbers are very useful in deflating

Classification of index numbers.

Index numbers may be classified in terms of what they measure. In economics and business the classification is;

- Price
- Quantity
- Value
- Special purpose

Problems in the constructing of index numbers

Before constructing index numbers a careful thought must be given to the following problems.

- i) The purpose of the index; as the very outset the purpose of constructing the must be very clearly decided.
- ii) Availability and comparability of data; it is needless to say that it is impossible to make appropriate comparison unless the necessary statistical data can be obtained
- iii) Selection of base period; whenever index number are constructs, a reference is made to some base period ,safe period should be normal , it should not be too distant.
- iv) Selection of number of items; every item cannot be included while constructing an index number and hence one has to select a sample.
- v) Price quotation; after the commodities have been selected the next problem is to obtain price quotation for these commodities pricing vary from place to place and even from shop to shop.

- vi) Choice of an average; since index numbers are specific average a decision has to be made as to which particular average (i.e. arithmetic mean, median , mode , geometric mean)
- vii) Selection of appropriate weight; the problem of selecting suitable weights are quite important and at the same time quite difficult to decide.
- viii) Selection of appropriate formula; a large number of formula have been devised for often constructing the index numbers. The problem very often is that in selecting the most appropriate formula.

An index number is an attempt to summarize a whole mass of data into one figure. The single figure shows how one year differs from another year. It is a statistical devise used to measure the change in the level of prices, wages output and other variables at given times, relative to their level at an earlier time which is taken as the base for comparison purposes

A simple price index = $\frac{\sum P_n}{\sum P_o} \times 100$ (an unweighted price index)

A simple quantity index = $\frac{\sum Q_n}{\sum Q_o} \times 100$ (an unweighted quantity index)

Where p_n is the price of a commodity in the current year (the year for which the price index to be calculated)

Where p_o is the price of the same commodity in the base year (the year for comparison purposes)

Similarly Q_n and Q_o are defined in the same way

Methods of constructing index numbers

- a) unweighed indices
- b) weighed indices

In the unweighted indices weight are not expressively assigned whereas in the weighed indices weights are assigned to the various items. Each of these types may further be divided under two heads.

- i) simple aggregate
- ii) simple average of price relations

Unweighted index numbers

- i) simple aggregate method

This is the simplest method of constructing index numbers. When this method is used to construct a price index number, the total of current year

prices for the various commodities is divided by total of base year and quotient is multiplied by 100.

$$P_{01} = \frac{\sum p_1}{\sum p_0} \times 100$$

Where $\sum p_1$ = total current price for various commodities

$\sum p_0$ = total of base year price for various commodities

Limitation of the method

- The units in which prices of commodities are given affect the price index.
- No consideration is given to the relative importance of the commodities.

ii) simple average of relative method

When this method is used to construct a price index, price relative are obtained for the various items include intercontinental he index and then an average of these relative

$$P_{01} = \frac{\sum p_1}{\sum p_0} \times 100 / N$$

Where N= refers to the number of items

Merits

- Extreme items do not influence the index
- The index is not influenced by the units in which prices are quoted.

Limitations

- Difficulty is faced with regard to the selection of an appropriate average.
- The relatives are assumed to have equal importance.

Weighed index numbers

Weighed index numbers are of two types

- i) weighed aggregate index numbers
- ii) weighed average of relative index numbers

Weighed aggregate method

An index is called a weighed aggregate index when it is constructed fro an aggregate of items (price) that have been weighed.

Some important formulae of construction a weighed aggregate index are

- i) Laspeyres Method
- ii) Paasche Method
- iii) Fishers Ideal Method
- iv) Marshall - Edgeworth

Laspeyres method

In this method the base year's quantities are taken as weights. The formula for constructing index is

$$P_{01} = \frac{\sum p_1 q_0}{\sum p_0 q_0} \times 100$$

Paasche method

In this method the current year quantities are taken as the formula for constructing index is

$$P01 = \frac{\sum p1q1}{\sum p0q1} \times 100$$

Fischer's ideal method

Prof Fischer has given a number of formulas for constructing index numbers

$$P01 = \sqrt{\left[\frac{\sum p1q0}{\sum p0q0} \times \frac{\sum p1q1}{\sum p0q1} \times 100 \right]} \text{ or}$$

$$p01 = \sqrt{L \cdot P}$$

Marshall Edge Worth Method

In this method also the current year as well as base year and quantities are considered

$$P01 = \frac{\sum (q0 + q1) p1}{\sum (q0 + q1) p0} \times 100$$

INDEX NUMBERS APPLICATION

The index of industrial production

It is a quantity index compiled by the government. It measures changes in the volume of production in major industries. The index is a good indication of the state of national economy.

It covers the following major industries in the UK

- i. Mining and quarrying
- ii. Manufacturing such as food, drinks and tobacco, chemicals, metal manufacture, engineering e.t.c
- iii. Textile
- iv. Construction
- v. Gas electricity, water e.t.c

It excludes agriculture, fishing, trade, transport, finance and other such industries.

Each industries order is given a weighting. The weighting is based on average monthly production in each industry in a fixed base year. It gives each item its relative importance amongst all other items and thus gives a better estimate of the index for comparison purposes.

The Geometric Index (Industrial Share index)

This index is an index of 30 selected top industrial companies. It is calculated by taking an unweighted geometric mean of the price relatives of the selected shares.

Example

The share prices of ordinary shares of four companies on 1st January 1990 and 1st January 1991 were as follows.

Share	Price on 1.1.1990	Price on 1.1.1991
Company A	Shs 10	Shs 12
Company B	Shs 12	Shs 15
Company C	Shs 20	Shs 25
Company D	Shs 5	Shs 6

Using an unweighted geometric index, calculate the index of share prices at 1.1.1991 if 1.1.1990 is the base date, index 100

Solution

$$\left(\frac{12}{10} \times \frac{15}{12} \times \frac{25}{20} \times \frac{6}{5}\right)^{\frac{1}{4}} = \left(\frac{27000}{12000}\right)^{\frac{1}{4}} = (2.25)^{\frac{1}{4}}$$

= 1.225

Percentage increase = 22.5% index = 122.5

Inflation

The inflation rate for a given period can be calculated using the following formula;

$$\text{Inflation} = \frac{\text{Current retail price index}}{\text{Retail price index in the base year}} \times 100$$

Marshal Hedge Worth Index

$$\text{Marshal Hedge worth index} = \frac{\sum p_n (p_o + q_n)}{\sum p_o (q_o + q_n)} \times 100$$

Tests for an Ideal Index Number

1. Factor Reversal Test

This test indicates that when the price index is multiplied with a quantity index i.e. factors are reversed), it should result in the value index.

2. The time reversal test

If we reverse the time subscripts of a price or quantity index, the result should be reciprocal of the original index.

PRACTICE QUESTION

QUESTION FIVE

Prodco PLC manufactures an item of domestic equipment which requires a number of components which have varied as various modifications of the model have been used. The following table shows the number of components required together with the price over the last three years of production.

COMPONENT	1981		1982		1983	
	Prices	Quantity	Prices	Quantity	Prices	Quantity
A	3.63	3	4.00	2	4.49	2
B	2.11	4	3.10	5	3.26	6
C	10.03	1	10.36	1	12.05	1
D	4.01	7	5.23	6	5.21	5

Required:

- Establish the base weighted price indices for 1982 and 1983 based on 1981 for the item of equipment. (8 marks)
- Establish the current weighted price indices for 1982 and 1983 based on 1981 for the item of equipment. (8 marks)
- Using the results of (a) and (b) as illustrations, compare and contrast Laspeyre's and Paasche price index numbers. (4 marks)

(Total: 20 marks)

QUESTION SIX

- a) A company manufacturing a product known as 257 uses five components in its assembly.

The quantities and prices of the components used to produce a unit of K257 in 1982, 1983 and 1984 are tabulated as follows:

COMPONENT	1982		1983		1984	
	Quantity	Prices	Quantity	Prices	Quantity	Prices
A	10	3.12	12	3.17	14	3.20
B	6	11.49	7	11.58	5	11.67
C	5	1.40	8	1.35	9	1.31
D	9	2.15	9	2.14	10	2.63
E	50	0.32	53	0.32	57	0.32

Required:

- Calculate Laspyere's type price index number for the cost of one unit of K257 for 1983 and 1984 based on 1982. (6 marks)
- Calculate Paasche type price index numbers for the cost of one unit of K257 for 1983 and 1984 based on 1982. (6 marks)
- Compare and contrast the Laspeyre and Paasche price-index numbers you have obtained in (i) and (ii) (3 marks)

A number of employers manufacturing plastic components used in plumbing have formed themselves into an association for the purpose of negotiating with the trade union for this industrial sector.

The negotiations cover pay and contributions in this sector.

Required:

Explain the usefulness of an index of Industrial Production and an index of retail prices to both sides in a series of pay negotiations.

(5 marks)

(Total: 20marks)

QUESTION SEVEN

The data given below indicates the prices and production of some horticultural products in Central Territory:

Produce	Production (1000 boxes)		Price per box (Shs)	
	1980	1990	1980	1990
Cabbages	48,600	62,000	100	150
Tomatoes	22,000	37,440	220	310
Onions	47,040	61,430	180	200
Spinach	43,110	55,720	130	170

Required:

Calculate the increase or decrease in prices from 1980 on the basis of the following indices:

- Mean relatives
- Laspeyres index
- Paasche index
- Marshall - Hedgeworth index
- Fishers index.

CHAPTER ELEVEN MATRIX ALGEBRA

SPECIFIC OBJECTIVES

At the end of this topic the trainee should be able to:

- Define matrices;
- Describe the types of matrices,
- Apply matrix operations
- Form matrix models from practical problems;
- Apply matrix to decision problems.

INTRODUCTION

A matrix is a rectangular array of items or numbers. These items or numbers are arranged in rows and columns to represent some information. The position of an element in one matrix is very important as well be seen later; therefore an element is located by the number of the row and column which it occupies. The size of a matrix is defined by the number of its rows (m) and column (n).

For example $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

are (2 x 2) and (3 x 3) matrices since A has 2 rows and 2 columns and B has 3 rows and 3 columns.

A matrix A with three rows and four columns is given by one of:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

or

$$A = (a_{ij}) \quad i = 1, 2, 3$$

$j = 1, 2, 3, 4$ where i represents the row number whereas j represents the column number

Types of matrices

Equal Matrices

Two matrices A and B are said to be equal, that is

$$A = B \quad \text{or} \quad (a_{ij}) = (b_{ij})$$

If and only if they are identical if they both have the same number of rows and columns and the elements in the corresponding locations in the two matrices should be the same, that is, $a_{ij} = b_{ij}$ for all i. And j.

Example

The following matrices are equal $\begin{pmatrix} 3 & 4 & 0 \\ 2 & 2 & 3 \\ 5 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 0 \\ 2 & 2 & 3 \\ 5 & 1 & 1 \end{pmatrix}$

Column Matrix or column vector

A column matrix, also referred to as column vector is a matrix consisting of a single column.

For example $x = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$

Row matrix or row vector

It is a matrix with a single row

For example $y = (y_1, y_2, y_3, \dots, y_n)$

Transpose of a Matrix

The transpose of an $m \times n$ matrix A is the $n \times m$ matrix A^T obtained by interchanging the rows and columns of A.

$$A = \begin{pmatrix} a_{ij} \end{pmatrix}$$

The transpose of A i.e. A^T is given by

$$A^T = \begin{pmatrix} a_{ij} \end{pmatrix} = \begin{pmatrix} a_{ji} \end{pmatrix}$$

mxn

nxm

Example

Find the transposes of the following matrices

$$A = \begin{pmatrix} 1 & 5 & 7 \\ 2 & 1 & 4 \\ 0 & 9 & 3 \end{pmatrix}$$

$$B = (b_1, b_2, b_3, b_4)$$

$$C = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Solution

$$\text{i. } A^T = \begin{pmatrix} 1 & 5 & 7 \\ 2 & 1 & 4 \\ 0 & 9 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 0 \\ 5 & 1 & 9 \\ 7 & 4 & 3 \end{pmatrix}$$

$$\text{ii. } B^T = (b_1, b_2, b_3, b_4)^T = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

$$\text{iii. } C^T = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^T = (x_1 \ x_2 \ x_3)$$

Square Matrix

A matrix A is said to be square when it has the same number of rows as columns

e.g.

$$A = \begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix} \quad \text{is a square matrix of order 2}$$

$B = n \times n$ is a square matrix of the order n

Diagonal matrices

It is a square matrix with zeros everywhere in the matrix except on the principal diagonal

e.g.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

An identity or unity matrix

It is a diagonal matrix in which each of the diagonal elements is a positive one (1)

e.g.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2×2 unit matrix 3×3 unit matrix

A null or zero matrix

A null or zero matrix is a matrix whose elements are all equal to zero.

Sub matrix

The sub matrix of the matrix A is another matrix obtained from A by deleting selected row(s) and/or column(s) of the matrix A .

e.g, if $A = \begin{pmatrix} 7 & 9 & 8 \\ 2 & 3 & 6 \\ 1 & 5 & 0 \end{pmatrix}$

then $A_1 = \begin{pmatrix} 2 & 3 & 6 \\ 1 & 5 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 7 & 9 \\ 1 & 5 \end{pmatrix}$

are both sub matrices of A

OPERATION ON MATRICES

Matrix addition and subtraction

We can add any number of matrices (or subtract one matrix from another) if they have the same sizes. Addition is carried out by adding together corresponding elements in the matrices. Similarly subtraction is carried out by subtracting the corresponding elements of two matrices as shown in the following example

Example: Given A and B, calculate A + B and A - B

$$A = \begin{pmatrix} 6 & -1 & 10 & 5 \\ 3 & 4 & 2 & -5 \\ -9 & -13 & -6 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 12 & 4 & -7 & 3 \\ 0 & -4 & 10 & -4 \\ 7 & -3 & 7 & 9 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 6 & -1 & 10 & 5 \\ 3 & 4 & 2 & -5 \\ -9 & -13 & -6 & 0 \end{pmatrix} + \begin{pmatrix} 12 & 4 & -7 & 3 \\ 0 & -4 & 10 & -4 \\ 7 & -3 & 7 & 9 \end{pmatrix} = \begin{pmatrix} 18 & 3 & 3 & 8 \\ 3 & 0 & 12 & -9 \\ -2 & -16 & 1 & 9 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 6 & -1 & 10 & 5 \\ 3 & 4 & 2 & -5 \\ -9 & -13 & -6 & 0 \end{pmatrix} - \begin{pmatrix} 12 & 4 & -7 & 3 \\ 0 & -4 & 10 & -4 \\ 7 & -3 & 7 & 9 \end{pmatrix} = \begin{pmatrix} -6 & -5 & 17 & 2 \\ 3 & 8 & -8 & -1 \\ -16 & -10 & -13 & -9 \end{pmatrix}$$

If it is assumed that A, B, C are of the same order, the following properties are fulfilled:

- a) Commutative law: $A + B = B + A$
b) Associative law: $(A + B) + C = A + (B + C) = A + B + C$

Multiplying a matrix by a number

In this case each element of the matrix is multiplied by that number

Example

$$\text{If } A = \begin{pmatrix} 6 & -1 & 10 & 5 \\ 3 & 4 & 2 & -5 \\ -9 & 13 & -6 & 0 \end{pmatrix}$$

$$\text{then } (10)A = \begin{pmatrix} 60 & -10 & 100 & 50 \\ 30 & 40 & 20 & -50 \\ -90 & 130 & -60 & 0 \end{pmatrix}$$

Matrix Multiplication

a) Multiplication of two vectors

Let row vector A represent the selling price in shillings of one unit of commodity P, Q, R respectively and let column vector B represent the number of units of commodities P, Q, R sold respectively. Then the vector product $A \times B$ will be equal to the total sales value

i. e. $A \times B =$ Total sales value

$$\text{Let } A = (4 \ 5 \ 6) \text{ and } B = \begin{pmatrix} 100 \\ 200 \\ 300 \end{pmatrix}$$

$$\text{then } (4 \ 5 \ 6) \begin{pmatrix} 100 \\ 200 \\ 300 \end{pmatrix} = 400 + 1,000 + 1,800 = \text{Shs } 3,200$$

Rules of multiplication

- i. The row vector must have the same number of elements as the column vector
- ii. The first vector is a row vector and the second is a column vector
- iii. The corresponding elements in each vector are multiplied together and the results obtained are added. This addition is always a single number

Going back to the example given before

$$A \times B = (4 \ 5 \ 6) \begin{pmatrix} 100 \\ 200 \\ 300 \end{pmatrix} = 4 \times 100 + 5 \times 200 + 6 \times 300 = \text{Shs } 3,200, \text{ a single number}$$

b) Multiplication of two matrices

Rules

- i. Multiplication is only possible if the first matrix has the same number of columns as the second matrix has rows. That is if A is the order $a \times b$, then B has to be of the order $b \times c$. If the $A \times B = D$, then D must be of the order $a \times c$.
- ii. The general method of multiplication is that the elements in row \underline{m} of the first matrix are multiplied by the corresponding elements in columns \underline{n} of the second matrix and the products obtained are then added giving a single number.

We can express this rule as follows

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and } b = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$\text{Then } A \times B = D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \end{pmatrix}$$

$$A = 2 \times 2 \text{ matrix} \quad B = 2 \times 3 \text{ matrix} \quad D = 2 \times 3 \text{ matrix}$$

Where

$$d_{11} = a_{11} \times b_{11} + a_{12} \times b_{21}$$

$$d_{12} = a_{11} \times b_{12} + a_{12} \times b_{22}$$

Example I

$$\begin{pmatrix} 6 & 1 \\ 2 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 0 & 2 \\ 4 & 5 & 8 \end{pmatrix} = \begin{pmatrix} 6 \times 3 + 1 \times 4 & 6 \times 0 + 1 \times 5 & 6 \times 2 + 1 \times 8 \\ 2 \times 3 + 3 \times 4 & 2 \times 0 + 3 \times 5 & 2 \times 2 + 3 \times 8 \end{pmatrix}$$
$$= \begin{pmatrix} 22 & 5 & 20 \\ 18 & 15 & 28 \end{pmatrix}$$

Example II

Matrix X gives the details of component parts used in the make up of two products P_1 and P_2 matrix Y gives details of products made on each day of the week as follows:

Matrix X			Matrix Y		
Parts			Products		
	A	B	C	P ₁	P ₂
Products	$\begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 3 \end{bmatrix}$			Mon	$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 2 \\ 2 & 2 \\ 1 & 1 \end{bmatrix}$
				P ₁	
	P ₂		Tues		
			Wed		
			Thur		
			Fri		

Use matrix multiplication to find the number of component parts used on each day of the week.

Solution:

After careful consideration, it will be easy to decide that the correct order of multiplication is YXX (Note the order of multiplication). This multiplication is compatible and also it gives the desired answer.

$$\begin{matrix}
 Y \times X = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 2 \\ 2 & 2 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 4 & 2 \\ 2 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 \times 3 + 2 \times 2 & 1 \times 4 + 2 \times 5 & 1 \times 2 + 2 \times 3 \\ 2 \times 3 + 3 \times 2 & 2 \times 4 + 3 \times 5 & 2 \times 2 + 3 \times 3 \\ 3 \times 3 + 2 \times 2 & 3 \times 4 + 2 \times 5 & 3 \times 2 + 2 \times 3 \\ 2 \times 3 + 2 \times 2 & 2 \times 4 + 2 \times 5 & 2 \times 2 + 2 \times 3 \\ 1 \times 3 + 1 \times 2 & 1 \times 4 + 1 \times 5 & 1 \times 2 + 1 \times 3 \end{pmatrix} \\
 5 \times 2 \text{ matrix} \quad 2 \times 3 \text{ matrix} = \quad 5 \times 3 \text{ matrix}
 \end{matrix}$$

	A	B	C
Mon	7	14	8
Tues	12	23	13
Wed	13	22	12
Thur	10	18	10
Fri	5	9	5

Interpretation

On Monday, number of component parts A used is 7, B is 14 and C is 8. in the same way, the number of component parts used for other days can be interpreted.

The determinant of a square matrix

The determinant of a square matrix $\det(A)$ or $|A|$ is a number associated to that matrix. If the determinant of a matrix is equal to zero, the matrix is called singular matrix otherwise it is called non-singular matrix. The determinant of a non square matrix is not defined.

Determination of a 2 x 2 matrix

$$(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

ii. Determinant of a 3 x 3 matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

$$a(ei - fh) - b(di - gf) + c(dh - eg)$$

simplify

iii. Determinant of a 4 x 4 matrix

$$A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

$$A = a \begin{pmatrix} f & g & h \\ j & k & l \\ n & o & p \end{pmatrix} - b \begin{pmatrix} e & g & h \\ i & k & l \\ n & o & p \end{pmatrix} + c \begin{pmatrix} e & f & h \\ i & j & l \\ m & o & p \end{pmatrix} - d \begin{pmatrix} e & f & g \\ i & j & k \\ m & n & o \end{pmatrix}$$

Simplify 3 x 3 determinants as in ii and then evaluate the 4 x 4 determinants.

Inverse of a matrix

If for an n (n square matrix A, there is another n (n square matrix B such that their product is the identity of the order n X n, In, that is A X B = B X A = I, then B is said to be inverse of A. Inverse is generally written as A-1

$$\text{Hence } AA^{-1} = I$$

Note: Only non singular matrices have an inverse and therefore the inverse of a singular matrix is non defined.

General method for finding inverse of a matrix

In order to introduce the rule to calculate the determinant as well as the inverse of a matrix, we should introduce the concept of minor and cofactor.

The minor of an element

Given a matrix $A = (a_{ij})$, the minor of an element a_{ij} in row i and column j (call it m_{ij}), is the value of the determinant formed by deleting row i and column j in matrix A .

Example

Let $A = \begin{bmatrix} 6 & 1 & 4 \\ 3 & 0 & 2 \\ 5 & 2 & 3 \end{bmatrix}$

The minors are,

$$m_{11} = \begin{vmatrix} 6 & 1 \\ 3 & 0 \end{vmatrix} = 6 \times 0 - 3 \times 1 = -3$$

$$m_{12} = \begin{vmatrix} 5 & 1 \\ 2 & 0 \end{vmatrix} = 5 \times 0 - 1 \times 2 = -2$$

Similarly

$$m_{13} = \begin{vmatrix} 5 & 6 \\ 2 & 3 \end{vmatrix} = 15 - 12 = 3$$

$$m_{21} = \begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} = 0 - 9 = -9$$

$$m_{22} = \begin{vmatrix} 4 & 3 \\ 2 & 0 \end{vmatrix} = 0 - 6 = -6$$

$$m_{23} = \begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix} = 12 - 4 = 8$$

$$m_{31} = \begin{vmatrix} 2 & 3 \\ 6 & 1 \end{vmatrix} = 2 - 18 = -16$$

$$m_{32} = \begin{vmatrix} 4 & 3 \\ 5 & 1 \end{vmatrix} = 4 - 15 = -11$$

$$m_{33} = \begin{vmatrix} 4 & 2 \\ 5 & 6 \end{vmatrix} = 24 - 10 = 14$$

The cofactor of an element

The cofactor of any element a_{ij} (known as c_{ij}) is the signed minor associated with that element.

The sign is not changed if $(i+j)$ is even and it is changed if $(i+j)$ is odd. Thus the sign alternated whether vertically or horizontally, beginning with a plus in the upper left hand corner.

i.e. 3 x 3 signed matrix will have signs $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

Hence the cofactor of element a_{11} is $m_{11} = -3$, cofactor of a_{12} is $-m_{12} = +2$ the cofactor of element a_{13} is $+m_{13} = 3$ and so on.

$$\text{Matrix of cofactors of A} = \begin{pmatrix} -3 & 2 & 3 \\ 9 & -6 & -8 \\ -16 & 11 & 14 \end{pmatrix}$$

$$\text{in general for a matrix M} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Cofactor of a is written as A, cofactor of b is written as B and so on. Hence matrix of cofactors of M is written as

$$= \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$$

The determinant of a $n \times n$ matrix

The determinant of a $n \times n$ matrix can be calculated by adding the products of the element in any row (or column) multiplied by their cofactors. If we use the symbol Δ for determinant.

$$\text{Then } \Delta = aA + bB + cC$$

or

$$= dD + eE + fF \text{ e.t.c}$$

Note: Usually for calculation purposes we take $\Delta = aA + bB + cC$

Hence in the example under discussion

$$\Delta = (4 \times -3) + (2 \times 2) + (3 \times 3) = 1$$

The ad joint or transposition of a matrix

$$\text{The ad joint of matrix } \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} \text{ is written as}$$

$$\begin{pmatrix} A & D & G \\ B & E & H \\ C & F & I \end{pmatrix}$$
 i.e. change rows into columns and columns into rows
 (transpose)

The inverse of the matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

is written as $\frac{1}{\text{determinant}} \times (\text{adjoint of the matrix of cofactors})$

$$\text{i.e. } A^{-1} = \frac{1}{\Delta} \times \begin{pmatrix} A & D & G \\ B & E & H \\ C & F & I \end{pmatrix}$$

Where $\Delta = aA + bB + cC$

Hence inverse of $\begin{pmatrix} 4 & 2 & 3 \\ 5 & 6 & 1 \\ 2 & 3 & 0 \end{pmatrix}$

is found as follows

$$\Delta = (4 \times -3) + (2 \times 2) + (3 \times 3) = 1$$

$$A = -3 \quad B = 2 \quad C = 3$$

$$D = 9 \quad E = -6 \quad F = -8$$

$$G = -16 \quad H = 11 \quad I = 14$$

□ EMBED Equation.DSMT4 □□□

(note: Check if $A (A^{-1} \times A = 1)$)

APPLICATION OF MATRICES

Solution of simultaneous equations

In order to determine the solutions of simultaneous equations, we may use either of the following 2 methods

i. The cofactor method

ii. Cramers rule

The cofactor method

This method requires that we obtain

- a) The minors and cofactors
- b) The adjoint of the matrix
- c) The inverse of the matrix
- d) Multiply the original by the inverse on both sides of the matrix equation

Example

Solve the following

a) $4x_1 + x_2 - 5x_3 = 8$
 $-2x_1 + 3x_2 + x_3 = 12$
 $3x_1 - x_2 + 4x_3 = 5$

b) $4x_1 + 3x_2 + 5x_3 = 27$
 $x_1 + 6x_2 + 2x_3 = 19$
 $3x_1 + x_2 + 3x_3 = 15$

c) $4x_1 + 2x_2 + 6x_3 = 28$
 $3x_1 + x_2 + 2x_3 = 20$
 $10x_1 + 5x_2 + 15x_3 = 70$

d) $2x_1 + 4x_2 - 3x_3 = 12$
 $3x_1 - 5x_2 + 2x_3 = 13$
 $-x_1 + 3x_2 + 2x_3 = 17$

Solution

a) From a, we have

$$\begin{pmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \\ 5 \end{pmatrix}$$

A X b

We need to determine the minors and the cofactors for the above matrix

Definition

A minor is a determinant of a sub matrix obtained when other elements are detected as shown below.

A cofactor is the product of $(-1)^{i+j}$ and a minor where

$i =$ lth row $i = 1, 2, 3, \dots$

$j =$ Jth row $j = 1, 2, 3, \dots$

$$\text{Cofactor of } 4 (a_{11}) = (-1)^{1+1} \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = 13$$

$$\text{Cofactor of } -2 (a_{21}) = (-1)^{2+1} \begin{vmatrix} 1 & -5 \\ -1 & 4 \end{vmatrix} = 1$$

$$\text{Cofactor of } 3 (a_{31}) = (-1)^{3+1} \begin{vmatrix} 1 & -5 \\ 3 & 1 \end{vmatrix} = 16$$

$$\text{Cofactor of } 1 (a_{12}) = (-1)^{1+2} \begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix} = 11$$

$$\text{Cofactor of } 3 (a_{22}) = (-1)^{2+2} \begin{vmatrix} 4 & -5 \\ 3 & 4 \end{vmatrix} = 31$$

$$\text{Cofactor of } -1 (a_{23}) = (-1)^{2+3} \begin{vmatrix} 4 & 5 \\ -2 & 1 \end{vmatrix} = 6$$

$$\text{Cofactor of } -5 (a_{13}) = (-1)^{1+3} \begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} = -7$$

$$\text{Cofactor of } +1 (a_{23}) = (-1)^{2+3} \begin{vmatrix} 4 & 1 \\ 3 & -1 \end{vmatrix} = 7$$

$$\text{Cofactor of } 4 (a_{33}) = (-1)^{3+3} \begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix} = 14$$

The matrix of C of cofactors is

$$\begin{pmatrix} 13 & 11 & -7 \\ 1 & 31 & 7 \\ 16 & 6 & 14 \end{pmatrix}$$

$$C^T = \begin{pmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{pmatrix} = \text{Adjoint of the original matrix of coefficients}$$

The original matrix of coefficients

$$= \begin{pmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{pmatrix}$$

Therefore

$$\begin{pmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{pmatrix} = \begin{array}{ccc} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{array}$$

$$\begin{aligned} &= (48 + 3 - 10) - (-45 - 4 - 8) \\ &= 41 + 57 \\ &= 98 \end{aligned}$$

The inverse of the matrix of coefficients, see (*) will be

$$= \frac{1}{98} \begin{pmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{pmatrix}$$

by multiplying the inverse on both sides of * we have,

$$\frac{1}{98} \begin{pmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{pmatrix} \begin{pmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \frac{1}{98} \begin{pmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{pmatrix} \begin{pmatrix} 8 \\ 12 \\ 5 \end{pmatrix}$$

$$= \frac{1}{98} \begin{pmatrix} 98 & 0 & 0 \\ 0 & 98 & 0 \\ 0 & 0 & 98 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{98} \begin{pmatrix} 196 \\ 490 \\ 98 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

$$\therefore X_1 = 2, X_2 = 5, X_3 = 1$$

c) $4x_1 + 2x_2 + 6x_3 = 28$

$$3x_1 + x_2 + 2x_3 = 20$$

$$10x_1 + 5x_2 + 15x_3 = 70$$

$$= \begin{pmatrix} 4 & 2 & 6 \\ 3 & 1 & 2 \\ 10 & 5 & 15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 28 \\ 20 \\ 70 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 & 6 \\ 3 & 1 & 2 \\ 10 & 5 & 15 \end{pmatrix} = \begin{array}{ccc|ccc} 4 & 2 & 6 & 4 & 2 & \\ \hline 3 & 1 & 2 & 3 & 1 & 2 \\ \hline 10 & 5 & 15 & 10 & 5 & 15 \end{array}$$

$$= (60 + 40 + 90) - (60 + 40 + 90)$$

$$= 0$$

Hence the solutions of x_1 , x_2 , and x_3 do not exist. The equations are independent

Now work out part (b) on your own.

Cramer's Rule in Solving Simultaneous Equations

Consider the following system of two linear simultaneous equations in two variables.

$$a_{11}x_1 + a_{12}x_2 = b_1 \dots\dots\dots(i)$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \dots\dots\dots(ii)$$

after solving the equations you obtain

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

and

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Solutions of x_1 and x_2 obtained this way are said to have been derived using Cramer's rule, practice this method over and over to internalize it. It is advisable for exam situation since it is shorter.

Example

Solve the following systems of linear simultaneous equations by Cramer's rule:

i) $2x_1 - 5x_2 = 7$

$$x_1 + 6x_2 = 9$$

ii) $x_1 + 2x_2 + 4x_3 = 4$

$$2x_1 + x_3 = 3$$

$$3x_2 + x_3 = 2$$

Solutions

i. $2x_1 - 5x_2 = 7$

$$x_1 + 6x_2 = 9$$

can be expressed in matrix form as

$$\begin{pmatrix} 2 & -5 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$$

$\bar{A} \qquad \qquad \bar{X} \qquad \qquad \bar{b}$

and applying cramer's rule

$$x_1 = \frac{\begin{vmatrix} 7 & -5 \\ 9 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 1 & 6 \end{vmatrix}} = \frac{87}{17} = 5\frac{2}{17}$$

$$x_2 = \frac{\begin{vmatrix} 2 & 7 \\ 1 & 9 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 1 & 6 \end{vmatrix}} = \frac{11}{17}$$

(ii) can be expressed in matrix form as

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$

and by cramer's rule

$$x_1 = \frac{\begin{vmatrix} 4 & 2 & 4 \\ 3 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix}} = \frac{22}{17}$$

$$x_3 = \frac{\begin{vmatrix} 1 & 2 & 4 \\ 2 & 0 & 3 \\ 0 & 3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix}} = \frac{7}{17}$$

$$x_2 = \frac{\begin{vmatrix} 1 & 4 & 4 \\ 2 & 3 & 1 \\ 0 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix}} = \frac{9}{17}$$

Solving simultaneous Equations using matrix algebra

i. Solve the equations

$$2x + 3y = 13$$

$$3x + 2y = 12$$

in matrix format these equations can be written as

$$\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ 12 \end{pmatrix}$$

Pre multiply both sides by the inverse of the matrix

$$\Delta = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5$$

and inverse of the matrix is

$$-\frac{1}{5} \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix}$$

Pre multiplication by inverse gives

$$\begin{pmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 13 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Therefore $x = 2$ $y = 3$

ii. Solve the equations

$$4x + 2y + 3z = 4$$

$$5x + 6y + 1z = 2$$

$$2x + 3y = -1$$

Solution:

Writing these equations in matrix format, we get

$$A \times \bar{X} = \bar{b}$$
$$\begin{pmatrix} 4 & 2 & 3 \\ 5 & 6 & 1 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$$

Pre-multiply both sides by the inverse

the inverse of A as found before is $A^{-1} = \begin{pmatrix} -3 & 9 & -16 \\ 2 & -6 & 11 \\ 3 & -8 & 14 \end{pmatrix}$

$$\begin{pmatrix} -3 & 9 & -16 \\ 2 & -6 & 11 \\ 3 & -8 & 14 \end{pmatrix} \begin{pmatrix} 4 & 3 & 2 \\ 5 & 6 & 1 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 & 9 & -16 \\ 2 & -6 & 11 \\ 3 & -8 & 14 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 22 \\ -15 \\ -18 \end{pmatrix}$$

Hence $x = 22$ $y = -15$ $z = -18$

(Note: under examination conditions it may be advisable to check the solution by substituting the value of x, y, z into any of the three original equations)

SIMPLE INPUT - OUTPUT ANALYSIS

The input output analysis is a topic which requires application of matrices. The technique analyses the flow of inputs from one sector of the economy to the other sectors thus the technique is quite useful in studying the interdependence of sectors within a single economy.

The input - output analysis was first developed by Prof Leontief hence the Leontief matrix has been developed. See the following example

Example (B)

INPUT OUTPUT TABLE

FROM	TO			Final Demand	Total Demand (output)
	Agric	Industry	Service		
Agric	300	360	320	1080	2060
Industry	450	470	410	800	2130
Service	610	500	520	270	1900
Primary inputs	700	800	650	-	-
Total inputs	2060	2130	1900	-	-

NB: In the above table, one should be able to interpret the table e.g. of the total demand of 2060 metric tones from the agricultural sector; 300 is produced for the agricultural sector, 360 for industrial sector, 320 for the service sector and 1080 metric tones makes up the final demand.

The final demand is the additional demand besides the sectoral demand which is normally made by other users e.g. government, foreign countries, other manufacturers not included in the other sectors.

For production if items besides the inputs from other sectors namely labour capital e.t.c

Technical coefficients: ('to' sectors)

$$\text{Agriculture } 300 = \frac{300}{2060} = 0.14$$

$$450 = \frac{450}{2060} = 0.22$$

$$610 = \frac{610}{2060} = 0.30$$

$$\text{Industry } 360 = \frac{360}{2130} = 0.17$$

$$470 = \frac{470}{2130} = 0.22$$

$$500 = \frac{500}{2130} = 0.23$$

$$\text{Service } 320 = \frac{320}{1900} = 0.17$$

$$410 = \frac{410}{1900} = 0.22$$

$$520 = \frac{520}{1900} = 0.27$$

The matrix of technical coefficients is:

FROM	TO			Final Demand	Total Demand (output)
	Agric	Industry	Service		
Agric	0.14	0.7	0.17	1080 (y ₁)	2060
Industry	0.22	0.22	0.22	800 (y ₂)	2130
Service	0.30	0.23	0.27	270(y ₃)	1900
Primary inputs	x	x	x	-	-
	2060(x ₁)	2130(x ₂)	1900(x ₃)	-	-

From the above table, we may develop the following equations

$$0.14x_1 + 0.7x_2 + 0.17x_3 + y_1 = x_1$$

$$0.22x_1 + 0.22x_2 + 0.22x_3 + y_2 = x_2$$

$$0.30x_1 + 0.23x_2 + 0.27x_3 + y_3 = x_3$$

$$\begin{pmatrix} 0.14 & 0.17 & 0.17 \\ 0.22 & 0.22 & 0.22 \\ 0.30 & 0.23 & 0.27 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \dots\dots\dots(*)$$

\bar{A} \bar{X} \bar{Y} \bar{X}

Let the coefficient matrix be represented by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

∴ Equation (*) may be written as

$$AX + Y = X$$

$$Y = X - AX$$

$$Y = X (I - A)$$

$$\Rightarrow (I - A)^{-1} Y = X$$

The matrix $I - A$ is known as Leontief Matrix

Technical Coefficients

These show the units required from each sector to make up one complete product in a given sector e.g. in the above matrix of coefficients it may be said that one complete product from the agricultural sector requires 0.14 units from the agricultural sector itself, 0.22 from the industrial sector and 0.30 from the service sector

NB: The primary inputs are sometimes known as “value added”

Example 1

Determine the total demand (x) for the industry 1, 2, 3 given the matrix of technical coefficients (A), Capital and the final demand vector B.

$$A = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0.3 & 0.4 & 0.1 \\ 0.5 & 0.2 & 0.6 \\ 0.1 & 0.3 & 0.1 \end{pmatrix} \end{matrix} \quad B = \begin{pmatrix} 20 \\ 10 \\ 30 \end{pmatrix}$$

From the input - output analysis

$$X = (I - A)^{-1} B, \text{ Where } X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \text{ is the demand vector}$$

$$\begin{aligned} I - A &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.3 & 0.4 & 0.1 \\ 0.5 & 0.2 & 0.6 \\ 0.1 & 0.3 & 0.1 \end{pmatrix} \\ &= \begin{pmatrix} 0.7 & -0.4 & -0.1 \\ -0.5 & 0.8 & -0.6 \\ -0.1 & -0.3 & 0.9 \end{pmatrix} \end{aligned}$$

$$\therefore (I - A)^{-1} = \begin{pmatrix} 0.7 & -0.4 & -0.1 \\ -0.5 & 0.8 & -0.6 \\ -0.1 & -0.3 & 0.9 \end{pmatrix}^{-1}$$

The matrix of cofactors of (I - A) is

$$= \begin{pmatrix} 0.54 & 0.51 & 0.23 \\ 0.39 & 0.62 & 0.25 \\ 0.32 & 0.47 & 0.36 \end{pmatrix}$$

The transpose (adjoint) of the above matrix is

$$= \begin{pmatrix} 0.54 & 0.39 & 0.32 \\ 0.51 & 0.62 & 0.47 \\ 0.23 & 0.25 & 0.36 \end{pmatrix}$$

$$\begin{aligned} \Delta \text{ of } (I - A) &= 0.495 - (0.008 + 0.126 + 0.18) \\ &= 0.809 \end{aligned}$$

$$\therefore (I - A)^{-1} = \frac{1}{0.809} \begin{pmatrix} 0.54 & 0.39 & 0.32 \\ 0.51 & 0.62 & 0.47 \\ 0.23 & 0.25 & 0.36 \end{pmatrix} = \begin{pmatrix} 0.66 & 0.48 & 0.40 \\ 0.63 & 0.77 & 0.58 \\ 0.23 & 0.31 & 0.44 \end{pmatrix}$$

$$\begin{pmatrix} 0.66 & 0.48 & 0.40 \\ 0.63 & 0.77 & 0.58 \\ 0.23 & 0.31 & 0.44 \end{pmatrix} \begin{pmatrix} 20 \\ 10 \\ 30 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

Therefore X =

$$X = \begin{pmatrix} 30 \\ 37.7 \\ 21.9 \end{pmatrix}$$

The total demand from the three industries 1, 2 and 3 is 30 from 1, 37.7 from 2 and 21.9 from 3.

Example 2

Three clients of Disrupt, Ltd P, Q and Rare direct competitors in the retail business. In the first week of the year P had 300 customers Q had 250 customers and R had 200 customers. During the second week, 60 of the original customers of P transferred to Q and 30 of the original customers of P transferred to R. similarly in the second week 50 of the original customers of Q transferred to P with no transfers to R and 20 of the original customers of R transferred to P with no transfers to Q.

Required

- a) Display in a matrix the pattern of retention and transfers of customers from the first to the second week
(4 marks)

- b) Re-express the matrix that you have obtained in part (a) showing the elements as decimal fractions of the original numbers of customers of P, Q and R (2 marks) Refer to this re expressed matrix as B
- c) Multiply matrix B by itself to determine the proportions of the original customers that have been retained or transferred to P, Q and R from the second to the third week. (4 marks)
- d) Solve the matrix equation $(xyz)B = (xyz)$ given that $x + y + z = 1$ (8 marks)
- e) Interpret the result that you obtain in part (d) in relation to the movement of customers between P, Q and R (2marks) **(Total 20 marks)**

Solution

- a). Think of each row element as being the point from which the customer originated and each column element as being the destination e.g. 210 customers move from P to P, 60 move from P to Q and 30 move from P to R. The sum of the elements of the first row totaling 300 that is the number of customers originally with P.

Hence required matrix is

	P	Q	R	To
P	210	60	30	(row total 300)
From Q	50	200	0	(row total 250)
R	20	0	180	(row total 200)

- b). the requirement of this part is to express each element as a decimal fraction of its corresponding row total. The second row, first element is therefore $50/250$, that is 0.2 and the second element is therefore $200/250$ that is 0.8.

Hence B =

$$\begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.1 & 0 & 0.9 \end{pmatrix}$$

c).

$$\begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.1 & 0 & 0.9 \end{pmatrix} \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.1 & 0 & 0.9 \end{pmatrix} = \begin{pmatrix} 0.54 & 0.30 & 0.16 \\ 0.30 & 0.68 & 0.02 \\ 0.16 & 0.02 & 0.82 \end{pmatrix}$$

The result can be checked by the normal rules of matrix multiplication.

$$d). \quad (x \ y \ z) X \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.1 & 0 & 0.9 \end{pmatrix} = (x \ y \ z)$$

This produces from the first row

$$0.7x + 0.2y + 0.1z = x$$

Which reduces to $0.2y + 0.1z = 0.3x$

$$\text{Or} \quad 2y + z = 3x \dots\dots\dots(i)$$

Or

The second row produces, $0.2x + 0.8y = y$

$$\text{Reducing to} \quad 0.2x = 0.2y$$

$$X = y \dots\dots\dots(ii)$$

Or

The third row produces $0.1x + 0.9z = z$

$$\text{Reducing to} \quad 0.1x = 0.1z$$

$$X = z \dots\dots\dots(iii)$$

At this point you will notice that condition h (ii) and condition (iii) produce $2x + x = 3x$ when substituted into condition (i), we therefore need extra condition $x + y + z = 1$ to solve the problem.

$$\text{Thus} \quad x + x + x = 1$$

$$\text{Or} \quad 3x = 1$$

$$\text{That is} \quad x = \frac{1}{3}$$

$$\text{Leading to} \quad x = \frac{1}{3}, \quad y = \frac{1}{3}, \quad z = \frac{1}{3}$$

e). In proportion terms this solution means that P, Q, and R will in the long term each have one third of the total customers

Example 4

There are three types of breakfast meal available in supermarkets known as brand BM1, brand BM2 and Brand BM3. In order to assess the market, a survey was carried out by one of the manufacturers. After the first month, the survey revealed that 20% of the customers purchasing brand BM1 switched to BM2 and 10% of the customers purchasing brand BM1 switched to BM3. similarly, after the first month of the customers purchasing brand

BM2, 25% switched to BM1 and 10% switched to BM3 and of the customers purchasing brand BM3 0.05% switched to BM1 and 15% switched to BM2

Required

- i. Display in a matrix S, the patterns of retention and transfers of customers from the first to the second month, expressing percentage in decimal form. (2marks)
- ii. Multiply matrix S by itself (that is form S²) (5 Marks)
- iii. Interpret the results you obtain in part ii with regard to customer brand loyalty (3 marks)

Solution

The objective of the first part of the question was to test the candidate’s ability to formulate and manipulate a matrix, then interpret the result of such manipulation.

- a. i. The matrix showing the pattern of retention and transfer from the first to the second month is

$$S = \begin{matrix} & \begin{matrix} \text{BM1} & \text{BM2} & \text{BM3} \end{matrix} \\ \begin{pmatrix} 0.70 & 0.20 & 0.10 \\ 0.25 & 0.65 & 0.10 \\ 0.05 & 0.15 & 0.80 \end{pmatrix} & \begin{matrix} \text{BM1} \\ \text{BM2} \\ \text{BM3} \end{matrix} \end{matrix}$$

(The second element in the first row shows the 20% movement from BM1 to BM2 and so on)

- i. The product of matrix S with itself is demonstrated as follows

$$\begin{pmatrix} 0.70 & 0.20 & 0.10 \\ 0.25 & 0.65 & 0.10 \\ 0.05 & 0.15 & 0.80 \end{pmatrix} \begin{pmatrix} 0.70 & 0.20 & 0.10 \\ 0.25 & 0.65 & 0.10 \\ 0.05 & 0.15 & 0.80 \end{pmatrix} = \begin{pmatrix} 0.5450 & 0.2850 & 0.1700 \\ 0.3425 & 0.4875 & 0.1700 \\ 0.1125 & 0.2275 & 0.6600 \end{pmatrix}$$

Where for example second element in the first row, that is 0.2850 is the result of multiplying the corresponding elements of the first row of S by the second column of S and summing the product.

$$0.2850 = 0.70 \times 0.20 + 0.20 \times 0.65 + 0.10 \times 0.15 \\ = 0.14 + 0.13 + 0.015 \text{ e.t.c.}$$

- ii. The resulting matrix may be interpreted in the following way
Of the original customers who buy BM1, 54.5% will remain loyal to the brand in month three, 28.5% will have switched to BM2 and 17% will have switched to BM3.

Of the original customers who buy BM2, 48.7% will remain loyal to the brand in month three 34.25% will have switched to BM1 and 17% will have switched to BM3

Of the original customers who buy BM3, 66% will remain loyal to the brand in month three 11.25% will have switched to BM1 and 22.75% will have switched to BM2

Alternatively

In month three 54.5% of the customers buying BM1 are original customers. 34.5% came from BM2 originally and remaining 11.25% have switched from BM3 and so on.

SIMPLE MARKOV PROCESS

MARKOV CHAINS/PROCESSES

The Markov processes are defined as a set of trials which follow a certain sequence which depend on a given set of probabilities known as transition probabilities. These probabilities indicate how a particular activity or product moves from one state to another.

Applications of Markov Chains in Business

The Markov processes or chains are frequently applied as follows:-

1. Brand Switching

By using the transitional probabilities we can be able to express the manner in which consumers switch their tastes from one product to another.

2. Insurance industry

Markov analysis may be used to study the claims made by the insured persons and also decide the level of premiums to be paid in future.

3. Movement of urban population

By formulating a transition matrix for the current population in the urban areas, one can be able to determine what the population will be in say 5 years.

4. Movement of customers from one bank to another.

It is a fact that customers tend to look for efficient banks. Therefore at a certain time when a given bank installs such machinery as computers it will tend to attract a number of customers who will move from certain banks to efficient ones.

PROPERTIES OF MARKOV CHAINS

1. Each outcome in a markov process belongs to a state space or transition matrix. E.g.

$$\begin{matrix} & S_1 & S_2 & S_3 \\ \begin{matrix} S_1 \\ S_2 \\ S_3 \end{matrix} & \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \end{matrix} \quad \text{Where } S_1, S_2, S_3 \text{ are states and } P_{11} P_{12} \text{ e.t.c are} \\ \text{probabilities}$$

2. The outcome of each trial depends on the immediate preceding activities but not on the previous activities

BASIC TERMS IN MARKOV CHAINS

a) Probability Vector

This is a row matrix whose elements are non-negative and also they add up to 1 e.g. $u = 0.2, 0.1, 0.2, 0.5$)

Example

Consider $u = (\frac{3}{4}, 0, -\frac{1}{4}, \frac{1}{2})$ Not because $-\frac{1}{4}$ is negative

$v = (\frac{3}{4}, \frac{1}{2}, 0, \frac{1}{4})$ Not because the sum of the element < 1

$w = (\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2})$ Adds up to 1, each element is non negative.

Therefore it's a prob, vector

State the ones which are probability vectors

b) Stochastic matrix

A matrix whose row elements are all non negative and also add up to 1.

$$\text{Example (i) } M = \begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.0 & 0.7 & 0.1 & 0.2 \\ 0.5 & 0.1 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.2 & 0.1 \end{pmatrix}$$

Example ii) = Consider the following matrices

$$A = \begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{3}{4} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \quad B = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}$$

A is not stochastic matrix because the element in the 2nd row and 3rd column is negative.

B is not stochastic matrix because the elements in the second row do not add up to 1

C is stochastic matrix because each element is non negative and they add up to 1 in each row.

c) Regular stochastic matrix

A matrix P is said to be regular stochastic matrix if all the elements in P^m are all positive, where m is a power, m = 1, 2, 3 e.t.c

$$\text{Let } A = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{Where A is a Stochastic Matrix}$$

$$A^2 = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \times \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{8} & \frac{5}{8} \end{pmatrix}$$

Since the elements in A^2 and A^3 are all positive then A is regular stochastic matrix.

ABSORBING STATES

A state S_i (i = 1, 2, 3 ...) of a markov chain is called absorbing if the system remains in the state, S_i once it enters there. Thus a state, S_i is absorbing if and only if the i^{th} row of the transition matrix p has a 1 on the main diagonal and zeroes every where else. See the following example.

The following matrix, P is a transition matrix of the markov chain.

$$P = \begin{matrix} & \begin{matrix} S_1 & S_2 & S_3 & S_4 & S_5 \end{matrix} \\ \begin{matrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{matrix} & \begin{pmatrix} \cancel{1/4} & 0 & 1/4 & 1/4 & 1/4 \\ 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & \cancel{1/4} & 1/4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cancel{1} \end{pmatrix} \end{matrix}$$

The States S2 and S5 are absorbing states since the 2nd and 4th rows have 1 on the main diagonal.

Probability Transition Matrices (Brand switching)

These are matrices in which the individual elements are in the form of probabilities. The probabilities represent the probability of one event following another event i.e. the probability of transition from one event to the next. The probabilities of the various changes applied to the initial state by matrix multiplication, give a forecast of the succeeding state. Normally a transition matrix is defined with its columns adding up to one and state vectors as column vectors.

In this case the succeeding state is found by pre-multiplying the transition matrix by the proceeding state (column) vector. If the transition matrices are given with their rows adding up to one, then the succeeding state is found by post multiplying the transition matrix by the preceding state (row) vector.

Example 1

The probability transition matrix of the switching probabilities, consider that two brands G and X share the market in the ratio of 60% to 40% respectively of customers. If in every week 70% of G's customers retain the brand but 30% switch to product x where as 80% of X's customers retain brand but 20% percent switch to brand G. Analyze the exchange in share market per week.

$$\begin{array}{l} \text{State the system next week} \end{array} \begin{array}{c} \text{G} \quad \text{X} \\ \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \end{array}$$

$$\text{Column vector for initial market share} \begin{array}{c} \text{G} \\ \text{X} \end{array} \begin{pmatrix} 60 \\ 40 \end{pmatrix}$$

$$\text{Share next week} \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} 60 \\ 40 \end{pmatrix} = \begin{pmatrix} 50 \\ 50 \end{pmatrix}$$

$$\text{Share week after} \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} 50 \\ 50 \end{pmatrix} = \begin{pmatrix} 45 \\ 55 \end{pmatrix}$$

and so on

This process can continue till equilibrium is reached.

$$\text{Let the market share be} \begin{pmatrix} \text{G} \\ \text{X} \end{pmatrix}$$

$$\therefore \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} G \\ X \end{pmatrix} = \begin{pmatrix} G \\ X \end{pmatrix}$$

$$\begin{array}{lcl} 0.7G + 0.2X = G & \text{or} & 0.3G + 0.8X = X \\ 0.2X = 0.3G & \text{or} & 0.3X = 0.2X \end{array}$$

$$\text{i.e. } \frac{G}{X} = \frac{0.3}{0.2} = \frac{3}{2}$$

Hence G's share is 60% and X's share is 40%

Example 2

A marketing division toothpaste manufacturing company has worked out the following transition probability matrices concerning the behaviors of customers before and after an advertising campaign.

Transition probability matrix
(before advertising campaign)

FROM	TO	
	Our Brand (State I)	Another Brand (State II)
Our brand (State I)	0.8	0.2
Another Brand (State II)	0.4	0.6

Transition probability matrix
(After advertisement)

FROM	TO	
	Our Brand (State I)	Another Brand (State II)
Our brand (State I)	0.9	0.1
Another Brand (State II)	0.5	0.5

If the advertising campaign costs Shs 20,000 per year, would it be worthwhile for the company to undertake the campaign?

You may suppose there are 60,000 buyers of toothpaste in the market and for each customer average annual profit of the company is Shs 2.50

Solution

Let P_1 be the fraction share of our brand and P_2 be the fraction share of another brand

$$(P_1 \ P_2) \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} = (P_1 \ P_2)$$

$$(0.8P_1 + 0.4P_2 \ 0.2P_1 + 0.6P_2) = (P_1 \ P_2)$$

$$0.8P_1 + 0.4P_2 = P_1 \text{ and } 0.2P_1 + 0.6P_2 = P_2$$

$$0.4P_2 = 0.2P_1 \text{ and } 0.2P_1 = 0.4P_2$$

Thus:

$$P_1 = \frac{2}{3} \text{ and } P_2 = \frac{1}{3}$$

After Advertising

$$(P_1 \ P_2) \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = (P_1 \ P_2)$$

$$(0.9P_1 + 0.5P_2 \ 0.1P_1 + 0.5P_2) = (P_1 \ P_2)$$

$$0.5P_2 = 0.1P_1 \text{ and } 0.5P_2 = 0.1P_1$$

Thus:

$$P_1 = \frac{5}{6} \text{ and } P_2 = \frac{1}{6}$$

Before Advertising

If there are 60,000 buyers

Before Advertising

$$P_1 = \frac{2}{3}$$

this implies that,

$\frac{2}{3} \times 60,000 = 40,000$ Customers
will buy our
brand

$$\begin{aligned} \text{contribution} &= 40,000 \times 2.5 \\ &= \text{Sh.}100,000 \end{aligned}$$

After Advertising

$$P_1 = \frac{5}{6}$$

this implies that,

$\frac{5}{6} \times 60,000 = 50,000$ customers will buy
our brand

$$\begin{aligned} \text{contribution} &= (50,000 \times 2.5) - 20,000 \\ &= \text{Sh.}105,000 \end{aligned}$$

www.masomomsingi.com

the difference between advertising and not advertising is

105,000 - 100,000 = Sh.5, 000 in favour of advertising,
Thus the advertising campaign is worthwhile.

PRACTICE QUESTIONS

QUESTION ONE

Because of increasing cost increasing cost energy, the population within Maueni district seem to be shifting from the north to the south the transition matrix S describes the migration behaviour observed between the regions.

$$S = \begin{pmatrix} 0.90 & 0.10 \\ 0.05 & 0.95 \end{pmatrix} \begin{matrix} \text{to north} & \text{to south} \\ \text{from north} \\ \text{from south} \end{matrix}$$

determine whether the populations will attain an equilibrium condition and if so, the population of the two regions.

QUESTION TWO

A simple hypothetical economy of three industries A, B and C is represented in the following table (data in millions of shillings).

User \ Producer	A	B	C	Final Demand	Total Output
A	80	100	100	40	320
B	80	200	60	60	400
C	80	100	100	20	300

Determine the output vector for the economy if the final demand changes to 60 for A, 60 for B and 60 for C

QUESTION THREE

A tea blender uses two types of tea, T_1 , and T_2 , to produce two blends, B_1 and B_2 for sale. B_1 uses 40% of available T_1 and 60% of the available T_2 whilst B_2 uses 50% of the available T_1 and 25% of the available T_2 .

Required:

- a) Given that t_1 kilos of T_1 and t_2 of T_2 are made available to produce b_1 kilos of B_1 and b_2 kilos of B_2 . Express the blending operation in the matrix format.
- b) If 400 kilos of T_1 and 700 kilos of T_2 were made available for blending, what quantities of B_1 and B_2 would be produced?
- c) If 600 kilos of B_1 and 450 kilos of B_2 were produced, use a matrix method to determine what quantities of T_1 and T_2 would be used to produce the blends.

QUESTION FOUR

Let $A = \begin{pmatrix} 2 & 2 \\ 3 & -3 \end{pmatrix}$

- a) Find A^2 and A^3
- b) If $F(x) = x^3 - 3x^2 - 2x + 4I$
Find $F(A)$
- c) Find the inverse of matrix A

QUESTION FIVE

A child's toy is marketed in three sizes standard size contains 10 squares (S), 15 triangles (T) and 6 hexagons (H). The deluxe set contains 15 S, 20 T, 6 H and 4 octagons (O). The super set contains 24 T, 8 H, 16 H, 16 S and 6 (O). Squares cost 12 pence to produce, triangles cost 8p, hexagons cost 18p and octagons 22p.

The standard set is sold at £6, the deluxe set for £10 and super for £15. The manufacturer produces 100 standard sets, 80 deluxe sets and 50 super sets per week.

Use matrix form and matrix multiplications to find:

- The cost of producing each set.
The number of each shape required each week
Total expenditure on shapes each week

QUESTION SIX

Matrix N below shows the number of items of type A, B, and C in warehouses Y and W. Matrix p shows the cost in pence per day of storing (S) and maintaining (M) one item each of A, B and C

$$N = \begin{matrix} & \text{A} & \text{B} & \text{C} \\ \text{Y} & 10 & 12 & 50 \\ \text{W} & 60 & 0 & 20 \end{matrix}$$

$$P = \begin{matrix} & \text{S} & \text{M} \\ \text{A} & 2 & 0.5 \\ \text{B} & 3 & 1.5 \\ \text{C} & 2 & 0.5 \end{matrix}$$

a) Evaluate the matrix (N×P) and say what it represents.

b) Stock movement occurs as follows:

At the start of the day 1:

Withdrawal of 2 type B from warehouse Y, 20 of type A from warehouse W.

At the start of day 2:

Delivery of 7 type B and 10 of type C to warehouse Y and 15 of type B to warehouse W.

Evaluate the total cost of storage and maintenance for days 1 and 2.

c) Write down without evaluating a matrix expression which could be used to evaluate the storage and maintenance cost of items A, B and C for the period from day 1 to 4. Allow for the stock movements on days 1 and 2, as described in part (b). There were no stock movements on days 3 and 4.

CHAPTER TWELVE

INTRODUCTION TO CALCULUS

SPECIFIC OBJECTIVES

At the end of this topic the trainee should be able to:

- Differentiate algebraic functions;
- Integrate algebraic functions

INTRODUCTION

Calculus is a branch of mathematics which explains how one variable changes in relationship to another variable. It enables us to find the rate of change of one variable with respect to another variable.

Example

- i. The rate at which business revenue is increasing at a particular stage when volume of sales is increasing.
- ii. The rate at which costs are changing at a particular stage when volume of sales is given
- iii. The evaluation of 'rate of change' can help us to identify when the change in one variable reaches a maximum or minimum.
- iv. Calculus may be used in production management when the production manager wants to know
 - a) How much is to be manufactured in order to maximize the profits, revenues e.t.c
 - b) How much is to be produced in order to minimize the production costs

Calculus is divided into two sections namely:

Differentiation and integration

Differentiation deals with the determination of the rates of change of business activities or simply the process of finding the derivative of a function.

Integration deals with the summation or totality of items produced over a given period of time or simply the reverse of differentiation

The derivative and differentiation

The process of obtaining the derivative of a function or slope or gradient is referred to as derivation or differentiation.

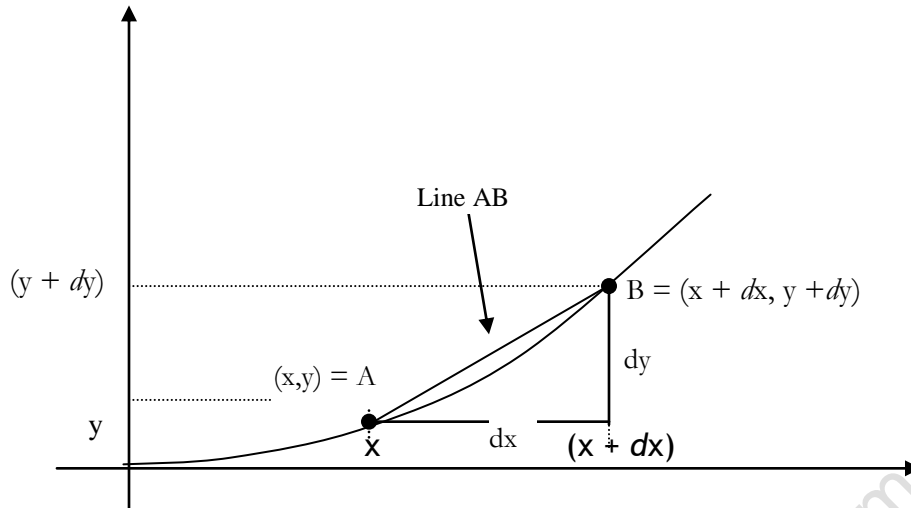
The derivative is denoted by $\frac{dy}{dx}$ or $f'(x)$ and is given by dividing the change in y variable by the change in x variable.

The derivative or slope or gradient of a line AB connecting points (x,y) and (x+dx, y + dy) is given by

$$\frac{\Delta y}{\Delta x} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{(y + dy) - y}{(x + dx) - x} = \frac{dy}{dx}$$

Where dy is a small change in y and dx is a small change in x variables.

Illustration



Rules of Differentiation

1. The constant function rule

If given a function $y = k$ where k is a constant then

$$\frac{dy}{dx} = 0$$

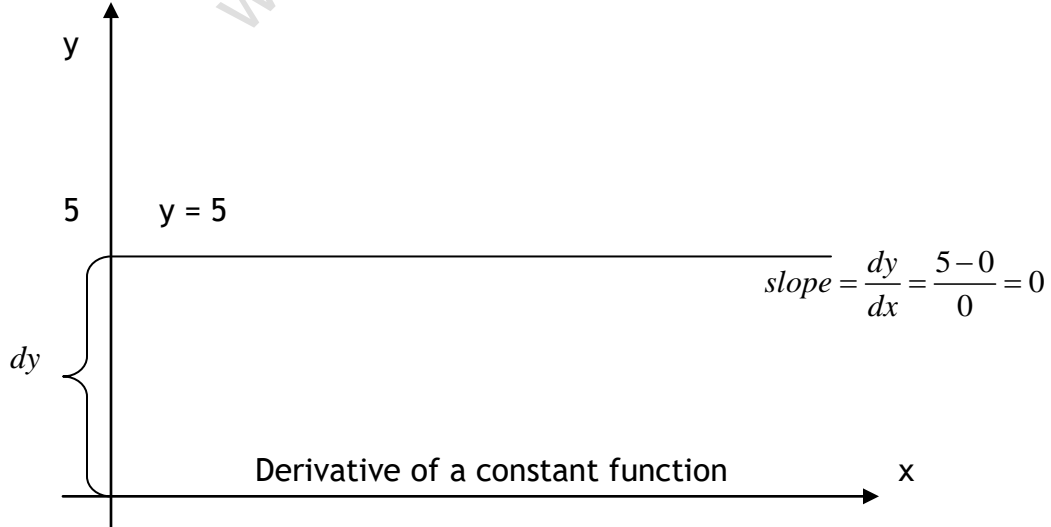
Example

Find the derivative of (i) $y = 5$

Solution

i. $y = 5$ $\frac{dy}{dx} = 0$

Illustration



2. Power function rule

Given a function $y = x^r$

$$\text{Then } \frac{dy}{dx} = rx^{r-1}$$

Example

Find $\frac{dy}{dx}$ for;

- (i). $y = x^7$
- (ii). $y = x^{2x}$
- (iii). $y = x^{-3}$
- (iv). $y = x$

Solution

i. $y = x^7$
 $\frac{dy}{dx} = 7x^{7-1} = 7x^6$

ii. $y = x^{2x}$
 $\frac{dy}{dx} = 2^x x^{(2x-1)}$

iii. $y = x^{-3}$
 $\frac{dy}{dx} = -3x^{-3-1} = -3x^{-4}$

iv. $y = x$
 $\frac{dy}{dx} = 1x^{1-1} = 1 \cdot x^0 = 1$ (since $x^0 = 1$)

3. Power function multiplied by a constant

If given $y = Ax^r$, then $\frac{dy}{dx} = rAx^{r-1}$

4. The sum rule

The derivative of the sum of two or more functions equals the sum of the derivatives of the functions.

For instance

If $H(x) = h(x) + g(x)$

Then $\frac{dy}{dx}$ or $H'(x) = h'(x) + g'(x)$

5. The difference rule

The derivative of the difference of two or more functions equals the difference of the derivatives of the functions

If $H(x) = h(x) - g(x)$

Then $H'(x) = h'(x) - g'(x)$

Examples.

Find the derivatives of

i. $y = 3x^2 + 5x + 7$

ii. $y = 4x^2 - 2x^b$

Solution

i. $y = 3x^2 + 5x + 7$

$$\frac{dy}{dx} = \frac{d(3x^2)}{dx} + \frac{d(5x)}{dx} + \frac{d(7)}{dx}$$

$$= 6x + 5 + 0$$

$$= 6x + 5$$

ii. $y = 4x^2 - 2x^b$

$$\frac{dy}{dx} = \frac{d(4x^2)}{dx} - \frac{d(2x^b)}{dx}$$

$$= 8x - 2bx^{b-1}$$

6. The product rule - both factors are functions

The derivative of the product of two functions equals the derivative of the first function multiplied by the second function PLUS the derivative of the second function multiplied by the first function.

given that $H(x) = h(x).g(x)$

Then $H'(x) = h'(x).g(x) + h(x).g'(x)$

Example

Find $\frac{dy}{dx}$ for

i. $y = x^2(x)$

ii. $y = (x^2 + 3)(2x^3 + x^2 - 3)$

Solution

i. $y = x^2(x)$

$$\frac{dy}{dx} = x \cdot \frac{d(x^2)}{dx} + x^2 \cdot \frac{d(x)}{dx}$$

$$= x \cdot 2x + x^2 \cdot 1$$

$$= 2x^2 + x^2$$

$$= 3x^2$$

Note that $y = x^2(x) = x^3$. Directly differentiating this we get $3x^2$.

ii. $y = (x^2 + 3)(2x^3 + x^2 - 3)$

$$\frac{dy}{dx} = \frac{d(x^2 + 3)}{dx} \cdot (2x^3 + x^2 - 3) + (x^2 + 3) \cdot \frac{d(2x^3 + x^2 - 3)}{dx}$$

$$2x \cdot (2x^3 + x^2 - 3) + (x^2 + 3) \cdot (6x^2 + 2x)$$

$$10x^4 + 4x^3 + 18x^2$$

7. Quotient Rule

The derivative of the quotient of two functions equals the derivative of the numerator times the denominator MINUS the derivative of the denominator times the numerator, all which are divided by the square of the denominator

If given $H(x) = \frac{h(x)}{g(x)}$

$$\text{then } H'(x) = \frac{h'(x) \cdot g(x) - h(x) \cdot g'(x)}{[g(x)]^2}$$

For example

Find $\frac{dy}{dx}$ for

i. $\frac{x}{3 + x^2}$

ii. $\frac{x}{3x + 7}$

Solutions

$$\begin{aligned}
 \text{i. } & \frac{x}{3+x^2} \\
 \frac{dy}{dx} &= \frac{\frac{d(x)}{dx} \cdot (3+x^2) - \frac{d(3+x^2)}{dx} \cdot x}{(3+x^2)^2} \\
 &= \frac{(3+x^2) - (2x) \cdot x}{(3+x^2)^2} \\
 &= \frac{3+x^2-2x^2}{(3+x^2)^2} = \frac{3-x^2}{(3+x^2)^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii. } & y = \frac{x^3}{(3x+7)} \\
 \frac{dy}{dx} &= \frac{(3x^2)(3x+7) - (3)(x^3)}{(3x+7)^2} = \frac{6x^3 + 21x^2 - 3x^3}{(3x+7)^2}
 \end{aligned}$$

Example

A farmer of a large farm of poultry announced that egg production per month follows the equation;

$$w = \frac{3m^3 - m^2}{m^2 + 10}$$

Where w - Total no of eggs produced per month

m - amount in kilograms of layers mash feed.

Required

Determine the rate of change of w with respect to m (i.e. the rate at which the number of eggs per month increase or decrease depending on the rate at which the kilos of layers marsh are increased).

Solution

$$\text{Let } u = 3m^3 - m^2$$

$$\therefore \frac{du}{dm} = 9m^2 - 2m$$

$$\text{Let } v = m^2 + 10$$

$$\therefore \frac{dv}{dm} = 2m$$

$$\therefore \frac{dw}{dm} = \frac{(m^2+10)(9m^2+2m) - (3m^3-m^2)2m}{(m^2+10)^2}$$

$$= \frac{9m^4 + 90m^2 - 2m^3 - 20m - (6m^4 + 2m^3)}{(m^2 + 10)^2}$$

$$= \frac{3m^4 + 90m^2 - 20m}{(m^2 + 10)^2}$$

8. Chain Rule

This rule is generally applied in the determination of the derivatives of composite functions, which can be defined as a function in which another function can be considered to have taken the place of the independent variable. The composite function is also referred to as a function of a function. It is normally of the form $y = (2x^2 + 3)^3$. If we let $u = (2x^2 + 3)$, then $y = u^3$. In order to differentiate such an equation we use the formula

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Solution

$$y = (2x^2 + 3)^3$$

$$\text{Let } u = 2x^2 + 3$$

$$\therefore \frac{du}{dx} = 4x$$

$$\text{Let } y = u^3$$

$$\therefore \frac{dy}{du} = 3u^2$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \times 4x = 12xu^2 \\ &= 12x(2x^2 + 3)^2 \end{aligned}$$

Example

Consider the function

$$y = (x^2 + 16x + 5)^2$$

which can be decomposed into

$$y = u^2 \text{ and } u = x^2 + 16x + 5. \text{ in this case } y \text{ is a function of } (x^2 + 16x + 5)$$

Hence $y = f(u)$ and $u = g(x)$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= (2u) (2x + 16) \end{aligned}$$

$$= 2(x^2 + 16x + 5)(2x + 16)$$

The derivative of a function raised to power r; the composite function rule.

The derivative of a function raised to power r equals to the power r times the function which is raised by power (r-1), all of which is multiplied by the derivative of the function

$$\text{If } y = [g(x)]^r$$

$$\text{Then } \frac{dy}{dx} = r[g(x)]^{r-1} \cdot g'(x)$$

For example

$$\text{Find } \frac{dy}{dx} \text{ given } y = (3x^2 + 4x)^5$$

Solution

$$\frac{dy}{dx} = 5(3x^2 + 4x)^4 \cdot (6x + 4)$$

Differentiation of an implicit function

An Implicit function is one of the $y = x^2 y + 3x^2 + 50$. it is a function in which the dependent variable (y) appears also on the right hand side.

To differentiate the above equation we use the differentiation method for a product, quotient or function of a function.

Solution

$$y = x^2 y + 3x^2 + 50$$

$$\frac{dy}{dx} = \frac{d(x^2 y)}{dx} + \frac{d(3x^2)}{dx} + \frac{d(50)}{dx}$$

$$\frac{dy}{dx} = \left[y(2x) + x^2 \frac{dy}{dx} \right] + 6x + 0$$

$$0 = 2xy + x^2 \frac{dy}{dx} - \frac{dy}{dx} + 6x$$

$$0 = 2xy + (x^2 - 1) \frac{dy}{dx} + 6x$$

$$-(x^2 - 1) \frac{dy}{dx} = 2xy + 6x$$

$$\frac{dy}{dx} = \frac{-(2xy + 6x)}{(x^2 - 1)}$$

Partial derivatives

These derivatives are used when we want to investigate the effect of one independent variable on the dependent variable.

For example, the revenues of a farmer may depend on two variables namely; the amount of fertilizer applied and also the type of the natural soil.

$$\text{Let } \pi = 30x^2y + y^2 + 50x + 60y$$

Where π = annual revenue in £ '000'

x = type of soil

y = amount of fertilizer applied

Required

Determine the rate of change of the π with respect to x and y

Solution

$$\pi = 30x^2y + y^2 + 50x + 60y$$

Differentiating π with respect to x keeping y constant we have

$$\frac{d\pi}{dx} = 60xy + 50$$

Differentiating with respect to y keeping x constant we have

$$\frac{d\pi}{dy} = 30x^2 + 2y + 60$$

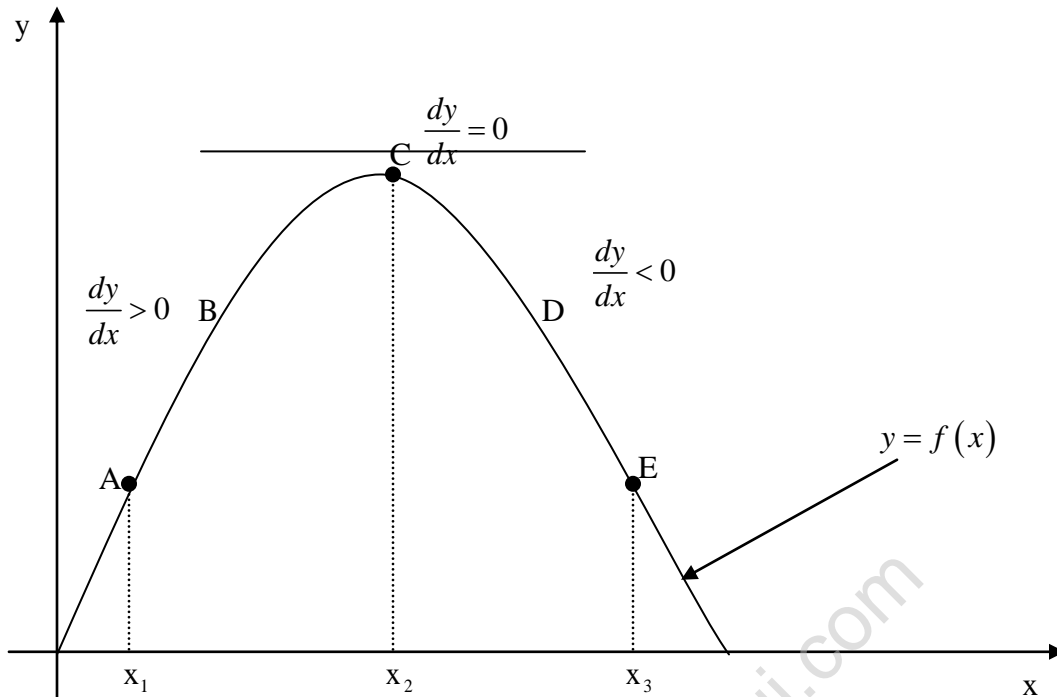
Maxima, minima and points of inflexion

a) Test for relative maximum

Consider the following function of x whose graph is represented by the figure below

$$y = f(x)$$

$$\frac{dy}{dx} = f'(x)$$



Relative maximum point

The graph of the function slopes upwards to the right between points A and C and hence has a positive slope between these two points. The function has a negative slope between points C and E. At point C, the slope of the function is Zero.

Between points X_1 and X_2 $\frac{dy}{dx} > 0$ Where $X_1 \leq X < X_2$

and between X_2 and X_3 $\frac{dy}{dx} < 0$ Where $X_2 < X \leq X_3$.

Thus the first test of the maximum points require that the first derivative of a function equals zero or

$$\frac{dy}{dx} = f'(x) = 0$$

The second test of a maximum point requires that the second derivative of a function is negative or

$$\frac{d^2y}{dx^2} = f''(x) < 0$$

Example

Determine the critical value for the following functions and find out the critical value that constitutes a maximum

$$y = x^3 - 12x^2 + 36x + 8$$

Solution

$$y = x^3 - 12x^2 + 36x + 8$$

$$\text{then } \underline{dy} = 3x^2 - 24x + 36 = 0$$

- iii. $\frac{dy}{dx}$ The critical values for the function are obtained by equating the first derivative of the function to zero, that is:

$$\frac{dy}{dx} = 0 \text{ or } 3x^2 - 24x + 36 = 0$$

$$\text{Hence } (x-2)(x-6) = 0$$

$$\text{And } x = 2 \text{ or } 6$$

The critical values for x are x = 2 or 6 and critical values for the function are y = 40 or 8

- ii. To ascertain whether these critical values of x will give rise to a maximum, we apply the second text, that is

$$\frac{d^2y}{dx^2} < 0$$

$$\frac{dy}{dx} = 3x^2 - 24x + 36 \text{ and}$$

$$\frac{d^2y}{dx^2} = 6x - 24$$

- a) When x = 2

$$\text{Then } \frac{d^2y}{dx^2} = -12 < 0$$

- b) When x = 6

$$\text{Then } \frac{d^2y}{dx^2} = +2 > 0$$

Hence a maximum occurs when x = 2, since this value of x satisfies the second condition. X = 6 does not give rise to a local maximum

b) Tests for relative minimum

There are two tests for a relative minimum point

- i. The first derivative, that is

$$\frac{dy}{dx} = f'(x) = 0$$

- ii. The second derivative, that is

$$\frac{d^2y}{dx^2} = f''(x) > 0$$

Example:

For the function

$$h(x) = \frac{1}{3}x^3 + x^2 - 35x + 10$$

Determine the critical values and find out whether these critical values are maxima or minima. Determine the extreme values of the function

Solution

- i. Critical values

$$h(x) = \frac{1}{3}x^3 + x^2 - 35x + 10 \text{ and}$$

$$h'(x) = x^2 + 2x - 35$$

by first text,

$$\text{then } h'(x) = x^2 + 2x - 35 = 0$$

or $(x-5)(x+7) = 0$
Hence $x = 5$ or $x = -7$

- ii. The determinant of the maximum and the minimum points requires that we test the value $x = 5$ and -7 by the second test

$$H''(x) = 2x + 2$$

a) When $x = -7$ $h''(x) = -12 < 0$

b) When $x = 5$ $h''(x) = 12 > 0$

There $x = -7$ gives a maximum point and $x = 5$ gives a minimum point.

- iii. Extreme values of the function

$$h(x) = \frac{1}{3}x^3 + x^2 - 35x + 10$$

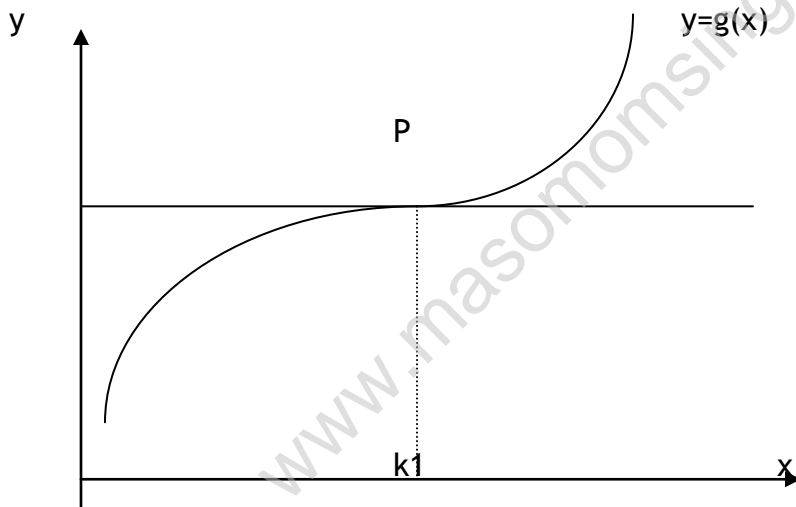
when $x = -7$, $h(x) = 189 \frac{2}{3}$

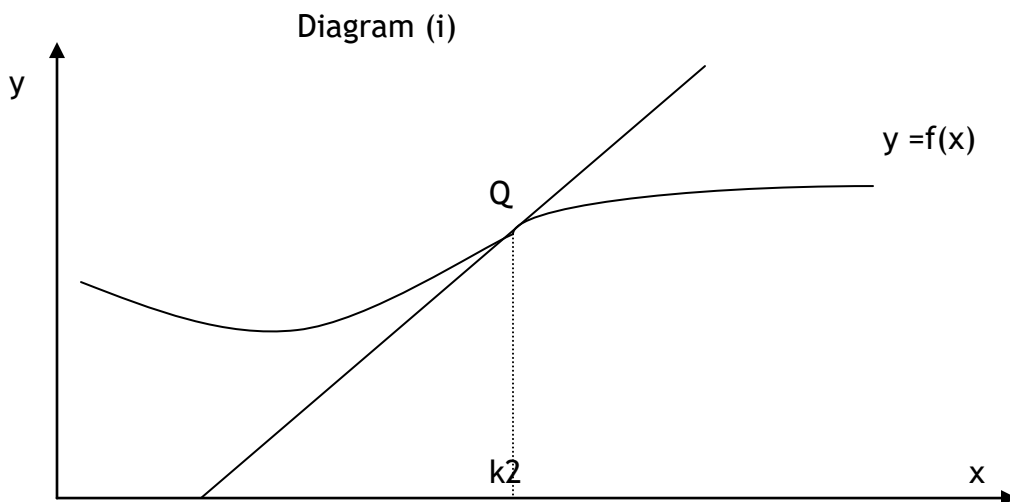
when $x = 5$, $h(x) = -98 \frac{1}{3}$

The extreme values of the function are $h(x) = 189 \frac{2}{3}$ which is a relative maximum and $h(x) = -98 \frac{1}{3}$, a relative minimum

c) Points of inflexion

Given the following two graphs, points of inflexion can be determined at points P and Q as follows:





The points of inflexion will occur at point P when

$$\begin{aligned} g''(x) &= 0 & \text{at} & \quad x = k_1 \\ g''(x) &< 0 & \text{at} & \quad x < k_1 \\ g''(x) &> 0 & \text{at} & \quad x > k_1 \end{aligned}$$

and at point Q when

$$\begin{aligned} f''(x) &= 0 & \text{at} & \quad x = k_1 \\ f''(x) &> 0 & \text{at} & \quad x < k_1 \\ f''(x) &< 0 & \text{at} & \quad x > k_1 \end{aligned}$$

Example

Find the points of inflexion on the curve of the function

$$y = x^3$$

Solution

The only possible inflexion points will occur where

$$\frac{d^2y}{dx^2} = 0$$

From the function given

$$\frac{dy}{dx} = 3x^2 \quad \text{and} \quad \frac{d^2y}{dx^2} = 6x$$

Equating the second derivative to zero, we have

$$6x = 0 \quad \text{or} \quad x = 0$$

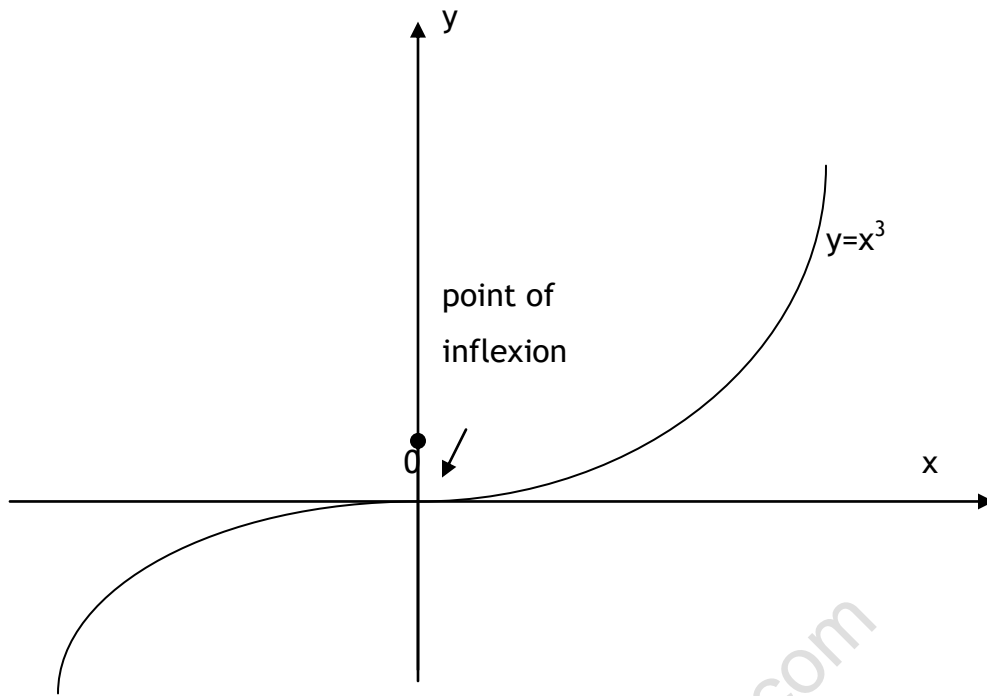
We test whether the point at which $x = 0$ is an inflexion point as follows

When x is slightly less than 0, $\frac{d^2y}{dx^2} < 0$ which means a downward concavity

When x is slightly larger than 0, $\frac{d^2y}{dx^2} > 0$ which means an upward concavity

Therefore we have a point of inflexion at point $x = 0$ because the concavity of the curve changes as we pass from the left to the right of $x = 0$

Illustration



Example

1. The weekly revenue Sh. R of a small company is given by

$$R = 14 + 81x - \frac{x^3}{12} \text{ Where } x \text{ is the number of units produced.}$$

Required

- i. Determine the number of units that maximize the revenue
- ii. Determine the maximum revenue
- iii. Determine the price per unit that will maximize revenue

Solution

- i. To find maximum or minimum value we use differential calculus as follows

$$R = 14 + 81x - \frac{x^3}{12}$$

$$\frac{dR}{dx} = 81 - \frac{1}{12} \cdot 3x^2$$

$$\frac{d^2R}{dx^2} = 0 - \frac{1}{12} \cdot 3 \cdot 2x = -\frac{x}{2}$$

$$\text{put } \frac{dR}{dx} = 0 \quad \text{i.e. } 81 - \frac{1}{4}x^2 = 0$$

which gives $x = 18$ or $x = -18$

$$\frac{d^2R}{dx^2} = -\frac{x}{2}$$

thus when $x = 18$; $\frac{d^2R}{dx^2} = -9$ which is negative – indicating a maximum value

Therefore at $x = 18$, the value of R is a maximum. Similarly at $x = -18$, the value of R is a minimum. Therefore, the number of units that maximize the revenue = 18 units

ii. The maximum revenue is given by

$$R = 14 + 81 + 18 - \frac{(18)^3}{12}$$

$$= \text{Shs. } 986$$

ii. The price per unit to maximize the revenue is

$$\frac{986}{18} = 54.78 \text{ or Shs. } 54.78$$

INTEGRATION

It is the reversal of differentiation. An integral can either be indefinite (when it has no numerical value) or definite (have specific numerical values)

It is represented by the sign $\int f(x)dx$.

Rules of integration

i. The integral of a constant

$$\int a dx = ax + c \quad \text{where } a = \text{constant}$$

Example

Find the following

a) $\int 23 dx$

b) $\int x^2 dx$. (where x is a variable independent of x , thus it is treated as a constant).

Solution

i. $\int 23 dx = 23x + c$

ii. $\int x^2 dx = \frac{x^3}{3} + c$

ii. The integral of x raised to the power n

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + c$$

Example

Find the following integrals

a) $\int x^2 dx$

b) $\int x^{-5/2} dx$

Solution

i) $\int x^2 dx = \frac{1}{3} x^3 + c$

ii) $\int x^{-5/2} dx = -\frac{2}{3} x^{-3/2} + c$

iii). Integral of a constant times a function

$$\int af(x) dx = a \int f(x) dx$$

Example

Determine the following integrals

i. $\int ax^3 dx$

ii. $\int 20x^5 dx$

Solution

a) $\int ax^3 dx = a \int x^3 dx$
 $= \frac{a}{4} x^4 + c$

b) $\int 20x^5 dx = 20 \int x^5 dx$
 $= \frac{10}{3} x^6 + c$

iv). Integral of sum of two or more functions

$$\int \{f(x) + g(x)\} dx = \int f(x) dx + \int g(x) dx$$

$$\int \{f(x) + g(x) + h(x)\} dx = \int f(x) dx + \int g(x) dx + \int h(x) dx$$

Example

Find the following

i. $\int (4x^2 + \frac{1}{2} x^{-3}) dx$

ii. $\int (x^{3/4} + \frac{3}{7} x^{-1/2} + x^5) dx$

Solution

i) $\int (4x^2 + \frac{1}{2} x^{-3}) dx = \int 4x^2 dx + \int \frac{1}{2} x^{-3} dx$
 $= \frac{4}{3} x^3 - \frac{1}{4} x^{-2} + c$

ii) $\int (x^{3/4} + \frac{3}{7} x^{-1/2} + x^5) dx = \int x^{3/4} dx + \int \frac{3}{7} x^{-1/2} dx + \int x^5 dx$
 $= \frac{4}{7} x^{7/4} + \frac{6}{7} x^{1/2} + \frac{1}{6} x^6 + c$

5. Integral of a difference

$$\int \{f(x) - g(x)\} dx = \int f(x) dx - \int g(x) dx$$

Definite integration

Definite integrals involve integration between specified limits, say a and b

The integral $\int_a^b f(x) dx$ is a definite integral in which the limits of integration are a and b

The integral is evaluated as follows

1. Compute the indefinite integral $\int f(x) dx$. Supposing it is $F(x) + c$
2. Attach the limits of integration
3. Substitute b (the upper limit) and then substitute a (the lower limit) for x.
4. Take the difference and the result is the numerical value for the definite integral.

Applying these steps to the definite integral

$$\begin{aligned}\int_a^b f(x) dx &= [F(x) + c]_a^b \\ &= \{[F(b) + c] - [F(a) + c]\} \\ &= F(b) - F(a)\end{aligned}$$

Example

Evaluate

i. $\int_1^3 (3x^2 + 3) dx$

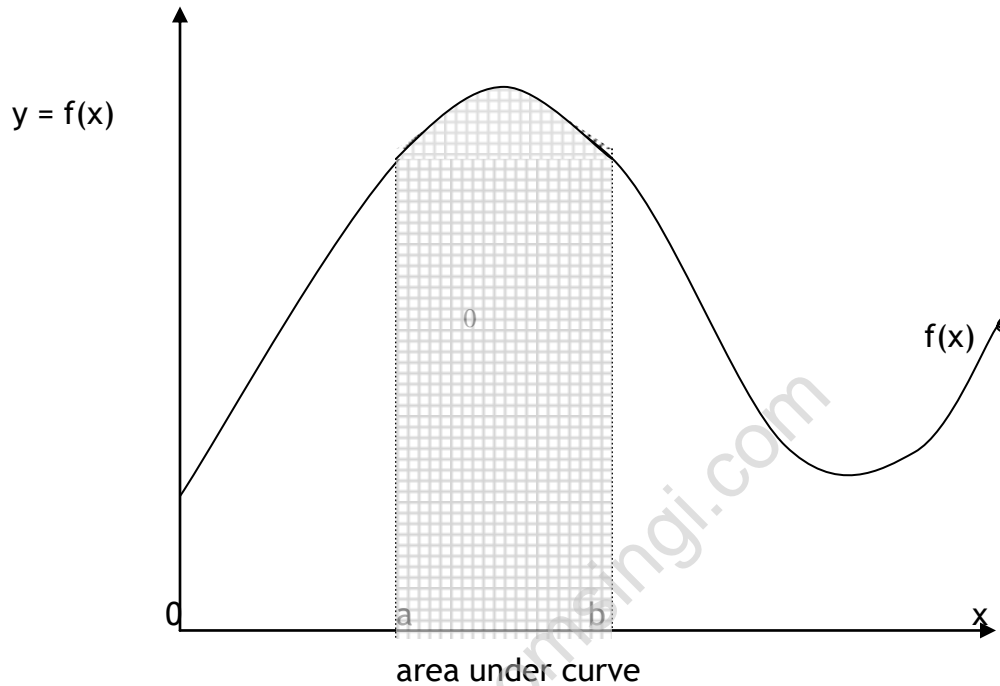
ii. $\int_0^5 (x + 15) dx$

Solution

$$\begin{aligned}\text{a. } \int_1^3 (3x^2 + 3) dx &= [(x^3 + 3x + c)] \\ &= (27 + 9 + c) - (1 + 3 + c) \\ &= 32\end{aligned}$$

$$\begin{aligned}\text{b. } \int_0^5 (x + 15) dx &= \left[\left(\frac{1}{2}x^2 + 15x + c\right)\right]_0^5 \\ &= (12\frac{1}{2} + 75 + c) - (0 + 0 + c) \\ &= 87\frac{1}{2}\end{aligned}$$

The numerical value of the definite integral $\int_a^b f(x)dx$ can be interpreted as the area bounded by the function $f(x)$, the horizontal axis, and $x=a$ and $x=b$ see figure below



Therefore $\int_a^b f(x)dx = A$ or area under the curve

Example

1. You are given the following marginal revenue function

$$MR = a + a_1q$$

Find the corresponding total revenue function

Solution

$$\begin{aligned} \text{Total revenue} &= \int MR \cdot dq = \int (a + a_1q) dq \\ &= aq - \frac{1}{2}a_1q^2 + c \end{aligned}$$

Example 2

A firm has the following marginal cost function

$$MC = a - a_1q + a_2q^2$$

Find its total cost function.

Solution

The total cost C is given by

$$\begin{aligned} C &= \int MC \cdot dq \\ &= \int (a - a_1q + a_2q^2) \cdot dq \\ &= aq - \frac{a_1}{2}q^2 + \frac{a_2}{3}q^3 + c \end{aligned}$$

Note: **Exams focus:** Note the difference between marginal function and total function. You differentiate total function to attain marginal function, this is common in exams,
total profit = total revenue - total cost.

Example 3.

Your company manufactures large scale units. It has been shown that the marginal (or variable) cost, which is the gradient of the total cost curve, is $(92 - 2x)$ Shs. thousands, where x is the number of units of output per annum. The fixed costs are Shs. 800,000 per annum. It has also been shown that the marginal revenue which is the gradient of the total revenue is $(112 - 2x)$ Shs. thousands.

Required

- i. Establish by integration the equation of the total cost curve
- ii. Establish by integration the equation of the total revenue curve
- iii. Establish the break even situation for your company
- iv. Determine the number of units of output that would
 - a) Maximize the total revenue and
 - b) Maximize the total costs, together with the maximum total revenue and total costs

Solution

- i. First find the indefinite integral limit points of the marginal cost as the first step to obtaining the total cost curve
Thus $\int(92 - 2x) dx = 92x - x^2 + c$
Where c is constant

Since the total costs are the sum of variable costs and fixed costs, the constant term in the integral represents the fixed costs, thus if T_c are the total costs then,

$$T_c = 92x - x^2 + 800$$
$$\text{or } T_c = 800 + 92x - x^2$$

- ii. As in the above case, the first step in determining the total revenue is to form the indefinite integral of the marginal revenue
Thus $\int(112 - 2x) dx = 112x - x^2 + c$
Where c is a constant

The total revenue is zero if no items are sold, thus the constant is zero and if T_r represents the total revenue, then

$$T_r = 112x - x^2$$

- iii. At break even the total revenue is equal to the total costs
Thus $112x - x^2 = 800 + 92x - x^2$
 $20x = 800$
 $x = 40$ units per annum

iv.

$$\text{a) } Tr = 112x - x^2$$

$$\frac{d(Tr)}{dx} = 112 - 2x$$

$$\frac{d^2(Tr)}{dx^2} = -2$$

at the maximum point

$$\frac{d^2(Tr)}{dx^2} = 0 \quad \text{that is } 112 - 2x = 0$$

$x = 56$ units per annum

Since $\frac{d^2(Tr)}{dx^2} = -2$ this confirms the maximum

The maximum total revenue is Shs. $(112 \times 56 - 56 \times 56) \times 1000$
= Shs. 3,136,000

$$\text{ii. } Tc = 800 + 92x - x^2$$

$$\frac{d(Tc)}{dx} = 92 - 2x$$

$$\frac{d^2(Tc)}{dx^2} = -2$$

At this maximum point

$$\frac{d(Tc)}{dx} = 0$$

$$92 - 2x = 0$$

$$92 = 2x$$

$x = 46$ units per annum

since

$\frac{d^2(Tc)}{dx^2} = -2$ this confirms the maximum

the maximum costs are Shs. $(800 + 92 \times 46 - 46 \times 46) \times 1000$
= Shs. 2,916,000

PRACTICE QUESTIONS

QUESTION ONE

Find the derivative of

a) $y = 6x - x$

b) $y = \frac{1}{x^2}$

c) $y = \sqrt{1+2x}$

d) $y = \frac{1}{\sqrt{x}}$

QUESTION TWO

A cost function is

$$\text{Ksh. (c)} = Q^2 - 30Q + 200$$

Where Q = quantity of units produced

Find the point of minimum cost.

QUESTION THREE

250 members of a certain society have voted to elect a new chairman. Each member may vote for either one or two candidates. The candidate elected is the one who polls most votes.

Three candidates x, y, z stood for election and when the votes were counted, it was found that,

59 voted for y only, 37 voted for z only

12 voted for x and y, 14 voted for x and z

147 voted for either x or y or both x and y but not for z

102 voted for y or z or both but not for x.

Required:

- i) How many voters did not vote?
- ii) How many voters voted for x only?
- iii) Who won the election?

QUESTION FOUR

The weekly revenue Ksh.R of a small company is given by:

$$R = 14 + 81x - \frac{x^3}{12} \text{ where } x \text{ is the number of units produced}$$

Required:

- a) Determine the number of units that maximize the revenue.

- b) Determine the maximum revenue.
- c) Determine the price per unit that will maximize the revenue

QUESTION FIVE

A furniture firm has two operating departments; Production and sales. The firm's operating costs are split between these two departments with the resultant period of fixed costs of Shs.20,000 and Shs.6,000 respectively. The production department has a basic variable cost per unit of Shs.6 plus additional variable cost per unit of Shs.0.0002 which relates to all the manufactured items during the period. The sales department has a variable cost per unit of Shs.2. The sales department receives the finished goods from the production department and pay the basic variable cost per unit plus 80% of the same.

NB: Demand Q is given by the following function:

$Q = 40,000 - 2,000P$, where P is the selling price of the sales department.

Required:

- a) Calculate the quantity that maximizes the profits of the production department.
- b) Calculate the selling price that maximizes the profits of the sales department.
- c) Determine the firm's profit as a result of adopting the quantity and selling prices in i and ii.
- d) Determine the quantity and selling price that maximize the firm's profit. What is the amount of this profit?

QUESTION SIX

- a) Describe how quadratic equations can be used in decision making.
- b) The demand for a commodity is given by $p = 400 - q$. The average total cost of producing the commodity is given by

$$ATC = \frac{1000}{q} + 100 - 5q + q^2$$

Where p is the price in shillings and q is the quantity in kilograms.

Required

- i) What does $\frac{1000}{q}$ in the ATC equation represent economically?
(1 mark)
- ii) Determine the output that leads to maximum profit and the profit at the level of output. (9 marks)

- c) Alpha industries sells two products, X and Y, in related markets, with demand functions given by:

$$P_x - 13 + 2X + Y = 0$$

$$P_y - 13 + X + 2Y = 0$$

The total cost, in shillings, is given by:

$$TC = X + Y$$

Required:

Determine the price and the output for each good which will maximize profits. (7 marks)

(Total: 20 marks)

QUESTION SEVEN

- a) The following table shows the Fixed Cost (F) and the variable cost (V) of producing 1 unit of X and 1 unit of Y:

	Product		
	X	Y	
Cost F	5	8	(Shs '000')
Cost V	4	12	

When x units of X and y units of Y are produced, the total fixed cost is Shs.640,000 and total variable cost is Shs.820,000. Express this information as a matrix equation and hence find the quantities of x and y produced using matrix algebra. (10 marks)

The marginal productivity of an industrial operation (the production of electric furnaces) is given by:

$$f(x) = \frac{60}{x^2} + 10$$

Where x is capitalization in millions of shillings. Given that, when the capitalization is Shs. Million they can produce 62 of the furnaces per week.

Required:

- a) How many furnaces they will be able to produce if their capitalization increased to Shs 10 million.
 b) What does the term marginal of productivity mean?

(10 marks)

(Total: 20 marks)

CHAPTER THIRTEEN

NETWORK ANALYSIS

Specific objectives

At the end of this topic the trainee should be able to:

- ❖ Describe networks and need for the analysis;
- ❖ Construct project networks;
- ❖ Determine the critical path;
- ❖ Apply network analysis to decision problems.

INTRODUCTION

This is a system of interrelationship between jobs and tasks for planning and control of resources of a project by identifying critical path of the project.

Terminology

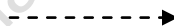
Activity. Task or job of work, which takes time and resources e.g building a bridge. It is represented by an arrow which indicates where the task begins and ends



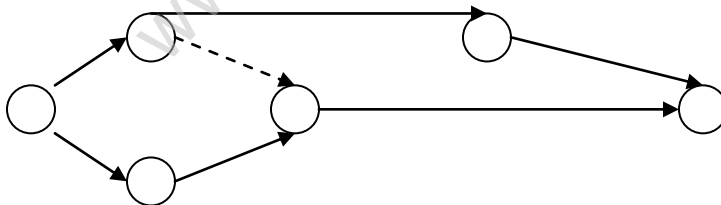
Event (node). This is a point in time and it indicates the start or finish of an activity e.g in building a bridge, rails installed. It is represented by a circle.



Dummy activity. An activity that doesn't consume time or resources, it is merely to show logical dependencies between activities so as abide by rules of drawing a network, it is represented by a dotted arrow



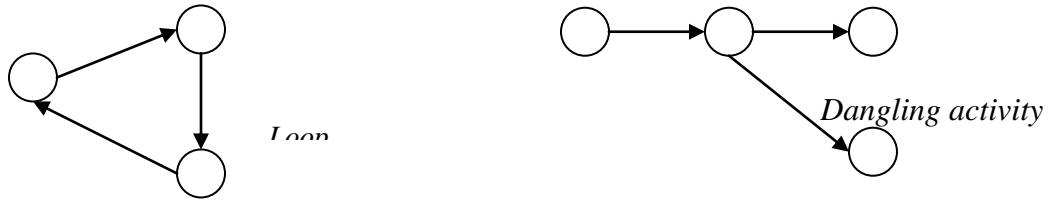
Network. This is a combination of activities and events (including dummy activities)



Rules for Drawing a Network

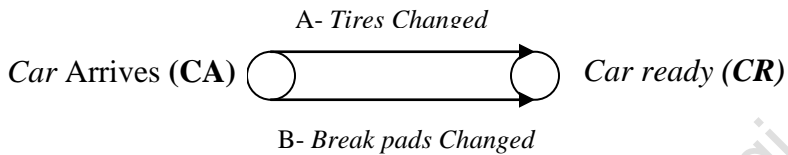
- a) A network should only have one start point and one finish point (*start event and finish event*)
- b) All activities must have at least one preceding event (*tail event*) and at least one succeeding event (*head event*), but an activity may not share the same tail event and head event.
- c) An activity can only start after its tail event has been reached
- d) An event is only complete after all activities leading to it are complete.
- e) Activities are identified by alphabetical or numeric codes i.e. A,B,C; 1,2,3 or identification by head or tail events 1-2, 2-4, 3-4,1-4...

- f) Loops (a series of activities leading back to the same event) and dangles (activities which do not link to the overall project) are not allowed

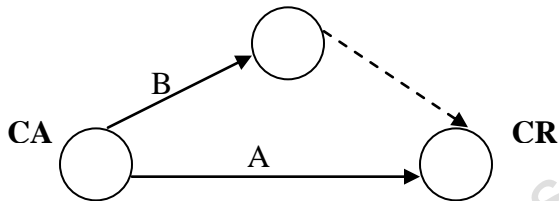


Dummy Events

This is an event that does not consume time or resources, it is represented by dotted arrow. Dummies are applied when two or more events occur concurrently and they share the same head and tail events e.g. when a car goes to a garage tires are changed and break pads as well, instead of representing this as;



These events are represented as;

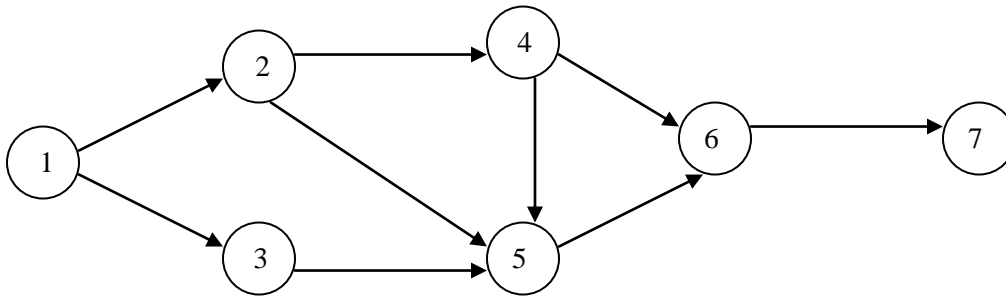


Example of a network.

Activities

1-2 - where 1 is the preceding event where as 2 is the succeeding event of the activity

- 1-3
- 2-4
- 2-5
- 3-5
- 4-5
- 4-6
- 5-6
- 6-6



Network Analysis-Time Analysis

Assessing the time

- a) After drawing the outline of the network time durations of the activities are then inserted.
- a) Time estimates. The analysis of the projects time can be achieved by using :
- i. Single time estimates for each activity. These estimates would be based on the judgment of the individual responsible or by technical calculations using data from similar projects
 - ii. Multiple time estimates for each activity. the most usual multiple time estimates are three estimates for each activity , i.e. optimistic (O), Most Likely (ML), and Pessimistic (P). These three estimates are combined to give an expected time and the accepted time formula is:

$$\text{Expected time} = \frac{O + P + 4ML}{6}$$

For example assume that the three estimates for an activity are

Optimistic	11 days
Most likely	15 days
Pessimistic	18 days

$$\begin{aligned} \text{Expected time} &= \frac{11 + 18 + 4(15)}{6} \\ &= \mathbf{14.8 \text{ days}} \end{aligned}$$

- b) Use of time estimates. as three time estimates are converted to a single time estimate. There is no fundamental difference between the two methods as regards the basic time analysis of a network. However, on completion of the basic time analysis, projects with multiple time estimates can be further analyzed to give an estimate of the probability of completing the project by a scheduled date.
- c) Time units. Time estimates may be given in any unit, i.e. minutes , hours, days depending on the project. All times estimates within a project must be in the same units otherwise confusion is bound to occur.

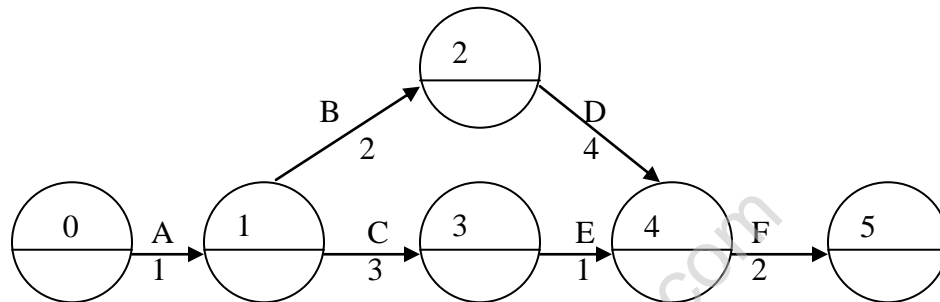
Basic time analysis - critical path

- b) The critical path of a network gives the **shortest time in which the whole project can be completed**. It is the chain of activities with the

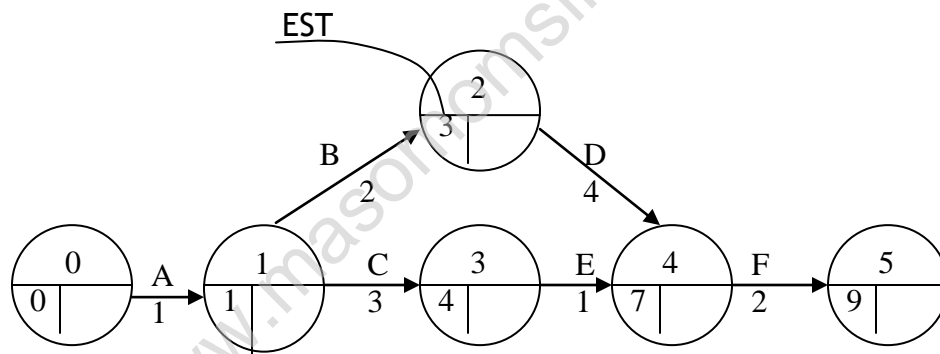
longest duration times. There may be more than one critical path which may run through a dummy.

Earliest start times (EST) - Forward pass, Once the activities have been timed we can assess the total project time by calculating the ESTs for each activity. The EST is the earliest possible time at which a succeeding activity can start.

Assume the following network has been drawn and the activity times estimated in days.



The ESTs can be inserted as follows.



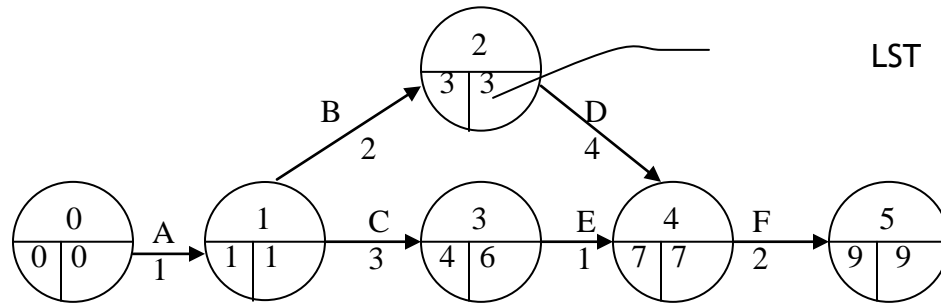
The method used to insert the ESTs is also known as the **forward pass**, this is obtained by;

$$\text{EST} = \text{The greater of [EST (tail event) + Activity duration]}$$

- Start from the start event giving it 0 values,
- For the rest of the events EST is obtained by summing the EST of the tail event and the activity duration
- Where two or more routes converge into an activity, calculate individual EST per route and then select the longest route (time)
- The EST of the finish event is the shortest time the whole project can be completed.

Latest Start Times (LST) - Backward pass. this is the latest possible time with which a preceding activity can finish without increasing the project duration. After this operation the critical path will be clearly defined.

From our example this is done as follows;



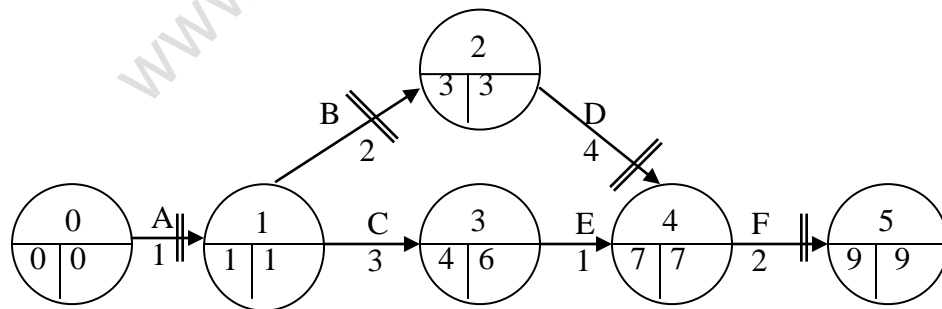
LST = Lowest of [LST (head event) - activity duration]

- Starting at the finish event, insert the LST (i.e. 9 for our example) ,and work backwards through the network.
- deduct each activity duration from the previously calculated LST (i.e. head LST).
- Where the tails of activities join an event, the lowest number is taken as the LST for that event

Critical path determination

Critical Path. . This is the chain of activities in a network with the longest duration Assessment of the resultant network shows that one path through the network (A, B, D, F) has EST's and LST's which are identical this is the critical path.

The critical path can be indicated on the network either by a different colour or by two small transverse lines across the arrows along the path, thus in our example we have;

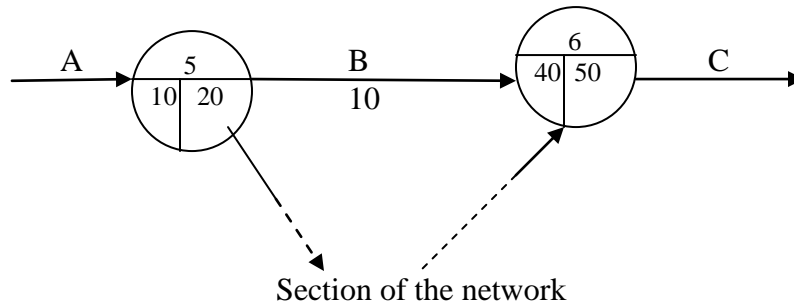


Activities along the critical path are vital activities which must be completed by their EST's/LST's otherwise the project will be delayed.

Non critical activities (in the example above, C and E) have spare time or *float* available. C and/ or E could take up to an additional 2 days in total without delaying the project duration. If it is required to reduce the overall project duration then the time of one or more of the activities on the critical path must be reduced perhaps by using more labour, or better equipment to reducing job times.

Float

Float or spare time can only be associated with activities which are non-critical. By definition, activities on the critical path cannot have float. There are three types of float, **Total Float**, **Free Float** and **Independent Float**. To illustrate these types of float we use the following example.



- a) Total float. Amount of time by which a path of activities could be delayed without affecting the overall project duration. The path in this example consists of one activity only i.e. B

Total Float = Latest Finish time (LFT) - Earliest Start time(EST) time - Activity Duration

$$\begin{aligned}\text{Total Float} &= 50 - 10 - 10 \\ &= \underline{30 \text{ days}}\end{aligned}$$

- b) Free float Amount of time an activity can be delayed without affecting the commencement of a subsequent activity at its earliest start time, but may affect float of a previous activity.

Free Float = Earliest Finish Time(EFT) - EST - Activity Duration

$$\begin{aligned}\text{Free Float} &= 40 - 10 - 10 \\ &= \underline{20 \text{ days}}\end{aligned}$$

- c) Independent float. Amount of time an activity can be delayed when all preceding activities are completed as late as possible and all succeeding activities commenced as early a possible. Independent float therefore does not affect the float of either preceding or subsequent activities.

Independent float = EFT- Latest Start time (EST) - Activity Duration

$$\begin{aligned}\text{Independent float} &= 40 - 20 - 10 \\ &= \underline{10 \text{ days}}\end{aligned}$$

Note:

- for examination purposes, float always refers to total float
- The total float can be calculated separately for each activity but it is often useful to find the total float over chains of non-critical activities *between critical events*

Example.

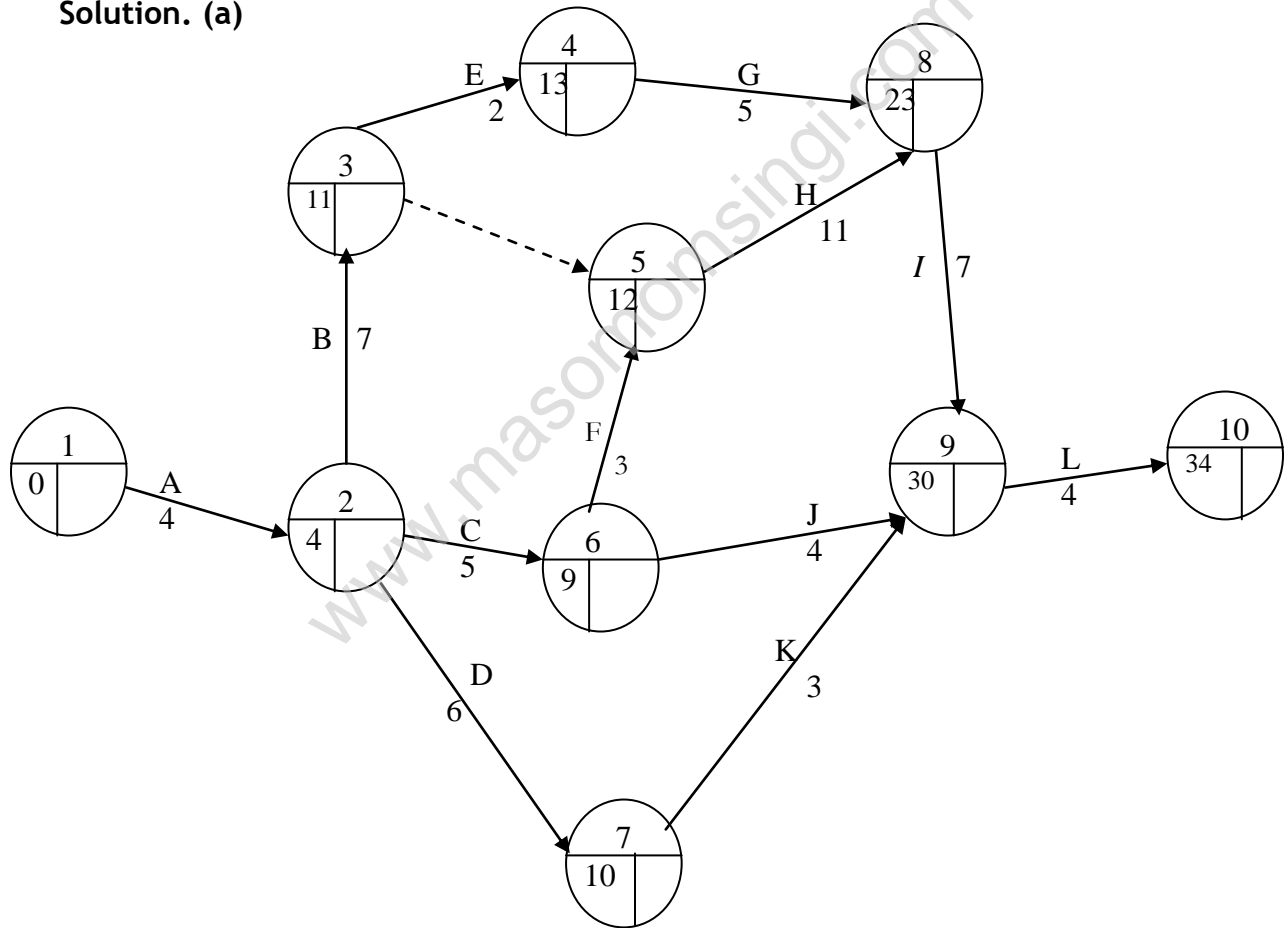
The following represents activities of a network.

Activity	Preceding Activity	Duration Days
A	-	4
B	A	7
C	A	5
D	A	6
E	B	2
F	C	3
G	E	5
H	B,F	11
I	G,H	7
J	C	4
K	D	3
L	I,J,K	4

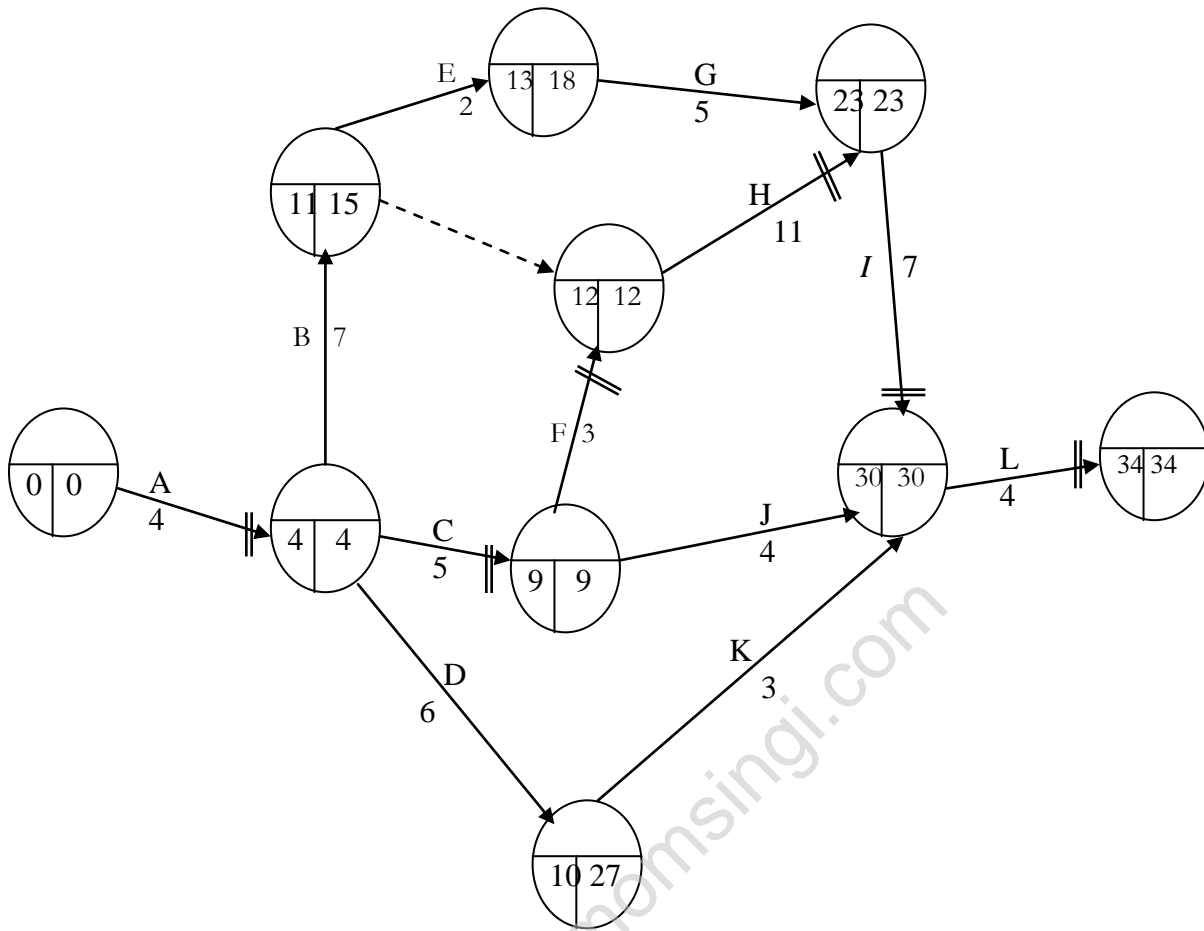
Required:

- Draw the network diagram and find the critical path
- Calculate the floats of the network in question

Solution. (a)



- First we draw the network structure ensuring it fits the data above
- We then label all activities from 1 to 12 and indicate activity duration
- Conduct a forward pass operation (to obtain the diagram above)
- Operate backward pass to establish the critical path, thus we have...



Therefore we get the critical path to be, A- C- F- H- I- L

b) The floats of the network,

Activity	EST	LST	EFT	LFT	Activity Duration	Total Float LFT - EST - D	Free Float EFT - EST - D	Independent Float EFT - LST - D
*A	0	0	4	4	4	-	-	-
B	4	4	11	15	7	4	-	-
*C	4	4	9	9	5	-	-	-
D	4	4	10	22	6	12	-	-
E	11	15	13	21	2	8	-	-
*F	9	9	15	15	3	-	-	-
G	13	21	23	23	5	5	5	-
*H	12	12	23	23	11	-	-	-
*I	23	23	30	30	7	-	-	-
J	9	9	30	30	4	17	17	17
K	10	22	30	30	3	17	17	5
*L	30	30	34	34	4	-	-	-

The total float on the non-critical chains are;

Non-critical chain	Time required (sum of)	Time available (LFT of last activity - EST of 1 st)	Total Float over chain
--------------------	------------------------	---	------------------------

	duration)	activity)	
B,E,G	14	19	5
B,Dummy	7	8	1
D,K	9	26	17
J	4	21	17

Slack

This is the difference between the EST and LST for each event. Strictly it does not apply to activities but on occasions the terms are confused in examination questions and unless the context makes it abundantly clear that event slack is required, it is likely that some form of activity float is required. Events on the critical path have zero slack.

APPLICATIONS

Cost Scheduling

This is done by calculating the cost of various project durations, cost analysis seeks to find the cheapest way of reducing the overall cost duration of a project by increasing labour hours, equipment e.t.c.

Terminologies

Normal cost. The costs associated with a normal time estimate for an activity. Often the normal time estimate is set at the point where resources (labour, equipment, etc.) are used in the most efficient manner.

Crash cost. The costs associated with the minimum possible time for an activity. Crash costs, because of extra wages, overtime premiums, extra facility costs are always higher than normal costs.

Crash time. The minimum possible time that an activity is planned to take. . The minimum time is invariably brought about by the application of extra resources, e.g. more labour or machinery.

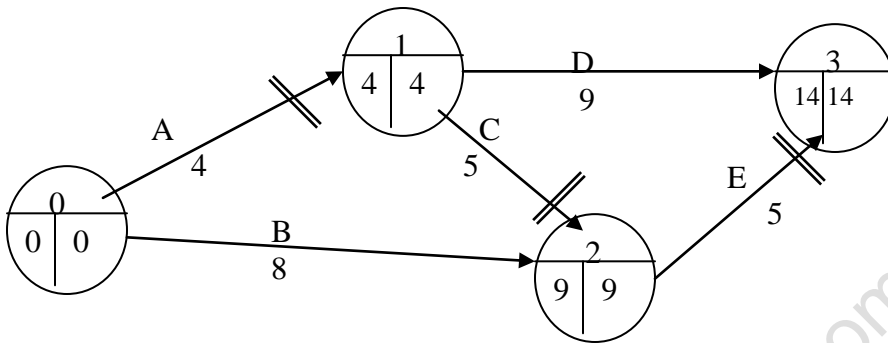
Cost slope. This is the average cost of shortening an activity by one time unit (day, week, month as appropriate). The cost slope is generally assumed to be linear and is calculated as follows:

$$\text{Cost slope} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}}$$

Example

A project has the following activities and costs. You are required to prepare the least cost schedules for all possible durations from *normal time - normal cost* to *crash time - crash cost*.

Activity	Preceding Activity	Duration days	Crash time	Cost (Shs).	Crash cost	Cost slope
A	-	4	3	360	420	60
B	-	8	5	300	510	70
C	A	5	3	170	270	50
D	A	9	7	220	300	40
E	B,C	5	3	200	360	80



Project duration and costs

- (a) Normal duration = 14 days
 Critical path = A,C,E
 Project cost (cost of all activities in normal time) = Shs. 1,250.
- (b) Reduce by 1 day the activity on the critical path with the lowest cost slope. Thus we reduce C at extra cost of Shs. 50.

Now

Project duration = 13 days
 Project cost = Shs. 1,300

Note: that all activities are now critical.

- (c) Further reducing the critical path by 1 day will require that more than one activity is affected because there exist several critical paths.

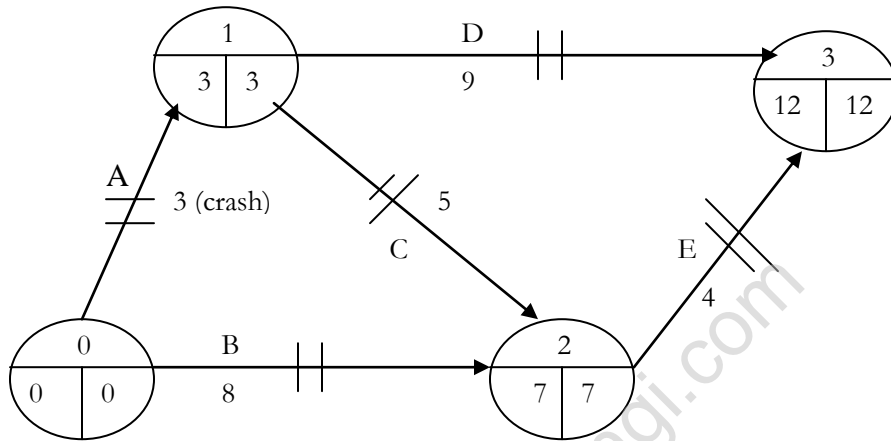
Reduce by 1 day	Extra cost	Activities critical
A and B	$60 + 70 = 130$	All
D and E	$40 + 80 = 120$	All
B, C and D	$70 + 50 + 40 = 160$	All
A and E	$60 + 80 = 140$	A, D, B, E

From this we realize that reducing D and E is the cheapest.

However closer examination of the fourth alternative reveals that C is now non-critical and has 1 day float. Since we earlier reduced C for Shs. 50, if we reduce A and E and increase C by a day which will save Shs. 50.

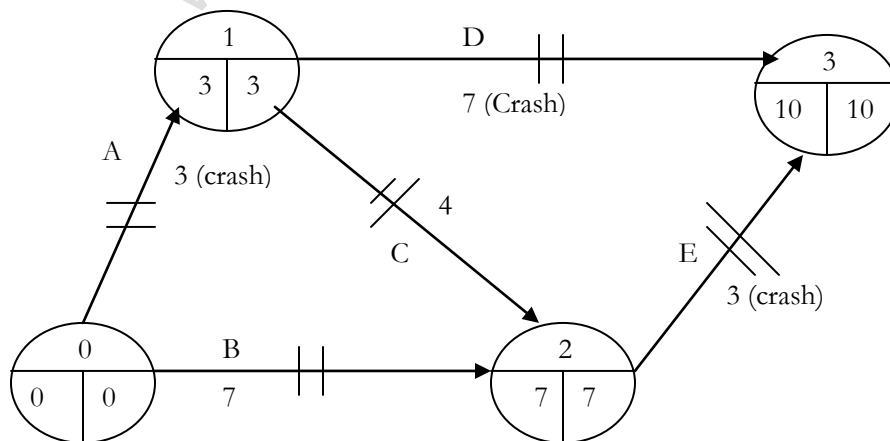
Then the net cost for 12 day duration = 1,300 + (140 - 50) = 1,390.

The network becomes.....



- (d) Next we reduce D & E
 Project duration = 11 days
 Project cost = 1,510
 Critical activities = All

- (e) Final reduction possible is by reducing B, C & D for Shs. 160 the network then becomes.



Duration = 10 days

Cost = Shs. 1,670
 Critical activities = All.

Note: only critical activities affect project duration.
 : Always look for a possibility of increasing the duration of a
 previously
 Crashed activity.

Scheduling Resources and Gantt Chart

Apart from time, cost network analysis also help in controlling and planning of resources.

Example

A project has the following activity durations and resource requirements.

Activity	Preceding activity	Duration (days)	Resource requirement (man power)
A	-	6	3
B	-	3	2
C	-	2	2
D	C	2	1
E	B	1	2
F	D	1	1

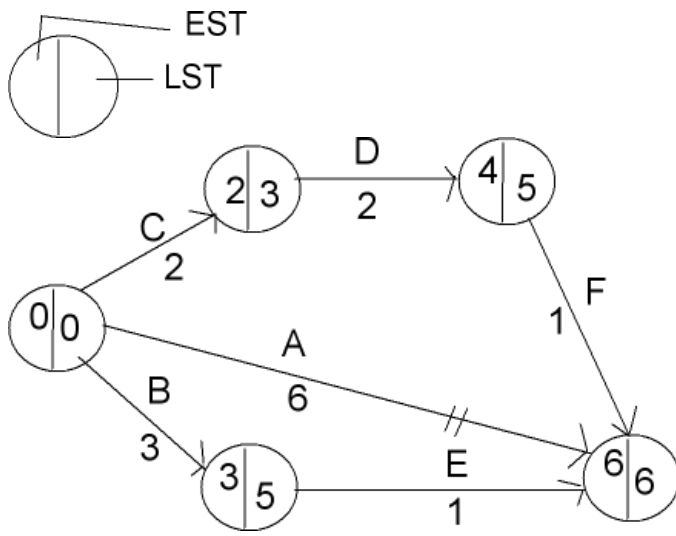
Required

- What is the networks critical path
- Draw a gantt chart diagram indicating activity times, using their estimate.
- Show resource requirement on a day to day basis assuming all events commence at their estimates.
- Assuming that only six employees are available, how will the activities be planned for?

Solution

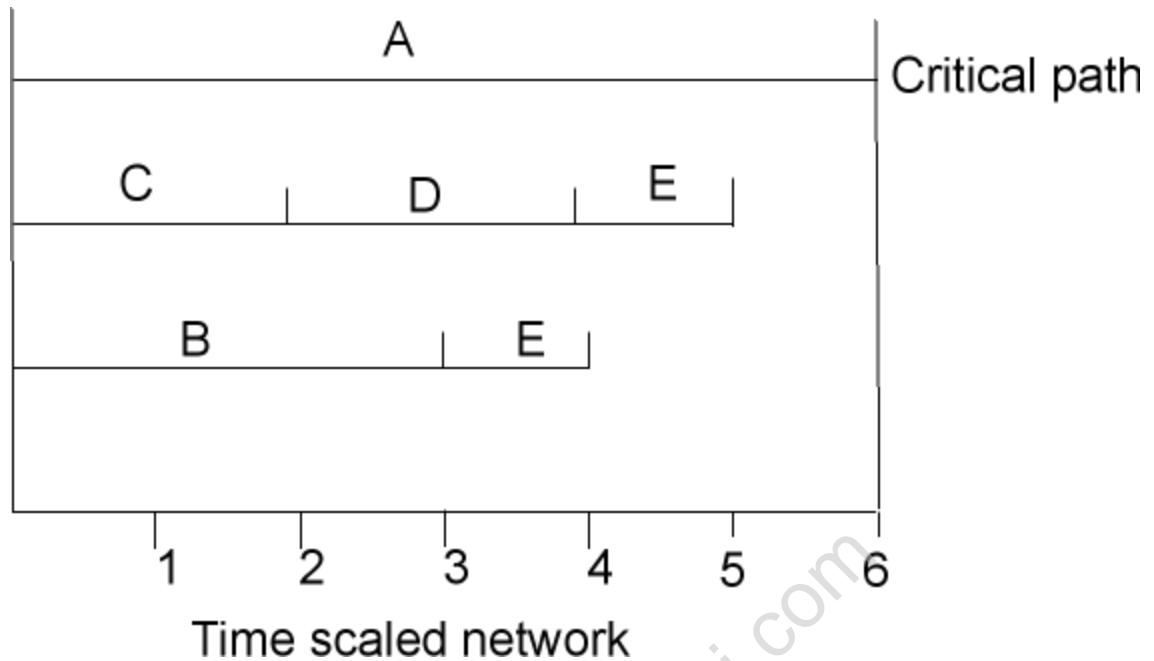
i)

Activities	Duration	EST	LST	Man power
A	6	0	0	3
B	3	0	0	2
C	2	0	0	2
D	2	2	3	1
E	1	3	5	2
F	1	4	5	1

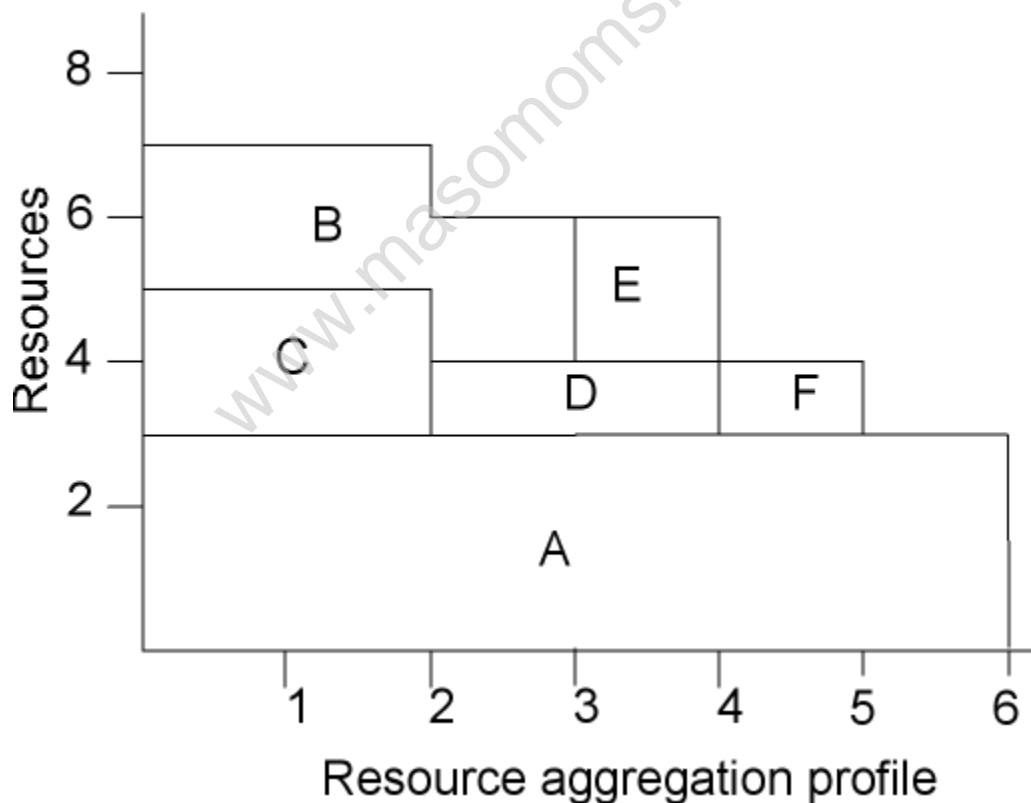


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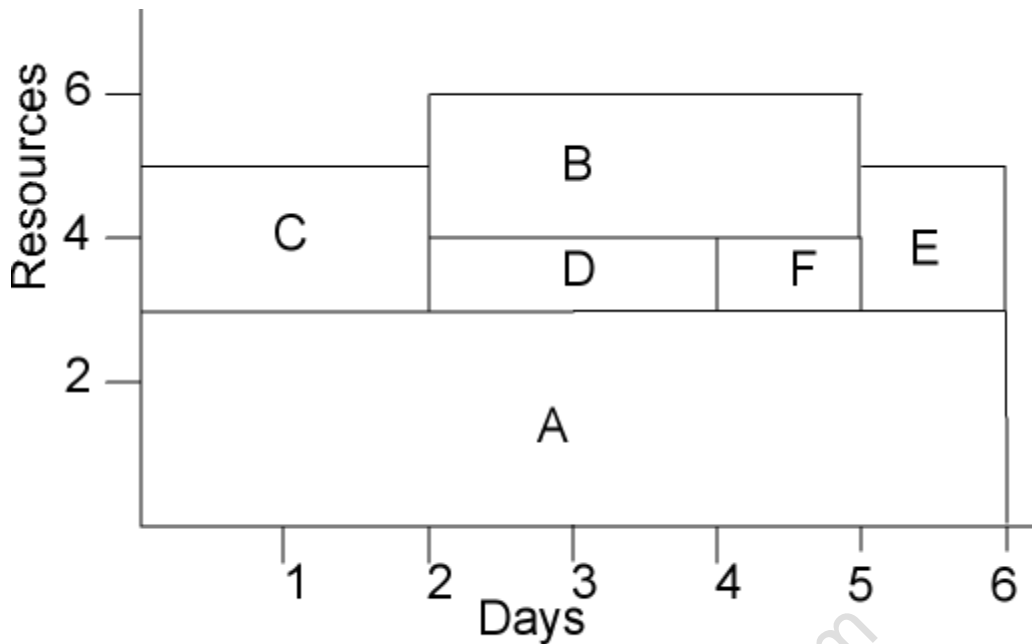
- ii) A gantt chart or a bar chart. This is a diagram indicating a resource scaled network.



- iii) Resource requirements on a day to day basis.



- iv) When on 6 manpower resources are available then we adjust the activities to accommodate this and still end at the given critical time duration i.e.



Smoothened Resource requirements

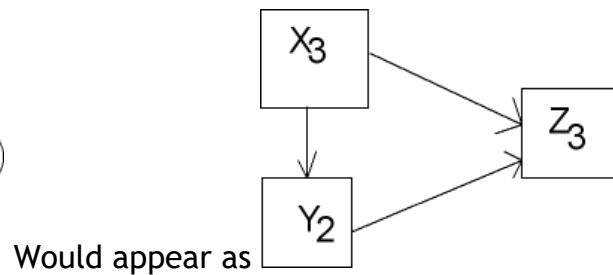
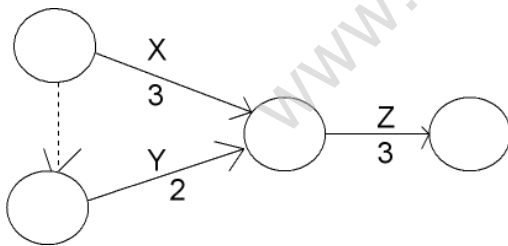
Node Networks

This network also known as a procedure diagram is represented with the same information as a network diagram.

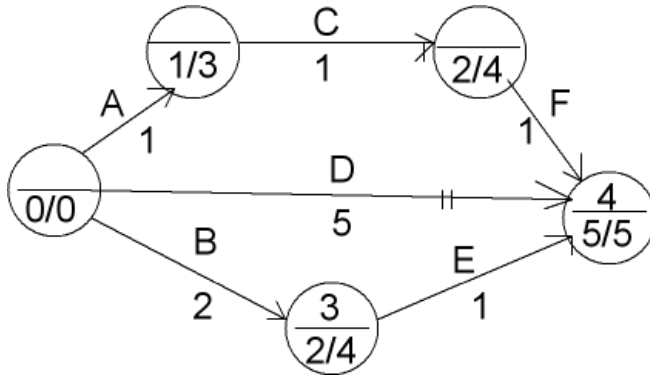
Its characteristics are;

- i) Activities are shown in boxes instead of arrows
- ii) Events are not represented.
- iii) The arrows linking boxes indicate the sequence precedence of activities.
- iv) Dummies aren't necessary.

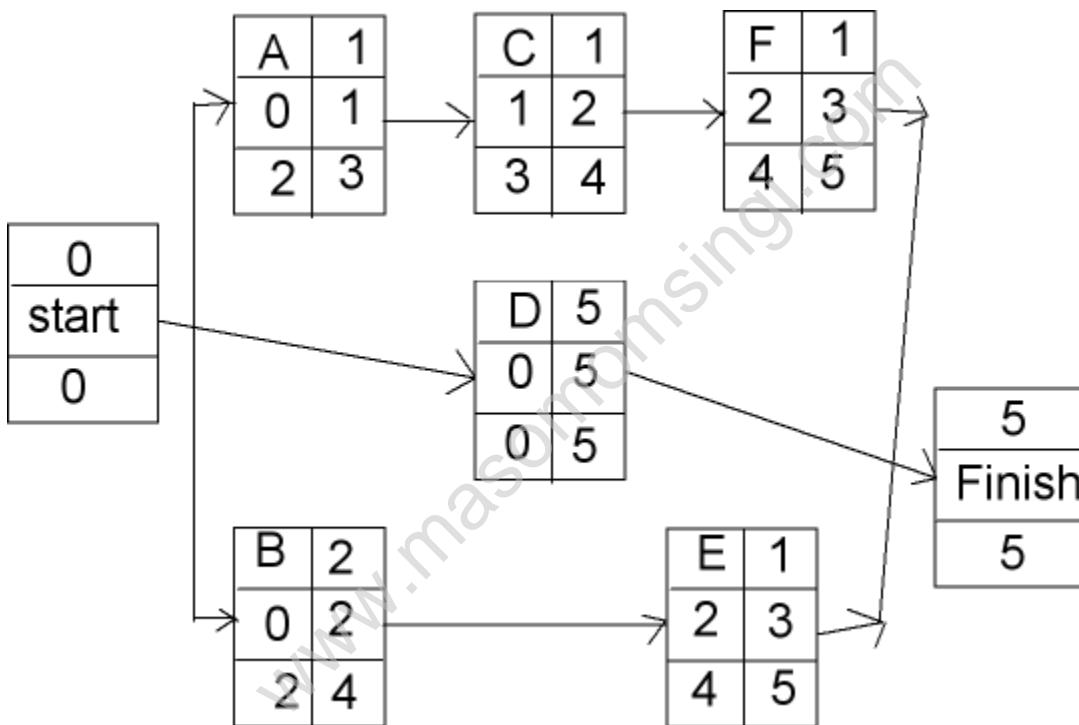
E.g.



A full activity node network is represented as;



This is represented as;



Note:

- i) EST and LST are calculated by the same process we learnt earlier.
- ii) EFT and LFT are calculated by adding the activity time duration to EST and LST respectively.
- iii) Critical path is similarly identified by identifying equal EST and LST throughout the path.

LESSON 8 REINFORCING QUESTIONS

QUESTION ONE

Regal Investments has just received instructions from a client to invest in two shares; one an airline share, the other an insurance share. The total maximum appreciation in share value over the next year is to be maximized subject to the following restrictions:

- the total investment shall not exceed Sh.100,000
- at most Sh.40,000 is to be invested in the insurance shares
- quarterly dividends must total at least Sh.2,600

The airline share is currently selling for Sh.40 per share and its quarterly dividend is Sh.1 per share. The insurance share is currently selling for Sh.50 per share and the quarterly dividend is Sh.1.50 per share. Regal's analysts predict that over the next year, the value of the airline share will increase by Sh.2 per share and the value of the insurance share will increase by Sh.3 per share.

A computer software provided the following part solution output:

Objective Function Value = 5,400

	Variable	Number	Reduced cost
Airline shares	1,500	0.000	
Insurance shares	800	0.000	
Constraint	Slack/Surplus		Dual prices
Total investment		0.000	0.050
Investment in insurance		0.000	0.010
Dividends		100.000	0.000

Objective Coefficient Ranges

Variable	Lower limit	Current value	Upper limit
Airline share	2.500	3.000	No upper limit
Insurance share	0.000	2.000	2.400

Right-hand Side Ranges

Constraint	Lower limit	Current value	Upper limit
Total investment	96,000.00	100,000	No upper limit
Investment in insurance	20,000.00	40,000	100,000.00
Dividends	No lower limit	2,600	2,700.00

Required:

- Formulate the above problem.
- Explain what reduced cost and dual prices columns above mean.
- How should the client's money be invested to satisfy the restrictions?
- Suppose Regal's estimate of the airline shares appreciation is an error, within what limits must the actual appreciation lie for the answer in (c) above to remain optimal?

(Q 6 Dec

2001)

QUESTION TWO

- A baker makes two products; large loaves and small round loaves. He can sell up to 280 of the large loaves and up to 400 small round loaves per day. Each large loaf occupies 0.01m^3 of shelf space, each small loaf occupies 0.008m^3 of space, and there is 4m^3 of shelf space available. There are 8 hours available each night for baking, and he can produce large loaves at the rate of 40 per hour, and small loaves at the rate of 80 per hour. The profit on each large loaf is Sh.5.00 and Sh.3.00 profit on the small round loaf.

Required:

In order to maximize profits, how many large and small round loaves should he produce?

- Summarize the procedure for solving the kind of quantitative technique you have used to solve part (a) above.

(Q 6 June

2001)

QUESTION THREE

- A small company will be introducing a new line of lightweight bicycle frames to be made from special aluminium alloy and steel alloy. The frames will be produced in two models, deluxe and professional. The anticipated unit profits are currently Sh.1,000 for a deluxe frame and Sh.1,500 for a professional frame. The number of kilogrammes of each alloy

needed per frame is summarized in the table below. A supplier delivers 100 kilogrammes of the aluminium alloy and 80 kilogrammes of the steel alloy weekly.

	Aluminium alloy	Steel alloy
Deluxe	2	3
Professional	4	2

Required:

- i) Determine the optimal weekly production schedule.
 - ii) Within what limits must the unit profits lie for each of the frames for this solution to remain optimal?
- b) Explain the limitations of the technique you have used to solve part (a) above.

(Q 6 Dec 2000)

QUESTION FOUR

- a) Define the following terms as used in linear programming:
 - i) Feasible solution
 - ii) Transportation problem
 - iii) Assignment problem
- b) The TamuTamu products company Ltd is considering an expansion into five new sales districts. The company has been able to hire four new experienced salespersons. Upon analysing the new salesperson's past experience in combination with a personality test which was given to them, the company assigned a rating to each of the salespersons for each of the districts. These ratings are as follows:

c)

		Districts				
		1	2	3	4	5
Salespersons	A	92	90	94	91	83
	B	84	88	96	82	81
	C	90	90	93	86	93
	D	78	94	89	84	88

The company knows that with four salespersons, only four of the five potential districts can be covered.

Required:

- i) The four districts that the salespersons should be assigned to in order to maximize the total of the ratings
- ii) Maximum total rating.

(Q 6 June 2002)

QUESTION FIVE

- a) Explain the value of sensitivity analysis in linear programming problems and show how dual values are useful in identifying the price worth paying to relax constraints.
- b) J.A Computers is a small manufacturer of personal computers. It concentrates on production of three models- a Desktop 386, a Desktop 286, and a Laptop 486, each containing one CPU Chip. Due to its limited assembly facilities JA Computers are unable to produce more than 500 desktop models or more than 250 Laptop models per month. It has one hundred and twenty 80386 chips (these are used in Desktop-386) and four hundred 80286 chips (used in desktop 286 and Laptop 486) for the month. The Desktop 386 model requires five hours of production time, the Desktop 286 model requires four hours of production time, and the Laptop 486 requires three hours of production time. J.A Computers have 2000 hours of production time available for the coming month. The company estimates that the profit on Desktop 386 is Sh. 5,000. for a desktop 286 the profit is Sh.3,400 and Sh.3,000 profit for a laptop 486.

Required:

Formulate this problem as a profit maximization problem and mention the basic assumptions that are inherent in such models.

- c) An extract of the output from a computer package for this problem is given below:

Output solution

$$X_1=120, X_2 = 200, X_3 = 200$$

Dual values Constraints 3 150

 Constraints 4 90

 Constraints 5 20

Sensitivity analysis of objective function coefficients:

Variable	Lower limit	Original value	Upper limit
X_1	100	250	No limit
X_2	150	170	200
X_3	127.5	150	170

Sensitivity analysis on R.H.S ranges.

Constraints	Lower limit	Original value	Upper limit
1	320	500	No limit
2	200	250	No limit
3	80	120	130
4	350	400	412.5
5	1950	2000	2180

X_1 = Monthly production level for Desktop 386.
 X_2 = Monthly production level for Desktop 286.
 X_3 = Monthly production level for Laptop 486.

Required:

- i) Interpret the output clearly, including optimum product mix, monthly profit, unused resources and dual values
- ii) Explain the purpose of upper limits and lower limits for the variables X_1, X_2, X_3 and constraints 1 to 5.
- iii) Calculate the increase in profit if the company is able to produce a further 10 CPU 80386 chips. (Q7 July 2000 Pilot paper)

QUESTION SIX

Preface Retailers is a high-technology retailer and mail order business. In order to improve its process the company decides to install a new microcomputer system to manage its entire operation (i.e. payroll, accounts, inventory).

Terminals for each of its many stores will be networked for fast, dependable service. The specific activities that Preface will need to accomplish before the system is up and running are listed below. The table also includes the necessary increased staffing to undertake the project.

Activity	Preceding Activities	Duration (Days)	Increase d Staff
A. Build insulated enclosure	-	4	1
B. Decide on computer system	-	1	3
C. Electrical wiring of room	A	3	2
D. Order and collect computer	B	2	1
E. Install air conditioning	A	4	2
F. Install computer	D, E	2	2
G. Staff testing	B	5	1
H. Install software	C, F	2	1
I. Staff training	G, H	3	1

Required:

- a) Draw a network diagram for the project and determine the critical path and its duration.
- b) Assuming that all activities start as soon as possible, draw a progress chart for the project, showing the times at which each activity takes place and the manpower requirements.
- c) The union has decided that any staff employed on the project must be paid for the duration of the project whether they work or not, at a rate of £500 per day.

Assuming that the same staff is employed on the different activities, determine the work schedule that will minimise labour costs though not necessarily the project time. What is the cost associated with this schedule?

Comment on the validity of the assumption.

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CHAPTER FOURTEEN

INVENTORY CONTROL MODELS

Specific objectives

At the end of this topic the trainee should be able to:

- Define inventory;
- Describe the control systems;
- Calculate the EOQ;
- Determine safety stock (ss) and re-order level.

INTRODUCTION

The activities of a business during a financial year combine investment projects in progress with new projects commencing and others terminate within the year. It would appear reasonable to presume, therefore, that business financial reports are presented in the cash-flow mode used to appraise investments, to facilitate comparison of actual with planned cash flows.

Some businesses do make such comparisons as part of their retrospective monitoring of investment decisions, but there is no obligation to do so. Cash-flow accounting, as it is called, has its supporters, but its introduction is frustrated by statutory and non-statutory regulations.

The Companies Act requires limited companies to produce profit and loss accounts and balance sheets in prescribed form. The Inland Revenue assumes that taxable profit has been computed by applying recognized accounting principles. The Accounting Standards Committee recommends the application of standard practices in the measurement of profit and portrayal of a company's financial position in its balance sheet. More compellingly, profit and loss reporting is compatible with the investors' objectives of stable and growing earnings.

Profit is measured conventionally by setting against the sales revenue for a period the costs expired in earning that revenue. That is, sales are *matched* against their relevant costs. Profit is therefore more evenly reported than it would be if all cash receipts and payments, capital and revenue, were fully reflected in the accounts of the period in which they are received and paid.

The management accountant also adopts the *matching principle* when preparing control information in both actual and budgeted form, and also ascertains full product cost as a starting point for setting selling prices.

This outlines the systems and methods used to control the flow of resources through production and service cost centres, for their eventual inclusion in product and period costs.

MATERIAL CONTROL

It is said that "any fool can sell"—it is buying at the right price that is more critical to the achievement of a satisfactory return on capital employed. Buying price is important of course, but buying the right materials, is equally important if production targets are to be achieved and investment in inventories to be minimized.

WHAT TO ORDER

This is governed by product specifications, but an efficient buyer will always have his ear to the ground to discover new and substitute materials and components of advantageous quality and price. Other economies can be realized by reducing the variety of materials purchased by *standardization*, e.g. reducing the variety of colors of paint stocked, or by introducing value analysis into the decision process.

VALUE ANALYSIS

Is a formalized technique involving a rigorous analysis of products at the design stage or at any time during the saleable lives, to determine their value characteristics. These are the attributes that a customer looks for in a product and include its use value (functional qualities), appeal value (color, style etc.) and second-hand value (e.g. trade-in-price). The object of value analysis is to build into the product the optimum of desired value at minimum cost, by introducing the most up-to-date designs, materials and methods of manufacture. No more value need be built into the product than is desired by the customer. For example, moulded plastic bumper bars are now fitted to many cars, because they are cheaper than and equally as functional as chromium-plated steel ones.

How much to order

Supposing the estimated annual usage of a component by Harambee Agricultural Machinery Ltd is 20,000 units. Usage is even throughout the year and only one order per annum is placed with the supplier. Because only one delivery is made, average stock will be high, i.e. $20,000 \div 2 = 10,000$ and consequently stockholding costs will be very high. On the other hand, the costs of ordering will be negligible. If two orders are placed there will be less in stock (i.e. average 5,000), which will reduce holding costs, but ordering costs will increase. Thus, the higher the number of orders placed, the lower are stockholding costs, but the higher are ordering costs.

Stockholding costs include interest on the capital invested in stocks, storage, insurance, rates, security, building maintenance, heating, etc. Ordering costs include buying-department staff costs, receiving and handling.

Assuming that the cost of each Harambee component is £10, that holding cost is 10% of stock value and the cost of placing an order is £1, the total annual cost of stockholding and ordering when different numbers of orders are placed, is as follows:

Number of orders	4	20	50	100	200	400
Size of order	5,000	1,000	400	200	100	50
Average Stock (50% order)	2,500	500	200	100	50	25
Holding cost	£2,500	£500	£200	£100	£50	£25
Ordering cost (£1 per order)	£4	£20	£50	£100	£200	£400
Total Annual Cost	£2,504	£520	£250	£200	£250	£425

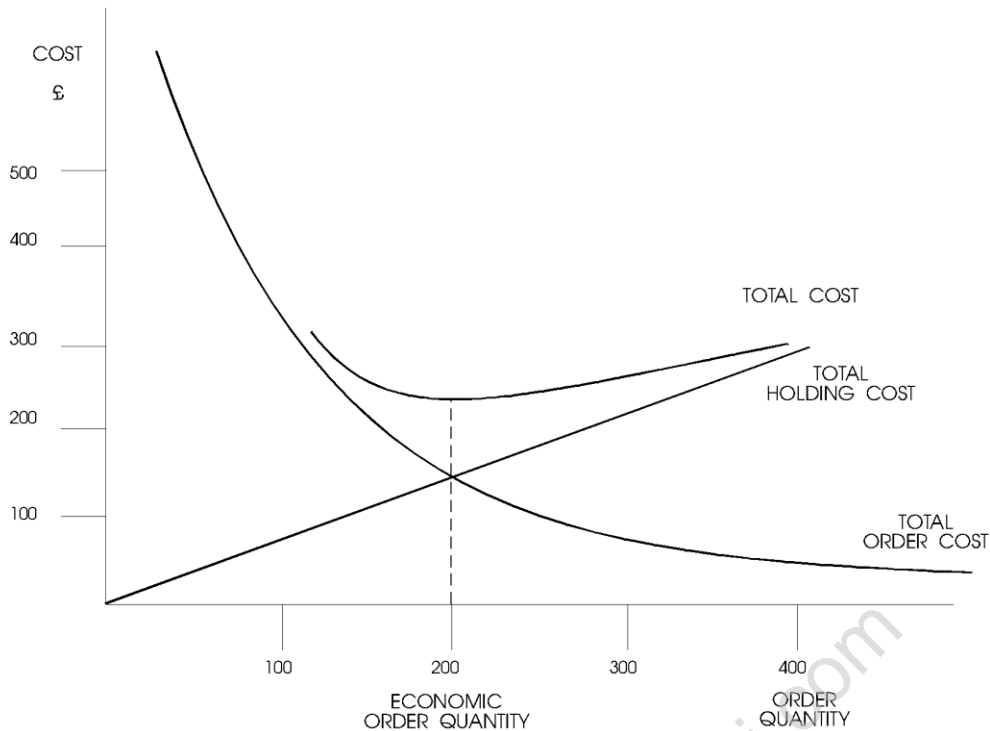


Figure 1: Economic Order Quantity

Placing 100 orders a year results in the lowest of ordering and holding cost of £200, therefore the economic order quantity is 200 units.

The same information is graphed in *Figure 1* above, showing that the *economic order quantity* (EOQ) is the point where ordering and holding costs are equal, and total £200.

As costs of ordering and holding stock are equal at the EOQ point, we can build a simple mathematical model to solve the problem, as follows:

$$\frac{Q}{2} \times H = \frac{A}{Q} \times P$$

Where Q = EOQ
 H = holding cost per unit
 A = annual demand
 P = cost of placing an order

$$finally Q = \sqrt{\frac{2AP}{H}}$$

Using the data in the previous example:

$$\begin{aligned} EOQ &= \sqrt{\frac{2 \times 20,000 \times 1}{1}} \\ &= \sqrt{40,000} \\ &= 200 \end{aligned}$$

Although the model assumes that holding and ordering costs are fixed, this simplification is acceptable given a relatively unchanging level of production activity. In addition, because the total cost curve in the *Figure 1* is relatively flat either side of the EOQ, minor errors and approximation in the variables used in the calculation may not affect the end result significantly.

Practical constraints on the use of the model include restrictions on the available storage space, the availability of quantity discounts (though the model can be modified in this respect), the seasonal nature of supplies, the shelf-life of products and delivery schedules imposed by suppliers.

WHEN TO ORDER

If deliveries from suppliers normally take two weeks to arrive, then replenishment orders should be placed with them when the level of stocks represents two weeks' supply. For example, if usage is 200 units a week, an order (the EOQ) will be placed when the stock level falls to 400 units. *Figure 6(a)* illustrates that, with *certain* knowledge of usage and lead time, delivery takes place just as stock is exhausted.

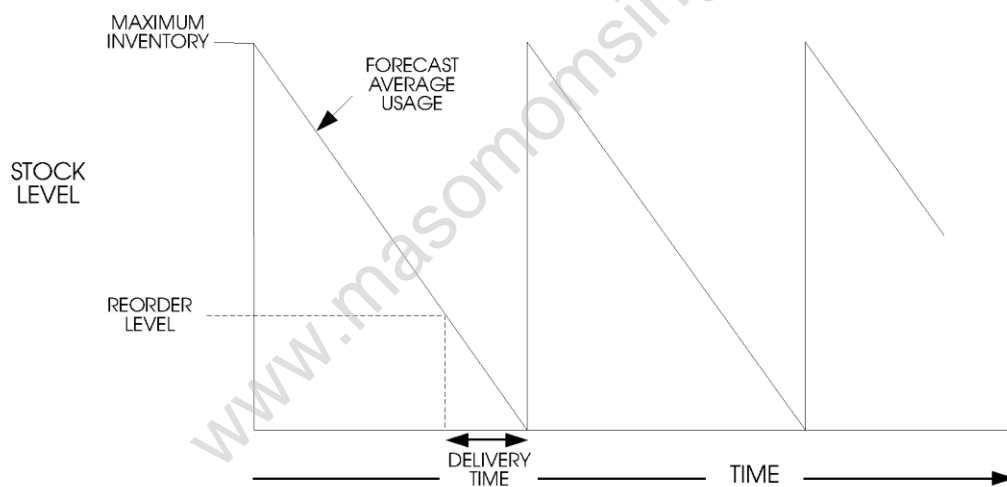


Figure 2(a): Stock levels when usage and replenishment times are known

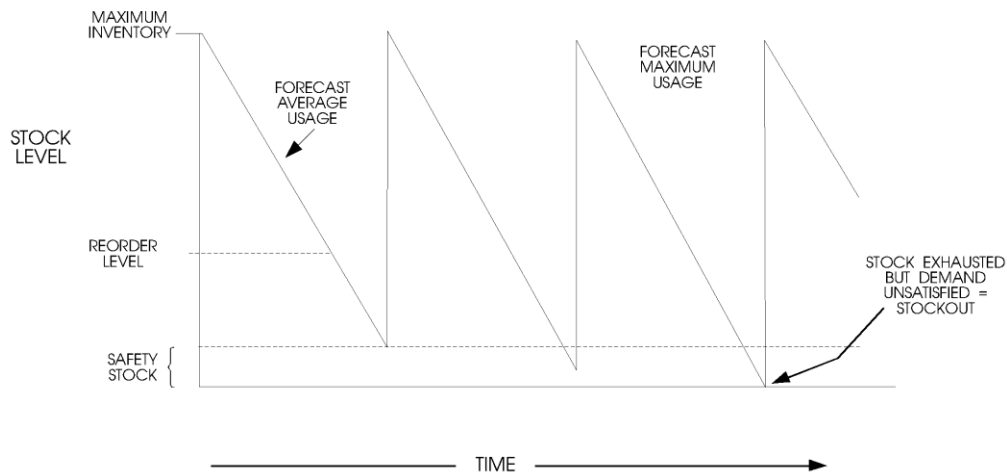


Figure 2(b): Stock levels when usage and replenishment are uncertain

Lead times and usage may not be stable and provision against running out of stock becomes necessary (*Figure 2 (b)*). *Safety stocks* have a cost, however, and this has to be balanced against the cost of running out of stock. 'Stock outs' may cause loss of customers and the probability of this happening at various levels of safety stock must be estimated. The point, at which the cost of carrying safety stocks plus the cost of 'stock outs' is lowest, indicates the safety stock level. Notice that uncertainty causes the reorder level to be at a higher level to include the required safety stock.

CONTROLLING MATERIAL FLOW

Figure 3 outlines the progressive stages in purchasing, issuing and recording materials in a manufacturing concern. An efficient system of documenting and recording is vitally necessary, not only for accounting purposes, but to ensure that the right materials arrive at the right place at the right time.

The *purchase requisition* submitted to the buyer may be triggered automatically if the system is computerised, by a message from the stores ledger that the reorder level has been reached. Other requests to purchase may be raised by the production planning department for new product materials not yet carried in stock, and also by any departmental head for supplies and equipment of any kind.

The buyer, ideally after making *enquiries* of several suppliers, sends a *purchase order*, and eventually the material is received, checked by the good-inwards department as to quality and quantity, and is detailed on a *goods-received note* (GRN). One copy of the GRN goes to the buyer to write off the outstanding order record; one to the accounts department for checking against the order and invoice—the latter authorizing payment to the supplier; and one to the stores department with the materials.

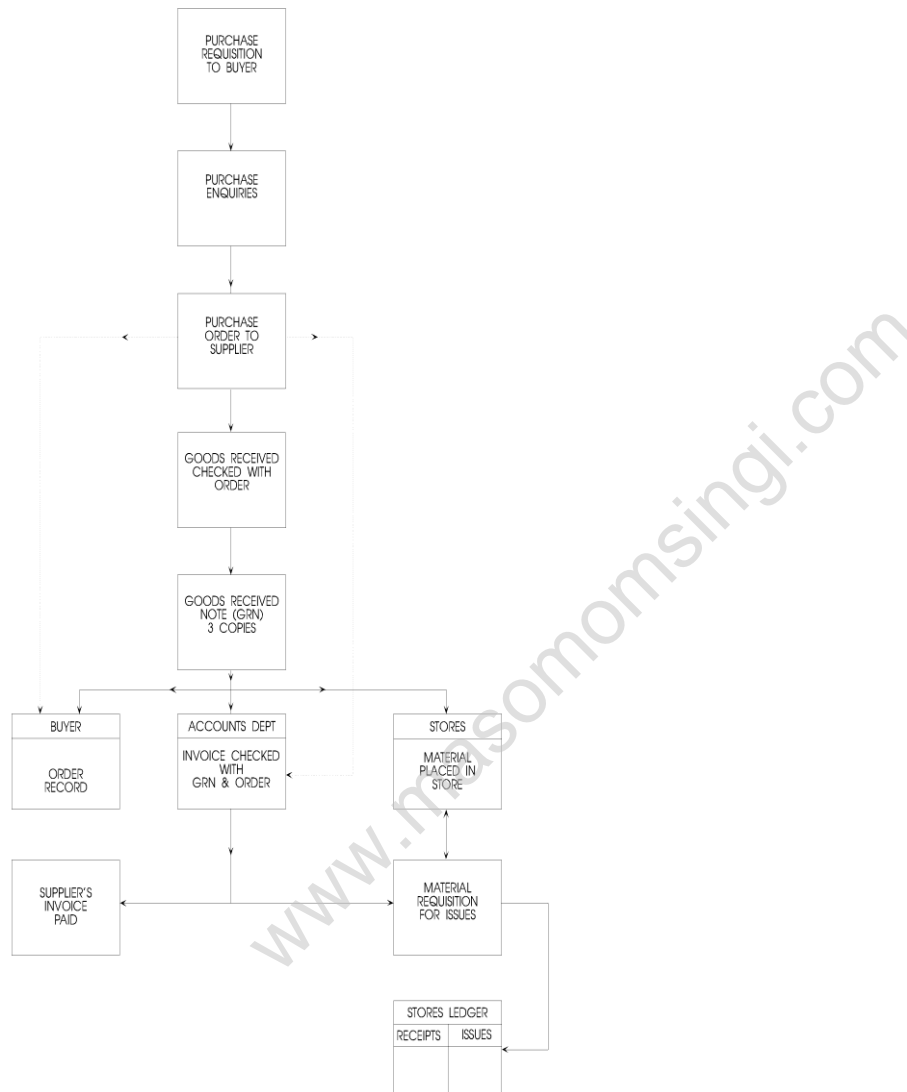


Figure 3: Procedure of material acquisition, recording and issue

A *stores record* is maintained into which the quantity and value of materials received is entered. Issues of materials to production are made by authorized materials requisitions which are also entered into the stores ledger to keep that record up to date continuously, and also into the appropriate job or process cost record.

As already indicated, all the above procedures may be integrated into a computerised stock record which can provide information at the press of a button to the storekeeper, buyer, production planner, financial manager or any

other person authorized to key into it. For example, information on slow-moving stock items can be obtained automatically and without delay.

MATERIAL STORAGE

Sophisticated mathematical models to control economic buying, and systems control the flow of material may all be for naught if the obvious—efficient storekeeping—ignored. Good practice in this respect implies:

- The employment of a well-trained stores staff
- Use of the most efficient equipment—for storage and handling
- Easy access to items—stored in logical order
- Siting of stores convenient to users
- Security against theft and fire
- Protection against deterioration
- A system of continuously checking physical with recorded stocks.

INVENTORY PLANNING AND CONTROL

The major goal of "inventory control" is to discover and maintain the optimum level of investment in all types of inventories, from raw materials and supplies to finished goods that helps to maximize long-run profits.

Two limits must be imposed in controlling inventory levels, because there are two danger points that management usually wants to avoid. They are:

- i. That inadequate inventories, disrupts production and may lose sales.
- ii. That excessive inventories, introduces unnecessary carrying costs and obsolescence risks.

MOTIVES FOR HOLDING INVENTORIES

If production and delivery of goods were instantaneous, there would be no need for inventories. However in reality, the manufacturing and purchasing processes do not function quickly enough to avoid the need for having inventories. Hence inventories must be maintained so that he or she does not turn to another source of supply. In turn, production operations cannot flow smoothly without having inventories of direct materials, work-in-progress, supplies, etc. In other words inventories must be viewed as cushions:

- i. To absorb planning errors and unforeseen fluctuations in supply and demand. (i.e. precautionary motive).
- ii. To facilitate smooth production and marketing operations.
Note: The fundamental questions to which answers are required are:
 - a. How much should the company order (when stocks are re-ordered)?
 - b. When should the company reorder?

- c. How much should the company produce, and when (if internal production is involved).

Accordingly regardless of their complexity, all inventory planning models focus on the twin problem of size and timing.

1. CHOOSING ORDER QUANTITY (SIZE–PROBLEM)

The objective of inventory decisions is usually to minimize total inventory costs to the company. Costs are ascribed to all elements which are of interest in reaching its inventory decisions (e.g. purchasing costs, stock out costs etc.), and solutions, are derived based on these costs.

Several inventory planning models exist. These models can be classified into two basic classified into two basic classifications:

- i. Deterministic Models:—whereby all parameters are known with certainty, e.g. lead-time, annual demand, etc.
- ii. Stochastic Models:—in which parameters (particularly demand and lead time) are not known with certainty, but follow known probability distributions (i.e. risks)

A. THE DETERMINISTIC MODELS

1. THE BASIC EOQ MODEL

This is the most simple of all the models discussed. In addition to the general assumptions which relate to all deterministic models (i.e. certainty of all parameters) it is further assumed that:

- a. Demand is continuous and constant over time.
- b. That suppliers lead time is zero i.e. stocks are delivered immediately on the day the order is made.
- c. That stock-outs are not allowed.
- d. There are *no* bulk quantity discounts.
- e. Holding costs per unit, ordering costs per order and costs per unit are constant.

Relevant costs of basic EOQ model

The relevant costs that should be considered when determining optimum inventory levels can be classified into two categories:

- i. Ordering costs.
 - ii. Holding (Carrying) Costs.
I.e. $TC = \text{Ordering Costs} + \text{Holding Costs}$
- i. Ordering Costs

These are incurred in getting purchased items into the company's inventory or stores, and usually consist of clerical costs of:

1. Making the purchase requisition.
2. Issuing of a purchase order
3. Follow-up action
4. Receiving the goods
5. Inspection for quality control
6. Placing goods in stores
7. Paying vendors (Suppliers)

Note:

The basic EOQ model assumes that these costs are fixed constant for each order made.

ii. Carrying costs of inventory

These are costs incurred because the firm has decided to maintain inventories. They usually consist of:

1. Stock-out costs
2. Insurance costs
3. Warehouse and storage costs
4. Material handling costs
5. Costs of obsolescence

$$\text{Total Ordering Cost} = \frac{\text{Total demand for period} \times \text{Ordering Costs per period}}{\text{Quantity Ordering}}$$

$$= \frac{DO}{Q}$$

$$\text{Total Holding Costs} = \frac{\text{Quantity Ordered} \times \text{Holding Costs per unit}}{2}$$

$$= \frac{Q H}{2}$$

Therefore total relevant costs (TC) for any order quantity can be expressed as:

$$TC = \frac{DO}{Q} + \frac{Q H}{2}$$

We can determine a minimum of this total cost function by:

- i. Differentiating the above formula with respect to Q and setting the derivative (1st) equal to zero.

$$\begin{aligned} \frac{dTC}{dQ} &= \frac{-DO}{Q^2} + \frac{H}{2} = 0 \\ \frac{H}{2} &= \frac{DO}{Q^2} \\ Q^2 &= \frac{2DO}{H} \end{aligned}$$

$$Q = \sqrt{\frac{2DO}{H}}$$

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ii. Equating ordering costs to holding costs.

$$\frac{DO}{Q} = \frac{QH}{2}$$

$$Q^2 = \frac{2DO}{H}$$

$$Q = \sqrt{\frac{2DO}{H}}$$

ILLUSTRATION

1. Assume a wholesaler has to supply his customer with 40,000 units of a given product every year.
2. Assume that demand is fixed and known
3. Assume the cost of placing each order is Shs 2.00 while the holding cost per unit is Shs 1.00.

Required:

- a. Determine the optimum order size using the basic EOQ model.
- b. Determine total costs incurred at optimum order size.

SOLUTION

a.
$$Q^2 = \sqrt{\frac{2 \times 40,000 \times 2}{1}}$$

$$Q^2 = \sqrt{160,000} = \sqrt{16 \times 10^4}$$

$$Q = \underline{400} \text{ units}$$

b.
$$TC = \frac{40,000}{400} (2) + \frac{400}{2} (1)$$

$$= 200 + 200$$

$$TC = \text{Shs } \underline{400}$$

1. EOQ Model with quantity discounts

Circumstances frequently occur where firms are able to obtain quantity discounts for large purchase orders. Buying in bulk has some advantages and disadvantages.

Advantages

- i. (A saving in purchase rule) Decreases in unit cost, which consists of the total amount of discount for the period.
- ii. A reduction in the total ordering costs because fewer orders are placed to take advantages of the discounts.

Disadvantages

Increased holding cost arising from higher stock levels when large quantities are purchased.

Such as:

- i. Stock out cost
- ii. Insurance.
- iii. Deterioration
- iv. Security etc.

Broadly there are two types of discount structures:

- i. Fixed discount on "all units" when the order placed is for a minimum quantity.
- ii. Variable discounts for given ranges.

1. The case of a fixed discount

When evaluating inventory decisions when a fixed discount rate exists, the appropriate procedure is to compare the total costs of the EOQ with the total costs when discounts are taken. The option giving lower costs is then chosen.

Note:

The Unit (variable) cost (i.e. Purchase Price) behaves in the following manner.

$$C = \begin{matrix} C_o & \text{if} & 0 \leq Q \leq Q_b \\ C_o (1 - P) & \text{if} & Q \geq Q_b \end{matrix}$$

Where C_o = basic unit cost without a discount
 P = Discount rate allowed.
 Q_b = Break-point (Quantity)—where discounts

become operational.

In order to determine the optimal ordering quantity, it is necessary to include the costs of the inventory with the carrying ordering costs.

$$\text{Total costs of Inventory cost} + \text{Total carrying cost} = \text{Total Purchase cost} + \text{Total order cost}$$

$$TC = DC_o + \frac{Q^* H}{2} + \frac{D_o}{Q^2} \quad \text{If } 0 \leq Q \leq Q_b \quad (i)$$

$$TC = DC \cdot (1 - P) + \frac{Q H}{2} + \frac{D_o}{Q} \quad \text{If } Q \geq Q_b \quad (ii)$$

Note:

Equation (ii) i.e. with discounts will give a lower TC than equation (i) for the same. The decision whether to go for the discount lies on a trade-off between extra carrying costs vs. a reduction in acquisition costs.

ILLUSTRATION

1. Assume X Ltd purchases a raw material from an outside supplier at cost of Shs 70 per unit.
2. Assume total annual demand for the product is 9,000 units.
3. Assume the holding cost is Shs 40 per unit and the ordering cost is Shs 50 per order.
4. Assume a quantity discount of 3% of the purchase price is available for orders in excess of 1,000 units.

Required:

- a. Calculate the EOQ and the associated costs.
- b. Calculate the total costs if the company purchased in batches of 1,000 units (N.B. It is not wise to buy in batches with more than 1,000 units because of the increase rate in carrying exceed the rate at which ordering costs decline).
- c. Advise the management on the appropriate inventory policy.

SOLUTION

$$a. \quad i. \quad EOQ = \sqrt{\frac{2 \times 9,000 \times 50}{40}}$$

$$= \sqrt{22,500} = \sqrt{225 \times 10^2}$$

$$= \underline{150} \text{ units}$$

$$ii. \quad TC = \frac{9,000}{150} (50) + \frac{150}{2} (40) + 9,000 (70)$$

$$= 3,000 + 3,000 + 630,000$$

$$= \text{Shs } \underline{636,000}$$

$$\begin{aligned}
 \text{c. } Q &= 1,000 \\
 \text{TC} &= \frac{9,000}{1,000} (50) + \frac{1,000}{2} (40) + 9,000 (70)(1 - 0.03) \\
 &= 450 + 20,000 + 611,100 \\
 &= \text{Shs } \underline{631,500}
 \end{aligned}$$

$$\text{Note: Discount price} = 70 (1 - 0.03) = \text{Shs } \underline{67.90}$$

$$\begin{aligned}
 \text{d. Decline in inventory costs} &= 631,550 - 636,000 \\
 &= \text{Shs } \underline{4,450}
 \end{aligned}$$

DECISION

The firm saves Shs 4,450 by taking the quantity discount.

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THE CASE OF VARIABLE QUANTITY DISCOUNTS

In practice, suppliers may offer different discounts for different quantities purchased. For example:

<i>SEGMENT</i>	<i>QUANTITY PURCHASED</i>	<i>UNIT PRICE</i>
1	0 – 500	Shs 100
2	501 – 1,000	Shs 90
3	1001 – 1,500	Shs 80
4	over – 1,500	Shs 70

The best approach to the solution in this case is to apply the price-breaks theorem. This works as follows:

1. For each segment an EOQ is calculated. There are two possible requests:
 - i. The EOQ is within the quantity segment (i.e. valid)
In this case, the EOQ is used as the minimum cost quantity for that segment.
 - ii. The EOQ is outside the quantity segment (i.e. invalid)
In this case the minimum cost quantity will be the quantity within the segment closest to the EOQ as calculated.
2. Select the quantity that leads to the lowest total inventory costs (i.e. Purchase, Ordering & Carrying).

Illustration:

1. Assume a manufacturer uses 3,300 drums of a certain chemical per year.
2. Assume delivery costs incurred per order are Shs 40 and inventory carrying costs are estimated to be 30% of stock value.
3. Assume the normal cost per drum is Shs 22 but the supplier offers discount of 1.5% on orders for 500 drums or more, and 3% on orders for 1,000 drums or more.

Required:

Determine the order quantity the manufacturer should adopt to minimize total costs.

Solution:

Note:

There are 3 discount levels (0, 1.5% & 3%) and hence 3 segments.

STEPS

1. Calculating the EOQ for each segment.

- a. For Segment 1 (0 - 499 drums)
C = Shs 22

$$EOQ = \frac{200}{7} \text{ drums}$$

$$EOQ = \sqrt{\frac{2 \times 3,300 \times 40}{0.3(22)}}$$

$$= \sqrt{\frac{264,000}{6.6}} = \sqrt{40,000}$$

$$= \underline{200}$$

- b. For Segment 2 (500 - 999 drums)

$$C = 22(1 - 0.015)$$

$$C = 22(0.985)$$

$$C = \text{Shs } \underline{21.67}$$

$$EOQ = \sqrt{\frac{2 \times 3,000 \times 40}{0.3(21.67)}}$$

$$= \sqrt{\frac{264,000}{6.50}} = \sqrt{40,615}$$

$$= \underline{201.5} \text{ drums (can round to 202 drums)}$$

- c. For Segment 3 (1,000 or more)

$$C = 22(1 - 0.03)$$

$$C = \text{Shs } \underline{21.34}$$

$$EOQ = \sqrt{\frac{264,000}{0.3(21.34)}}$$

$$= \sqrt{\frac{264,000}{6.40}} = \sqrt{41,237}$$

= 203 drums

Segment	Price Break	Unit Quantity	Price	Valid EOQ	Quantity
1	0 - 499	Shs 22	200	200	(Valid)
2	500 - 999	Shs 21.67	202	500	(Invalid)
3	over 1000	Shs 21.34	203	1,000	(Invalid)

2. Calculate total costs

SEGMENT	UNITS ORDERED	TOTAL COSTS
1	200	$(3,300 \times 22) + (200 \div 2 \times 6.6) + (3,300 \div 200 \times 40) =$ Shs 73.920
2	500	$(3,300 \times 21.67) + (500 \div 2 \times 6.5) + (3,300 \div 500 \times 40)$ = Shs 73.400
3	1,000	$(3,300 \times 21.34) + (1,000 \div 2 \times 6.4) + (3,300 \div 1000 \times 40)$ = Shs 73.754

Decision:

The firm should order 500 drums per order per annum and incur Shs 73.400 total costs.

NON-ZERO LEAD TIME (DETERMINING REORDER POINT)

This basic EOQ model assumes that the suppliers lead time is zero (i.e. goods are delivered immediately on the day the order was made). In reality, however, supplies are rarely ordered and received on the same day. Accordingly, orders must be placed some time before stocks reach zero. In world of certainty (when demand is continuous and constant) the reorder point will be the number of days/weeks lead time multiplied by the daily/weekly usage during the period.

i.e. Reorder point = Average daily usage x Lead time in days.



Note:

The reorder point has no cost implications, since it does not affect the EOQ.

Illustration:

1. Assume X Ltd uses 50,000 kg of a raw material annually.
2. Assume ordering costs are Shs 160 per order and stock holding costs are Shs 0.25 per kg per annum.
3. Assume the purchase price is Shs 20 per kg and no quantity discounts are offered.
4. Assume lead time for delivery of orders is 4 weeks.
5. Assume working time is 50 weeks a year.

Required:

- a. Calculate the EOQ
- b. Calculate the inventory reorder level.
- c. Calculate the total costs per annum.

SOLUTION

$$\begin{aligned}
 \text{a. } EOQ &= \sqrt{\frac{2 \times 50,000 \times 160}{0.25}} \\
 &= \sqrt{\frac{16,000,000}{0.25}} = \sqrt{64,000,000}
 \end{aligned}$$

$$EOQ = \underline{8000} \text{ kgs}$$

Make 6.25 orders (50,000÷8,000) per annum.

b. Reorder level = Demand per week x Lead time.

$$= \left(\frac{50,000}{50} \right) \times 4$$

$$= 1,000 \times 4$$

$$= \underline{4,000} \text{ units.}$$

c. Total costs = (50,000 x 20) + (8,000÷2 x 0.25) + 50,000÷8,000 (160)
= 1,000,000 + 1,000 + 1,000
= Shs 1,002,000

INVENTORY PLANNING & CONTROL UNDER UNCERTAINTY

The basic EOQ model assumes that all the parameters (elements) in the model are certain (i.e. can be predicted accurately in advance). These parameters are:

- i. Demand or usage of stocks
- ii. Lead times.
- iii. Holding costs per unit, ordering costs per order and costs per unit.

In reality however, stock demand, supplies lead times and cost data are not known with certainty. Accordingly to make the models applicable to real situations we must consider uncertainty when planning for inventory levels.

To protect itself from conditions of uncertainty, a firm will maintain a level of safety stocks for raw materials, work-in-progress and finished goods stocks. Thus safety stocks are the amount of stocks that are carried in excess of the expected use during the lead time to provide a cushion against running out of stocks. Thus the reorder point is computed as safety stock plus the average usage during the lead time

i.e. reorder point = Average usage during lead time + safety (buffer) stock.

DETERMINING THE SAFETY STOCKS LEVEL

1. Uncertainty of demand

Demand is the most troublesome variable to predict accurately. Actually, demand may fluctuate from day to day, from week to week or from

month to month. Thus, the firm takes the risk of running out of stocks if there are sudden increases in demand. Hence safety stock is the extra inventory held as a buffer of protection against the possibility of stock due to higher demand.

However, a larger inventory of safety stock will involve a higher inventory carrying costs, and on the other hand, the higher safety stock will decrease stock-out costs. Therefore one has to make a balance between these two costs in order to find out an optimal safety costs.

Note:

The optimum safety-stock level exists where the costs of carrying an extra unit are exactly counter balanced by the expected stock-out costs. This would be the level that minimizes the annual total stock-out and carrying costs.

Stock-out costs

These are the opportunity costs of running out of stock. They include:

- i. The costs of lost customer sales, and therefore lost contribution to fixed costs.
- ii. Potential loss of goodwill with customers whose demand cannot be met.
- iii. Acquiring emergency supplies at higher prices to meet demand.
- iv. Cost production of finished goods, where raw material stock-outs occur.

The computation of safety stocks lingers on demand forecasts. The manager will have some notion (usually based on past experience) of the range of daily demand. That is the probability that exists for usage of various quantities.

Hence total inventory costs will be as follows:

Total inventory costs = Purchase price cost + carrying costs + stock-out cost + order costs.

= Purchase price costs + "normal~ carrying costs (Q÷2 H) + Buffer Stock holding costs (B x H) + Stock-out costs + order costs.

Total inventory costs = D·C + Q÷2·H + (B x H) + stock-out costs + D÷Q · Co

Where: D = Total annual demand
H = Holding costs per unit
B = Buffer stock
Q = EOQ

Note:

1. The normal EOQ formula is used to compute order quantity Q. Hence purchase costs, carrying costs, and ordering costs remain unchanged.

Only buffer stock holding costs and stock-out costs change. Accordingly, the minimization of these two costs will also mean that total/overall costs will be minimized.

2. Stock-out costs = Number of units short x Probability of being short.

= {(Number of units short x Stock-out costs per unit Probability of being short)–Stock-out costs for every usage duration} x Number of orders per year.

3. Buffer Stock-holding costs = Bx H

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ILLUSTRATION

A company has an annual demand for material X of 250 tonnes per annum. Order lead time is 4 days and usage during lead time as shown by past record is

<i>USAGE</i>	<i>PROBABILITY</i>
0	0
1	0.01
2	0.05
3	0.15
4	0.25
5	0.30
6	0.10
7	0.09
8	0.05

The cost per tonne is £20 and stock holding cost is 25% per annum of the stock value. Delivery cost per batch is £4. The costs of stock out also estimated to be £4.

You are required to:

- Calculate the economic batch quantity and the expected number of orders per annum
- Ascertain the re-order level taking the information given above into consideration.

Solution:

- EBQ or EOQ where stock-outs are permitted

$$EBQ \text{ or } EOQ = \sqrt{\frac{2DK}{h}} \sqrt{\frac{h + C_s}{C_s}}$$

Where D is annual demand
K is order cost

h is holding cost

and C_s is stock out cost

$$EBQ = \sqrt{\frac{2 \times 250 \times 4}{5}} \sqrt{\frac{5 + 4}{4}}$$

$$= \sqrt{400} \sqrt{\frac{9}{4}}$$

$$= 30$$

$$\text{No. of orders} = \frac{250}{30}$$

(Note: Stock holding cost $b = 25\%$ of $\text{£}20 = \text{£}5$)

We are required to find out expected demand during lead time.

Usage x	Prob $p(x)$	$x p(x)$
0	0	0.00
1	0.01	0.01
2	0.05	0.10
3	0.15	0.45
4	0.25	1.00
5	0.30	1.50
6	0.10	0.60
7	0.09	0.63
8	0.05	0.40
	$\Sigma xp(x)$	4.69

Hence normal usage during lead time is 4.69 tonnes but maximum usage during lead time can be as large as 8 tonnes with a probability of 0.05.

Average usage in four days = 4.69 tonnes and at this level there is no buffer stock. The problem is how much buffer stock we should have, so that the cost of holding stock together with the cost of expected stock-out cost is minimum.

$$\begin{aligned} \text{Let } \mu_1 &= 4.69 \text{ tonnes.} \\ B &= \text{Buffer stock} \\ S &= \text{Re-order level (for lead time)} = \mu_1 + B \end{aligned}$$

$$\begin{aligned} \text{Hence } B &= S - \mu_1 \\ \text{Here } B &= S - 4.69 \end{aligned}$$

1	2	3	4	5	6	7
Re-order Levels S	Buffer Stock B	Expected Stock-out per order	Expected Annual Shortage	Expected stock-out cost (£)	Holding Cost (£)	Total cost (£)
4.69	0	0.5974	4.978	19.912	0	19.912
5	0.31	0.43	3.58	14.32	1.55	15.87
6	1.31	0.19	1.583	6.332	6.55	12.88 ← minimum cost
7	2.31	0.05	0.4166	1.664	11.55	13.214
8	3.31	0	0	0	16.55	16.55
			Working column 3 x 250 30	Column 4 x 4	Column 2 x 5 i.e. B x h	

Hence for minimum cost $S = 6$
 Buffer stock $S = 6 - 4.69$
 $= 1.31$ tones
 and minimum cost $= £12.88$

WORKINGS AND EXPLANATIONS

1. When Re-order Level $S = 4.69$
 $B = 0$

Hence possible shortages will occur when demand is 5, 6, 7 or 8 units.

Demand	Shortage x	Prob p(x)	Expected Value xp(x)
5	5 - 4.69	0.30	0.31 x 0.30 = 0.93
6	6 - 4.69	0.10	1.31 x 0.10 = 0.131
7	7 - 4.69	0.09	2.31 x 0.09 = 0.2079
8	8 - 4.69	0.05	3.31 x 0.05 = 0.1655
		$\Sigma xp(x)$	0.5974

This is expected stock out per order when demand is 5, 6, 7 or 8

2. Similarly if Re-order level is 5
 Shortages will occur if demand is 6, 7, 8

Working in the way as above

Demand	Shortage x	Prob p(x)	Expected Stock- out cost
6	1	0.10	0.10
7	2	0.09	0.18
8	3	0.05	0.15
		$\Sigma xp(x)$	0.43

3. If re-order level is 6, shortage will occur if demand is 7 or 8

Demand	Shortage x	Prob p(x)	Expected Stock- out cost
7	1	0.09	0.09
8	2	0.05	0.10
		$\Sigma xp(x)$	0.19

4. If re-order level is 7, shortage if demand is 8
 Expected shortage cost = $(8 - 7) \times 0.05 = 0.05$

5. If re-order level is 8, there is no probability of shortage

- a. Expected Annual Shortage = (Expected Stock-out per order) x
 No. of orders

order) x 250 i.e (Expected stock-out per

3
 3

- b. Expected stock-out cost = (Expected annual shortage) x
 Shortage cost per item

- c. Holding cost = Buffer stock x cost of holding per item

Important Note:

If shortage C_s is not taken into account

$$EOQ = \sqrt{\frac{2 \times 250 \times 4}{5}}$$

$$= \sqrt{400} = 20$$

$$\text{and number of orders} = \frac{250}{20} = 12.5$$

Using this model, we can still develop a table similar to the table worked out in this problem.

Graph of total cost against re-order level can be drawn using the table. The minimum cost and the re-order level can then read from the graph. Once re-order level is known, safety stock can be worked out.

In the next example, the initial EOQ value has been taken using the formula

$$EOQ = \sqrt{\frac{2DK}{h}}$$

ILLUSTRATION

1. Assume a manufacturer has experienced trouble from stock shortages (stock-outs) of raw materials X which is required in a manufacturing process. Usage of X averages 6,000 units per year where a year consists of 50 weeks.
2. Assume the costs of ordering each batch of X is Shs 30 and the lead time is 2 weeks (known for certain). The annual holding costs amount to Shs 1 per unit of X held. The cost of a stock-out has been estimated to be Shs 5 per unit short.
3. Assume the demand (usage) is unknown. However, the total usage of raw material X over the 2 week lead time is expected to be as follows:

Usage in units	Probability
60	0.07
120	0.08
180	0.20
240	0.30
300	0.20
360	0.08
420	0.07

Required:

- a. Calculate the EOQ, and the expected number of orders per annum
- b. Calculate the average usage per 2 weeks (Lead-time)

SOLUTION

$$\text{Annual demand} = 6,000 \text{ units}$$

Order cost = Shs 30
 Holding cost = Shs 1
 Shortage cost = Shs 5 per unit short

$$EOQ = \sqrt{\frac{2 \times 6,000 \times 30}{1}} = 600 \text{ units}$$

$$\text{Number orders} = \frac{6,000}{600} = 10$$

$$\begin{aligned} \text{Average usage for two weeks} &= 60 \times 0.07 + 120 \times 0.08 + 180 \times 0.20 + 240 \\ &\times 0.30 + 300 \times 0.20 + 360 \times 0.08 + 420 \times 0.07 \\ &= 240 \text{ units} \end{aligned}$$

Re-order Levels	Buffer Stock	Expected Stock-out per order	Expected Annual Shortage	Expected stock-out cost	Holding Cost at Shs 1	Total cost
240	0	34.2	342	1,710	0	1,710
300	60	13.2	132	660	60	720
360	120	4.2	42	210	120	330
420	180	0	0	0	180	180

i. Expected Stock out per order where re-order level is 240

$$(300 - 240) \times 0.2 + (360 - 240) \times 0.08 + (420 - 240) \times 0.07 = 34.2$$

ii. similarly for re-order level = 300

$$(360 - 300) \times 0.08 + (420 - 300) \times 0.07 = 13.2$$

iii. and for re-order level = 360

$$(420 - 360) \times 0.07 = 4.2$$

iv. No shortage when re-order level is 420

Hence total cost is minimum when re-order level is 420 units.

Best policy is to have safety stock of 180 units.

SENSITIVITY ANALYSIS OF EOQ MODEL

Sensitivity Analysis is concerned with the way in which those results of solutions change in response to change in model parameters.

$$EOQ = \sqrt{\frac{2DO}{H}}$$

Note:

It is important to appreciate that in formulating our inventory models, we have really been performing a planning exercise. Thus we have made certain assumptions and estimates (e.g. annual demand D, holding costs H, & ordering costs O) and our solutions have obviously been affected by these. When for example calculating a deterministic EOQ value with maximization of total inventory costs as the objective, expected annual demand (D) is taken into account. If we subsequently find that annual demand has differed from that expected then we will find that the EOQ we selected was not the optimum and, as a result, the total inventory cost was not actually minimum.

Illustration:

1. Assume X Ltd expected annual demand for 1991 for 62,500 units of raw material X per annum.
2. Assume holding costs are Shs 15 per unit per annum and each order costs Shs 10.
3. Assume that the end of the year actual demand has been found to have been for 90,000 units, not the 62,500 expected.

Required:

Calculate the additional costs borne by X Ltd through basing the size and frequency of orders on expected figures.

Solution:

- a. Calculating EOQ based on expected figures, and the associated inventory costs.

$$EOQ = \sqrt{\frac{2 \times 62,500 \times 10}{5}}$$

$$EOQ = \sqrt{\frac{1,250,000}{5}} = \sqrt{250,000}$$

$$= \underline{500} \text{ units}$$

$$\text{Total inventory costs} = \frac{62,500}{500} (10) + \frac{500}{2} (5)$$

$$= 1,250 + 1,250$$

$$= \text{Shs } \underline{2,500}$$

b. Calculating the total inventory costs based on actual demand.

$$TC = \frac{90,000 (10)}{500} + \frac{500 (5)}{2}$$

$$= 1,800 + 1,250$$

$$= \text{Shs } \underline{3,050}$$

c. Calculating the total inventory costs had the company forecasted the actual demand accurately.

$$EOQ = \sqrt{\frac{2 \times 90,000 \times 10}{5}}$$

$$EOQ = \sqrt{\frac{1,800,000}{5}} = \sqrt{360,000}$$

$$= \underline{600} \text{ units}$$

$$TC = \frac{90,000 (10)}{500} + \frac{600 (5)}{2}$$

$$= 1,500 + 1,500$$

$$= \text{Shs } \underline{3,000}$$

Therefore, X Ltd total inventory costs are actually Shs 50 (3,050 - 3,000) higher than they would have been if the EOQ had been set in accordance with perfect information.

Observation:

Hence the change in demand by 44% $\frac{(99,000 - 62,500)}{(600 - 500)}$ has result in a 20% $\frac{62,000}{500}$

Increase in EOQ but only 91.64% $\frac{(3,050 - 3,000)}{3,000}$ increase in total costs.

Hence we can conclude that total cost is relatively insensitive to changes in demand.

SELECTIVE INVENTORY MANAGEMENT

The inventory of an industrial firm generally comprises thousands of items with diverse prices, usage and lead time, as well as procurement and/or technical problems. It is neither desirable nor possible to exercise the same degree of control over all those items. The organisation should pay more attention and care to those items whose usage value is high and less attention to those whose usage and consumption value is low. The organisation has, therefore, to be selective in its approach to control its investment in various types of stocks and inventories. Such a system is known as '*selective inventory control*' system.

ABC ANALYSIS (PARETO ANALYSIS)

In ordinary parlance, *ABC analysis* can be best compared with our class society where the population is categorised into Top, Middle and Lower classes. In the case of inventories also, it has been noticed that out of a large number of items (in a million-tonne capacity steel plant there would be usually about 50,000 items of inventory of various types) that are generally held in stock, some of the items are quite significant whereas the others are not that important. Through *ABC* plan which is in fact an analytical approach based on common statistical techniques, the relative importance of the various items is established for the purpose of individual scrutiny and subsequent control. Through this technique '*VIP*' or the '*privileged few*' and the '*trivial many*' are distinguished and treated as such.

ABC analysis contemplates to classify all the inventory items in a number of categories, generally in three categories based on their values. Items of high value but small in number are classified as '*A*' items which would be under a strict control. '*C*' items represent relatively small value items and would be under simple control. Items of moderate value and size are classified as '*B*' items and would attract reasonable attention of the management. Since this plan concentrates attention on the basis of the relative importance of the various items of inventory, it is also known as '*control by importance and exception*'. As items are classified in order of their relative importance in terms of value, it is also known as the '*proportional value analysis*'.

It has been found that normal inventory items in most organisations show the following distribution pattern:

- A – 5 to 10% of total number of items account for about 70% of the total consumption value. These items may be called "A" items.
- B – 10 to 20% account for 20% of total consumption value.
- C – the remaining large number of items account for the balance of 15% of the consumption value.

Remark When a detailed scrutiny was conducted in respect of inventories held by the Ford Motor Company, the following results were obtained.

1. 9% of the total items (in number) were accounting for 57% of the total value of the inventory. These were classed as `A' items.
2. 10% of the items (in number) were found to be accounting for 18% of the total value. These were categorized as `B' items.
3. 81% of the items (in number) were found to be accounting for only 25% of the total value. These were classified as `C' items.

Similarly, when ABC ANALYSIS was done in the case of G.E.C., the results obtained were as under:

`A' Category—Items accounting for 8% of the total number but 75% of total value.

`B' Category—Items accounting for 25% of the total number and 20% of total value.

`C' Category—Items accounting for 67% of the total number and only 5% of the total value.

It would thus be observed that substantial and effective controls are made possible if greater attention is focused on `A' category items since these would be covering quite a substantial part of the inventory in terms of shilling value.

Some Remarks:

1. There is no hard and fast rule that all the inventory items should be classified only in these three categories. There can be a large number of classifications based on the requirements of the company and the nature of the items. For example `A' items may be further sub-classified as A₁, A₂, A₃, etc. The same principle may be extended to `B' items also or alternatively all the inventory items may be classified into A, B, C, D, E, F, etc.
2. All items that the company consumes must be considered together while classifying into ABC classes. Separate classification of raw materials, spares and consumable is not really meaningful.

While classifying as ABC items, what counts is the consumption shillings and not the unit price of an item or its consumptions in terms of units. Thus of the three items given below, the last one is most important since its annual consumption in terms of value is more than the other two.

<i>Item</i>	<i>Price/U nit</i>	<i>Annual Consumption</i>	<i>Annual Consumption</i>
-------------	------------------------	-------------------------------	-------------------------------

	<i>Shs</i>	<i>in units</i>	<i>in Shs</i>
X	20,000	2	40,000
Y	0.02	100,000,000	200,000
Z	1,000	500	500,000

4. Even though, so far we have referred to annual consumption, it is not at all necessary that the consumption figures should be taken only for one year. It can be for 6 months or even 3 months. But the period should be so selected that the consumption figures would be representative. However, annual figures are far more convenient and are universally followed.
5. If a firm follows ABC analysis, it will devote much time and effort on the control of 'A' items. For example, extra care will be taken in the determination of minimum, maximum, reorder level, etc. of the 'A' items, whereas so much control may not be exercised on 'C' items. 'A' items may be purchased only once in a year. For 'A' items perpetual inventory system may be applied whereas in the case of 'C' items, only a bin card may be maintained. In the same way an appropriate accounting method for 'B' items may be devised. However, in the classification of items into ABC categories if there are some critical items which are of small value whose non-availability may hamper the production, may in the normal situation, be classified as 'C' items but, due to the critical nature of these items extra care may be taken so that these may not go out of stock.
6. The objective of classifying inventory items into 'A', 'B' and 'C' categories is to develop policy guidelines for selective control. Such a policy can be designed in a variety of ways. In general 'A' items merit a tightly controlled inventory system with periodic attention, and 'C' items to subject to loose control with casual attention.

7. TABLE SHOWING FEATURES OF ABC ANALYSIS

Nature	A items (High value)	B items (Moderate value)	C items (Low value)
1. Extent of control	Rigid control (close day to day control)	Moderate control (Regular review)	Loose control (infrequent review)
2. Safety stock coverage	Low safety stocks	Medium safety stock	Large safety stocks
3. Frequency of order	Frequently	Less frequently	Bulk ordering
4. Degree of posting	Individual posting	Small group postings	Group postings
5. Period of	Every fortnight	Quarterly	Yearly

Nature	A items (High value)	B items (Moderate value)	C items (Low value)
review			
6. Sources of supplies	Good number of sources	Few reliable sources	One or two sources
7. Follow up	Vigorous	Periodic	Occasional
8. Control statements	Weekly control statements	Monthly control reports	Quarterly control reports
9. Forecasting	Emphasis on accurate forecast	Focus on past trend	Rough estimate
10. Level of management	Senior management	Middle management	Stores supervisor
11. Lead time	Maximum efforts to reduce lead time	Moderate efforts	Minimum clerical efforts
12. Value % and item percentage (Approximation)	80% of the value in 20% of the items	15% of the value in 30% of the items	5% of the value in 50% of the items

ADVANTAGES OF "ABC ANALYSIS"

The benefits derived from this analysis and its subsequent follow up are summarized below:

- Facilities selective control and thereby save valuable time of busy executives.
- Eliminates lot of unnecessary paper work involved in various other control procedures. Tangible savings can be affected in this behalf by following Two-Bin System which is very closely related with this technique.
- Facilitates Inventory Control and control over usage of stores materials which ultimately results in cost control.

LIMITATIONS OF ABC ANALYSIS

Although ABC analysis is a fundamental tool for exercising selective control over numerous inventory items, it does not, in its present form, permit precise consideration of all relevant problems of inventory management. For instance, a never-ending problem in inventory management is that of adequately handling thousands of low-value 'C' items. Low-value purchases frequently require more items and thereby reduce the time allowance available to purchasing personnel for value analysis, vendor investigation, and other 'B' items.

Besides, if ABC analysis is not periodically reviewed and updated, the very approach of control may be defeated. For example, 'C' items like diesel oil in a firm, will become most high-value items during power crisis should, therefore,

deserve more attention, but this point may be overlooked if classification of items is not reviewed and updated.

The following steps are involved in implementing the ABC analysis:

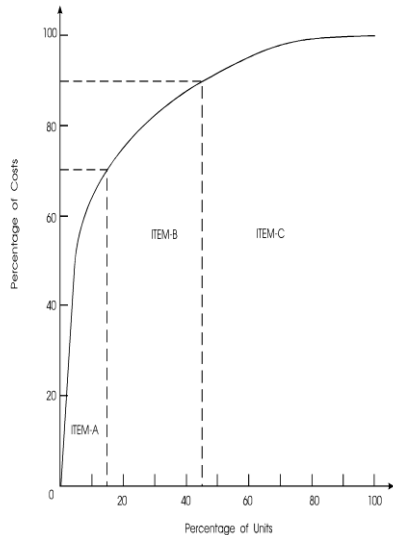
1. Classify the items of inventories, determining the expected use in units and price per unit for each item.
2. Determine the total value of each item by multiplying the expected units by its unit price.
3. Rank the item in accordance with the total value, giving first rank to the item with highest total value and so on.
4. Compute the ratios (percentage) of number of units of each item to total units of all items and the ratio of total value of each item total value of all items.

Combine items on the basis of their relative value to form three categories—A, B and C.

The data in the table below illustrates the ABC analysis.

TABLE: ABC ANALYSIS									
Item	Units	% of total		Cumulati ve	Unit price	Total cost	% of total		Cumulativ e
1	10,000	10	}		30.4	304,000	38.0	}	
				15					70
2	5,000	5			51.20	256,000	32.0		
3	16,000	16	}		5.50	88,000	11.0	}	
				45					90
4	14,000	14			5.14	72,000	9.00		
5	30,000	30	}		1.70	51,000	6.38	}	
				100					100
6	15,000	15			1.50	22,500	2.81		
7	10,000	10			0.65	6,500	0.81		
TOTAL	100,000					800,000			

The tabular and graphic representation indicates that "Item A" forms a minimum proportion, 15 per cent of total units of all items, but represents the higher value, 70 per cent. On the other hand, "Item C" represents 55 per cent of the total units and only 10 per cent of the total value. "Item B" occupies the middle place. Items A and B jointly represent 45 per cent of the total units and 90 per cent of the investment. More than half of the total units are item C, representing merely 10 per cent of the investment. Thus, a tighter control should be exercised on "Item A" in order to maximize profitability on its investment. In case of "Item C" simple controls will be sufficient.



JUST-IN TIME (JIT) INVENTORY MANAGEMENT

JIT is a system whose objective is to produce or to purchase products or components as they are required by customers or for use rather than for stock. A JIT system is a pull system which responds to demand as opposed to a push system in which stocks acts as buyers* between the different element of the system such as purchasing, production and sales

JIT PRODUCTION

Is a production system, which is driven by demand for the finished products whereby each component on the production line is produced only when needed for the next stage.

JIT PURCHASING

_On the other hand is a purchasing system in which material purchased are contracted so as that the receipt and usage of materials to the maximum extent possible, coincide.

JIT concept can be traced back to the Japanese company whose success in the international market generates interest among many western companies as to how this success was achieved.

The implementation of JIT production methods was considered to be pursuit of excellence in all phase of manufacturing systems design and operations.

The JIT are to produce the required items at the required quality and in the required quantities, at the precise time that they are required.

JIT seeks to achieve the following goals-

- (1). Elimination of non-value adding activities.
- (2). Zero inventory
- (3). Zero defects
- (4). Batch size of one.
- (5). Zero break-downs
- (6). 100% on time delivery services.

The above goals represent perfection and are most unlikely to be archived in practice. They do however offer targets and create a climate for continuous improvement and excellence.

Major features of JIT.

(1). Elimination of non-value added activity

JIT manufacturing can be described as a philosophy of management, dedicate to the elimination of waste. Waste is defined as anything that does not add value to a product.

The cycle time involved in manufacturing and selling a product consist of -

- Process time-add values
- Inspection time
- More time
- Queue time
- Storage time

Of these 5 steps only process time actually adds value to the products. All the other activities add cost and No value to the production and therefore are deemed as non-value within the JIT philosophy.

Usually in many companies, process time is less than 10% of total manufacturing, lead and cycle time. Therefore 90% of the manufacturing lead time disassociated with the product, adds cost but no-value to the product by adapting a JIT philosophy and forecasting, on reducing lead time, it is claimed that total cost can be significantly reduced.

The ultimate products with lead-time = processing time, and eliminating all non-value adding activities.

(2). Factory Layout

The first stage of implementing the JIT manufacturing techniques is to rearrange the factory floor away from the batch production functional layout towards a production layout using low lines with a functional plant layout production through a No. of special departments that normally contain a group of similar machines.

Products are processed in stage batches so as to minimize the set times when machine settings are changed between processing batches of different products. Batches move via different and complex routes through the various departments, traveling over much of the factory floor before they are complete.

Each process normally involves a considerable amount of waiting time and which much time is taken transporting items from one process to another.

A further problem is that it is not easy at any point in time to determine what progress has been made on individual batches and therefore detailed cost accumulation records are necessary to track work in progress. This results in long manufacturing cycles and high work in progress levels.

The JIT solution is to reorganize the production process by dividing the many different products that an organization makes into families of similar products or component. All the products in a particular group will have similar production requirements and routing. Production is necessary used* so that each production family is manufactured in a well-defined production cell based on flow line principles. In a production flow lines specialist department containing similar markets no longer exist. Instead groups of dissimilar markets are organized into products or component family flow lines that function like an assembly line.

For each production lines the market are placed close together in the order in which they are required by the group of products to be processed. Items in each product family can now move one at a time from process to process more easily, thereby reducing wip, and lead time.

The aim is to produce products or component from start to finish without returning to the stock room or stores

(3). Batch size of one.

Set up time is the amount of time required to adjust equipment and to retool for a different product. Long set ups a change over time make the production of batches with a small no. of units uneconomical.

However, the creation of large batches lead to substantial lead time delays and the creation of high inventory levels. The JIT philosophy is to reduce and eventually eliminate set-up times. E.g. by investing in advanced manufacturing technology some machines setting can be adjusted automatically instead of manually.

Alternatively some set up times can be eliminated entirely by reducing products, so that markets do not have to be reset each time a different product has to be made.

If the set up times are approaching zero, then there's no advantage production in batches and therefore the optimal batch size can be one. With a batch size of one, the work can flow smoothly to the next stage without the need for and to schedule the next machine to accept this item.

JIT PURCHASING ARRANGEMENTS

JIT philosophy also extends to adapting JIT purchasing techniques whereby delivery of material immediately precedes their use. By arranging with suppliers for more frequent deliveries stocks can be out to a **minimum**. Considerable savings in material handling expenses can be obtained by requiring suppliers to inspect materials before their delivery and guarantee their quality.

This improved service is obtained by giving more business to fewer suppliers and **placing long term purchasing orders**, therefore the suppliers has an assurance of long term sales and can plan to meet this demand.

Companies that have implemented JIT purchasing techniques have claimed to substantially reduce **their investment in raw materials and work in progress stocks**.

Other advantages include

1. Substantial savings in factory space.
2. Large quantity discount.
3. Savings in time from negotiating with fewer suppliers
4. Reduction in paper work arising from issuing (long term orders) to a few suppliers rather than individual purchase order to many suppliers.

JIT and Management Accounting.

Management accountants in many organisations have been criticised because of their failure to change their managing accounting system to reflect the mode from a traditional manufacturing to a JIT manufacturing system.

Conventional management accounting systems can encourage behaviour that is inconsistent with JIT phylosopy, management, accounting must support JIT manufacturing by monitoring, identifying, and communicating to decision makers any delays errors and waste in the system.

Modern management accounting systems are now placing greater emphasis on providing information on suppliers reliability set up times cycle times, percentage of deliveries that are on time and defect rates

All these measures are critical in supporting JIT manufacturing philosophy.

FINANCIAL MANAGER'S ROLE IN INVENTORY MANAGEMENT

The techniques of inventory management, discussed above are very useful in determining the optimum level of inventory and finding answers to the problems of the economic order quantity, the re-order point and the safety stock. The techniques are very essential to economize the use of resources by minimizing

the total inventory cost. Although out treatment of inventory management has been simple, it indicates the broad framework of managing inventories. Many sophisticated techniques have been evolved to handle inventory management problems more efficiently and effectively and the improvements are still continuing. For the majority of the companies, inventory represents a substantial investment. Thus, the goal of the wealth maximization is related to the efficiency with which inventory is managed. Consequently, the financial manager has an important role to play in the management of inventory, although it is not his operating responsibility to control inventory. The financial manager should see that only an optimum amount is invested in inventory. He should be familiar with the inventory control techniques and ensure that inventory is managed well. He should introduce the policies which reduce the lead time, regulate usage and thus, minimize safety stock. The net effect would be to reduce inventory investment and increase the firm's prospects of making more profits.

PRACTICE QUESTIONS

QUESTION

Boots Ltd. manufactures a range of five similar products, A, B, C, D and E. The table below shows the quantity of each of the required inputs necessary to produce one unit of each product, together with the weekly inputs available and selling prices of each product.

Inputs	A	B	C	D	E	Weekly inputs available
Raw materials (Kg)	6.0	6.5	6.1	6.1	6.4	35,000 Kgs
Forming (hours)	1.00	0.75	1.25	1.00	1.00	6,000 hours
Firing (hours)	3.00	4.50	6.00	6.00	4.50	30,000 hours
Packing (hours)	0.50	0.50	0.50	0.75	1.00	4,000 hours
Selling price (Sh.)	40	42	44	48	52	

The costs of each input are as follows:

Material	Sh.2.10 per Kg
Forming	Sh.3.00 per hour
Firing	Sh.1.30 per hour
Packing	Sh.8.00 per hour

Required:

- a) Formulate this problem as a Linear Programming problem. (7 marks)

- b) The problem has been solved using a computer package and the following final tableau of a simplex solution has been produced:

Basis	A	B	C	D	E	X	S	T	U	Value
A	1	1.18	1.04	0.46	0	0.36	0	0	-2.29	3,357
B	0	-0.34	0.23	0.02	0	-0.18	1	0	0.14	321
T	0	1.37	2.97	2.28	0	-0.27	0	1	-2.79	9,482
E	0	-0.09	-0.02	0.52	0	-0.18	0	0	2.14	2,321
Zj	0	1.26	1.06	0.51	0	2.02	0	0		105,791
									8.81	

Where A, B, C, D and E are the weekly production levels for the five products; X is the amount of raw material that falls short of the maximum available; S, T and U are the respective number of hours short of maximum weekly input of forming, firing and packing time.

- Use this tableau to find the optimum weekly production plan. (4 marks)
- Describe the implications of using this plan in terms of unused resources and overall contribution to profit. (3 marks)
- In the context of this problem explain the meaning of "The dual or shadow price of a resource" (3 marks)
- There is a proposition that the company manufactures an additional product which would sell at Sh.50 per unit. Each unit will need 6 kg of raw material, one hour of forming time, five hours of firing time and one hour of packing time. Is it worthwhile proposition? (3 marks)

(Total: 20 marks)

CHAPTER SIXTEEN

INTRODUCTION TO LINEAR PROGRAMMING

Specific Objectives

At the end of this to the trainee should be able to:

- Explain the model concepts;
- Describe the various types of models;
- State the requirements of LP models;
- Model decision problems as linear models;
- Solve LP models.

INTRODUCTION

These are mathematical technique that deals with optimism of linear function variables known as objective functions. It is a method of determine an optimum program of independent activities in view of available resources. Linear programming is a technique of decision making used by managers to allocate limited resources e.g. machinery, raw materials and labor in order to minimize costs or maximize production. Decision variable are the amounts of each product to be made in a given time period. Linear programming assumes that the variable has a linear relationship.

N.B; it is a technique concerned with scarce resources.

Advantages

1. It helps in retaining the optimum use of production factor e.g. efficient use of man power.
2. it improves quality of decision
3. it gives possible practice positively
4. It improves the knowledge and skills of executive.

Limitation

1. it assumes that the activity is infinitely divisible
2. it treats all relationships as linear
3. it takes into account single object only.
4. It can be adopted only under the condition of certainty i.e. when the quantity of resources is known.

Assumption of linear programming

1. linearity
2. Divisibility- assumes that quantities, costs and revenues are infinitely divisible.
3. certainty- the technique assumes that there is certainty in all
4. positive solutions- non negative constraints are introduced to ensure the values are considered
5. Interdependent between demands for products is ignored.
6. Time factor is ignored. It assumes that production is instantaneous.

Application of linear programming

Production department to decide the quantity of pots to be produced subject to limited resources (constraints) e.g. labor, power, machine hours, raw materials etc.

Marketing department: Allocation of salesmen to different sales regions subject to their expected performance.

Human Resource: Scheduling personnel's work hours and job description to either maximize production or minimize cost.

Steps in solving linear programming problems (problem formulation)

1. Identify variables (e.g. product x and product y)
2. Identify the objective (To maximize contribution or to minimize cost), and write down its mathematical presentation in terms of variables.
3. Identify the constraints (i.e. the limited resources shared among the variables), and write down its mathematical representation in terms of variables.
4. Write down the objectives and the constraints in terms of the variables.

These steps apply regardless of the number of the variables.

NOTE: If only two variables are involved, a graphical solution can be used otherwise for multivariable problems, an algebraic method is applied to find the solution.

Example 1:

Long Castling Breweries manufactures two brands of beer, Benko lager and Benoni lager. Benko has a contribution of Sh.4 per unit and Benoni has a contribution of Sh.3 per unit. Benko requires 30 machine minutes and 30 labor minutes to manufacture a unit. Total available machine hours per day are 12hrs whereas total available labor hours per day are 14hrs.

Required:

1. Formulate linear programming model.
2. How much of each brand should Long Castling produce if it wishes to maximize its daily contribution assuming that all the lager produced is sold.

Solution:

1. Formulating a linear programming model

Step 1: Identifying variables:

The variables here are the number of units of Benko and Benoni lager produced by Long castling breweries per day; we can represent them as:

X_1 = a unit of Benko lager.

X_2 = a unit of Benoni lager.

Step 2: Identify the objective:

Definition: An **objective** is the desired result i.e. optimization of a function dependent on decision variable and subject to some constraints.

The objective of Long Castling Breweries is to maximize daily contribution. Objective function is the formula that will give us the total contribution in a day for both Benko lager and Benoni lager.

The information above can thus be represented in a tabular form as:

PRODUCT			Maximum available
(PER DAY)	X_1	X_2	hours/day
Machine hours	0.5	0.33	12
Labor hours	0.5	0.5	14
Contribution	4	3	

Objective function = $4X_1 + 3X_2$

The objective is to maximize $4X_1 + 3X_2$

Step 3: Identifying constraints (constraints formulation)

Definition: **Constraints** are circumstances that govern achievement of an objective.

Limitations must be quantified mathematically and they must be linear.

For Long castling breweries we have limited machine hours (12hrs/day), which must be shared among production of Benko and Benoni lagers.

Therefore production must be such that the numbers of machine hours required is less than or equal to 12 hours per day.

$$0.5X_1 + 0.33X_2 \leq 12\text{hrs}$$

Similarly for labor hours we have:

$$0.5X_1 + 0.5X_2 \leq 14\text{hrs}$$

Non-Negativity: It is logical assumption to assume that the company cannot manufacture negative amounts of a product, thus it can only manufacture either zero product or more. Therefore we have:

$$X_1 \geq 0 \\ X_2 \geq 0 \quad \text{or} \quad X_1, X_2 \geq 0$$

Thus the complete linear programming model is;

$$\text{Maximize } 4X_1 + 3X_2$$

Subject to the constraints;

$$0.5X_1 + 0.33X_2 \leq 12$$

$$0.5X_1 + 0.5X_2 \leq 14$$

$$X_1, X_2 \geq 0$$

Solving linear programming problems

The question requires us to optimize (in our case, maximize) the objective (the contribution function), or in simple terms we are required to solve the linear programming model.

Solving linear programming model entails finding the values of variables that satisfy all inequalities simultaneously and optimize the objective.

Graphical solution

This method is used to solve LP models in case where only two variables are involved. For more than two variables (multivariable) then the simplex technique (algebraic method may be used).

Now in solving the problem above we first draw the axis, taking X_1 to be the y axis and X_2 to be the X axis.



Next we plot the scales on each axis to approximate the scales to use them, we consider each constraint equation. We get the value of one of variables putting the other variable to be zero and by substituting the inequality \leq or \geq with equality sign (=).

For: $0.5X_1 + 0.33X_2 \leq 12$

When $X_1 = 0$

$$0.5(0) + 0.33X_2 = 12$$

$$0.33X_2 = 12$$

$$X_2 = 12/0.33 = 36$$

Therefore point to plot is $(\rightarrow 36, 0)$

... Implying that when

$X_2 = 36$ then $X_1 = 0$

When $X_2 = 0$

$$0.5X_1 + 0.33(0) = 12$$

$$0.5X_1 = 12$$

$$X_1 = 12/0.5 = 24$$

Therefore the point is (0, 24)

For $0.5X_1 + 0.5X_2 \leq 14$

When $X_1 = 0$

$$0.5(0) + 0.5X_2 = 14$$

$$0.5X_2 = 14$$

$$X_2 = 14/0.5 = 28$$

Therefore the point is (28, 0)

When $X_2 = 0$

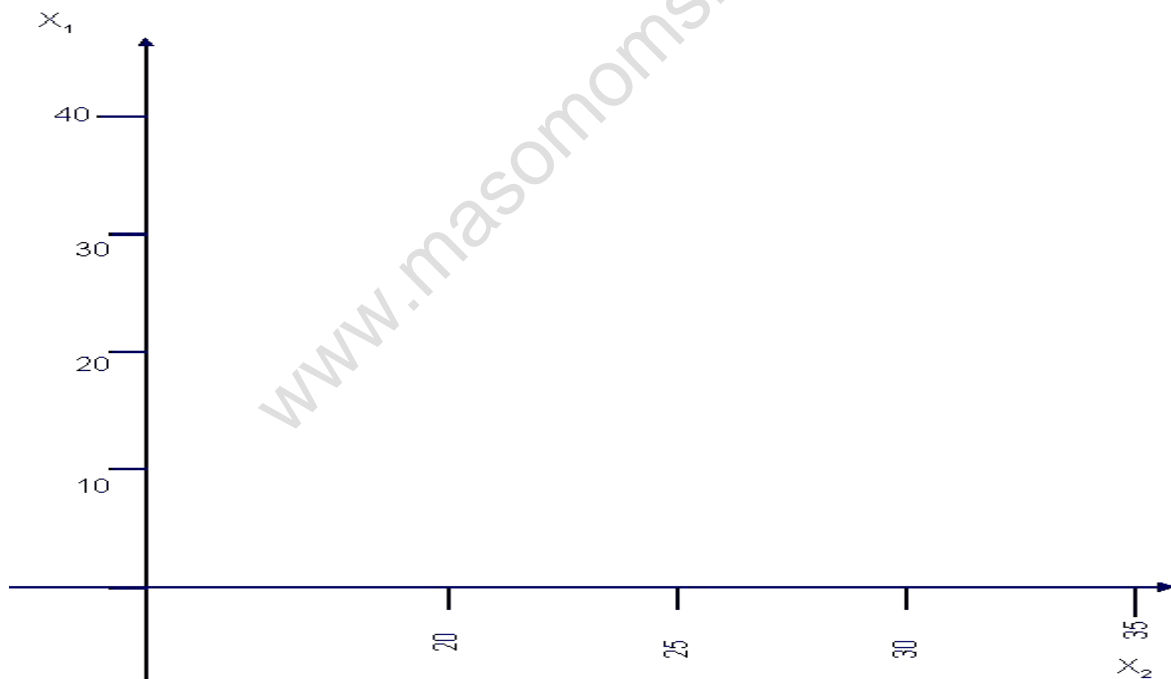
$$0.5X_1 + 0.5(0) = 14$$

$$0.5X_1 = 14$$

$$X_1 = 14/0.5 = 28$$

Therefore the point is (0, 28)

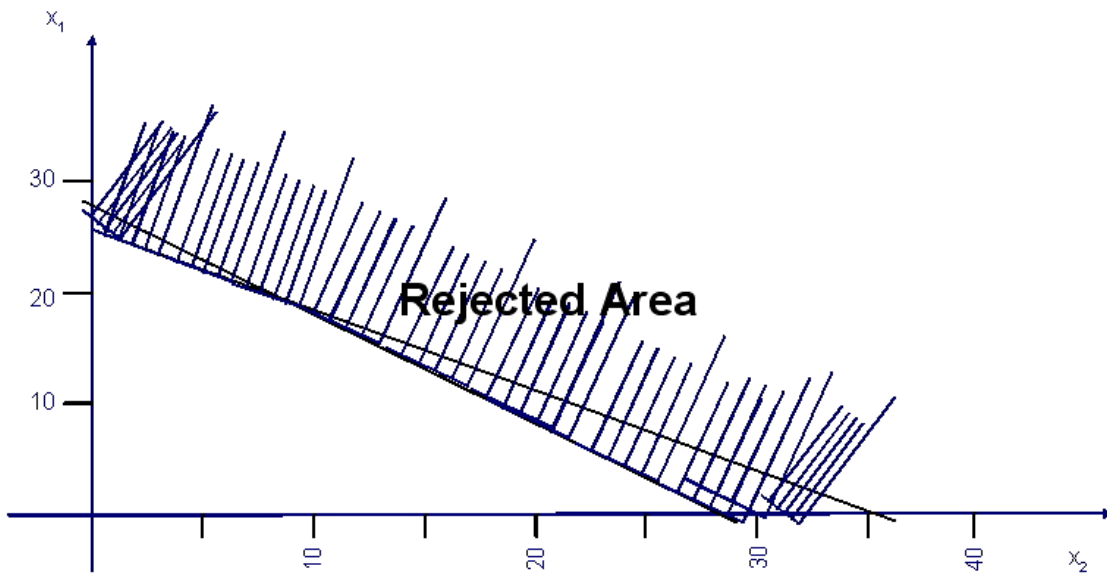
Comparing these values we see that X_2 ranges between 0 - 28, therefore we can have the graph plotted as:



Next draw each limitation (constraint) as separate line on the graph.

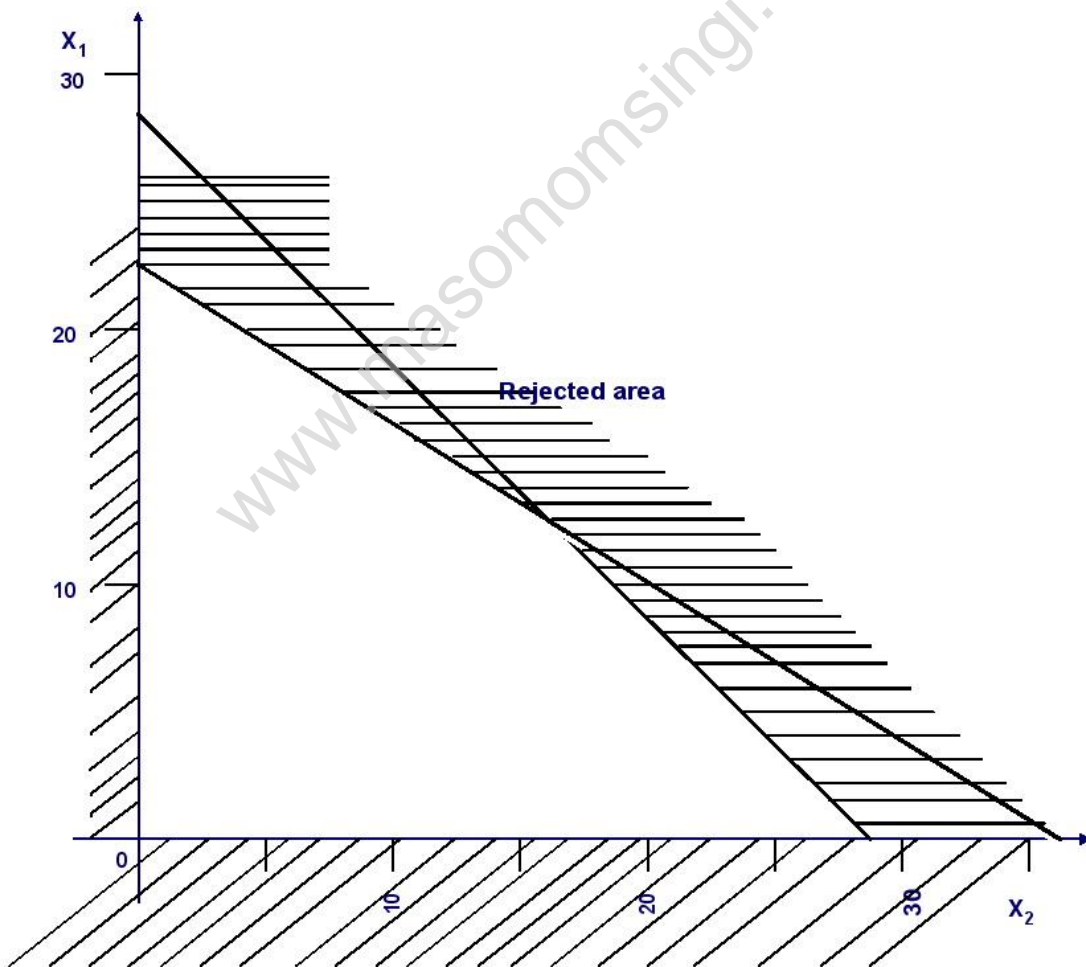
For $0.5X_1 + 0.33X_2 \leq 12$

The two points that represent this line are (36, 0) and (0, 24). This is plotted as a straight line from 36 on X_2 axis to 24 on X_1 axis.



Now including the Non-Negativity constraints since no negative product can be produced;

$$x_1 \geq 0; x_2 \geq 0$$



We must now consider how to choose the production which will maximize contribution. This we do by plotting a line representing the objective function ($4x_1 + 3x_2$).

First choose a convenient point inside the feasible region

$$\begin{aligned} \text{E.g. } X_2(10) + 3X_1(20) &= 40 + 60 \\ &= \text{Sh } 100 \end{aligned}$$

All of the other product mixes that give a contribution of Sh.100 lies on the line:

$$100 = 4X_1 + 3X_2 \dots\dots\dots (i)$$

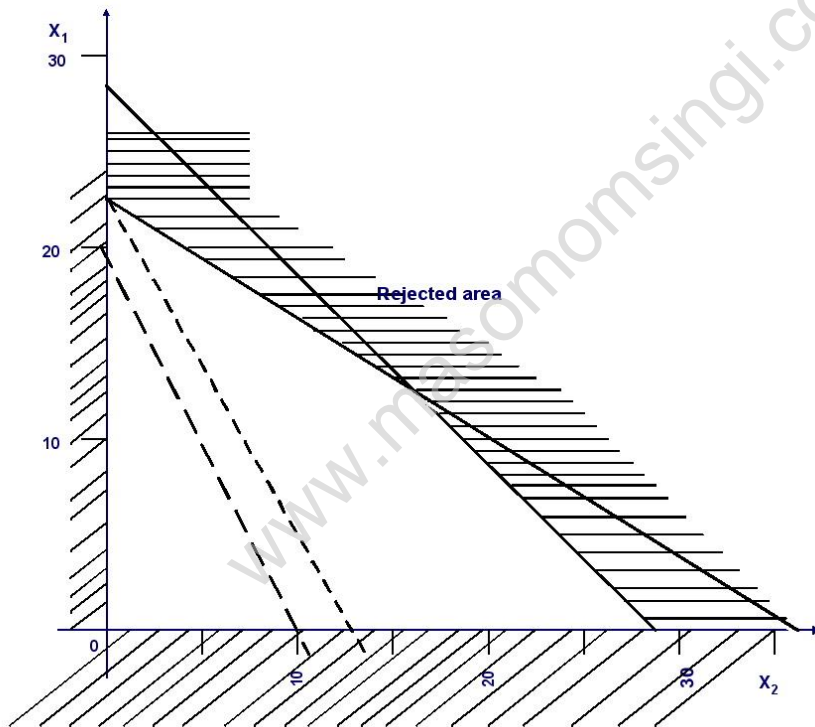
<<This line is called a contribution line>>

Picking another point, say $X_2=10$ and $X_1=20$

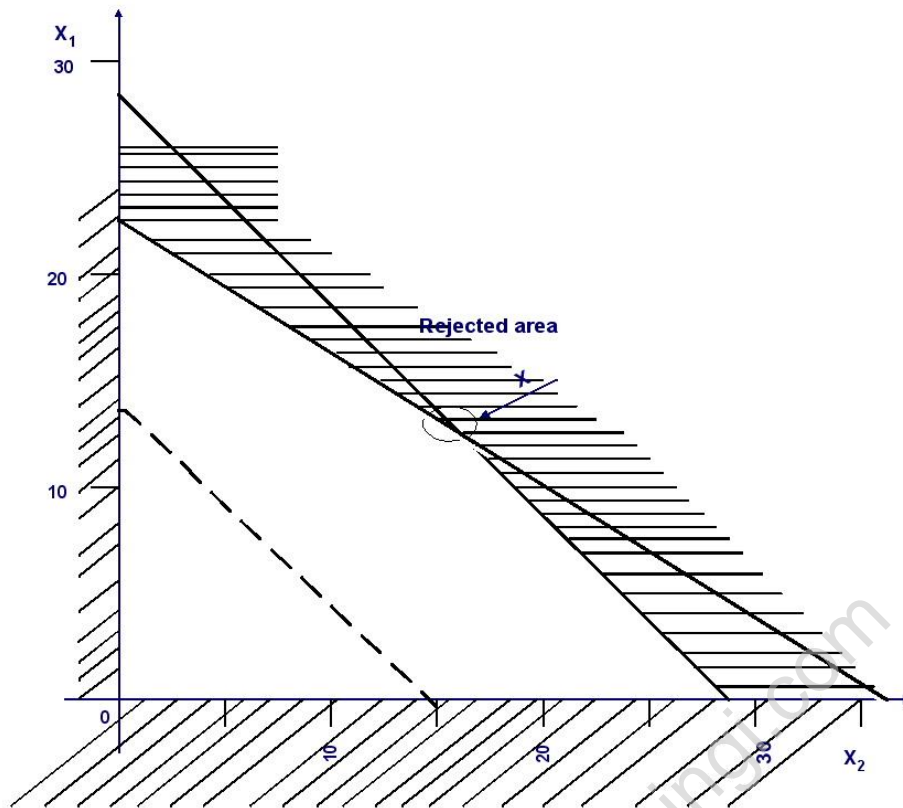
Its contribution value is SH 110, thus give a contribution line of

$$110 = 4X_1 + 3X_2 \dots\dots\dots (ii)$$

Plotting these two contribution lines to our graph we get two parallel lines.



Until we reach the last feasible solution(s) before the line moves entirely out of the feasible region.



Point X is the last feasible solution. Coordinates of this point give a combination of the two lager's production volumes that fetches the highest contribution.

Coordinates of point X can be read from the graph, but for precision they are calculated by solving simultaneously the equations of the two lines that intersect at point X.

The two constraints are called binding or limiting constraints. They are the resources being fully used thus preventing daily contribution from increasing further.

Therefore to get point x we solve:

$$0.5X_1 + 0.33X_2 = 12 \dots (i)$$

$$0.5X_1 + 0.5X_2 = 14 \dots (ii)$$

Since X is the intersection of these two constraints, solving by deducting (i) from (ii) we get

$$0.17X_2 = 2$$

$$X_2 = 11.76$$

And substituting $X_2 = 11.76$ to equation (i) we get

$$X_1 = 16.24$$

Therefore 11.76 units of Benko lager and 16.24 units of Benoni lager need to be produced for maximum contribution.

$$\text{Contribution} = 4(16.24) + 3(11.76) = 100.24$$

Requirements in linear programming

Assumptions that are made to solve these types of problems are that:.

- Proportionality: all activities in linear programming problems are proportional to the level of decision variables.
- Divisibility: the solution to a linear programming problem does not have to be an integer but for strictly whole number solutions, use integer programming.
- Non-negativity: no decision variable can be negative.
- Additivity: the total of all activities in linear programming problems are assumed to equal to the sum of individual activities.

Special cases in linear programming

- Infeasibility: This is when all constraints don't satisfy a particular point thus there is no feasible solution.
- Redundancy: A constraint is considered redundant if it does not affect the feasible region. This happens in cases of excess resources since it does not limit attainment of the objective.
- Multiple optimal solutions: This occurs when the objective function has the same slope as a binding constraint.

Minimizing problem

Example

A manufacturing company has acquired new machine for producing product P at a rate of 25 units per hour with a 98% rate of efficiency. The company requires producing at least 1800 units of P per day. The 10 old machines that the company has produce 15 units of P with 95% efficiency.

The cost of operating the new machine is Sh. 4 per hour and Sh. 3 per hour for the old ones. The cost incurred due to inefficiency is Sh. 2 per unit; It is government policy that at least 2 of the new machines must be indulged into production.

The company wishes to optimally allocate the machines in order to minimize the total manufacturing cost if the total available hours for production in a day are 8 hours.

Solution

In minimizing problems we use \geq (greater or equal to) type inequality.

Step 1: Identifying variables.

Since the problem requires us to appropriately allocate the machines in order to minimize the costs thus our variables are the new machines and the old machines, we can let;

X_1 =new machines

X_2 =old machines

Step 2: Identify objectives:

The objective is to minimize manufacturing costs.

Total manufacturing cost per machine=Operating cost + (inefficiency rate × number of units × cost of loss)

Therefore cost for new machines

$$\text{Cost} = 4 + (0.02 \times 25 \times 2) = \text{Sh. 5 per hour} \\ = 5 \times 8 = \text{Sh. 40 per day}]$$

Similarly for old machines

$$\text{Cost} = 3 + (0.05 \times 15 \times 2) = \text{Sh. 4.5 per hour} \\ = 4.5 \times 8 = \text{Sh. 36 per day}$$

Therefore the objective function is to minimize $40X_1 + 36X_2$

Step 3: Identifying constraint functions.

$$X_1 \leq 8$$

$$X_2 \leq 10$$

$$(25 \times 8) X_1 + (8 \times 15) X_2 \geq 1800 \quad (\text{This can be simplified further as})$$

$$200 X_1 + 120 X_2 \geq 1800 \quad (\text{dividing through by 40})$$

$$5X_1 + 3 X_2 \geq 45$$

$$X_1 \geq 2$$

$$X_1, X_2 \geq 0$$

Thus the LP model is:

$$\text{Minimize} \quad 40X_1 + 36X_2$$

$$\text{Subject to:} \quad X_1 \leq 8$$

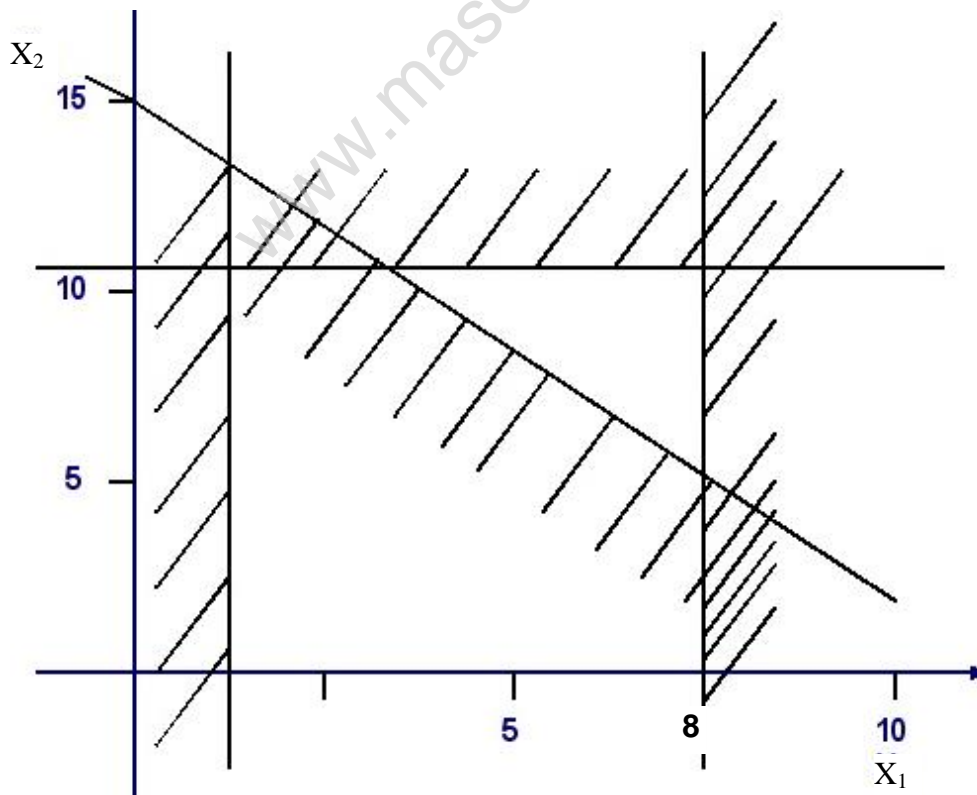
$$X_2 \leq 10$$

$$5X_1 + 3 X_2 \geq 45$$

$$X_1 \geq 2$$

$$X_1, X_2 \geq 0$$

Plotting this on a graph we get:



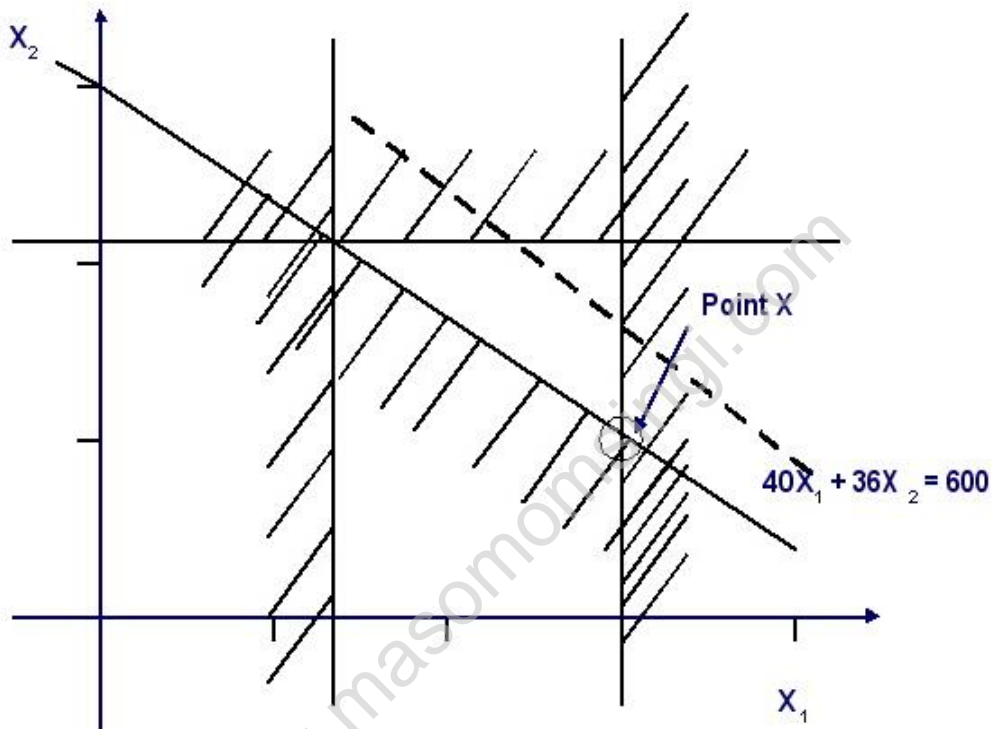
The line $X_1 \geq 2$ does not affect the feasible region (doesn't cause reduction of the feasible region), this constraint doesn't limit attainment of the objective, thus its known as a redundant constraint.

Now picking a convenient point inside the feasible region, say (6, 10)

We get a total cost of $600 = (6(40)+10(36))$

Thus the objective function line of $40X_1 + 36X_2 = 600$

Moving this line parallel toward the origin to locate the last apex before the line completely fall off the feasible region, we get:



Point X is the point of optimal solution

The binding constraints here are:

$$X_1 \leq 8$$

And $5X_1 + 3X_2 \geq 45$

Solving this to get coordinates of point X

$$5X_1 + 3X_2 = 45 \text{ (when } X_1 = 8\text{)}$$

$$40 + 3X_2 = 45$$

$$X_2 = 5/3$$

Thus the solution is $X_1 = 8$

$$X_2 = 5/3$$

Shadow or dual prices

Definition: A shadow price or a dual price is the amount increase (or decrease) of the objective function when one more (or one less) of the binding constraints is made available.

Consider **example 1**.

Maximize $4X_1 + 3X_2$

Subject to: $0.5X_1 + 0.33X_2 \leq 12$ (Machine hours)

$0.5X_1 + 0.5X_2 \leq 14$ (labor hours)

Starting with machine hours; let's assume that one more machine hour is available (with labor hours remaining constant)

We get:

$$0.5X_1 + 0.33X_2 = 13$$

$$0.5X_1 + 0.5X_2 = 14$$

Solving this simultaneously we get the values of X_1 and X_2 as

$$0.17 X_2 = 1$$

$$X_2 = 5.88$$

$$X_1 = 22.12$$

Thus the contribution is

$$4(22.12) + 3(5.88) = \text{Sh.}106.12$$

Comparing this with its original contribution of Sh.100.24 (see example 1) we see increasing machine hours by one unit has increased contribution by Sh.5.88, which is the shadow price per machine hour.

Note: This figure is also arrived at if we assume that machine hours are reduced by 1 unit i.e. 12-1.

Similarly assuming that one more labor hour is made available, and then contribution change is:

$$0.5X_1 + 0.33X_2 = 12$$

$$0.5X_1 + 0.5X_2 = 15$$

Solving this simultaneously gives:

$$0.17 X_2 = 3$$

$$X_2 = 17.65$$

$$X_1 = 12.35$$

Which give a contribution of.

$$4(12.35) + 3(17.65) = \text{Sh.}102.35$$

The contribution change is Sh.2.11 which is the shadow price per labor hour.

Note:

The shadow prices apply in so far as the constraint is binding for example if more and more labor hours are available it will reach a point where labor hours are no longer scarce thus labor hours cease to be a binding constraint and its shadow price becomes a zero. (All non-binding constraints have zero shadow price). Logically its senseless to pay more to increase a resource, which is already abundant.

Interpretation of shadow prices

A shadow price of a binding constraint indicates to management how much extra contribution will be gained by increasing a unit of the scarce resource.

In the example above Sh.2.11 is the shadow price for labor hours. This implies that management is ready to pay up to Sh.2.11 extra per hour for the extra hours i.e. say an employee is paid sh.5 per hour and one day he works for two hours extra (overtime), the management is prepared to pay up to sh.7.11 per hour for the two hours overtime worked.

Sensitivity Analysis

Definition: Sensitivity analysis is the test of how certain changes in resources affect the optimal solution.

In sensitivity analysis we consider the effect of additional limiting or non-limiting constraints.

We already know that adding more non-limiting constraints does not change the optimal solution.

We also know that adding more binding constraints affects the objective function.

It is very important for the management to know how much of a limit resource can be made available until it has no effect on the objective function (ie ceases to be a binding resource)

SIMPLEX METHOD

When analyzing linear programming problems with three or more variables the graphical method becomes inadequate, in such cases we employ *simplex method*. Simplex method is an algebraic procedure for solving systems of equations requiring optimization of the objective function..

This method can be applied to any number of variables, the more they are the more complex it becomes to work out a solution on paper. Computer programs e.g. Tora are used to solve the most intricate problems.

The first step in simplex method is conversion of inequalities to linear equations

Example

Consider the linear problem.

$$\begin{aligned} &\text{Maximize } 45x_1 + 80x_2 \\ &\text{Subject to } 5x_1 + 20x_2 \leq 400 \\ &\quad 10x_1 + 15x_2 \leq 450 \\ &\quad x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Solution.

1. To convert this problem to a system of linear equation, we introduce slack variables to each constraint.

$$\begin{aligned} Z &= 45x_1 + 80x_2 \\ \text{Subject to } 5x_1 + 20x_2 + x_3 &= 400 \\ 10x_1 + 15x_2 + x_4 &= 450 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Where the structural variables x_3 and x_4 are slack variables

2. we then place this information in a tabular form known as a **tableau**
- Initial tableau

Solution Variable	Products		Slack Variables		Solution Quantit
	x_1	x_2	x_3	x_4	

e				y	
x_3	5	20	1	0	400
x_4	10	15	0	1	450
Z	45	80	0	0	0

The tableau shows that

$$x_3 = 400$$

$$x_4 = 450$$

$$Z = 0 \quad (\text{Total contribution})$$

It also shows that unused capacity is at maximum i.e. the value for the slack x_3 and x_4 is 0.

After several operations and when an optimal solution has been attained, these values will change to give an optimal feasible solution.

3. Select the column with the highest value of Z (i.e. 80), then divide the positive numbers in that column (i.e. the x_2 column) into the quantity column.

$$\text{i.e. } 400 \div 20 = 20$$

$$450 \div 15 = 30$$

Choose the row giving the lowest result (in our case the row with x_3 gives 20) and mark the element falling on the intersection of the selected row and selected column (i.e. 20: selected element)

4. Divide all the elements in the selected row by the value of the selected element and change the solution variable to the heading of the identified column (from x_3 to x_2)

Thus the tableau will appear as follows.

Solution Variable	Products		Slack Variables		Solution Quantity
	x_1	x_2	x_3	x_4	
x_2	$\frac{1}{4}$	1	$\frac{1}{20}$	0	20
x_4	10	15	0	1	450
Z	45	80	0	0	0

5. Next we conduct row operations that aim to reduce elements falling in the same column as the previously marked element to zero. These row operations may sometimes necessitate multiplying or dividing the selected row with an arbitrary number.

Therefore:

$$\text{Row 2} - 15 \times \text{Row 1}$$

$$\begin{array}{r}
 x_4 \quad 10 \quad 15 \quad 0 \quad 1 = 450 \\
 15 \times \frac{1}{4} \quad x_2 \quad \frac{15}{4} \quad 15 \quad \frac{15}{20} \quad 0 = 300 \\
 \hline
 \text{new} \quad x_4 \quad 6\frac{1}{4} \quad 0^* \quad \frac{3}{4} \quad 1 = 150
 \end{array}$$

Note that the aim was to attain the zero.

$$\begin{array}{r}
 Z \quad 45 \quad 80 \quad 0 \quad 0 = 0 \\
 80 \times \frac{1}{4} \quad x_2 \quad 20 \quad 80 \quad 4 \quad 0 = 1600 \\
 \hline
 \text{new} \quad Z \quad 25 \quad 0^* \quad -4 \quad 0 = -1600
 \end{array}$$

On replacing the new rows to the tableau we get 2nd tableau.

Solution Variable	Products		Slack Variables		Solution Quantity
	x_1	x_2	x_3	x_4	
x_2	$\frac{1}{4}$	1	$\frac{1}{20}$	0	20
x_4	$6\frac{1}{4}$	0	$\frac{3}{4}$	1	150
Z	25	0	-4	0	-1600

Since in the Z row under products column we still have values greater than zero, we conduct another operation.

Taking the column with a Z value of 25, we repeat the process in the same manner.

$$20 \div \frac{1}{4} = 80$$

$$150 \div 6\frac{1}{4} = 24$$

Thus we pick the x_4 row and mark the element $6\frac{1}{4}$, the row solution variable is changed to x_1 and we divide the row by $6\frac{1}{4}$ to convert the marked element to 1.

$$\text{Therefore; } \quad x_1 \quad 1 \quad 0 \quad \frac{3}{25} \quad \frac{4}{25} \quad = 24$$

Next we do the row operations

$$\begin{array}{r}
 x_2 \quad \frac{1}{4} \quad 1 \quad \frac{1}{20} \quad 0 = 20 \\
 (\div 4) \quad x_1 \quad \frac{1}{4} \quad 0 \quad \frac{3}{100} \quad \frac{4}{100} = 6 \\
 \hline
 \text{new} \quad x_2 \quad 0^* \quad 1 \quad \frac{1}{50} \quad -\frac{1}{25} = 24
 \end{array}$$

$$\begin{array}{r}
 Z \quad 25 \quad 0 \quad -4 \quad 0 = -1600 \\
 25 \times \frac{1}{4} \quad x_1 \quad 25 \quad 0 \quad 3 \quad 4 = 600 \\
 \hline
 \text{new} \quad Z \quad 0^* \quad 0 \quad -7 \quad -4 = -2200
 \end{array}$$

the tableau becomes, 3rd Tableau

Solution Variable	Products	Slack Variables	Solution Quantity
x_2	$\frac{1}{4}$	1	$\frac{1}{20}$
x_4	$6\frac{1}{4}$	0	$\frac{3}{4}$
Z	25	0	-4

n Variable	Variables				n Quantity
	x_1	x_2	x_3	x_4	
x_2	0	1	$\frac{1}{50}$	$-\frac{1}{25}$	24
x_1	1	0	$\frac{3}{25}$	$\frac{4}{25}$	24
Z	0	0	-7	-4	-2200

This is the final tableau since the Z row has no values greater than zero thus we have the optimal solution.

Interpretation

- to maximize Z we need to produce 24 units x_2 and 24 units of x_1 , we obtain these values from the solutions quantity column
Thus, $Z = 24(45) + 24(80) = 3000$
- We have zero slack (unused quantities of constraints).
Assume tableau 2 is the final tableau and let the constraint with variable x_3 be labor hours and x_4 are raw materials, the slack would've been interpreted as:
 - 150 units of raw materials were unused
 - To maximize Z we produce 20 units of x_2 and none of x_1 .
- The value represents in Z row under slack variable column represents **shadow prizes**. Thus the shadow prize for the first constraint with x_3 is 7 and the shadow prize for the second constraint with the variable x_4 is 4.

INTERPRETATION OF COMPUTER GENERATED SOLUTION

Example

Maximize $25x_1 + 20x_2 + 24x_3$
 where: $x_1 = \text{Xtragrow}$, $x_2 = \text{Youngrow}$, $x_3 = \text{Zupergrow}$
 Subject to $0.3x_1 + 0.2x_3 \leq 500$
 $0.5x_2 + 0.4x_3 \leq 1000$
 $0.2x_1 + 0.1x_2 + 0.1x_3 \leq 800$
 $0.4x_2 + 0.3x_3 \leq 600$
 $x_1 \geq 1500$
 $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

The computer generated solution for this problem is as follows;

Objective value = 71666.7			
Variable	Value	Obj. Coeff	Obj Value Contribution
X1: Xtragrow	1666.7	25	41666.7
X2: Youngrow	1500	20	30000
X3: Zupergrow	0	24	0

Constraint	RHS	Slack- /Surplus+
1(<)	500	0
2(<)	1000	250-
3(<)	800	316.67-
4(<)	600	0
5(>)	1500	166.7+

Sensitivity Analysis				
Variable	Current obj coeff	Min obj coeff	Max Obj Coeff	Reduced cost
X1: Xtragrow	25	13.50	Infinity	0
X2: Youngrow	20	9.78	Infinity	0
X3: Zupergrow	24	-Infinity	31.67	7.67
Constraint	Current RHS	Min RHS	Max RHS	Dual price
1(<)	500	450	975	83.33
2(<)	1000	750	Infinity	0
3(<)	800	483.3	Infinity	0
4(<)	600	0	800	50
5(>)	1500	-Infinity	1666.7	0

Required

Interpret the data generated by the computer.

Solution.

Table 1:

Objective value, is the solution to objective function (e.g. the solution to this example is 71,436)

The four columns of table 1 are to be interpreted as follows;

- Variable: these are the variables of the model. In our example we have $x_1 = \text{Xtragrow}$, $x_2 = \text{Young row}$ and $x_3 = \text{Zupergrow}$
- Value: this is value that the variables assume at optimal solution (to optimize the objective function one needs to produce this amounts of the variables). In our example we are required to produce 1,666.67 of x_1 and 1,750 of x_2 and none of x_3
- Objective coefficient: these are the coefficients of the objective function
- Objective value contribution: this is the value contributed by each variable to the objective function (for $x_1=25 \times 1,666.67$), the total of this is equal to our objective value (i.e. $41,666.67+35,000=76,666.67$).

The 3 columns of the second part of table1 can be interpreted as follows;

- Constraints: this is constraints of the model representing the limited resources.
- RHS: the Right hand side value is the limiting value of the constraint. E.g. for the first constraint the maximum amount of material A is 500 tons.

- Slack/surplus: at optimal production not all the materials for some of the constraints will be fully utilized, slack is the amount of material that is left over after production. For constraint 1 and 4 no material remained, this also implies that these are the binding constraints i.e. their adjustment directly affects the objective solution

Sensitivity Analysis

This is the analysis of the effect of adjusting variables or constraint, whether the objective solution will be affected. How much of the objective coefficient (or the maximum available amount of a constraint) can be reduced or increased without affecting the objective solution.

The columns of this table can be interpreted as follows;

Variable: as explained above

Current objective coefficient: this is the value of coefficients of the objective function

Minimum objective coefficient: this is how low the coefficient can be reduced without affecting the optimal basis. The coefficient for x1 can be reduced from 25 to 13.50 (the prize for Xtragrow fall to as low as \$13.50 from \$25) but the optimal solution will remain the same

Maximum objective coefficient: this is how high the coefficient can be increased without affecting the *optimal basis*

Reduced cost: this is amount by which the coefficient of the variable has to be adjusted with for it to become a basic variable (included to the objective optimal solution). X1 and x2 have 0 reduced costs implying that they already make part of the optimal solution, x3 will require to be increased by 7.67 for it to make part of the basic variable.

The second part of the table is interpreted as follows;

- Constraints: as described above
- Current Right Hand Side: the limiting value of the constraint. E.g. for the first constraint the maximum amount of material A is 500 tons.
- Minimum RHS: the lowest the available amount of the constraint can be reduced without affecting the optimal basis
- Maximum RHS: the highest the available amount of the constraint can be increased without affecting the optimal solution
- Dual price: this is amount increase to the objective contribution due to a unit increase of the available constraint. Since there is only 500 tons of material A, if management decides to increase it by a unit to 501tons then the objective optimal solution will be $76,666.67+83.33$.

PRACTICE QUESTIONS

QUESTION ONE

Regal Investments has just received instructions from a client to invest in two shares; one an airline share, the other an insurance share. The total maximum appreciation in share value over the next year is to be maximized subject to the following restrictions:

- the total investment shall not exceed Sh.100,000
- at most Sh.40,000 is to be invested in the insurance shares
- quarterly dividends must total at least Sh.2,600

The airline share is currently selling for Sh.40 per share and its quarterly dividend is Sh.1 per share. The insurance share is currently selling for Sh.50 per share and the quarterly dividend is Sh.1.50 per share. Regal's analysts predict that over the next year, the value of the airline share will increase by Sh.2 per share and the value of the insurance share will increase by Sh.3 per share.

Computer software provided the following part solution output:

Objective Function Value = 5,400

	Variable	Number	Reduced cost
Airline shares	1,500	0.000	
Insurance shares	800	0.000	
Constraint	Slack/Surplus		Dual prices
Total investment		0.000	0.050
Investment in insurance		0.000	0.010
Dividends		100.000	0.000

Objective Coefficient Ranges

Variable	Lower limit	Current value	Upper limit
Airline share	2.500	3.000	No upper limit
Insurance share	0.000	2.000	2.400

Right-hand Side Ranges

Constraint	Lower limit	Current value	Upper limit
Total investment	96,000.00	100,000	No upper limit
Investment in insurance	20,000.00	40,000	100,000.00
Dividends	No lower limit	2,600	2,700.00

Required:

- Formulate the above problem.
- Explain what reduced cost and dual prices columns above mean.
- How should the client's money be invested to satisfy the restrictions?
- Suppose Regal's estimate of the airline shares appreciation is an error, within what limits must the actual appreciation lie for the answer in (c) above to remain optimal?

(Q 6 Dec

2001)

QUESTION TWO

- A baker makes two products; large loaves and small round loaves. He can sell up to 280 of the large loaves and up to 400 small round loaves per day. Each large loaf occupies 0.01m^3 of shelf space, each small loaf occupies 0.008m^3 of space, and there is 4m^3 of shelf space available. There are 8 hours available each night for baking, and he can produce large loaves at the rate of 40 per hour, and small loaves at the rate of 80 per hour. The profit on each large loaf is Sh.5.00 and Sh.3.00 profit on the small round loaf.

Required:

In order to maximize profits, how many large and small round loaves should he produce?

- Summarize the procedure for solving the kind of quantitative technique you have used to solve part (a) above.

(Q 6 June

2001)

QUESTION THREE

- A small company will be introducing a new line of lightweight bicycle frames to be made from special aluminum alloy and steel alloy. The frames will be produced in two models, deluxe and professional. The anticipated unit profits are currently Sh.1, 000 for a deluxe frame and Sh.1,500 for a professional frame. The number of kilograms of each alloy needed per

frame is summarized in the table below. A supplier delivers 100 kilogram's of the aluminum alloy and 80 kilogram's of the steel alloy weekly.

	Aluminum alloy	Steel alloy
Deluxe	2	3
Professional	4	2

Required:

- iii) Determine the optimal weekly production schedule.
 - iv) Within what limits must the unit profits lie for each of the frames for this solution to remain optimal?
- d) Explain the limitations of the technique you have used to solve part (a) above.

(Q 6 Dec 2000)

QUESTION FOUR

- a) Define the following terms as used in linear programming:
 - i) Feasible solution
 - ii) Transportation problem
 - iii) Assignment problem
- b) The TamuTamu products company ltd is considering an expansion into five new sales districts. The company has been able to hire four new experienced salespersons. Upon analysing the new salesperson's past experience in combination with a personality test which was given to them, the company assigned a rating to each of the salespersons for each of the districts .These ratings are as follows:

c)

		Districts				
		1	2	3	4	5
Salespersons	A	92	90	94	91	83
	B	84	88	96	82	81
	C	90	90	93	86	93
	D	78	94	89	84	88

The company knows that with four salespersons, only four of the five potential districts can be covered.

Required:

- iii) The four districts that the salespersons should be assigned to in order to maximize the total of the ratings
- iv) Maximum total rating. (Q 6 June 2002)

QUESTION FIVE

- d) Explain the value of sensitivity analysis in linear programming problems and show how dual values are useful in identifying the price worth paying to relax constraints.
- e) J.A Computers is a small manufacturer of personal computers. It concentrates on production of three models- a Desktop 386, a Desktop 286, and a Laptop 486, each containing one CPU Chip. Due to its limited assembly facilities JA Computers are unable to produce more than 500 desktop models or more than 250 Laptop models per month. It has one hundred and twenty 80386 chips (these are used in Desktop-386) and four hundred 80286 chips (used in desktop 286 and Laptop 486) for the month. The Desktop 386 model requires five hours of production time, the Desktop 286 model requires four hours of production time, and the Laptop 486 requires three hours of production time. J.A Computers have 2000 hours of production time available for the coming month. The company estimates that the profit on Desktop 386 is Sh. 5,000. For a desktop 286 the profit is Sh.3, 400 and Sh.3,000 profit for a laptop 486.

Required:

Formulate this problem as a profit maximization problem and mention the basic assumptions that are inherent in such models.

- f) An extract of the output from a computer package for this problem is given below:

Output solution

$$X_1=120, X_2 = 200, X_3 = 200$$

Dual values Constraints 3 150

 Constraints 4 90

 Constraints 5 20

Sensitivity analysis of objective function coefficients:

Variable	Lower limit	Original value	Upper limit
X_1	100	250	No limit
X_2	150	170	200
X_3	127.5	150	170

Sensitivity analysis on R.H.S ranges.

Constraints	Lower limit	Original value	Upper limit
1	320	500	No limit
2	200	250	No limit
3	80	120	130
4	350	400	412.5
5	1950	2000	2180

X_1 =Monthly production level for Desktop 386.

X_2 =Monthly production level for Desktop 286.

X_3 =Monthly production level for Laptop 486.

Required:

- iv) Interpret the output clearly, including optimum product mix, monthly profit, unused resources and dual values
- v) Explain the purpose of upper limits and lower limits for the variables X_1 , X_2 , X_3 and constraints 1 to 5.
- vi) Calculate the increase in profit if the company is able to produce a further 10 CPU 80386 chips. (Q7
July 2000 Pilot paper)

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