CHAPTER TWELVE

INTRODUCTION TO CALCULUS

SPECIFIC OBJECTIVES
At the end of this topic the trainee should be able to:

- Differentiate algebraic functions;
- Integrate algebraic functions

INTRODUCTION

Calculus is a branch of mathematics which explains how one variable changes in relationship to another variable. It enables us to find the rate of change of one variable with respect to another variable.

Example

i. The rate at which business revenue is increasing at a particular stage when volume of sales is increasing.
ii. The rate at which costs are changing at a particular stage when volume of sales is given
iii. The evaluation of 'rate of change' can help us to identify when the change in one variable reaches a maximum or minimum.
iv. Calculus may be used in production management when the production manager wants to know
    a) How much is to be manufactured in order to maximize the profits, revenues e.t.c
    b) How much is to be produced in order to minimize the production costs

Calculus is divided into two sections namely:

Differentiation and integration
Differentiation deals with the determination of the rates of change of business activities or simply the process of finding the derivative of a function.
Integration deals with the summation or totality of items produced over a given period of time or simply the reverse of differentiation

The derivative and differentiation
The process of obtaining the derivative of a function or slope or gradient is referred to as derivation or differentiation.

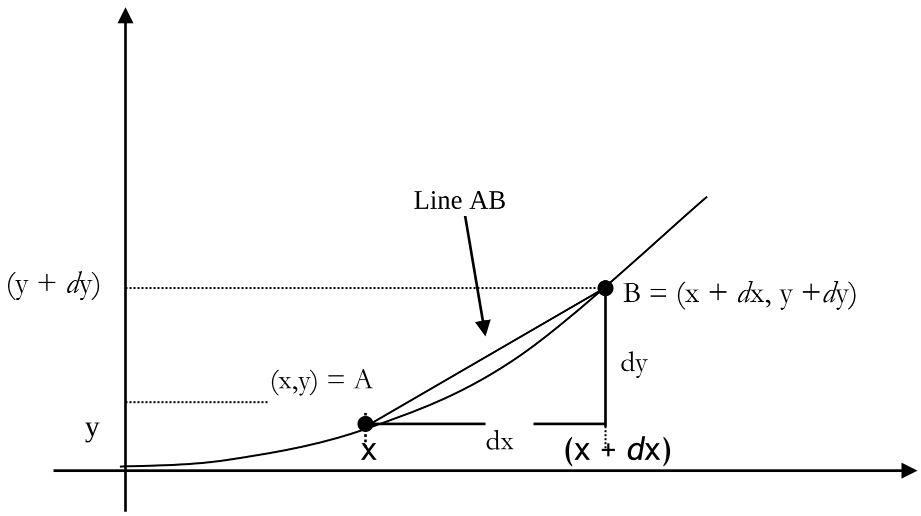
The derivative is denoted by $\frac{dy}{dx}$ or f´(x) and is given by dividing the change in y variable by the change in x variable.

The derivative or slope or gradient of a line AB connecting points (x,y) and (x+dx, y + dy) is given by

$$\frac{\Delta y}{\Delta x} = \frac{\text{Change in y}}{\text{Change in x}} = \frac{(y+dy)-y}{(x+dx)-x} = \frac{dy}{dx}$$

Where dy is a small change in y and dx is a small change in x variables.

Illustration



Rules of Differentiation

1. <u>The constant function rule</u>

If given a function y = k where k is a constant then $\frac{dy}{dx} = 0$

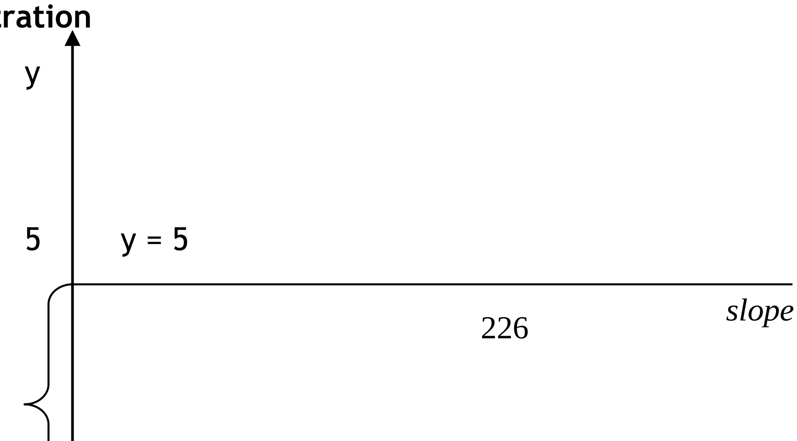
Example

Find the derivative of (i) y = 5

Solution

i. y = 5 $\frac{dy}{dx} = 0$

Illustration



$$slope = \frac{dy}{dx} = \frac{5-0}{0} = 0$$

dy

Derivative of a constant function x

2. Power function rule

Given a function $y = x^r$

$$Then \ \frac{dy}{dx} = rx^{r-1}$$

Example

Find $\frac{dy}{dx}$ for;

(i). $y = x^7$
(ii). $y = x^{2ˠ}$
(iii). $y = x^{-3}$
(iv). $y = x$

Solution

i. $y = x^7$

$\frac{dy}{dx} = 7x^{7-1} = 7x^6$

ii. $y = x^{2ˠ}$

$\frac{dy}{dx} = 2ˠ\, x^{(2ˠ - 1)}$

iii. $y = x^{-3}$

$\frac{dy}{dx} = -3x^{-3-1} = -3x^{-4}$

iv. $y = x$

$\frac{dy}{dx} = 1x^{1-1} = 1.x^0 = 1$ (*since* $x^0 = 1$)

3. Power function multiplied by a constant

If given $y = Ax^r$, then $\frac{dy}{dx} = rAx^{r-1}$

4. The sum rule

The derivative of the sum of two or more functions equals the sum of the derivatives of the functions.
For instance
If H(x) = h(x) + g(x)

Then $\frac{dy}{dx}$ or H´(x) = h´(x) + g´(x)

5. <u>The difference rule</u>

The derivative of the difference of two or more functions equals the difference of the derivatives of the functions
If H (£) = h(x) – g(x)

Then H´ (£) = h´(x) – g´(x)

Examples.

Find the derivatives of

i. $y = 3x^2 + 5x + 7$

ii. $y = 4x^2 - 2x^b$

Solution

i. $y = 3x^2 + 5x + 7$

$$\frac{dy}{dx} = \frac{d\left(3x^2\right)}{dx} + \frac{d\left(5x\right)}{dx} + \frac{d\left(7\right)}{dx}$$

$$= 6x + 5 + 0$$
$$= 6x + 5$$

ii. $y = 4x^2 - 2x^b$

$$\frac{dy}{dx} = \frac{d\left(4x^2\right)}{dx} - \frac{d\left(2x^b\right)}{dx}$$

$$= 8x - 2bx^{b-1}$$

6. <u>The product rule – both factors are functions</u>

The derivative of the product of two functions equals the derivative of the first function multiplied by the second function PLUS the derivative of the second function multiplied by the first function.
given that $H\left(x\right) = h\left(x\right).g\left(x\right)$

Then $H'\left(x\right) = h'\left(x\right).g\left(x\right) + h\left(x\right).g'\left(x\right)$

Example

Find $\frac{dy}{dx}$ for

i. y = $x^2(x)$

ii. y = $(x^2+3)(2x^3+x^2-3)$

Solution

i. y = $x^2(x)$

$$\frac{dy}{dx} = x.\frac{d\left(x^2\right)}{dx} + x^2.\frac{d\left(x\right)}{dx}$$

$$= x.2x + x^2.1$$
$$= 2x^2 + x^2$$
$$= 3x^2$$

Note that y = $x^2(x) = x^3$. Directly differentiating this we get $3x^2$.

ii. y = $(x^2+3)(2x^3+x^2-3)$

$$\frac{dy}{dx} = \frac{d\left(x^2+3\right)}{dx}.\left(2x^3+x^2-3\right)+\left(x^2+3\right).\frac{d\left(2x^3+x^2-3\right)}{dx}$$

$$2x.\left(2x^3+x^2-3\right)+\left(x^2+3\right).\left(6x^2+2x\right)$$
$$10x^4+4x^3+18x^2$$

7. Quotient Rule

The derivative of the quotient of two functions equals the derivative of the numerator times the denominator MINUS the derivative of the denominator times the numerator, all which are divided by the square of the denominator

If given H (x) = $\frac{h(x)}{g(x)}$

then $H'(x) = \frac{h'(x).g(x) - h(x).g'(x)}{\left[g(x)\right]^2}$

For example

Find $\frac{dy}{dx}$ for

i. $\frac{x}{3+x^2}$

ii. $\frac{x}{3x+7}$

Solutions

i. $\frac{x}{3+x^2}$

$$\frac{dy}{dx} = \frac{\frac{d(x)}{dx}.(3+x^2) - \frac{d(3+x^2)}{dx}.x}{(3+x^2)^2}$$

$$= \frac{(3+x^2)-(2x).x}{(3+x^2)}$$

$$= \frac{3+x^2-2x^2}{(3+x^2)} = \frac{3-x^2}{(3+x^2)}$$

ii. $y = \frac{x^3}{(3x+7)}$

$$\frac{dy}{dx} = \frac{(3x^2)(3x+7)-(3)(x^3)}{(3x+7)^2} = \frac{6x^3+21x^2}{(3x+7)^2}$$

Example

A farmer of a large farm of poultry announced that egg production per month follows the equation;

$$w = \frac{3m^3 - m^2}{m^2 + 10}$$

Where w – Total no of eggs produced per month
m – amount in kilograms of layers mash feed.

Required

Determine the rate of change of w with respect to m (i.e. the rate at which the number of eggs per month increase or decrease depending on the rate at which the kilos of layers marsh are increased).

Solution

Let $u = 3m^3 - m^2$

$\therefore \quad \frac{du}{dm} = 9m^2 - 2m$

Let $v = m^2 + 10$

$\therefore \quad \frac{dv}{dm} = 2m$

$$\therefore \quad \frac{dw}{dm} = \frac{(m^2+10)(9m^2+2m)-(3m^3-m^2)2m}{(m^2+10)^2}$$

$$= \frac{9m^4+90m^2-2m^3-20m-(6m^4+2m^3)}{(m^2+10)^2}$$

$$= \frac{3m^4+90m^2-20m}{(m^2+10)^2}$$

8. <u>Chain Rule</u>

This rule is generally applied in the determination of the derivatives of composite functions, which can be defined as a function in which another function can be considered to have taken the place of the independent variable. The composite function is also referred to as a function of a function. It is normally of the form $y = (2x^2 + 3)^3$. If we let $u = (2x^2 + 3)$, then $y = u^3$.
In order to differentiate such an equation we use the formula

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Solution

$$y = (2x^2 + 3)^3$$

Let $u = 2x^2 + 3$

$\therefore \quad \frac{du}{dx} = 2x$

Let $y = u^3$

$\therefore \quad \frac{dy}{du} = 3u^2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \text{ x } 4x = 12xu^2$$

$$= 12x(2x^2 + 3)^2$$

Example

Consider the function

$$y = (x^2 + 16x + 5)^2$$

which can be decomposed into

$y = u^2$ and $u = x^2 + 16x + 5$. in this case y is a function of $(x^2 + 16x + 5)$

Hence y = f(u) and u = g(x)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= (2u)(2x + 16)$$

$$= 2(x^2 + 16x + 5)(2x + 16)$$

The derivative of a function raised to power r; the composite function rule.

The derivative of a function raised to power r equals to the power r times the function which is raised by power (r-1), all of which is multiplied by the derivative of the function

If $y = [g(x)]^r$

Then $\frac{dy}{dx} = r[g(x)]^{r-1} \cdot g'(x)$

For example

Find $\frac{dy}{dx}$ given $y = \left(3x^2 + 4x\right)^5$

Solution

$$\frac{dy}{dx} = 5\left(3x^2 + 4x\right)^4 . \left(6x + 4\right)$$

Differentiation of an implicit function

An Implicit function is one of the $y = x^2 y + 3x^2 + 50$. it is a function in which the dependent variable (y) appears also on the right hand side.

To differentiate the above equation we use the differentiation method for a product, quotient or function of a function.

Solution

$$y = x^2 y + 3x^2 + 50$$

$$\frac{dy}{dx} = \frac{d\left(x^2 y\right)}{dx} + \frac{d\left(3x^2\right)}{dx} + \frac{d\left(50\right)}{dx}$$

$$\frac{dy}{dx} = \left[y\left(2x\right) + x^2 \frac{dy}{dx}\right] + 6x + 0$$

$$0 = 2xy + x^2 \frac{dy}{dx} - \frac{dy}{dx} + 6x$$

$$0 = 2xy + \left(x^2 - 1\right)\frac{dy}{dx} + 6x$$

$$-\left(x^2 - 1\right)\frac{dy}{dx} = 2xy + 6x$$

$$\frac{dy}{dx} = \frac{-\left(2xy + 6x\right)}{\left(x^2 - 1\right)}$$

Partial derivatives

These derivatives are used when we want to investigate the effect of one independent variable on the dependent variable.

For example, the revenues of a farmer may depend on two variables namely; the amount of fertilizer applied and also the type of the natural soil.

Let $\pi = 30x^2y + y^2 + 50x + 60y$

Where π = annual revenue in £ '000'

x = type of soil

y = amount of fertilizer applied

Required

Determine the rate of change of the π with respect to x and y

Solution

$\pi = 30x^2y + y^2 + 50x + 60y$

Differentiating π with respect to x keeping y constant we have

$$\frac{d\pi}{dx} = 60xy + 50$$

Differentiating with respect to y keeping x constant we have

$$\frac{d\pi}{dy} = 30x^2 + 2y + 60$$

Maxima, minima and points of inflexion

a) Test for relative maximum

Consider the following function of x whose graph is represented by the figure below

$y = f(x)$

$$\frac{dy}{dx} = f'(x)$$

y

$$\frac{dy}{dx} < 0$$

<u>Relative maximum point</u>

The graph of the function slopes upwards to the right between points A and C and hence has a positive slope between these two points. The function has a negative slope between points C and E. At point C, the slope of the function is Zero.

Between points X_1 and X_2 $\frac{dy}{dx} > 0$ *Where* $X_1 \le X < X_2$

and between X_2 and X_3 $\frac{dy}{dx} < 0$ *Where* $X_2 < X \le X_3$.

Thus the first test of the maximum points require that the first derivative of a function equals zero or

$$\frac{dy}{dx} = f'(x) < 0$$

The second text of a maximum point requires that the second derivative of a function is negative or

$$\frac{d^2y}{dx^2} = f''(x) < 0$$

Example

Determine the critical value for the following functions and find out the critical value that constitutes a maximum

$y = x^3 - 12x^2 + 36x + 8$

Solution

$y = x^3 - 12x^2 + 36x + 8$

then $\frac{dy}{dx} = 3x^2 - 24x + 36 + 0$

iii. The critical values for the function are obtained by equating the first derivative of the function to zero, that is:

$\frac{dy}{dx} = 0$ or $3x^2 - 24x + 36 = 0$

Hence $(x-2)(x-6) = 0$

And $x = 2$ or 6

The critical values for x are x = 2 or 6 and critical values for the function are y = 40 or 8

ii. To ascertain whether these critical values of x will give rise to a maximum, we apply the second text, that is

$\frac{d^2y}{d^2x} < 0$

$\frac{dy}{dx} = 3x^2 - 24x + 36$ and

$\frac{d^2y}{d^2x} = 6x - 24$

a) When x = 2

Then $\frac{d^2y}{d^2x} = -12 < 0$

b) When x = 6

Then $\frac{d^2y}{d^2x} = +2 > 0$

Hence a maximum occurs when x = 2, since this value of x satisfies the second condition. X = 6 does not give rise to a local maximum

b) Tests for relative minimum

There are two tests for a relative minimum point

i. The first derivative, that is

$\frac{dy}{dx} = f'(x) = 0$

ii. The second derivative, that is

$\frac{d^2y}{dx^2} = f'(x) > 0$

Example:

For the function

$h(x) = 1/3\, x^3 + x^2 - 35x + 10$

Determine the critical values and find out whether these critical values are maxima or minima. Determine the extreme values of the function

Solution

i. Critical values

$h(x) = 1/3\, x^3 + x^2 - 35x + 10$ and

$h'(x) = x^2 + 2x - 35$

by first text,

then $h'(x) = x^2 + 2x - 35 = 0$

or $(x-5)(x+7) = 0$

Hence x = 5 or x = -7

ii. The determinant of the maximum and the minimum points requires that we test the value x = 5 and –7 by the second text

$H''(x) = 2x + 2$

a) When x = -7 $h''(x) = -12 < 0$

b) When x = 5 $h''(x) = 12 > 0$

There x = -7 gives a maximum point and x = 5 gives a minimum point.

iii. Extreme values of the function

$h(x) = 1/3\ x^3 + x^2 - 35x + 10$

when x = -7, h(x) = 189 2/3

when x = 5, h(x) = -98 1/3

The extreme values of the function are h(x) = 189 2/3 which is a relative maximum and h(x) = -98 1/3 , a relative minimum

c) Points of inflexion

Given the following two graphs, points of inflexion can be determined at points P and Q as follows:

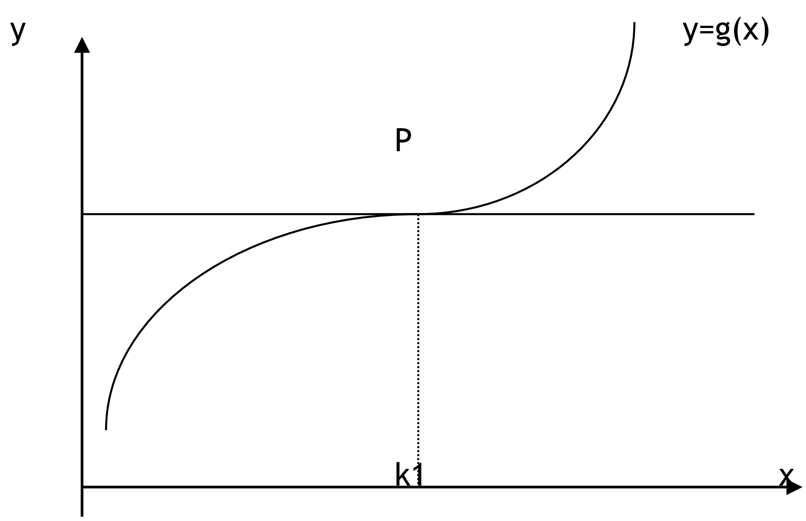
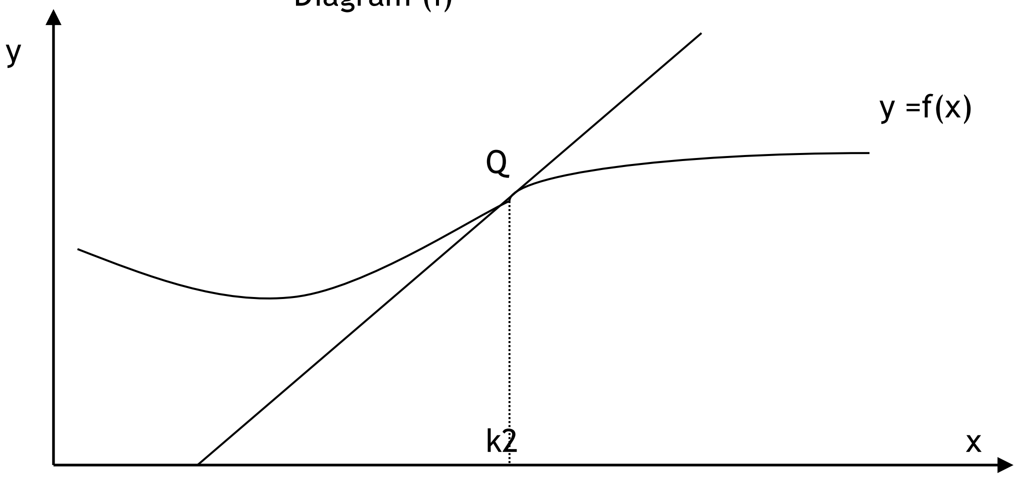


Diagram (i)



The points of inflexion will occur at point P when

$g''(x) = 0$ at $x = k_1$
$g''(x) < 0$ at $x < k_1$
$g''(x) > 0$ at $x > k_1$

and at point Q when

$f''(x) = 0$ at $x = k_1$
$f''(x) > 0$ at $x < k_1$
$f''(x) < 0$ at $x > k_1$

Example

Find the points of inflexion on the curve of the function

$y = x^3$

Solution

The only possible inflexion points will occur where

$$\frac{d^2y}{dx^2} = 0$$

From the function given

$$\frac{dy}{dx} = 3x^2 \text{ and } \frac{d^2y}{dx^2} = 6x$$

Equating the second derivative to zero, we have

$6x = 0$ or $x = 0$

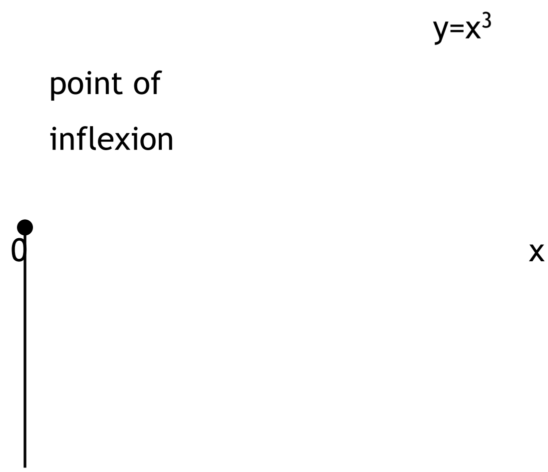
We test whether the point at which x = 0 is an inflexion point as follows

When x is slightly less than 0, $\frac{d^2y}{dx^2} < 0$ which means a downward concavity

When x is slightly larger than 0, $\frac{d^2y}{dx^2} > 0$ which means an upward concavity

Therefore we have a point of inflexion at point x = 0 because the concavity of the curve changes as we pass from the left to the right of x = 0

Illustration



Example

1. The weekly revenue Sh. R of a small company is given by

$R = 14 + 81x - \frac{x^3}{12}$ Where x is the number of units produced.

Required

i. Determine the number of units that maximize the revenue
ii. Determine the maximum revenue
iii. Determine the price per unit that will maximize revenue

Solution

i. To find maximum or minimum value we use differential calculus as follows

$$R = 14 + 18x - \frac{x^3}{12}$$

$$\frac{dR}{dx} = 81 - \frac{1}{12}.3x^2$$

$$\frac{d^2R}{dx^2} = 0 - \frac{1}{12}.3.2x = -\frac{x}{2}$$

$$put \quad \frac{dR}{dx} = 0 \quad i.e. \quad 81 - \frac{1}{4}x^2 = 0$$

which gives $x = 18$ or $x = -18$

$$\frac{d^2R}{dx^2} = -\frac{x}{2}$$

thus when $x = 18;$ $\frac{d^2R}{dx^2} = -9$ *which is negative – indicating a* maximum *value*

Therefore at x = 18, the value of R is a maximum. Similarly at x = -18, the value of R is a minimum. Therefore, the number of units that maximize the revenue = 18 units

ii. The maximum revenue is given by

$$R = 14 + 81 + 18 - \frac{(18)^3}{12}$$

= Shs. 986

ii. The price per unit to maximize the revenue is

$\frac{986}{18}$ = 54.78 or Shs.54.78

INTEGRATION

It is the reversal of differentiation. An integral can either be indefinite (when it has no numerical value) or definite (have specific numerical values)
It is represented by the sign $\int f(x)dx$.

Rules of integration

i. The integral of a constant

$\int adx = ax + c$ where a = constant

Example
Find the following

a) $\int 23dx$
b) $\int \gamma^2 dx$. (where γ is a variable independent of x, thus it is treated as a constant).

Solution

i. $\int 23dx = 23x + c$
ii. $\int \gamma^2 dx. = \gamma^2 x + c$

ii. The integral of x raised to the power n

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + c$$

Example

Find the following integrals

a) $\int x^2 dx$

b) $\int x^{-5/2}\, dx$

Solution

$$i) \quad \int x^2 dx = \tfrac{1}{3} x^3 + c$$

$$ii) \quad \int x^{-\frac{5}{2}} dx = -\tfrac{2}{3} x^{-\frac{3}{2}} + c$$

iii). Integral of a constant times a function

$$\int af(x)dx = a\int f(x)dx$$

Example

Determine the following integrals

i. $\int ax^3 dx$

ii. $\int 20x^5 dx$

Solution

$$a) \quad \int ax^3 dx = a\int x^3 dx$$

$$= \tfrac{a}{4} x^4 + c$$

$$b) \quad \int 20x^5 dx = 20\int 20x^5 dx$$

$$= \tfrac{10}{3} x^6 + c$$

iv). Integral of sum of two or more functions

$\int\{f(x) + g(x)\}\, dx = \int f(x)dx + \int g(x)\, dx$

$\int\{f(x) + g(x) + h(x)\}dx = \int f(x)dx + \int g(x)dx + \int h(x)dx$

Example

Find the following

i. $\int(4x^2 + ½\, x^{-3})\, dx$

ii. $\int(x^{3/4} + 3/7\, x^{-½} + x^5)$

Solution

$$i) \quad \int\left(4x^2 + \tfrac{1}{2} x^{-3}\right)dx = \int 4x^2 dx + \int \tfrac{1}{2} x^{-3} dx$$

$$= \tfrac{4}{3} x^3 - \tfrac{1}{4} x^2 + c$$

$$ii) \quad \int\left(x^{\frac{3}{4}} + \tfrac{3}{7} x^{-\frac{1}{2}} + x^5\right)dx = \int x^{\frac{3}{4}} dx + \int \tfrac{3}{7} x^{-\frac{1}{2}} dx + \int x^5 dx$$

$$= \tfrac{4}{7} x^{\frac{7}{4}} + \tfrac{6}{7} x^{\frac{1}{2}} + \tfrac{1}{6} x^6 + c$$

5. Integral of a difference

$\int\{f(x) - g(x)\} dx = \int f(x)dx - \int g(x) dx$

Definite integration

Definite integrals involve integration between specified limits, say a and b

The integral $\int_a^b f(x)dx$ Is a definite integral in which the limits of integration are a and b

The integrals is evaluated as follows

1. Compute the indefinite integral $\int f(x)dx$. Supposing it is F(x) + c
2. Attach the limits of integration
3. Substitute b(the upper limit) and then substitute a (the lower limit) for x.
4. Take the difference and the result is the numerical value for the definite integral.

Applying these steps to the definite integral

$$\int_a^b f(x)dx = \left[F(x)+c\right]_a^b$$

$$= \left\{\left[F(b)+c\right] - \left[F(a)+c\right]\right\}$$

$$= F(b) - F(a)$$

Example

Evaluate

i. $\int_1^3 (3x^2 + 3)dx$

ii. $\int_0^5 (x + 15)dx$

Solution

a. $\int_1^3 (3x^2 + 3)dx = [(x^3 + 3x + c)]$

$= (27 + 9 + c) - (1 + 3 + c)$

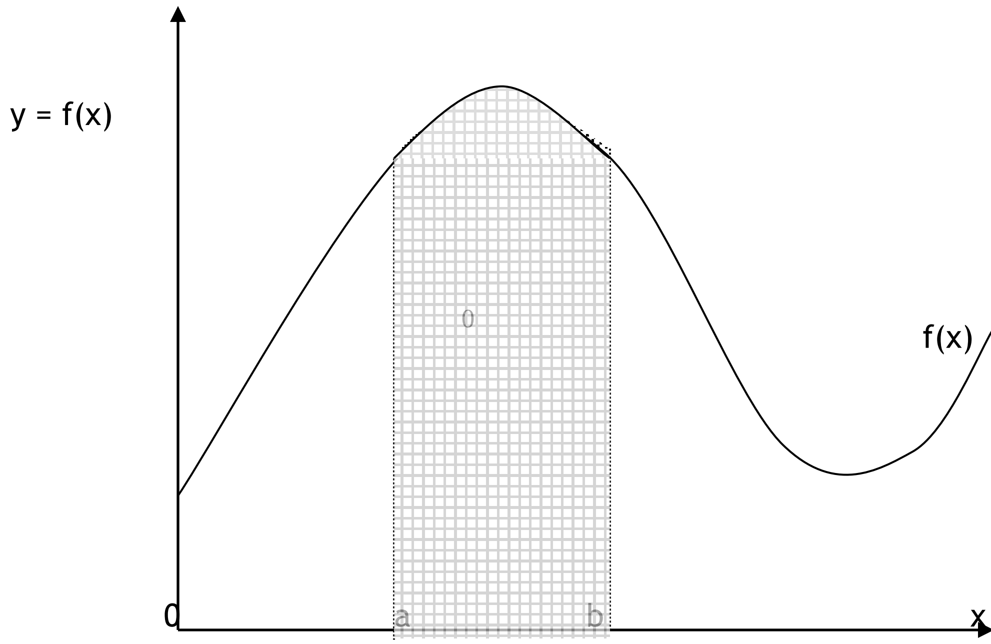
$= 32$

b. $\int_0^5 (x + 15)dx = [(½ x^2 + 15x + c)]_0^5$

$= (12 ½ + 75 + c) - (0 + 0 + c)$

$= 87 ½$

The numerical value of the definite integral $\int_a^b f(x)dx$ can be interpreted as the area bounded by the function f(x), the horizontal axis, and x=a and x=b see figure below



area under curve

Therefore $\int_a^b f(x)dx$ = A or area under the curve

Example

1. You are given the following marginal revenue function

$$MR = a + a_1 q$$

Find the corresponding total revenue function

Solution

Total revenue $= \int MR.dq = \int (a + a_1 q)dq$

$$= aq - \tfrac{1}{2} a_1 q^2 + c$$

Example 2

A firm has the following marginal cost function

$$MC = a - a_1 q + a_2 q^2$$

Find its total cost function.

Solution

The total cost C is given by

$$C = \int MC.dq$$

$$= \int (a + a_1 q + a_2 q^2).dq$$

$$= aq + \tfrac{a_1}{2} q^2 + \tfrac{a_2}{3} q^3 + c$$

Note: ***Exams focus***: Note the difference between marginal function and total function. You differentiate total function to attain marginal function, this is common in exams,
total profit = total revenue – total cost.

Example 3.
Your company manufacturers large scale units. It has been shown that the marginal (or variable) cost, which is the gradient of the total cost curve, is (92 – 2x) Shs. thousands, where x is the number of units of output per annum. The fixed costs are Shs. 800,000 per annum. It has also been shown that the marginal revenue which is the gradient of the total revenue is (112 – 2x) Shs. thousands.

Required

i. Establish by integration the equation of the total cost curve
ii. Establish by integration the equation of the total revenue curve
iii. Establish the break even situation for your company
iv. Determine the number of units of output that would
   a) Maximize the total revenue and
   b) Maximize the total costs, together with the maximum total revenue and total costs

Solution

i. First find the indefinite integral limit points of the marginal cost as the first step to obtaining the total cost curve
   Thus $\int(92 - 2x)\,dx = 92x - x^2 + c$
   Where c is constant

Since the total costs are the sum of variable costs and fixed costs, the constant term in the integral represents the fixed costs, thus if Tc are the total costs then,

$$Tc = 92x - x^2 + 800$$
$$\text{or } Tc = 800 + 92x - x^2$$

ii. As in the above case, the first step in determining the total revenue is to form the indefinite integral of the marginal revenue
   Thus $\int(112 - 2x)\,dx = 112x - x^2 + c$
   Where c is a constant

The total revenue is zero if no items are sold, thus the constant is zero and if Tr represents the total revenue, then

$$Tr = 112x - x^2$$

iii. At break even the total revenue is equal to the total costs

   Thus $112x - x^2 = 800 + 92x - x^2$

   $20x = 800$

   $x = 40$ units per annum

iv.

a) Tr = 112x – x^2

$$\frac{d(Tr)}{dx} = 112x - 2x$$

$$\frac{d^2(Tr)}{dx^2} = -2$$

at the maximum point

$$\frac{d^2(Tr)}{dx^2} = 0 \quad \text{that is } 112 - 2x = 0$$

x = 56 units per annum

Since $\frac{d^2(Tr)}{dx^2} = -2$ this confirms the maximum

The maximum total revenue is Shs. (112 x 56 – 56 x 56) x 1000

= Shs. 3,136,000

ii. Tc = 800 + 92 x – x2

$$\frac{d(Tc)}{dx} = 92 - 2x$$

$$\frac{d^2(Tc)}{dx^2} = -2x$$

At this maximum point

$$\frac{d(Tc)}{dx} = 0$$

92 – 2x = 0

92 = 2x

x = 46 units per annum

since

$$\frac{d^2(Tc)}{dx^2} = -2x \text{ this confirms the maximum}$$

the maximum costs are Shs. (800 + 92 x 46 - 46 x 46) x 1000

= Shs. 2,916,000

PRACTICE QUESTIONS

QUESTION ONE

Find the derivative of

a) y = 6x – x

b) $y = \frac{1}{x^2}$

c) $y = \sqrt{1+2x}$

d) $y = \frac{1}{\sqrt{x}}$

QUESTION TWO

A cost function is

$$\text{Ksh.}(c) = Q^2 - 30Q + 200$$

Where Q = quantity of units produced

Find the point of minimum cost.

QUESTION THREE

250 members of a certain society have voted to elect a new chairman. Each member may vote for either one or two candidates. The candidate elected is the one who polls most votes.

Three candidates x, y, z stood for election and when the votes were counted, it was found that,

59 voted for y only, 37 voted for z only
12 voted for x and y, 14 voted for x and z
147 voted for either x or y or both x and y but not for z
102 voted for y or z or both but not for x.

Required:

i) How many voters did not vote?
ii) How many voters voted for x only?
iii) Who won the election?

QUESTION FOUR

The weekly revenue Ksh.R of a small company is given by:

$R = 14 + 81x - \frac{x^3}{12}$ where x is the number of units produced

Required:

a) Determine the number of units that maximize the revenue.
b) Determine the maximum revenue.
c) Determine the price per unit that will maximize the revenue

QUESTION FIVE

A furniture firm has two operating departments; Production and sales. The firms's operating costs are split between these two departments with the resultant period of fixed costs of Shs.20,000 and Shs.6,000 respectively. The production department has a basic variable cost per unit of Shs.6 plus additional variable cost per unit of Shs.0.0002 which relates to all the manufactured items during the period. The sales department has a variable cost per unit of Shs.2. The sales department receives the finished goods from the production department and pay the basic variable cost per unit plus 80% of the same.

NB: Demand Q is given by the following function:

Q = 40,000 – 2,000P, where P is the selling price of the sales department.

Required:

a) Calculate the quantity that maximizes the profits of the production department.
b) Calculate the selling price that maximizes the profits of the sales department.
c) Determine the firm's profit as a result of adopting the quantity and selling prices
in i and ii.
d) Determine the quantity and selling price that maximize the ship's profit. What is the amount of this profit?

QUESTION SIX

a) Describe how quadratic equations can be used in decision making.

b) The demand for a commodity is given by ***p = 400 - q***. The average total cost of producing the commodity is given by

$$ATC = \frac{1000}{q} + 100 - 5q + q^2$$

Where ***p*** is the price in shillings and ***q*** is the quantity in kilograms.

Required

i) What does $\frac{1000}{q}$ in the ATC equation represent economically? (1 mark)
ii) Determine the output that leads to maximum profit and the profit at the level of output. (9 marks)

c) Alpha industries sells two products, X and Y, in related markets, with demand functions given by:

$P_x - 13 + 2X + Y = 0$
$P_y - 13 + X + 2Y = 0$

The total cost, in shillings, is given by:
$TC = X + Y$

Required:
Determine the price and the output for each good which will maximize profits. (7 marks)
(Total: 20 marks)

QUESTION SEVEN

a) The following table shows the Fixed Cost (F) and the variable cost (V) of producing 1 unit of X and 1 unit of Y:

	Product		
	X	Y	
Cost F	5	8	(Shs '000')
Cost V	4	12	

When x units of X and y units of Y are produced, the total fixed cost is Shs.640,000 and total variable cost is Shs.820,000. Express this information as a matrix equation and hence find the quantities of x and y produced using matrix algebra. (10 marks)

The marginal productivity of an industrial operation (the production of electric furnaces) is given by:

$$f(x) = \frac{60}{x^2} + 10$$

Where x is capitalization in millions of shillings. Given that, when the capitalization is Shs. Million they can produce 62 of the furnaces per week.

Required:
a) How many furnaces they will be able to produce if their capitalization increased to Shs 10 million.
b) What does the term marginal of productivity mean?
(10 marks)
(Total: 20 marks)