QUESTION SEVEN

The data given below indicates the prices and production of some horticultural products in Central Territory:

Produce	Production (1000 boxes)		Price per box (Shs)	
	1980	**1990**	**1980**	**1990**
Cabbages	48,600	62,000	100	150
Tomatoes	22,000	37,440	220	310
Onions	47,040	61,430	180	200
Spinach	43,110	55,720	130	170

Required:

Calculate the increase or decrease in prices from 1980 on the basis of the following indices:

a) Mean relatives
b) Laspeyres index
c) Paasche index
d) Marshall – Hedgeworth index
e) Fishers index.

CHAPTER ELEVEN
MATRIX ALGBRA

SPECIFIC OBJECTIVES

At the end of this topic the trainee should be sable to:

- Define matrices;
- Describe the types of matrices,
- Apply matrix operations
- Form matrix models from practical problems;
- Apply matrix to decision problems.

INTRODUCTION

A matrix is a rectangular array of items or numbers. These items or numbers are arranged in rows and columns to represent some information. The position of an element in one matrix is very important as well be seen later; therefore an element is located by the number of the row and column which it occupies. The size of a matrix is defined by the number of its rows (m) and column (n).

$$\text{For example} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

are (2 x 2) and (3 x 3) matrices since A has 2 rows and 2 columns and B has 3 rows and 3 columns.

A matrix A with three rows and four columns is given by one of:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

or

$$A = (a_{ij}) \quad i = 1, 2, 3$$

$$j = 1, 2, 3, 4 \quad \textit{where i represents the row number whereas j represents the column number}$$

Types of matrices

Equal Matrices

Two matrices A and B are said to be equal, that is

$$A = B \qquad \text{or} \quad (a_{ij}) = (b_{ij})$$

If and only if they are identical if they both have the same number of rows and columns and the elements in the corresponding locations in the two matrices should be the same, that is, $a_{ij} = b_{ij}$ for all i. And j.

Example

The following matrices are equal $\begin{pmatrix} 3 & 4 & 0 \\ 2 & 2 & 3 \\ 5 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 0 \\ 2 & 2 & 3 \\ 5 & 1 & 1 \end{pmatrix}$

Column Matrix or column vector

A column matrix, also referred to as column vector is a matrix consisting of a single column.

For example x = $\begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix}$

Row matrix or row vector

It is a matrix with a single row

For example $y = (y_1, y_2, y_3 \ldots\ldots\ldots y_n)$

Transpose of a Matrix

The transpose of an mxn matrix A is the nxm matrix A^T obtained by interchanging the rows and columns of A.

$$A = (a_{ij})$$

The transpose of A i.e. A^T is given by

$$A^T = \underset{mxn}{(a_{ij})} = \underset{nxm}{(a_{ji})}$$

Example

Find the transposes of the following matrices

$$A=\begin{pmatrix} 1 & 5 & 7 \\ 2 & 1 & 4 \\ 0 & 9 & 3 \end{pmatrix}$$

$$B=\begin{pmatrix} b_1, b_2, b_3, b_4 \end{pmatrix}$$

$$C=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

<u>Solution</u>

i. $A^T = \begin{pmatrix} 1 & 5 & 7 \\ 2 & 1 & 4 \\ 0 & 9 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 0 \\ 5 & 1 & 9 \\ 7 & 4 & 3 \end{pmatrix}$

ii. $B^T = \begin{pmatrix} b_1, b_2, b_3, b_4 \end{pmatrix}^T = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$

iii. $C^T = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^T = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$

<u>Square Matrix</u>

A matrix A is said to be square when it has the same number of rows as columns

e.g.

$A = \begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix}$ is a square matrix of order 2

B = n × n is a square matrix of the order n

Diagonal matrices

It is a square matrix with zeros everywhere in the matrix except on the principal diagonal

e.g.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

An identity of unity matrix

It is a diagonal matrix in which each of the diagonal elements is a positive one (1)

e.g.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2×2 unit matrix $\qquad$ 3×3 unit matrix

A null or zero matrix

A null or zero matrix is a matrix whose elements are all equal to zero.

Sub matrix

The sub matrix of the matrix A is another matrix obtained from A by deleting selected row(s) and/or column(s) of the matrix A.

$$\text{e.g, if } A = \begin{pmatrix} 7 & 9 & 8 \\ 2 & 3 & 6 \\ 1 & 5 & 0 \end{pmatrix}$$

$$\text{then } A_1 = \begin{pmatrix} 2 & 3 & 6 \\ 1 & 5 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 7 & 9 \\ 1 & 5 \end{pmatrix}$$

are both sub matrices of A

OPERATION ON MATRICES

Matrix addition and subtraction

We can add any number of matrices (or subtract one matrix from another) if they have the same sizes. Addition is carried out by adding together corresponding elements in the matrices. Similarly subtraction is carried out by subtracting the corresponding elements of two matrices as shown in the following example
Example: Given A and B, calculate A + B and A – B

$$A = \begin{pmatrix} 6 & -1 & 10 & 5 \\ 3 & 4 & 2 & -5 \\ -9 & -13 & -6 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 12 & 4 & -7 & 3 \\ 0 & -4 & 10 & -4 \\ 7 & -3 & 7 & 9 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 6 & -1 & 10 & 5 \\ 3 & 4 & 2 & -5 \\ -9 & -13 & -6 & 0 \end{pmatrix} + \begin{pmatrix} 12 & 4 & -7 & 3 \\ 0 & -4 & 10 & -4 \\ 7 & -3 & 7 & 9 \end{pmatrix} = \begin{pmatrix} 18 & 3 & 3 & 8 \\ 3 & 0 & 12 & -9 \\ -2 & -16 & 1 & 9 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 6 & -1 & 10 & 5 \\ 3 & 4 & 2 & -5 \\ -9 & -13 & -6 & 0 \end{pmatrix} - \begin{pmatrix} 12 & 4 & -7 & 3 \\ 0 & -4 & 10 & -4 \\ 7 & -3 & 7 & 9 \end{pmatrix} = \begin{pmatrix} -6 & -5 & 17 & 2 \\ 3 & 8 & -8 & -1 \\ -16 & -10 & -13 & -9 \end{pmatrix}$$

If it is assumed that A, B, C are of the same order, the following properties are fulfilled:

a) Commutative law: A + B = B + A

b) Associative law: (A + B) + C = A + (B + C) = A + B + C

<u>Multiplying a matrix by a number</u>

In this case each element of the matrix is multiplied by that number

Example

$$\text{If } A = \begin{pmatrix} 6 & -1 & 10 & 5 \\ 3 & 4 & 2 & -5 \\ -9 & 13 & -6 & 0 \end{pmatrix}$$

$$\text{then } (10)A = \begin{pmatrix} 60 & -10 & 100 & 50 \\ 30 & 40 & 20 & -50 \\ -90 & 130 & -60 & 0 \end{pmatrix}$$

Matrix Multiplication

a) Multiplication of two vectors

Let row vector A represent the selling price in shillings of one unit of commodity P, Q, R respectively and let column vector B represent the number of units of commodities P, Q, R sold respectively. Then the vector product $A \times B$ will be equal to the total sales value

i. e. $A \times B =$ Total sales value

$$\text{Let } A = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \text{ and } B = \begin{pmatrix} 100 \\ 200 \\ 300 \end{pmatrix}$$

$$\text{then } \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 100 \\ 200 \\ 300 \end{pmatrix} = 400 + 1{,}000 + 1{,}800 = \text{Shs } 3{,}200$$

Rules of multiplication

i. The row vector must have the same number of elements as the column vector
ii. The first vector is a row vector and the second is a column vector
iii. The corresponding elements in each vector are multiplied together and the results obtained are added. This addition is always a single number

Going back to the example given before

$$A \times B = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 100 \\ 200 \\ 300 \end{pmatrix} = 4 \times 100 + 5 \times 200 + 6 \times 300 = \text{Shs}3{,}200, \text{ a single number}$$

b) Multiplication of two matrices

Rules

i. Multiplication is only possible if the first matrix has the same number of columns as the second matrix has rows. That is if A is the order a×b, then B has to be of the order b×c. If the A×B = D, then D must be of the order a×c.

ii. The general method of multiplication is that the elements in row m of the first matrix are multiplied by the corresponding elements in columns n of the second matrix and the products obtained are then added giving a single number.

We can express this rule as follows

Let A = $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and b = $\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$

Then A $\times$ B = D = $\begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \end{pmatrix}$

A = 2 x 2 matrix B = 2 x 3 matrix D = 2 x 3 matrix

Where

$d_{11} = a_{11} \times b_{11} + a_{12} \times b_{21}$

$d_{12} = a_{11} \times b_{12} + a_{12} \times b_{22}$

Example I

$$\begin{pmatrix} 6 & 1 \\ 2 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 0 & 2 \\ 4 & 5 & 8 \end{pmatrix} = \begin{pmatrix} 6\times3+1\times4 & 6\times0+1\times5 & 6\times2+1\times8 \\ 2\times3+3\times4 & 2\times0+3\times5 & 2\times2+3\times8 \end{pmatrix}$$

$$= \begin{pmatrix} 22 & 5 & 20 \\ 18 & 15 & 28 \end{pmatrix}$$

Example II

Matrix X gives the details of component parts used in the make up of two products P_1 and P_2 matrix Y gives details of products made on each day of the week as follows:

$$\text{Matrix X}\quad \text{Parts}$$

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \text{Products} \begin{array}{c} P_1 \\ P_2 \end{array} & \begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 3 \end{bmatrix} \end{array}$$

$$\text{Matrix Y}\quad \text{Products}$$

$$\begin{array}{cc} & \begin{array}{cc} P_1 & P_2 \end{array} \\ \begin{array}{c} \text{Mon} \\ \text{Tues} \\ \text{Wed} \\ \text{Thur} \\ \text{Fri} \end{array} & \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 2 \\ 2 & 2 \\ 1 & 1 \end{bmatrix} \end{array}$$

Use matrix multiplication to find the number of component parts used on each day of the week.

Solution:

After careful consideration, it will be easy to decide that the correct order of multiplication is YXX (Note the order of multiplication). This multiplication is compatible and also it gives the desired answer.

$$Y \times X = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 2 \\ 2 & 2 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 4 & 2 \\ 2 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 1\times3+2\times2 & 1\times4+2\times5 & 1\times2+2\times5 \\ 2\times3+3\times2 & 2\times4+3\times5 & 2\times2+3\times3 \\ 3\times3+2\times2 & 3\times4+2\times5 & 3\times2+2\times3 \\ 2\times3+2\times2 & 2\times4+2\times5 & 2\times2+2\times3 \\ 1\times3+1\times2 & 1\times4+1\times5 & 1\times2+1\times3 \end{pmatrix}$$

5 x 2 matrix   2 x 3 matrix = 5 x 3 matrix

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} \text{Mon} \\ \text{Tues} \\ \text{Wed} \\ \text{Thur} \\ \text{Fri} \end{array} & \begin{pmatrix} 7 & 14 & 8 \\ 12 & 23 & 13 \\ 13 & 22 & 12 \\ 10 & 18 & 10 \\ 5 & 9 & 5 \end{pmatrix} \end{array}$$

Interpretation

On Monday, number of component parts A used is 7, B is 14 and C is 8. in the same way, the number of component parts used for other days can be interpreted.

The determinant of a square matrix

The determinant of a square matrix det (A) or $|A|$ is a number associated to that matrix. If the determinant of a matrix is equal to zero, the matrix is called singular matrix otherwise it is called non-singular matrix. The determinant of a non square matrix is not defined.

Determination of a 2 x 2 matrix

$$(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

ii. Determinant of a 3 x 3 matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

$$a\,(ei - fh) - b(di - gf) + c(dh - eg)$$

simplify

iii. Determinant of a 4 x 4 matrix

$$A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

$$A = a\begin{pmatrix} f & g & h \\ j & k & l \\ n & o & p \end{pmatrix} - b\begin{pmatrix} e & g & h \\ i & k & l \\ m & o & p \end{pmatrix} + c\begin{pmatrix} e & f & h \\ i & j & l \\ m & o & p \end{pmatrix} - d\begin{pmatrix} e & f & g \\ i & j & k \\ m & n & o \end{pmatrix}$$

Simplify 3 x 3 determinants as in ii and then evaluate the 4 x 4 determinants.

<u>Inverse of a matrix</u>

If for an n (n square matrix A, there is another n (n square matrix B such that there product is the identity of the order n X n, In, that is A X B = B X A = I, then B is said to be inverse of A. Inverse if generally written as A-1

Hence AA-1 = I

Note: Only non singular matrices have an inverse and therefore the inverse of a singular matrix is non defined.

<u>General method for finding inverse of a matrix</u>

In order to introduce the rule to calculate the determinant as well as the inverse of a matrix, we should introduce the concept of minor and cofactor.

The minor of an element

Given a matrix A = (aij), the minor of an element aij in row i and column j (call it mij), is the value of the determinant formed by deleting row i and column j in matrix A.

Example

Let A = EMBED Equation.DSMT4

The minors are,

$$m_{11} = \begin{vmatrix} 6 & 1 \\ 3 & 0 \end{vmatrix} = 6\times 0 - 3\times 1 = -3$$

$$m_{12} = \begin{vmatrix} 5 & 1 \\ 2 & 0 \end{vmatrix} = 5\times 0 - 1\times 2 = -2$$

Similarly

$$m_{13} = \begin{vmatrix} 5 & 6 \\ 2 & 3 \end{vmatrix} = 15 - 12 = 3 \qquad m_{21} = \begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} = 0 - 9 = -9 \qquad m_{22} = \begin{vmatrix} 4 & 3 \\ 2 & 0 \end{vmatrix} = 0 - 6 = -6 \qquad m_{23} = \begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix} = 12 - 4 = 8$$

$$m_{31} = \begin{vmatrix} 2 & 3 \\ 6 & 1 \end{vmatrix} = 2 - 18 = -16 \qquad m_{32} = \begin{vmatrix} 4 & 3 \\ 5 & 1 \end{vmatrix} = 4 - 15 = -11 \qquad m_{33} = \begin{vmatrix} 4 & 2 \\ 5 & 6 \end{vmatrix} = 24 - 10 = 14$$

The cofactor of an element

The cofactor of any element a_{ij} (known as c_{ij}) is the signed minor associated with that element.
The sign is not changed if (i+j) is even and it is changed if (i+j)is odd. Thus the sign alternated whether vertically or horizontally, beginning with a plus in the upper left hand corner.

i.e. 3 x 3 signed matrix will have signs $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

Hence the cofactor of element a_{11} is m_{11} = -3, cofactor of a_{12} is $-m_{12}$ = +2 the cofactor of element a_{13} is $+m_{13}$ = 3 and so on.

Matrix of cofactors of A $= \begin{pmatrix} -3 & 2 & 3 \\ 9 & -6 & -8 \\ -16 & 11 & 14 \end{pmatrix}$

in general for a matrix M $= \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

Cofactor of a is written as A, cofactor of b is written as B and so on.
Hence matrix of cofactors of M is written as

$$= \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$$

<u>The determinant of a n×n matrix</u>

The determinant of a n×n matrix can be calculated by adding the products of the element in any row (or column) multiplied by their cofactors. If we use the symbol Δ for determinant.

Then $\Delta = aA + bB + cC$

or

$= dD + eE + fF$ e.t.c

Note: Usually for calculation purposes we take $\Delta = aA + bB + cC$

Hence in the example under discussion

$$\Delta = (4 \times -3) + (2 \times 2) + (3 \times 3) = 1$$

<u>The ad joint or transposition of a matrix</u>

The ad joint of matrix $\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$ is written as

$\begin{pmatrix} A & D & G \\ B & E & H \\ C & F & I \end{pmatrix}$ i.e. change rows into columns and columns into rows (transpose)

The inverse of the matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

is written as $\dfrac{1}{\text{determinant}} \times (\text{adjoint of the matrix of cofactors})$

$$\text{i.e. } A^{-1} = \frac{1}{\Delta} \times \begin{pmatrix} A & D & G \\ B & E & H \\ C & F & I \end{pmatrix}$$

Where $\Delta = aA + bB + cC$

Hence inverse of $\begin{pmatrix} 4 & 2 & 3 \\ 5 & 6 & 1 \\ 2 & 3 & 0 \end{pmatrix}$

is found as follows

$\Delta = (4 \times -3) + (2 \times 2) + (3 (3) = 1$

A = -3 B = 2 C = 3

D = 9 E = -6 F = -8

G = -16 H = 11 I = 14

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(note: Check if A (A-1 = $A^{-1} \times A = 1$)

APPLICATION OF MATRICES

Solution of simultaneous equations

In order to determine the solutions of simultaneous equations, we may use either of the following 2 methods

i. The cofactor method

ii. Cramers rule

The cofactor method

This method requires that we obtain

a) The minors and cofactors
b) The adjoint of the matrix
c) The inverse of the matrix
d) Multiply the original by the inverse on both sides of the matrix equation

Example

Solve the following

a) $4x_1 + x_2 - 5x_3 = 8$

$-2x_1 + 3x_2 + x_3 = 12$

$3x_1 - x_2 + 4x_3 = 5$

b) $4x_1 + 3x_3 + 5x_3 = 27$

$x_1 + 6x_2 + 2x_3 = 19$

$3x_1 + x_2 + 3x_3 = 15$

c) $4x_1 + 2x_2 + 6x_3 = 28$

$3x_1 + x_2 + 2x_3 = 20$

$10x_1 + 5x_2 + 15x_3 = 70$

d) $2x_1 + 4x_2 - 3x_3 = 12$

$3x_1 - 5x_2 + 2x_3 = 13$

$-x_1 + 3x_2 + 2x_3 = 17$

Solution

a) From a, we have

$$\underset{A}{\begin{pmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{pmatrix}} \underset{X}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \underset{b}{\begin{pmatrix} 8 \\ 12 \\ 5 \end{pmatrix}}$$

We need to determine the minors and the cofactors for the above matrix

Definition

A minor is a determinant of a sub matrix obtained when other elements are detected as shown below.
A cofactor is the product of (-1) i + j and a minor where

i = Ith row i = 1, 2, 3 …….

j = Jth row j = 1, 2, 3 …….

Cofactor of 4 (a_{11}) = $(-1)^{1+1}$ $\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = 13$

Cofactor of -2 (a_{21}) = $(-1)^{2+1}$ $\begin{vmatrix} 1 & -5 \\ -1 & 4 \end{vmatrix} = 1$

Cofactor of 3 (a_{31}) = $(-1)^{3+1}$ $\begin{vmatrix} 1 & -5 \\ 3 & 1 \end{vmatrix} = 16$

Cofactor of 1 (a_{12}) = $(-1)^{1+2}$ $\begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix} = 11$

Cofactor of 3 (a_{22}) = $(-1)^{2+2}$ $\begin{vmatrix} 4 & -5 \\ 3 & 4 \end{vmatrix} = 31$

Cofactor of -1 (a_{23}) = $(-1)^{2+3}$ $\begin{vmatrix} 4 & 5 \\ -2 & 1 \end{vmatrix} = 6$

Cofactor of -5 (a_{13}) = $(-1)^{1+3}$ $\begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} = -7$

Cofactor of +1 (a_{23}) = $(-1)^{2+3}$ $\begin{vmatrix} 4 & 1 \\ 3 & -1 \end{vmatrix} = 7$

Cofactor of 4 (a_{33}) = $(-1)^{3+3}$ $\begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix} = 14$

The matrix of C of cofactors is

$$\begin{pmatrix} 13 & 11 & -7 \\ 1 & 31 & 7 \\ 16 & 6 & 14 \end{pmatrix}$$

$$C^T = \begin{pmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{pmatrix}$$ = Adjoint of the original matrix of coefficients

The original matrix of coefficients

$$= \begin{pmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{pmatrix}$$

Therefore

$$\begin{pmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{pmatrix} = \begin{matrix} 4 & 1 & -5 & 4 & 1 \\ -2 & 3 & 1 & -2 & 3 \\ 3 & -1 & 4 & 3 & -1 \end{matrix}$$

= (48 + 3 – 10) – (-45 – 4 – 8)

= 41 + 57

= 98

The inverse of the matrix of coefficients, see (*) will be

$$= \frac{1}{98}\begin{pmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{pmatrix}$$

by multiplying the inverse on both sides of * we have,

$$\frac{1}{98}\begin{pmatrix}13 & 1 & 16\\ 11 & 31 & 6\\ -7 & 7 & 14\end{pmatrix}\begin{pmatrix}4 & 1 & -5\\ -2 & 3 & 1\\ 3 & -1 & 4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

$$= \frac{1}{98}\begin{pmatrix}13 & 1 & 16\\ 11 & 31 & 6\\ -7 & 7 & 14\end{pmatrix}\begin{pmatrix}8\\ 12\\ 5\end{pmatrix}$$

$$= \frac{1}{98}\begin{pmatrix}98 & 0 & 0\\ 0 & 98 & 0\\ 0 & 0 & 98\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \frac{1}{98}\begin{pmatrix}196\\ 490\\ 98\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}2\\ 5\\ 1\end{pmatrix}$$

$$= \begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}2\\ 5\\ 1\end{pmatrix}$$

$\therefore \quad X_1 = 2,\ X_2 = 5,\ X_3 = 1$

c) $4x_1 + 2x_2 + 6x_3 = 28$

$3x_1 + x_2 + 2x_3 = 20$

$10x_1 + 5x_2 + 15x_3 = 70$

$$= \begin{pmatrix}4 & 2 & 6\\ 3 & 1 & 2\\ 10 & 5 & 15\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}28\\ 20\\ 70\end{pmatrix}$$

$$\begin{pmatrix}4 & 2 & 6\\ 3 & 1 & 2\\ 10 & 5 & 15\end{pmatrix} = \begin{matrix}4 & 2 & 6 & 4 & 2\\ 3 & 1 & 2 & 3 & 1\\ 10 & 5 & 15 & 10 & 5\end{matrix}$$

$= (60 + 40 + 90) - (60 + 40 + 90)$

$= 0$

Hence the solutions of x_1, x_2, and x_3 do no exist. The equations are independent

Now work out part (b) on your own.

Cramers Rule in Solving Simultaneous Equations

Consider the following system of two linear simultaneous equations in two variables.

$a_{11}x_1 + a_{12}x_2 = b_1$ ....(i)

$a_{21}x_1 + a_{22}x_2 = b_2$ ....(ii)

after solving the equations you obtain

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

and

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Solutions of x_1 and x_2 obtained this way are said to have been derived using Cramers rule, practice this method over and over to internalize it. It is advisable for exam situation since it is shorter.

Example

Solve the following systems of linear simultaneous equations by Cramers' rule:

i) $2x_1 - 5x_2 = 7$

$x_1 + 6x_2 = 9$

ii) $x_1 + 2x_2 + 4x_3 = 4$

$2x_1 + x_3 = 3$

$3x_2 + x_3 = 2$

Solutions

i. $2x_1 - 5x_2 = 7$

$x_1 + 6x_2 = 9$

can be expressed in matrix form as

$$\begin{pmatrix} 2 & -5 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$$

$$\overline{A} \qquad \overline{X} \qquad \bar{b}$$

and applying cramers' rule

$$x_1 = \frac{\begin{vmatrix} 7 & -5 \\ 9 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 1 & 6 \end{vmatrix}} = \frac{87}{17} = 5\frac{2}{17}$$

$$x_2 = \frac{\begin{vmatrix} 2 & 7 \\ 1 & 9 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 1 & 6 \end{vmatrix}} = \frac{11}{17}$$

(ii) can be expressed in matrix form as

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$

and by cramers' rule

$$x_1 = \frac{\begin{vmatrix} 4 & 2 & 4 \\ 3 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix}} = \frac{22}{17}$$

$$x_3 = \frac{\begin{vmatrix} 1 & 2 & 4 \\ 2 & 0 & 3 \\ 0 & 3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix}} = \frac{7}{17}$$

$$x_2 = \frac{\begin{vmatrix} 1 & 4 & 4 \\ 2 & 3 & 1 \\ 0 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix}} = \frac{9}{17}$$

<u>Solving simultaneous Equations using matrix algebra</u>

i. Solve the equations

2x + 3y = 13

3x + 2y = 12

in matrix format these equations can be written as

$$\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ 12 \end{pmatrix}$$

Pre multiply both sides by the inverse of the matrix

$$\Delta = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5$$

and inverse of the matrix is

$$-\frac{1}{5}\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix}$$

Pre multiplication by inverse gives

$$\begin{pmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 13 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Therefore x = 2 y = 3

ii. Solve the equations

$4x + 2y + 3z = 4$

$5x + 6y + 1z = 2$

$2x + 3y = -1$

Solution:

Writing these equations in matrix format, we get

$$\begin{array}{cccc} A & \times\ B\bar{X} & = & \bar{b} \end{array}$$

$$\begin{pmatrix} 4 & 2 & 3 \\ 5 & 6 & 1 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$$

Pre-multiply both sides by the inverse

the inverse of A as found before is $A^{-1} = \begin{pmatrix} -3 & 9 & -16 \\ 2 & -6 & 11 \\ 3 & -8 & 14 \end{pmatrix}$

$$\begin{pmatrix} -3 & 9 & -16 \\ 2 & -6 & 11 \\ 3 & -8 & 14 \end{pmatrix} \begin{pmatrix} 4 & 3 & 2 \\ 5 & 6 & 1 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 & 9 & -16 \\ 2 & -6 & 11 \\ 3 & -8 & 14 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 22 \\ -15 \\ -18 \end{pmatrix}$$

Hence x = 22 y = -15 z = -18

(Note: under examination conditions it may be advisable to check the solution by substituting the value of x, y, z into any of the three original equations)

SIMPLE INPUT – OUTPUT ANALYSIS

The input output analysis is a topic which requires application of matrices
The technique analyses the flow of inputs from one sector of the economy to the other sectors thus the technique is quite useful in studying the interdependence of sectors within a single economy.
The input – output analysis was first developed by Prof Leontief hence the Leontief matrix has been developed. See the following example

Example (B)

INPUT OUTPUT TABLE

	TO			Final	Total
FROM	Agric	Industry	Service	Demand	Demand (output)
Agric	300	360	320	1080	2060
Industry	450	470	410	800	2130
Service	610	500	520	270	1900
Primary inputs	700	800	650	-	-
Total inputs	2060	2130	1900	-	-

NB: In the above table, one should be able to interpret the table e.g. of the total demand of 2060 metric tones from the agricultural sector; 300 is produced for the agricultural sector, 360 for industrial sector, 320 for the service sector and 1080 metric tones makes up the final demand.
The final demand is the additional demand besides the sectoral demand which is normally made by other users e.g. government, foreign countries, other manufacturers not included in the other sectors.
For production if items besides the inputs from other sectors namely labour capital e.t.c

Technical coefficients: ('to' sectors)

Agriculture 300 = $\frac{300}{2060}$ = 0.14

450 = $\frac{450}{2060}$ = 0.22

610 = $\frac{610}{2060}$ = 0.30

Industry 360 = $\frac{360}{2130}$ = 0.7

470 = $\frac{470}{2130}$ = 0.22

500 = $\frac{500}{2130}$ = 0.23

Service 320 = $\frac{320}{1900}$ = 0.17

410 = $\frac{410}{1900}$ = 0.22

$$520 = \frac{520}{1900} = 0.27$$

The matrix of technical coefficients is:

FROM	TO Agric	Industry	Service	Final Demand	Total Demand (output)
Agric	0.14	0.7	0.17	1080 (y_1)	2060
Industry	0.22	0.22	0.22	800 (y_2)	2130
Service	0.30	0.23	0.27	270(y_3)	1900
Primary inputs	x	x	x	-	-
	2060(x_1)	2130(x_2)	1900(x_3)	-	-

From the above table, we may develop the following equations

$0.14x_1 + 0.7x_2 + 0.17x_3 + y_1 = x_1$

$0.22x_1 + 0.22x_2 + 0.22x_3 + y_2 = x_2$

$0.30x_1 + 0.23x_2 + 0.27x_3 + y_3 = x_3$

$$\underset{\overline{A}}{\begin{pmatrix} 0.14 & 0.17 & 0.17 \\ 0.22 & 0.22 & 0.22 \\ 0.30 & 0.23 & 0.27 \end{pmatrix}} \underset{\overline{X}}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} + \underset{\overline{Y}}{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}} = \underset{\overline{X}}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(*)$$

Let the coefficient matrix be represented by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\therefore$ Equation (*) may be written as

AX + Y = X

Y = X – AX

Y = X (I-A)

$\Rightarrow$ $(I - A)^{-1}\, Y = X$

The matrix I – A is known as *<u>Leontief Matrix</u>*

Technical Coefficients

These show the units required from each sector to make up one complete product in a given sector e.g. in the above matrix of coefficients it may be said that one complete product from the agricultural sector requires 0.14 units from the agricultural sector itself, 0.22 from the industrial sector and 0.30 from the service sector
NB: The primary inputs are sometimes known as "value added"

Example 1

Determine the total demand (x) for the industry 1, 2, 3 given the matrix of technical coefficients (A), Capital and the final demand vector B.

$$A = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 0.3 & 0.4 & 0.1 \\ 0.5 & 0.2 & 0.6 \\ 0.1 & 0.3 & 0.1 \end{pmatrix} \qquad B = \begin{pmatrix} 20 \\ 10 \\ 30 \end{pmatrix}$$

From the input – output analysis

$X = (I - A)^{-1} B$, Where $X = \begin{pmatrix} X_1 \\ X_2 \\ x_3 \end{pmatrix}$ is the demand vector

$$I - A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.3 & 0.4 & 0.1 \\ 0.5 & 0.2 & 0.6 \\ 0.1 & 0.3 & 0.1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.7 & -0.4 & -0.1 \\ -0.5 & 0.8 & -0.6 \\ -0.1 & -0.3 & 0.9 \end{pmatrix}$$

$$\therefore (I - A)^{-1} = \begin{pmatrix} 0.7 & -0.4 & -0.1 \\ -0.5 & 0.8 & -0.6 \\ -0.1 & -0.3 & 0.9 \end{pmatrix}^{-1}$$

The matrix of cofactors of $(I - A)$ is

$$= \begin{pmatrix} 0.54 & 0.51 & 0.23 \\ 0.39 & 0.62 & 0.25 \\ 0.32 & 0.47 & 0.36 \end{pmatrix}$$

The transpose (adjoint) of the above matrix is

$$= \begin{pmatrix} 0.54 & 0.39 & 0.32 \\ 0.51 & 0.62 & 0.47 \\ 0.23 & 0.25 & 0.36 \end{pmatrix}$$

$$\Delta \text{ of } (I - A) = 0.495 - (0.008 + 0.126 + 0.18$$

$$= 0.809$$

$$\therefore (I - A)^{-1} = \frac{1}{0.809}\begin{pmatrix} 0.54 & 0.39 & 0.32 \\ 0.51 & 0.62 & 0.47 \\ 0.23 & 0.25 & 0.36 \end{pmatrix} = \begin{pmatrix} 0.66 & 0.48 & 0.40 \\ 0.63 & 0.77 & 0.58 \\ 0.23 & 0.31 & 0.44 \end{pmatrix}$$

Therefore X =

$$\begin{pmatrix} 0.66 & 0.48 & 0.40 \\ 0.63 & 0.77 & 0.58 \\ 0.23 & 0.31 & 0.44 \end{pmatrix}\begin{pmatrix} 20 \\ 10 \\ 30 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$$X = \begin{pmatrix} 30 \\ 37.7 \\ 21.9 \end{pmatrix}$$

The total demand **from** the three industries 1, 2 and 3 is 30 from 1, 37.7 from 2 and 21.9 from 3.

Example 2

Three clients of Disrupt, Ltd P, Q and Rare direct competitors in the retail business. In the first week of the year P had 300 customers Q had 250 customers and R had 200 customers. During the second week, 60 of the original customers of P transferred to Q and 30 of the original customers of P transferred to R. similarly in the second week 50 of the original customers of Q transferred to P with no transfers to R and 20 of the original customers of R transferred to P with no transfers to Q.

Required

a) Display in a matrix the pattern of retention and transfers of customers from the first to the second week
(4 marks)

b) Re-express the matrix that you have obtained in part (a) showing the elements as decimal fractions of the original numbers of customers of P, Q and R (2 marks) Refer to this re expressed matrix as B
c) Multiply matrix B by itself to determine the proportions of the original customers that have been retained or transferred to P, Q and R from the second to the third week. (4 marks)
d) Solve the matrix equation (xyz)B = (xyz) given that x + y + z = 1 (8 marks)
e) Interpret the result that you obtain in part (d) in relation to the movement of customers between P, Q and R (2marks **(Total 20 marks)**

Solution

a). Think of each row element as being the point from which the customer originated and each column element as being the destination e.g. 210 customers move from P to P, 60 move from P to Q and 30 move from P to R. The sum of the elements of the first row totaling 300 that is the number of customers originally with P.

Hence required matrix is

$$\begin{array}{cc} & \begin{array}{ccc} P & Q & R \end{array} \qquad \text{To} \\ \begin{array}{r} P \\ \text{From } Q \\ R \end{array} & \begin{pmatrix} 210 & 60 & 30 \\ 50 & 200 & 0 \\ 20 & 0 & 180 \end{pmatrix} \begin{pmatrix} \text{row total } 300 \\ \text{row total } 250 \\ \text{row total } 200 \end{pmatrix} \end{array}$$

b). the requirement of this part is to express each element as a decimal fraction of its corresponding row total. The second row, first element is therefore 50/250, that is 0.2 and the second element is therefore 200/250 that is 0.8.

Hence B = $\begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.1 & 0 & 0.9 \end{pmatrix}$

c). $\begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.1 & 0 & 0.9 \end{pmatrix} \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.1 & 0 & 0.9 \end{pmatrix} = \begin{pmatrix} 0.54 & 0.30 & 0.16 \\ 0.30 & 0.68 & 0.02 \\ 0.16 & 0.02 & 0.82 \end{pmatrix}$

The result can be checked by the normal rules of matrix multiplication.

d). $\begin{pmatrix} x & y & x \end{pmatrix} X \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.1 & 0 & 0.9 \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix}$

This produces from the first row

0.7x + 0.2y + 0.1z = x

Which reduces to 0.2y + 0.1z = 0.3x

Or 2y + z = 3x ………………………..(i)

Or

The second row produces, 0.2x + 0.8y = y

Reducing to 0.2x = 0.2 y

X = y ………………………..(ii)

Or

The third row produces 0.1x + 0.9z = z

Reducing to 0.1x = 0.1z

X = z ………. (iii)

At this point you will notice that condition h (ii) and condition (iii) produce 2x + x = 3x when substituted into condition (i), we therefore need extra condition x + y + z = 1 to solve the problem.

Thus x + x + x = 1

Or 3x = 1

That is x = $1/3$

Leading to x = $1/3$, y = $1/3$, z = $1/3$

e). In proportion terms this solution means that P, Q, and R will in the long term each have one third of the total customers

Example 4

There are three types of breakfast meal available in supermarkets known as brand BM1, brand BM2 and Brand BM3. In order to assess the market, a survey was carried out by one of the manufacturers. After the first month, the survey revealed that 20% of the customers purchasing brand BM1 switched to BM2 and 10% of the customers purchasing brand BM1 switched to BM3. similarly, after the first month of the customers purchasing brand

BM2, 25% switched to BM1 and 10% switched to BM3 and of the customers purchasing brand BM3 0.05% switched to BM1 and 15% switched to BM2

Required

i. Display in a matrix S, the patterns of retention and transfers of customers from the first to the second month, expressing percentage in decimal form. (2marks)

ii. Multiply matrix S by itself (that is form S^2) (5 Marks)

iii. Interpret the results you obtain in part ii with regard to customer brand loyalty (3 marks)

Solution

The objective of the first part of the question was to test the candidate's ability to formulate and manipulate a matrix, then interpret the result of such manipulation.

a. i. The matrix showing the pattern of retention and transfer from the first to the second month is

$$S = \begin{array}{cc} \begin{array}{ccc} BM1 & BM2 & BM3 \end{array} & \\ \begin{pmatrix} 0.70 & 0.20 & 0.10 \\ 0.25 & 0.65 & 0.10 \\ 0.05 & 0.15 & 0.80 \end{pmatrix} & \begin{array}{c} BM1 \\ BM2 \\ BM3 \end{array} \end{array}$$

(The second element in the first row shows the 20% movement from BM1 to BM2 and so on)

i. The product of matrix S with itself is demonstrated as follows

$$\begin{pmatrix} 0.70 & 0.20 & 0.10 \\ 0.25 & 0.65 & 0.10 \\ 0.05 & 0.15 & 0.80 \end{pmatrix} \begin{pmatrix} 0.70 & 0.20 & 0.10 \\ 0.25 & 0.65 & 0.10 \\ 0.05 & 0.15 & 0.80 \end{pmatrix} = \begin{pmatrix} 0.5450 & 0.2850 & 0.1700 \\ 0.3425 & 0.4875 & 0.1700 \\ 0.1125 & 0.2275 & 0.6600 \end{pmatrix}$$

Where for example second element in the first row, that is 0.2850 is the result of multiplying the corresponding elements of the first row of S by the second column of S and summing the product.

$$0.2850 = 0.70 \times 0.20 + 0.20 \times 0.65 + 0.10 \times 0.15$$
$$= 0.14 + 0.13 + 0.015 \text{ e.t.c.}$$

ii. The resulting matrix may be interpreted in the following way
Of the original customers who buy BM1, 54.5% will remain loyal to the brand in month three, 28.5% will have switched to BM2 and 17% will have switched to BM3.

Of the original customers who buy BM2, 48.7% will remain loyal to the brand in month three 34.25% will have switched to BM1 and 17% will have switched to BM3

Of the original customers who buy BM3, 66% will remain loyal to the brand in month three 11.25% will have switched to BM1 and 22.75% will have switched to BM2

Alternatively

In month three 54.5% of the customers buying BM1 are original customers. 34.5% came from BM2 originally and remaining 11.25% have switched from BM3 and so on.

SIMPLE MARKOV PROCESS

MARKOV CHAINS/PROCESSES

The Markov processes are defined as a set of trials which follow a certain sequence which depend on a given set of probabilities known as transition probabilities. These probabilities indicate how a particular activity or product moves from one state to another.

Applications of Markov Chains in Business

The Markov processes or chains are frequently applied as follows:-

1. Brand Switching

By using the transitional probabilities we can be able to express the manner in which consumers switch their tastes from one product to another.

2. Insurance industry

Markov analysis may be used to study the claims made by the insured persons and also decide the level of premiums to be paid in future.

3. Movement of urban population

By formulating a transition matrix for the current population in the urban areas, one can be able to determine what the population will be in say 5 years.

4. Movement of customers from one bank to another.

It is a fact that customers tend to look for efficient banks. Therefore at a certain time when a given bank installs such machinery as computers it will tend to attract a number of customers who will move from certain banks to efficient ones.

PROPERTIES OF MARKOV CHAINS

1. Each outcome in a markov process belongs to <u>a state space</u> or transition matrix. E.g.

$$\begin{array}{c|ccc} & S_1 & S_2 & S_3 \\ \hline S_1 & P_{11} & P_{12} & P_{13} \\ S_2 & P_{21} & P_{22} & P_{23} \\ S_3 & P_{31} & P_{32} & P_{33} \end{array}$$

Where S_1, S_2, S_3 are states and P_{11} P_{12} e.t.c are probabilities

2. The outcome of each trial depends on the immediate preceding activities but not on the previous activities

<u>BASIC TERMS IN MARKOV CHAINS</u>

a) Probability Vector

This is a row matrix whose elements are non-negative and also they add up to 1 e.g. u = 0.2, 0.1, 0.2, 0.5)

Example

Consider u = (¾, 0, - ¼, 1/2) Not because - ¼ is negative

v = (¾, ½, 0, ¼) Not because the sum of the element <1

w= (¼, ¼, 0, ½) Adds up to 1, each element is non negative.

Therefore it's a prob, vector

State the ones which are probability vectors

b) Stochastic matrix

A matrix whose row elements are all non negative and also add up to 1.

Example (i) M $= \begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.0 & 0.7 & 0.1 & 0.2 \\ 0.5 & 0.1 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.2 & 0.1 \end{pmatrix}$

Example ii) = Consider the following matrices

A = $\begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{3}{4} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ $B = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ $C = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}$

A is not stochastic matrix because the element in the 2^{nd} row and 3^{rd} column is negative.
B is not stochastic matrix because the elements in the second row do not add up to 1
C is stochastic matrix because each element is non negative and they add up to 1 in each row.

c) Regular stochastic matrix

A matrix P is said to be regular stochastic matrix if all the elements in P^m are all positive, where m is a power, m = 1, 2, 3 e.t.c

Let A = $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ Where A is a Stochastic Matrix

$$A^2 = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \times \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{8} & \frac{5}{8} \end{pmatrix}$$

Since the elements in A^2 and A^3 are all positive then A is regular stochastic matrix.

ABSORBING STATES

A state S_i (I = 1, 2, 3 …) of a markov chain is called absorbing if the system remains in the state, S_i once it enters there. Thus a state, S_i is absorbing if and only if the i^{th} row of the transition matrix p has a 1 on the main diagonal and zeroes every where else. See the following example.
The following matrix, P is a transition matrix of the markov chain.

$$P= \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \\ \begin{pmatrix} 1/4 & 0 & 1/4 & 1/4 & 1/4 \\ 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/4 & 1/4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

The States S2 and S5 are absorbing states since the 2^{nd} and 4^{th} rows have 1 on the main diagonal.

Probability Transition Matrices (Brand switching)

These are matrices in which the individual elements are in the form of probabilities. The probabilities represent the probability of one event following another event i.e. the probability of transition from one event to the next. The probabilities of the various changes applied to the initial state by matrix multiplication, give a forecast of the succeeding state. Normally a transition matrix is defined with its columns adding up to one and state vectors as column vectors.

In this case the succeeding state is found by pre-multiplying the transition matrix by the proceeding state (column) vector. If the transition matrices are given with their rows adding up to one, then the succeeding state is found by post multiplying the transition matrix by the preceding state (row) vector.

Example 1

The probability transition matrix of the switching probabilities, consider that two brands G and X share the market in the ratio of 60% to 40% respectively of customers. If in every week 70% of G's customers retain the brand but 30% switch to product x where as 80% of X's customers retain brand but 20% percent switch to brand G. Analyze the exchange in share market per week.

State the system next week
$$\begin{array}{cc} & \begin{array}{cc} G & X \end{array} \\ \begin{array}{c} G \\ X \end{array} & \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \end{array}$$

Column vector for initial market share
$$\begin{array}{c} G \\ X \end{array} \begin{pmatrix} 60 \\ 40 \end{pmatrix}$$

Share next week $\begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} 60 \\ 40 \end{pmatrix} = \begin{pmatrix} 50 \\ 50 \end{pmatrix}$

Share week after $\begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} 50 \\ 50 \end{pmatrix} = \begin{pmatrix} 45 \\ 55 \end{pmatrix}$

and so on

This process can continue till equilibrium is reached.

Let the market share be $\begin{pmatrix} G \\ X \end{pmatrix}$

$$\therefore \quad \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} G \\ X \end{pmatrix} = \begin{pmatrix} G \\ X \end{pmatrix}$$

0.7G + 0.2 X = G or 0.3G + 0.8X = X

0.2X = 0.3G or 0.3X = 0.2X

i.e. $\frac{G}{X} = \frac{0.3}{0.2} = \frac{3}{2}$

Hence G's share is 60% and X's share is 40%

Example 2

A marketing division toothpaste manufacturing company has worked out the following transition probability matrices concerning the behaviors of customers before and after an advertising campaign.

Transition probability matrix
(before advertising campaign)

FROM	TO: Our Brand (State I)	TO: Another Brand (Sate II)
Our brand (State I)	0.8	0.2
Another Brand (sate II)	0.4	0.6

Transition probability matrix
(After advertisement)

FROM	TO: Our Brand (State I)	TO: Another Brand (Sate II)
Our brand (State I)	0.9	0.1
Another Brand (sate II)	0.5	0.5

If the advertising campaign costs Shs 20,000 per year, would it be worthwhile for the company to undertake the campaign?

You may suppose there are 60,000 buyers of toothpaste in the market and for each customer average annual profit of the company is Shs 2.50

Solution

Let P_1 be the fraction share of our brand and P_2 be the fraction share of another brand

$$\begin{pmatrix} P_1 & P_2 \end{pmatrix}\begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} = \begin{pmatrix} P_1 & P_2 \end{pmatrix}$$

$$\begin{pmatrix} 0.8P_1 + 0.4P_2 & 0.2P_1 + 0.6P_2 \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \end{pmatrix}$$

$$0.8P_1 + 0.4P_2 = P_1 \text{ and } 0.2P_1 + 0.6P = P_2$$

$$0.4P_2 = 0.2P_1 \text{ and } 0.2P_1 = 0.4P_2$$

Thus:
$P_1 = \frac{2}{3}$ and $P_2 = \frac{1}{3}$

After Advertising

$$\begin{pmatrix} P_1 & P_2 \end{pmatrix}\begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = \begin{pmatrix} P_1 & P_2 \end{pmatrix}$$

$$\begin{pmatrix} 0.9P_1 + 0.5P_2 & 0.1P_1 + 0.5P_2 \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \end{pmatrix}$$

$$0.5P_2 = 0.1P_1 \text{ and } 0.5P_2 = 0.1P_1$$

Thus:
$P_1 = \frac{5}{6}$ and $P_2 = \frac{1}{6}$

Before Advertising

If there are 60,000 buyers

Before Advertising
$P_1 = \frac{2}{3}$

$\frac{2}{3} \times 60,000 = 40,000$ Customers will buy our brand

this implies that,

$contribution = 40,000 \times 2.5$
= Sh.100, 000

After Advertising

$P_1 = 5/6$

this implies that,

$5/6 \times 60,000 = 50,000$ customers will buy our brand

$contribution = (50,000 \times 2.5) - 20,000$

= Sh.105, 000

the difference between advertising and not advertising is

105,000 – 100,000 = Sh.5, 000 in favour of advertising,
Thus the advertising campaign is worthwhile.

PRACTICE QUESTIONS

QUESTION ONE

Because of inreasing cost increasing cost energy, the population within Maueni district seem to be shifting from the north to the south the transition matrix S describes the migration behaviour observed between the regions.

$$S = \begin{array}{cc} \text{to north} & \text{to south} \\ \end{array}$$
$$S = \begin{pmatrix} 0.90 & 0.10 \\ 0.05 & 0.95 \end{pmatrix} \begin{array}{l} \text{from north} \\ \text{from south} \end{array}$$

determine whether the populations will attain an equillibrium condition and if so, the population of the two regions.

QUESTION TWO

A simple hypothetical economy of three industries A, B and C is represented in the following table (data in millions of shillings).

User / Producer	A	B	C	Final Demand	Total Output
A	80	100	100	40	320
B	80	200	60	60	400
C	80	100	100	20	300

Determine the output vector for the economy if the final demand changes to 60 for A, 60 for B and 60 for C

QUESTION THREE

A tea blender uses two types of tea, T_1, and T_2, to produce two blends, B_1 and B_2 for sale. B_1 uses 40% of available T_1 and 60% of the available T_2 whilst B_2 uses 50% of the available T_1 and 25% of the available T_2.

Required:

a) Given that t_1 kilos of T_1 and t_2 of T_2 are made available to produce b_1 kilos of B_1 and b_2 kilos of B_2. Express the blending operation in the matrix format.

b) If 400 kilos of T_1 and 700 kilos of T_2 were made available for blending, what quantities of B_1 and B_2 would be produced?

c) If 600 kilos of B_1 and 450 kilos of B_2 were produced, use a matrix method to determine what quantities of T_1 and T_2 would be used to produce the blends.

QUESTION FOUR

Let A = $\begin{pmatrix} 2 & 2 \\ 3 & -3 \end{pmatrix}$

a) Find A^2 and A^3

b) If $F(x) = x^3 - 3x^2 - 2x + 4I$
Find F (A)

c) Find the inverse of matrix A

QUESTION FIVE

A child's toy is marketed in three sizes standard size contains 10 squares (S), 15 triangles (T) and 6 hexagons (H). The deluxe set contains 15 S, 20 T, 6 H and 4 octagons (O). The super set contains 24 T, 8 H, 16 H, 16 S and 6 (O). Squares cost 12 pence to produce, triangles cost 8p, hexagons cost 18p and octagons 22p.

The standard set is sold at £6, the deluxe set for £10 and super for £15. The manufacturer produces 100 standard sets, 80 deluxe sets and 50 super sets per week.

Use matrix form and matrix multiplications to find:

The cost of producing each set.
The number of each shape required each week
Total expenditure on shapes each week

QUESTION SIX

Matrix N below shows the number of items of type A, B, and C in warehouses Y and W. Matrix p shows the cost in pence per day of storing (S) and maintaining (M) one item each of A, B and C

$$N = \begin{array}{c} \\ Y \\ W \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 10 & 12 & 50 \\ 60 & 0 & 20 \end{pmatrix} \end{array} \qquad P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{cc} S & M \end{array} \\ \begin{pmatrix} 2 & 0.5 \\ 3 & 1.5 \\ 2 & 0.5 \end{pmatrix} \end{array}$$

a) Evaluate the matrix (N×P) and say what it represents.

b) Stock movement occurs as follows:

At the start of the day 1:
Withdrawal of 2 type B from warehouse Y, 20 of type A from warehouse W.

At the start of day 2:
Delivery of 7 type B and 10 of type C to warehouse Y and 15 of type B to warehouse W.

Evaluate the total cost of storage and maintenance for days 1 and 2.

c) Write down without evaluating a matrix expression which could be used to evaluate the storage and maintenance cost of items A, B and C for the period from day 1 to 4. Allow for the stock movements on days 1 and 2, as described in part (b). There were no stock movements on days 3 and 4.