PORTFOLIO MANAGEMENT

HEDGING STRATEGIES

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This document is dedicated to the use of derivatives in portfolio management. This is done principally to serve two purposes: hedging and portfolio insurance. In hedging, we try to reduce or eliminate the volatility of the global position. We try to eliminate the downside of return distribution while keeping the upside using portfolio insurance.

1. Combining options and traditional assets

In this section, we review some popular strategies combining options with traditional (i.e. non derivative) assets. We will specially focus on guaranteed equity investments. Such financial products often offer exposure to an equity market while guaranteeing the safety of most of the original investment.

1.1 Covered call strategy

Covered call writing **(**also called "buy write" or "overwrite"**)** is either the simultaneous purchase of stock and the sale of a call option on this stock, or the sale of a call option against a stock currently held by an investor. The writer receives cash for selling the call but will be obliged to sell the stock at the strike price if required to.

Covered call writing appears to be one of the most popular institutional strategies. It is often described as an income enhancement strategy, as the primary objective for most investors entering into covered call writing is to increase income of stock ownership through the received option premium. These premiums reduce the potential downside loss when compared to stock-only position and if the market falls, they provide a performance cushion for managers with market benchmarks. Furthermore, the option writer keeps any dividends issued on the stock.

Which investors should consider using covered calls? An investor who is neutral to moderately bullish on some equities in his portfolio and who is willing to limit his upside potential in exchange for some sort of downside protection.

The covered call writer is often seen as a seller of traditional insurance: he receives a premium to protect the buyer from a decline in price[1]. Thus, the writer hopes that he will not have to pay any claim, that is, the stock will not decline (the larger the decline, the larger his loss), but if the stock rises, then he is indifferent as to how large the rise is. Such a strategy is ideal for a bullish investor who believes the market will be less active than generally anticipated. In order to increase the likelihood of a profit (the stock must trade above the covered call break even point), analysis should be performed to eliminate potential « losers » from consideration. This differs from traditional analysis, which typically attempts to pick up winners.

If one considers only the option premium as an extra-income, he should then sell long-dated options on high-priced high-volatility stocks. These options have the biggest premium, but are rarely the best overwrites. What overwriters really want is the highest rate of premium erosion, that is, the highest decay over a given time period. Note that there are two common approaches to managing call risk:

- the first approach presumes some market timing skills: if you believe that you can forecast price, you might sell calls only when you expect the stock price not to rise significantly above the option's strike price;

1 In that sense, selling covered calls has the same risk as selling naked puts (this derives directly from the put-call parity for European options); but curiously, selling naked puts is often seen as too risky by institutional investors who prefer selling covered calls...

- the second approach presumes that on average, the market overprices out-of-the-money call options: in that case, you might regularly sell out-of-the-money calls.

In fact, covered call writing should be seen as a risk-reducing strategy, as it converts the prospects for uncertain future capital gains into immediate cash flows. If giving up capital gains for current income makes sense, covered call writing is certainly not the only way to achieve that goal: in a sense, these immediate cash flows are similar to dividends, and the investor may be better off selecting stocks that pay high dividends. Thus, there may be good reasons for occasionally entering in a covered call strategy, but a systematic program of covered call writing might well be no better than simply buying high dividend stocks instead. Furthermore, one should not forget that systematic covered call writing generates a huge amount of commissions (initial sale of the call, occasional exercise of the call, delivery of the stock, rolling over into new calls as the old ones expire). Note also that covered call writing can also be seen as a way to execute an investment decision which cannot at present be accommodated by the stock market, for instance selling an illiquid stock.

The payoff of a covered call position is represented in the following figure:

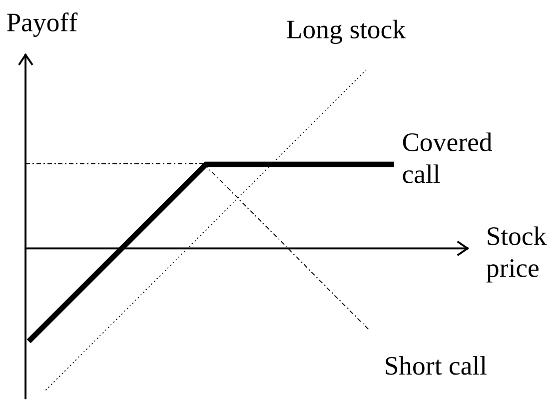


Figure 1-1: Covered call payoff at expiration

The profit potential of the total position is limited.

Example:
An investor owns 15 Deutsche Bank shares (quoted at EUR 31) and expects stable prices over the ensuing months. He decides to sell 3 December 33 calls on his shares, at EUR 3.6.
The profit and loss (per share) of the call writer at different share price at expiration are the following:

Stock price at expiration	Call writer's obligation in term of value	Profit/Loss per unit of option	Profit/Loss per stock at 31	Total Profit/loss
26	0	3.6	-5	-1.4
27	0	3.6	-4	-0.4
27.4	0	3.6	-3.6	0
28	0	3.6	-3	0.6
30	0	3.6	-1	2.6
31	0	3.6	0	3.6
33	0	3.6	2	5.6
35	-2	1.6	4	5.6
36	-3	0.6	5	5.6
37	-4	-0.4	6	5.6
38	-5	-1.4	7	5.6

If the share price on expiration date is between EUR 27.4 and the strike price of EUR 33, there will be a profit. Below EUR 27.4, there will be a loss equal to the loss on the share portfolio, plus the premium received. Above EUR 33, there will still be a profit, but it will be limited to EUR 5.6.

1.2 Enhanced indexing

In broad terms enhanced index funds aim to outperform a specific benchmark index while still closely following the shape of the benchmark returns over time; that is, keeping a low tracking error. They aim to deliver an additional constant alpha return above the benchmark. This is also to a large degree the aim of an active mutual fund. Enhanced index funds, however, distinguish themselves by mainly using quantitative methods to enhance performance. And as the argument goes by enhanced index providers: quantitative methods minimize the risk of human errors based on judgments in fundamental analysis.

There are several quantitative methods used in enhanced indexing, and they are often difficult to categorize since many providers of enhanced index products have different goals with their products[2]. Here we partly follow Ahmed and Nanda (2005) classification of enhanced index methods into two main categories: securities selection and futures contract indexing. The latter is dived into three subcategories.

1) *Security selection*. The methods use various quantitative models for stock selection such as factor analysis or technical analysis to enhance the returns and add alpha.

2) *Futures contract indexing. Index exposure is achieved through futures contracts*. Margins requirements often stay at around 5% for a 100% exposure. Thus, the remaining 95% of assets are used for further enhancement, these are:

2 AHMED P. and NANDA S., 2005, " Performance of Enhanced Index and Quantitative Equity Funds", The Financial Review, Vol. 40, pp.459-479

i) *Fixed income securities*. Returns are enhanced by investing into fixed income securities. Returns can be further boosted by taking on more credit risk (i.e. going from government bonds to corporate bonds) or moving up the yield curve to take on more duration risk.

ii) *Equity market neutral*. The proceeds are invested into a quantitative market-neutral long-short portfolio.

iii) *Leverage*. The index exposure is easily leveraged with futures contracts. Many enhanced funds in this category aim to hold a leverage factor of 1.5-2.

It is apparent that enhanced index funds come with several additional risk factors in order to beat their benchmarks. These risk factors may be credit risk, duration risk or simply leveraged risk exposure. Thus, arguing that the enhanced performance is a constant alpha return is misleading.

1.3 130/30 funds

A 130/30 fund invests 100% of their assets into a classical long-only portfolio. Additionally they hold a market neutral portfolio which is 30% short (proportional to the long-only portfolio) and 30% long different securities. The net exposure, and hence the name, is 130% long and 30% short. The short position is often achieved through derivatives, instead of actually selling borrowed stocks.

There are several reasons why this will make a good investment proposition. Studies suggest that it is more common with overvalued stocks than with undervalued stocks[3]. This is explained by among other things an overall bias from long-only funds to target undervalued stocks and that brokers favour buy-recommendations. Hence the short side of the market is more inefficient and open to more investment opportunities. Holding short positions will enable the manager to make more complete use of all his or her available information. It also enables the fund manager to underweight securities easier. There are clear theoretical benefits of allowing short-sales since it expands the efficient frontier of an investment opportunity set.

There are however some reasons to be cautious when considering investing in 130/30 funds.

Trading on the short side of the market comes with additional risks. When a short position starts losing money the position will increase, which is opposite of a long position losing money. Not only that but the loss of a short position has no bounds as opposed to a long position where only the initial capital can be lost. Thus it requires a seasoned manager with experience in *both* long and short trading to be a successful 130/30 fund manager. Last but not least—as with every active long-only manger—a 130/30 fund manager has to be superior at securities selection compared to the rest of the market. A skill that *per se* is rare and hard to find.

3 See e.g. MILLER E.M., 2005 “Why the Low Returns to Beta and Other Forms of Risk?”, Vol. 27, pp. 40-56

1.4 Using interest rates OTC products

Strategies using interest rate products can be implemented using call and put options on interest rate futures. However, the interbank market supplemented them by caps, floors and collars, which are tailor-made instruments.

Example:
Your client has a portfolio of USD 10 million invested in floating rate notes. He expects a decrease of interest rates below their current level (3%), and would like to hedge.

A hedge would consist in buying an interest rate floor on a USD 10 million notional capital with a reference rate set as the 6-month Libor, a strike price of 3% p.a. and a maturity of 2 years. Assume that the premium for such a floor is 0.30%, so that the total cost of purchasing the floor is USD 30’000.
Assume the following development for the interest rate:

- in 6 months (first settlement date), the 6 month Libor is at 2% p.a. The investor receives from his floor an amount of (3%-2%)·10’000’000/2 = USD 50’000.
- in 12 months (second settlement date), the 6 month Libor is at 2.5% p.a. The investor receives from his floor an amount of (3%-2.5%)·10’000’000/2 = USD 25’000.
- in 18 months (third settlement date), the 6 month Libor is at 3.5% p.a. The investor receives nothing from his floor.

The floor guarantees a minimal investment rate of return of 3%. If we include the hedging costs (and exclude the refinancing costs for the premium), the minimum semi-annual interest payment will be 3-(0.30·2/3) = 2.80% p.a.

The interest rate floor on the USD 10 million notional capital with a reference rate set as the 6-month Libor, a strike price of 3% p.a. and a maturity of 2 years costs a premium of 0.30%.
This cost has to be interpreted per annum, so that the total cost of purchasing the floor is USD 30’000 p.a.
Therefore, if we include the hedging costs (and exclude the refinancing costs for the premium), the minimum semi-annual interest payment will be 3 - (0.30·2/3) = 2.80% p.a.
In fact, 0.30% p.a. results in a total cost of 0.6% over the 2 years. Now, in 2 years there are 4 legs (each one six months in length). However, the first leg does not pay anything since the Libor has already been fixed so only 3 are left.
So the total cost split over 3 legs results in 0.20% per leg. Therefore the break even rate is 2.8%.

By purchasing a cap (selling a floor), the investor can hedge against higher (lower) interest rates by fixing a maximum (minimum) rate. By purchasing a collar, the investor holds both a cap and a floor, that is he fixes a certain investment range.

2. Portfolio insurance

This section develops static portfolio insurance and dynamic portfolio insurance.

2.1 Stop-loss approach

The stop-loss approach is a very basic methodology. In its first variant (SL1), if the manager wants to insure a floor Φ at time T, he invests an amount equal to $\Phi \cdot e^{-r \cdot (T-t)}$ in a risk-free asset paying a continuous return r and the rest in the risky asset portfolio.

Example:
An investor has 1 share with an initial value of EUR 100. He wants to insure his portfolio at EUR 80 for two years. The risk-free rate is 5% (zero-coupon bond with 2 years to maturity). The following conventions will be adopted:

- $P_{stock,t}$ denotes the stock price at time t
- $P_{bond,t}$ denotes the bond price at time t
- $A_{t-1,t}$ denotes the value of the stock portfolio when arriving at time t, before rebalancing
- $B_{t-1,t}$ denotes the value of the bond portfolio when arriving at time t, before rebalancing
- A_t denotes the value of the stock portfolio at time t, after rebalancing
- B_t denotes the value of the bond portfolio at time t, after rebalancing
- V_t denotes the total value of the portfolio at time t.

We have, S_0 = EUR 100, r = 5%, T = 2, Φ = EUR 80, V_0 = EUR 100. The following table illustrates SL1 insurance over 2 years in the case of a bear market. There is no rebalancing.

t	$P_{stock,t}$	$P_{bond,t}$	PV(Φ)	$A_{t-1,t}$	$B_{t-1,t}$	V_t	A_t	B_t
0	100.00	90.48	72.39			**100.00**	27.61	72.39
1	97.06	90.86	72.69	26.80	72.69	**99.49**	26.80	72.69
2	94.85	91.24	72.99	26.19	72.99	**99.18**	26.19	72.99
3	94.08	91.62	73.30	25.98	73.30	**99.28**	25.98	73.30
4	90.73	92.00	73.60	25.05	73.60	**98.66**	25.05	73.60
5	87.58	92.39	73.91	24.18	73.91	**98.09**	24.18	73.91
6	83.22	92.77	74.22	22.98	74.22	**97.20**	22.98	74.22
7	78.23	93.16	74.53	21.60	74.53	**96.13**	21.60	74.53
8	74.97	93.55	74.84	20.70	74.84	**95.54**	20.70	74.84
9	73.19	93.94	75.15	20.21	75.15	**95.36**	20.21	75.15
10	70.29	94.33	75.47	19.41	75.47	**94.88**	19.41	75.47
11	66.52	94.73	75.78	18.37	75.78	**94.15**	18.37	75.78
12	64.84	95.12	76.10	17.90	76.10	**94.00**	17.90	76.10
13	63.00	95.52	76.42	17.40	76.42	**93.81**	17.40	76.42
14	59.41	95.92	76.74	16.40	76.74	**93.14**	16.40	76.74
15	57.24	96.32	77.06	15.81	77.06	**92.86**	15.81	77.06
16	54.36	96.72	77.38	15.01	77.38	**92.39**	15.01	77.38
17	52.64	97.13	77.70	14.53	77.70	**92.23**	14.53	77.70
18	50.42	97.53	78.02	13.92	78.02	**91.95**	13.92	78.02
19	48.86	97.94	78.35	13.49	78.35	**91.84**	13.49	78.35
20	45.53	98.35	78.68	12.57	78.68	**91.25**	12.57	78.68
21	42.99	98.76	79.01	11.87	79.01	**90.88**	11.87	79.01
22	39.02	99.17	79.34	10.77	79.34	**90.11**	10.77	79.34
23	35.64	99.58	79.67	9.84	79.67	**89.51**	9.84	79.67
24	33.33	100.00	80.00	9.20	80.00	**89.20**		

- In its second variant (SL2), the manager invests everything in the risky portfolio. As soon as the value of the risky portfolio falls to $\Phi \cdot e^{-r \cdot (T-t)}$, everything is invested in the risk-free asset. Note that if, by bad luck, this level is reached too early, the manager does not have any chance of participating in a new rise.

Example:
An investor has 1 share with an initial value of EUR 100. He wants to insure his portfolio at EUR 80 for two years. The risk-free rate is 5% (zero-coupon bond with 2 years to maturity). We will use the same conventions as above.
We have: S_0 = EUR 100, r = 5%, T = 2, Φ = EUR 80, V_0 = EUR 100. The following table illustrates SL2 insurance over 2 years in the case of a bear market.

t	$P_{stock,t}$	$P_{bond,t}$	PV(Φ)	$A_{t-1,t}$	$B_{t-1,t}$	V_t	A_t	B_t
0	100.00	90.48	72.39			**100.00**	100.00	0.00
1	97.06	90.86	72.69	97.06	0.00	**97.06**	97.06	0.00
2	94.85	91.24	72.99	94.85	0.00	**94.85**	94.85	0.00
3	94.08	91.62	73.30	94.08	0.00	**94.08**	94.08	0.00
4	90.73	92.00	73.60	90.73	0.00	**90.73**	90.73	0.00
5	87.58	92.39	73.91	87.58	0.00	**87.58**	87.58	0.00
6	83.22	92.77	74.22	83.22	0.00	**83.22**	83.22	0.00
7	78.23	93.16	74.53	78.23	0.00	**78.23**	78.23	0.00
8	74.97	93.55	74.84	74.97	0.00	**74.97**	74.97	0.00
9	73.19	93.94	75.15	73.19	0.00	**73.19**	0.00	73.19
10	70.29	94.33	75.47	0.00	73.50	**73.50**	0.00	73.50
11	66.52	94.73	75.78	0.00	73.81	**73.81**	0.00	73.81
12	64.84	95.12	76.10	0.00	74.11	**74.11**	0.00	74.11
13	63.00	95.52	76.42	0.00	74.42	**74.42**	0.00	74.42
14	59.41	95.92	76.74	0.00	74.73	**74.73**	0.00	74.73
15	57.24	96.32	77.06	0.00	75.05	**75.05**	0.00	75.05
16	54.36	96.72	77.38	0.00	75.36	**75.36**	0.00	75.36
17	52.64	97.13	77.70	0.00	75.67	**75.67**	0.00	75.67
18	50.42	97.53	78.02	0.00	75.99	**75.99**	0.00	75.99
19	48.86	97.94	78.35	0.00	76.31	**76.31**	0.00	76.31
20	45.53	98.35	78.68	0.00	76.63	**76.63**	0.00	76.63
21	42.99	98.76	79.01	0.00	76.95	**76.95**	0.00	76.95
22	39.02	99.17	79.34	0.00	77.27	**77.27**	0.00	77.27
23	35.64	99.58	79.67	0.00	77.59	**77.59**	0.00	77.59
24	33.33	100.00	80.00	0.00	77.91	**77.91**		

There is only one rebalancing at period 9 when all the wealth is shifted from the risky to the risk-free asset. Note that the final portfolio value is below the floor in this example as we have rebalanced only after a drop in the market and not exactly when the portfolio value reached the present value of the floor.

2.2 Static portfolio insurance

Hayne Leland and Mark Rubinstein originally developed the concept of **portfolio insurance** in September 1976. Let us consider the case of an investor holding a well-diversified stock portfolio which is highly correlated with a given stock market index. We will assume that options and futures are available on this index. The investor wants to protect his portfolio against an unanticipated fall of the market.

A first solution would consist of controlling the beta coefficient of the portfolio with respect to the market using the techniques that will be discussed later. The investor can decrease (increase) the β when he expects a fall (rise) in the market, for example by using futures on the market index. A total immunisation can be obtained by bringing the β to zero. The resulting portfolio would only bear non-systematic risk which can be diversified away. If markets are efficient, the return on such a portfolio will be equal to the risk-free interest rate. The value of the portfolio is immunised and thus insensitive to changes in the market: if the market falls, the portfolio will keep its value; if the market rises, the portfolio will not appreciate.

But in many cases, the investor will prefer an **asymmetric protection** of his portfolio, so that its value will be guaranteed not to fall under a **floor value** when the market goes down while it will be free to appreciate when the market goes up. This second approach to the portfolio protection is provided by **portfolio insurance**.

To illustrate the portfolio insurance concept, let us compare an insured and an un-insured portfolio: the following figure shows the payoff of purchasing now (time 0) a risky portfolio and holding it over a specified time-horizon (until time T) as a function of the portfolio price at time T:

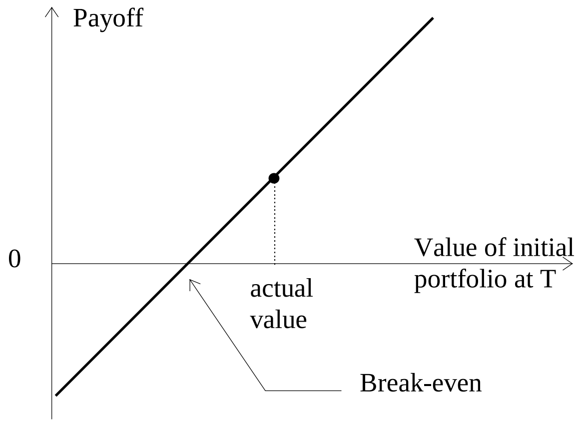


Figure 2-1: Payoff of a position as a function of portfolio final price

What will happen if the manager decides to protect the portfolio by purchasing insurance? The insurance strategy **has the goal to eliminate the risk that the portfolio may fall below a given floor.** In terms of payoffs, this reduction will be performed in an asymmetrical way around the current value of the position, as the upside potential will be retained. Graphically, this can be represented as follows:

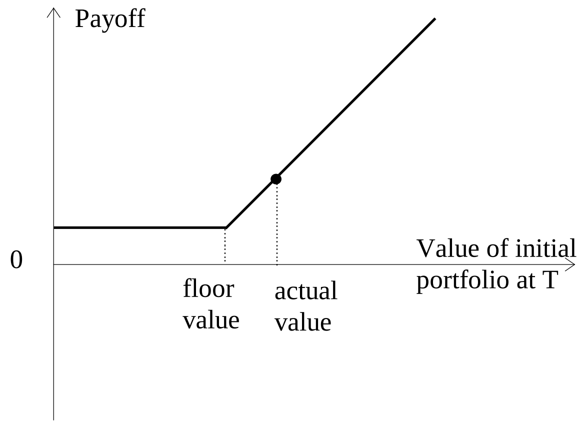


Figure 2-2: Payoff of an insured position as a function of portfolio final price

The insured portfolio value at maturity will not go below a given floor level which can be selected by the manager.

2.2.1 The protective put strategy

In its simplest form, portfolio insurance is equivalent to a position in a risky asset (S), plus an insurance policy on this same asset that guarantees the risky asset against loss under a pre-specified floor (Φ) and through a specified policy expiration date (T).

Thus, at time 0, to insure his portfolio containing one unit of a risky asset (S_0), an investor can hold a **European put option on the risky asset** maturing at time T and with a strike price K equal to the pre-specified floor (Φ). We will denote such a put option as P(S,T,K). At time T, the option provides the right (but not the obligation) to sell the underlying risky asset at a predetermined value $K=\Phi$:

- if $S_T < K$, the investor will exercise the put option, i.e. will receive an amount K against the delivery of the risky asset that he holds;
- if $S_T > K$, the put option is worthless, as the risky asset can be sold on the market for S_T which is higher than the strike price K.

The value of the insured portfolio at time T is depicted hereafter.

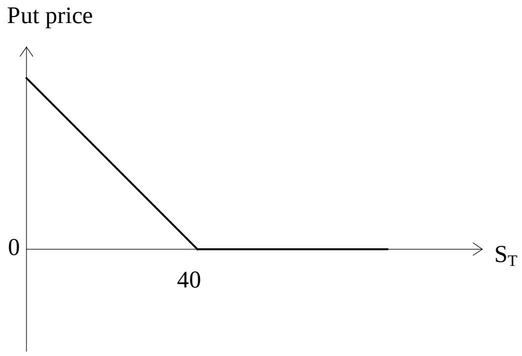
Position	Terminal Value	
	$S_T < K$	$S_T > K$
Risky asset	S_T	S_T
Index put	$K - S_T$	0
Net portfolio value	K	S_T

Table 2-1: Terminal value of stock portfolio with static portfolio insurance

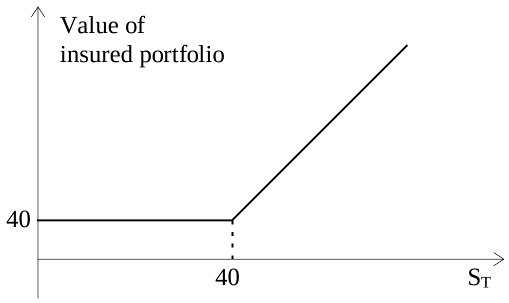
It should be clear that the value of the insured portfolio does not fall below the exercise value of the put (K), or in the context of this application the face amount of the insurance policy (Φ).

Example:
This simple example illustrates the payoff from insuring a single stock portfolio with a put. The current stock price is assumed to be EUR S_0 = 40. A put option on the stock with a strike price of EUR K = 40 and a size of 1 is available for $P(S_0,T,K)$ = EUR 4. By paying these EUR 4, the investor purchases protection against price declines below EUR 40.

The value of the put at maturity is shown as below:



Therefore, the value of the total position at maturity is the following:



This shows that the insured portfolio value cannot fall below EUR 40.

The put can be seen as an insurance policy on the risky asset price and such a strategy is called the **protective put option**.

Example:
The market is still in an up-trend in August. Signals about inflation, employment, and economic growth continue to be generally favourable yet there is lingering uncertainty. Market participants are already questioning whether inflation will pick up and the Federal Reserve will tighten short-term interest rates further in the coming months. You have USD 100’000 invested in the DJIA portfolio, and the DJIA is at 10’000.

A strategy is to purchase a put option on September futures as a way of insuring your portfolio against a possible market downturn. The put you purchase will reflect the extent to which you believe the market may fall and your risk tolerance if your opinion is incorrect. You decide to buy a 9’700 put at a premium of USD 66.10. Your cost is 10 times 66.10, or USD 661.00.

Buying the put places a floor on the value of the portfolio at the strike price. Buying a put with a strike price of 9’700 effectively locks in the value of your portfolio at USD 97’000. Above its strike price, a put is not exercised and the portfolio value is unconstrained. If you are wrong, and the market goes up, you lose the premium paid for the put. Depending on which strike price you choose, you increase or decrease your downside risk. You break even when the DJIA reaches a value of 9’633.90 (9’700 - 66.10), the strike price less the put premium. At this point, the unprotected and put-protected portfolio are equally profitable.

Note that

1. The intermediate cash-flows on the risky asset portfolio (such as dividend payments, etc.) are assumed to be zero for the sake of simplicity.
2. In a more general case, if the initial value of the portfolio to be insured is S_0, the put is on an index with initial level I_0 (and the portfolio moves like the index, i.e. beta = 1), and the option contract size is k, the number of options to be purchased is

$$N_P = \frac{\text{Portfolio value}}{\text{Index level} \cdot \text{Option contract size}} = \frac{S_0}{I_0 \cdot k}$$

Let us illustrate this by an example.

Example:
A EUR 9.6 Mio. portfolio is indexed on the Dow Jones STOXX 50 index that we will assume stand at 4800. The manager wishes to insure his portfolio at Φ = EUR 9.2 Mio. In the next six months. Option contracts on the index are available with a size of 10 times the index.

The protective put option would consist of buying 200 6-month put options contracts on the index with an exercise price of EUR 4600. If the index drops below 4600, the put options will become in the money and compensate the manager for the decline in the value of the portfolio. Suppose, for example, that the index drops to 4400 at the end of 6 months. The value of the manager's stock portfolio would be EUR 8.8 Mio. Since each option contract is on 10 times the index, the total value of the put options is EUR 0.4 Mio. This brings the value of the entire holding back up to EUR 9.2 Mio.

Of course, insurance is not free and the manager would have to pay an initial insurance premium.

The cost of establishing the insurance is simply the cost of the put. For a higher premium, an insurance policy that provides better protection (i.e. a higher Φ) against price declines can be obtained. Similarly, a cheaper insurance policy will imply bearing more downside risk, but this risk is compensated by a higher reward should the risky asset price rise.

2.2.2 Paying insurance on managed funds

Up to now we have neglected the initial insurance cost (the cost of purchasing the put). But which option should we use, if insurance cost is to be paid on managed funds?

We initially assume that the portfolio to be insured moves like the index which underlies the options, that is β=1. We first determine the portion of the portfolio which needs to be invested in the options. Then, we determine the necessary strike of the option.

In order to simplify things, we will consider the case of an individual who has a portfolio with an initial value V_0. The investor wants the end-of-year value of his portfolio to be higher than a certain floor Φ, equal to a fraction f of the initial total portfolio value: $\Phi = f \cdot V_0$. The total initial value of the portfolio is equal to the sum of the value S_0 invested in the risky asset and the premium paid for the options. This premium is equal to the number N_P of option contracts, multiplied by the contract size k and by the option quoted price P:

$$V_0 = S_0 + N_P \cdot k \cdot P(I_0, T, K).$$

We know that for a total hedge $N_P = \frac{\text{Investment value in risky asset}}{\text{Index level} \cdot \text{Option contract size}} = \frac{S_0}{k \cdot I_0}$, therefore

$$\boxed{V_0 = S_0 + P(I_0, T, K) \cdot \frac{S_0}{I_0}}.$$

In other words,

- $\dfrac{S_0}{S_0 + P(I_0,T,K)\cdot\dfrac{S_0}{I_0}} = \dfrac{I_0}{I_0 + P(I_0,T,K)}$ percent of the funds managed are invested in the risky asset and
- $\dfrac{P(I_0,T,K)}{I_0 + P(I_0,T,K)}$ percent of the funds managed are invested in the puts.

In order to establish the strike to be selected, we know that the final portfolio value V_T at expiration will be

$$V_T = S_T(\text{if } I_T \geq K), \text{ and } V_T = S_0 \cdot \frac{K}{I_0}\left(\text{if } I_T < K\right).$$

On the other hand, the floor is defined as

$$\Phi = f \cdot V_0 = f \cdot S_0 \cdot \left(1 + \frac{P(I_0,T,K)}{I_0}\right).$$

Equating the floor to the lower value attainable by V_T we obtain

$$f \cdot S_0 \cdot \left(1 + \frac{P(I_0,T,K)}{I_0}\right) = S_0 \cdot \frac{K}{I_0} \quad \Rightarrow \quad \boxed{K = f \cdot \left(I_0 + P(I_0,T,K)\right)}$$

This relation does not give a closed-form solution for K, because the put price is also a function of K. Nonetheless, in order to establish K it can be solved by numerical analysis.

Example:
Let us consider a general problem: our goal is to insure a diversified portfolio on a one year horizon by combining a stock index (I_0 = EUR 100), with a put on this index (K = EUR 105, $P(I_0,T,K)$ = EUR 12). The initial investment is EUR 100 and we want the final investment to be at least Φ = EUR 100 (which implies f = 1). The risk-free interest rate is R=2% p.a.

To obtain such a protection, a part of the portfolio has to be invested in the risk-free asset (cash). Assume that x is the percentage amount invested in the stock index and y the percentage amount invested in the money market (cash). These two unknown quantities have to satisfy the following equations:

$$\begin{cases} \text{today:} & 100 \cdot x + 12 \cdot x + 100 \cdot y = 100 \\ \text{in one year, if } I_T < K\text{:} & I_T \cdot x + (K\text{-}I_T) \cdot x + 1.02 \cdot 100 \cdot y = 100 \end{cases}$$

The first equation states that today's total position [i.e. the position on the stock index plus put option plus cash] is equal to the total initial investment of 100. The second one states that in one year, if the stock index terminates below the strike, the portfolio value must be equal to the insured value of 100. The solution of the equation system is: x=21.65%, y=75.76%. Hence EUR 21.65 is invested in the stock index, EUR 2.59 in the put option and EUR 75.76 in cash (money market). In one year, the cash has become EUR 75.76·(1+2%)= EUR 77.27. The insured portfolio value at maturity is:

Index value S_T	Stock Index + Option	Cash	Insured portfolio value (EUR)
75	22.73	77.27	100.00
80	22.73	77.27	100.00
85	22.73	77.27	100.00
90	22.73	77.27	100.00
95	22.73	77.27	100.00
100	22.73	77.27	100.00
105	22.73	77.27	100.00
110	23.81	77.27	101.08
115	24.89	77.27	102.16
120	25.97	77.27	103.25
125	27.06	77.27	104.33
130	28.14	77.27	105.41

Note that if the Index value I_T at maturity is lower than K=105, you sell the position in the stock index at 105 using the put, obtaining EUR $22.73 = 21.65 \cdot \frac{105}{100}$. If it is higher than K=105, let's say I_T =110, the option expires worthless but the stock position has value EUR $23.81 = 21.65 \cdot \frac{110}{100}$.

Note that participation in the upside begins only beyond a value of 105 for the index and that the upside capture is only 21.65%.

As we can see in the previous example, the **upside capture** is limited to:

$$\text{upside capture} = \frac{I_0}{I_0 + P(I_0, T, K) + \text{cash}}$$

that is, to the percentage of portfolio value allocated to the stock.

Example:
The Euro STOXX 50® is an index that was created by Dow Jones, the Swiss Exchange, the Deutsche Börse, and the Paris Bourse. It includes the 50 largest and most traded European stocks, that is, a diversified European portfolio. Here is an example of a certificate on this index that includes a downside risk protection.

Product	DJ EURO STOXX 50® Capital Protected Unit
Issuer	CSFB, London Branch
Underlying asset	Dow Jones Euro STOXX 50® index
Denomination	1 Unit= EUR 1'000 (Nominal Amount)
Issue price	100%
Issue date	November 5, 2001
Initial fixing Level	3'606.50 (closing of EuroStoxx50 at Issue date)
Capital Protection	90%
Upside participation	84.5%
Final Fixing	October 29, 2004
Expiration date	November 19, 2004

At issuance the issuer receives EUR 1'000 from the investor. In three years he has to guarantee a repayment of EUR 900; in order to do this he invests EUR 814.05 (assuming a yearly interest rate of 3.4024%) in zero coupon bonds. The remaining EUR 185.95 can be used to buy European calls on the EuroStoxx50 with strike 3'606.50 and maturity in three years. One certificate represents $\frac{1'000}{3'606.50} = 0.2773$ times the index. Assume that one call on the index (multiplier=1) with the above characteristics costs EUR 793.64. To assure a 100% participation one should spend 793.64·0.2773= EUR 220.06. Hence with only EUR 185.95 the upside participation is $\frac{185.95}{220.06} = 84.5\%$.

2.2.3 Static portfolio insurance using calls

Using put options was a pretty straightforward way to implement portfolio insurance. The same protection profile can be obtained by using call options, as shown in the preceding example.

At time 0, to create an insured portfolio (equivalent to one unit of a risky asset (S_0) plus the insurance), one can **buy a European call option on the risky asset** maturing at time T and with a strike price K equal to the desired floor (Φ). We will denote such a call option as C(S,T,K). In addition, one has to **invest the present value of the guaranteed floor** ($K \cdot e^{-r_f \cdot T}$) in risk-free zero-coupon bonds giving a continuous return of r_f with a maturity corresponding to the horizon of the protection.

At time T, the call option provides the right (but not the obligation) to buy the underlying risky asset at a predetermined value $K=\Phi$:

- if $S_T < K$, the call option is worthless as we can get the risky asset at a lower price by purchasing it directly in the market. Thus the value of the portfolio is equal to K, that is paid by the zero-coupon bond.
- if $S_T > K$, the call option is exercised, and we pay K (coming from the repayment of the zero-coupon bond) to receive a risky asset valued S_T. Thus the value of the portfolio is equal to S_T that is larger than K.

The value of the call and bond portfolio at time T is depicted in the following table:

Position	Initial Value	Terminal Value $S_T<K$	Terminal Value $S_T>K$
Buy zero-coupon	$K \cdot e^{-r_f \cdot T}$	K	K
Buy call	$C(S_0,T,K)$	0	S_T -K
Net portfolio value	$K \cdot e^{-r_f \cdot T} + C(S_0,T,K)$	K	S_T

Table 2-2: Terminal value of stock portfolio with static portfolio insurance

It should be clear that the value of this portfolio does not fall below the exercise value of the call (K), or in the context of this application the face amount of the insurance policy (Φ). The call plus the bond can be seen as the risky asset plus an insurance policy with an end profile that corresponds to the one obtained with the protective put strategy.

Example:
Suppose that an investor with a wealth of EUR 100 wants to reproduce the same payoffs as a given market index. Furthermore, he wants to have a minimum value of his portfolio of Φ = EUR 97.27 after one year. The index is at S_0 = EUR 100, a put option with a one year time to maturity and a strike price of K = EUR 100 costs EUR 2.81, a call option with a one year time to maturity and a strike price of 100 costs EUR 7.69. A one-year risk-free bond pays a continuous coupon rate of r_F = 5%.

If the investor wants to implement a protective put strategy, he will:

- invest 100/102.81=97.27% of his wealth in the spot asset, that is, purchase 0.9727 "units" of the index
- invest 2.81/102.81=2.73% of his wealth in the put option, that is, purchase 0.9727 put options.

The patterns of returns of his index plus put portfolio after one year are the following:

Index value in one year (EUR)	Value of insured portfolio (EUR)	Value of uninsured portfolio (EUR)
75	97.27	75
80	97.27	80
85	97.27	85
90	97.27	90
95	97.27	95
100	97.27	100
105	102.13	105
110	107.00	110
115	111.86	115
120	116.72	120
125	121.59	125

The minimum value of the portfolio is EUR 97.27, the upside potential will be 0.9727 times the value of the index.

The investor can also invest in the risk-free zero-coupon bond and in the call option. According to the above data, the investor would invest:

- EUR 92.52 in the risk-free zero-coupon bond, which will guarantee him a floor of Φ = EUR 97.27 in one year, as $92.52 \cdot e^{0.05}$ = EUR 97.27.
- EUR 7.47 in one call option (buy 0.972 call option)

The patterns of returns of his call and bond portfolio after one year are the following:

Index value in one year (EUR)	Value of insured portfolio (EUR)	Value of uninsured portfolio (EUR)
75	97.27	75
80	97.27	80
85	97.27	85
90	97.27	90
95	97.27	95
100	97.27	100
105	102.13	105
110	107.00	110
115	111.86	115
120	116.71	120
125	121.57	125

which are the same (except for rounding errors) as in the protective put strategy.

Indeed we just rediscovered the put-call parity of European options:

$$S_0 + P(S_0,T,K) = C(S_0,T,K) + K \cdot e^{-r_f \cdot T}$$

where S_0 is the underlying asset price, $P(S_0,T,K)$ the put premium, $C(S_0,T,K)$ the call premium, K the strike price of the put and the call, r_f the risk-free rate and T the time to maturity. According to it, holding a risky asset and a put is equivalent to holding a call plus a bond.

The insurance strategy based on calls corresponds to certificate of deposits (CDs) indexed to the Standard and Poor's 500 (as proposed for example by Chase Manhattan). The certificate of deposit guarantees a minimal return to the investor and a participation to the S&P 500 index appreciation (participation and minimal return being of course inversely related). This type of product is an alternative to a stock investment covered by puts and thus to portfolio insurance.

The two strategies studied, whether based on calls or on puts, are called **static**. Once initiated, they do not necessitate any intervention and guarantee the chosen protection whatever the evolution of the market happens to be.

2.2.4 Insurance when the portfolio characteristics differ from the index

When the portfolio characteristics differ from the index (that is, $\beta \neq 1$ or when the dividend yield of the index differs from the dividend yield of the portfolio), an additional step must be taken to construct the correct portfolio insurance (as there is no one-to-one relationship between the portfolio and the index returns).

Since index options are based on **price indices** (and not on performance indices), the portfolio insurance can only insure the capital. The dividend return of the index r_{MD} and the portfolio r_{PD} will be assumed as known.

Using the SML, the corresponding capital index return ($r_{MC} = r_M - r_{MD}$) can be calculated from the insured portfolio return ($r_P = r_{PC} + r_{PD}$). The CAPM's SML links the excess return of the portfolio to the index excess return:

$$r_P - r_F = \beta \cdot (r_M - r_F)$$

If we decompose these returns between a capital appreciation (r_{PC}, r_{MC}) and a dividend yield (r_{PD}, r_{MD}), we have:

$$r_{PC} + r_{PD} - r_F = \beta \cdot (r_{MC} + r_{MD} - r_F)$$

That is,

$$r_{MC} = \frac{r_{PC} + r_{PD} - r_F \cdot (1-\beta) - \beta \cdot r_{MD}}{\beta}$$

which defines the relationship between the capital market return (that can be used to insure) and the capital portfolio return (that we want to insure). Let us illustrate this.

Example:
You are managing a portfolio of USD 9.0 million indexed on the S&P 500. The index value is 1200. You want to insure your portfolio against a decline of its capital value of 5% over the next 6 months.

If the risk free rate is r_F= 6% p.a., the dividend yield of the portfolio and the S&P 500 is r_{PD}=r_{MD}=3% p.a. and β = 1, we have r_{PC}= -5% for 6 months, that is, r_{PC}= -10% p.a.

$$r_{MC} = \frac{r_{PC} + r_{PD} - r_F \cdot (1-\beta) - \beta \cdot r_{MD}}{\beta} = \frac{-10\% + 3\% - 6\% \cdot (1-1) - 1 \cdot 3\%}{1} = -10\% \text{ p.a.}$$

Thus a drop in the portfolio value of r_{PC}=-5% over the next 6 months is equivalent to a drop of r_{MC}=-5% of the index over the next six months. We have to insure against a drop of the index of 5% below its current value, that is 1'140.
Now, let us change the portfolio characteristics. If the risk free rate is r_F= 6% p.a., the dividend yield of the portfolio is r_{PD} = 4% p.a., the dividend yield of the market is r_{MD} = 3% p.a. and β = 1.5, we have r_{PC}= -5% for 6 months, that is, r_{PC}= -10% p.a.

$$r_{MC} = \frac{r_{PC} + r_{PD} - r_F \cdot (1-\beta) - \beta \cdot r_{MD}}{\beta} = \frac{-10\% + 4\% - 6\% \cdot (1-1.5) - 1.5 \cdot 3\%}{1.5} = -5\% \text{ p.a.}$$

Thus a drop in the portfolio value of r_{PC}= -5% for the next 6 months is the same as a drop of r_{MC}= –2.5% of the index over the next six months. We have to insure against a drop of the index of 2.5% below its current value, that is 1'170.

The number of options to be purchased is given by

$$N_P = \beta \cdot \frac{\text{Portfolio value}}{\text{Index level} \cdot \text{Option contract size}} = \beta \cdot \frac{S_0}{I_0 \cdot k}$$

Here also we can ask ourselves which portion of the portfolio needs to be invested in the options and which strike should be used for the option, if the insurance cost is to be paid on managed funds. To answer these questions, let's call V_0 the initial wealth. The investor wants the end-of-year value of his portfolio to be higher than a certain floor Φ, equal to a fraction f of the initial total portfolio value: $\Phi = f \cdot V_0$[4]. The total initial value of the portfolio V_0, is equal to the sum of the value S_0 of the investment in the risky asset and the premium paid for the options, which is equal to the number N_P of option contracts, multiplied by the contract size k and by the option price P:

$$V_0 = S_0 + N_P \cdot k \cdot P(I_0, T, K).$$

Since $N_P = \beta \cdot \frac{\text{Investment value in risky asset}}{\text{Index level} \cdot \text{Option contract size}} = \beta \cdot \frac{S_0}{k \cdot I_0}$, we obtain for V_0:

$$V_0 = S_0 + P(I_0, T, K) \cdot \frac{\beta \cdot S_0}{I_0}.$$

4 We assume that the portfolio floor value Φ comprises final capital plus income earned on the portfolio.

In other words,

- $\frac{I_0}{I_0 + \beta \cdot P(I_0, T, K)}$ percent of the funds managed are invested in the risky asset and
- $\frac{\beta \cdot P(I_0, T, K)}{I_0 + \beta \cdot P(I_0, T, K)}$ percent of the funds managed are invested in the puts.

The minimum value of the portfolio at expiration will be observed when the Index falls down from the initial value I_0 to the strike level K, i.e. when $(1+r_{MC}) = \frac{K}{I_0}$. In this case

$$r_P = r_F \cdot (1-\beta) + \beta \cdot r_M = r_F \cdot (1-\beta) + \beta \cdot r_{MD} + \beta \cdot (\frac{K}{I_0} - 1)$$, therefore

$$(1 + r_P) = (1 + r_F) \cdot (1-\beta) + \beta \cdot r_{MD} + \beta \cdot \frac{K}{I_0}.$$

So the final portfolio value V_T will be equal to

- $V_T = S_T$ (when $I_T \geq K$), and to
- $V_T = S_0 \cdot (1 + r_P) = S_0 \cdot \left[(1 + r_F) \cdot (1-\beta) + \beta \cdot r_{MD} + \beta \cdot \frac{K}{I_0} \right]$ (when $I_T < K$).

On the other hand, the defining condition for the floor is:

$$\Phi = f \cdot V_0 = f \cdot S_0 \left(1 + \frac{\beta \cdot P(I_0, T, K)}{I_0} \right)$$

Therefore, equating the two last equations we get:

$$\boxed{\left[(1 + r_F) \cdot (1-\beta) + \beta \cdot r_{MD} + \beta \cdot \frac{K}{I_0} \right] = f \cdot \left(1 + \frac{\beta \cdot P(I_0, T, K)}{I_0} \right)}$$

This relation is an equation which, given f (i.e. the floor in percentage terms), enables us to determine the strike K. It is an implicit equation as the put price is also a function of K, therefore there is no closed form solution, and numerical methods have to be used to solve it.

2.2.5 Rolling over a static portfolio insurance

What happens when the expiration date of the insurance is later than the expiration dates of all the option contracts that can be used (i.e., the time-horizon of the investor is longer than the time-to-maturity of the latest expiring option)? In that case, portfolio insurance has to be implemented using a sequence of short-maturity options that must be **rolled over** as they expire. Various strategies can be implemented concerning the strike price of the put option.

The simplest is the **fixed strike strategy**: the initial strike price remains constant for all the insurance period. When an option expires, the investor repurchases another option with the same strike price.

Example:
On January 1, an investor holds one stock with a quoted price of EUR 100. He starts a rolling fixed-strike portfolio insurance strategy and purchases a one month put option, with a strike of EUR 90 and an expiration date of February 1. We will assume that only one-month options are available and that the insurance period is 12 months. At the beginning of each month, he will roll-over his insurance by purchasing a new one-month put option with a strike of EUR 90, whatever the stock price.

The **fixed percentage strategy** involves in setting the strike price at a given percentage of the current stock price at the time of a roll-over.

Example:
On January 1, an investor holds one stock with a quoted price of EUR 100. He starts a rolling fixed percentage strike portfolio insurance strategy and purchases a one month put option with a strike of EUR 90 and an expiration date of February 1. We will assume that only one-month options are available and that the insurance period is 12 months. At the end of each month, he will roll-over his insurance by purchasing a new one-month put option with a strike of 0.9 times the current stock price.

For example, on February 1, if the stock price has risen to EUR 120, he will purchase a one-month put option with a strike of EUR 108. If the stock price has declined to EUR 95, he will purchase a one-month put option with a strike of EUR 85.5.

The **ratchet strategy** is a mixture of the first two: initially, the strike price is still set at a fixed percentage of the current stock price. At the time of roll-over, the strike price can never be lowered; it can only increase, if the fixed percentage of the current stock price is higher than the previous month strike price.

Example:
On January 1, an investor holds one stock with a quoted price of EUR 100. He starts a rolling ratchet portfolio insurance strategy and purchases a one month put option with a strike of EUR 90 and an expiration date of February 1. We will assume that only one-month options are available and that the insurance period is 12 months. At the end of each month, he will roll-over his insurance by purchasing a new one-month put option with a strike of 0.9 times the current stock price but with a minimum strike price equal to the strike price of the preceding month.

For example, on February 1, if the stock price has climbed to EUR 120, he will purchase a one-month put option with a strike of EUR 108. On March 1, if the stock price has declined to EUR 95, he will purchase a one-month put option with a strike of EUR 108 since the put strike cannot be lowered.

However, it should be clear that market conditions have a dramatic impact on the cost of rolling the insurance.

2.2.6 Practical problems with static portfolio insurance

Static portfolio insurance suffers from a series of major problems:

- the maximum maturity quoted is often much shorter than the desired horizon for the protection and that maximum maturity is rarely liquid enough. Suppose that there is no liquidity on the desired option beyond 6 months. After each 6-month period, a roll over can be effected but this practice is costly. The effective cost of the protection is only known when the last premium is paid and depends largely on the moves of the underlying asset until maturity of the protection (this is called a **path dependant insurance** and this property is clearly undesirable);

- sometimes only American options are available. American options used on the markets can be more expensive than the European ones, but the investor cares only about the value of his/her portfolio at expiration and would be happier with European options;
- options on the exchanges are standardised: only a few exercise prices and maturities are used. Moreover, the underlying asset used for the options may differ from the portfolio to be insured;
- liquidity on the desired options may fall short;
- option on the underlying may not exist. In Switzerland, for example, listed puts do not exist for all securities;
- in the case of a portfolio, the total risk is not the sum of the individual risks and it would be expensive and unnecessary to purchase puts for each individual risk; what is needed is in fact a put for the whole portfolio[5].

For these reasons, other insurance strategies have been developed in a dynamic (rather than static) framework.

2.3 Dynamic portfolio insurance

As we have seen, an alternative approach to the static put option buying involves creating the put option synthetically by taking a position in the underlying asset (or futures on the underlying asset) so that the delta of the position is maintained equal to the delta of the required option. The synthetic option can be created from trades in stocks themselves or from trades in index futures contracts. We will illustrate this in the context of a stock portfolio.

2.3.1 Dynamic insurance with stocks

We will first examine the creation of a put by trades in the stocks themselves. Note that the price of a put option on an index paying a known continuous dividend yield of y is given by:

$$P = Ke^{-r(T\text{-}t)}N(-d_2) - Se^{-y(T\text{-}t)}N(-d_1)$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r_f - y)\cdot(T-t)}{\sigma\cdot\sqrt{T-T}} + \frac{1}{2}\sigma\cdot\sqrt{T-t}$$

$$d_2 = d_1 - \sigma\cdot\sqrt{\tau}$$

Thus, from the above formula, it should be clear that the delta of a European put on an index is given by:

$$\Delta_P = -e^{-y\cdot(T-t)}\cdot N(-d_1) = e^{-y\cdot(T-t)}\cdot[N(d_1) - 1]$$

5 Forming a portfolio reduces the unsystematic risk of all the assets. This means that the total risk of the portfolio (i.e. its volatility) is smaller than the sum of the risks (i.e. volatility) of the individual assets. As a major price factor of an option is the expected value of the volatility, a decrease of the volatility will lead to a lower price of the option, that is, a lower price of the insurance. This implies a smaller option price on the portfolio than the sum of the options prices on each individual asset.

Since in this case, the fund manager's portfolio mirrors the index, Δ_P is also the delta of a put on the portfolio when it is regarded as a single security. Thus, in order to create the put option synthetically, the fund manager should ensure that at any time a proportion:

$$e^{-y\cdot(T-t)} \cdot [1 - N(d_1)]$$

of the stocks in the original portfolio have been sold ($\Delta_P < 0$), and the proceeds invested in risk-free assets.

Using this strategy to create portfolio insurance means that at any given time, funds are divided between the stock portfolio on which insurance is required and risk-free assets. Then, trading is done on a continuous-time basis:

- as the value of the original portfolio declines, the delta of the put becomes more negative, the position in the stock portfolio is decreased (i.e. some of the original portfolio must be sold) and risk-free assets are purchased. The maximum limit will be reached when the entire original portfolio has been sold ($\Delta_P = -1$), that is, when the portfolio is made up of only risk free assets;
- as the value of the original portfolio increases, the delta of the put becomes less negative, risk-free assets are sold and the position in the stock is increased (the proportion of the portfolio sold must be decreased, i.e. some of the original portfolio must be repurchased). The maximum limit will be reached when the entire original portfolio has been repurchased ($\Delta_P = 0$), that is, when the portfolio is made up of only risky assets.

The cost of the insurance arises from the fact that the portfolio manager is always selling **after** a decline in the market and buying **after** a rise in the market.

2.3.2 Dynamic insurance with futures

Using futures to create portfolio insurance may be preferable to using the underlying asset, as the transaction costs associated with trades in futures are generally lower. Furthermore, it allows keeping the original portfolio intact (if it exists) by selling futures contracts.

Note that if the underlying asset is a non dividend paying stock (or index), the futures price $F_{t,T}$ is given by:

$$F_{t,T} = S_t \cdot e^{r_f \cdot (T-t)}$$

where T is the maturity of the futures contract. The delta of the futures contract is therefore $e^{r_f \cdot (T-t)}$. When the underlying stock (or index) pays a continuous dividend yield y,

$$F_{t,T} = S_t \cdot e^{(r_f - y)\cdot(T-t)}$$

and the delta of the futures contract is $e^{(r_f - y)\cdot(T-t)}$. Therefore, $e^{-(r_f - y)\cdot(T-t)}$ futures contracts have the same sensitivity to stock price movements as one stock. The required position in futures contract at time t for delta hedging, knowing the required position in the underlying asset for delta hedging h^* is:

$$h_F^* = e^{-(r_f - y)\cdot(T-t)} \cdot h^*$$

Therefore the EUR amount of futures contracts sold as a proportion of the value of the portfolio should be:

$$e^{-(r_f - y)\cdot(T^* - t)} \cdot \left[e^{-y\cdot(T-t)} \cdot \left[1 - N(d_1) \right] \right] = e^{y\cdot(T^* - T)} \cdot e^{-r_f \cdot (T^* - t)} \cdot \left[1 - N(d_1) \right]$$

where, T^* is the maturity date of the futures contract and T is the maturity date of the option. If the portfolio is worth k_1 times the index and each index futures contract is on k_2 times the index, this means that the EUR amount of futures contracts sold at any given time as a proportion of the value of the portfolio should be

$$N_F = e^{y\cdot(T^* - T)} \cdot e^{-r_f \cdot (T^* - t)} \cdot \left[1 - N(d_1) \right] \cdot \frac{k_1}{k_2}$$

Let us illustrate this.

Example:

A manager has a USD 90 Mio. portfolio that mirrors the S&P 500 index returns. The S&P 500 index is at 900, and the manager wants to insure his portfolio value against a drop below USD 87 million in the next 6 months. The volatility of the index is 25% p.a., the continuously compounded risk-free rate is 3% p.a., and the continuously compounded dividend yield on the index, *y*, is 3% p.a.

- The manager can buy 1’000 6-month put options contracts on the S&P 500 with an exercise price of 870. If the index drops below 870, the put options become in the money and provide the manager with compensation for the decline in the value of the portfolio. If, for instance, the index drops to 850, the portfolio value will drop to USD 85 Mio. Since each option is on 100 times the index, the total value of the put option is USD 2 Mio. This brings the value of the entire holding back to USD 87 Mio. Note that we did not consider the initial cost of establishing the insurance.
- The manager can also create a synthetic option. The delta of the required put option is given by $d_1 = 0.2802$:

$$e^{-y\cdot(T-t)} \cdot \left(N(d_1) - 1 \right) = -0.3839$$

Hence, if trades in the portfolio are used to create the option, 38.39% of the portfolio should be sold initially (and other adjustments will be needed in the future).

- Finally, the manager can use 9-month futures on the S&P 500. In that case, $T^* - T = 0.25$, $T^* - t = 0.75$, $k_1 = 100'000$, $k_2 = 500$, so that the number of futures shorted is given by:

$$e^{y\cdot(T^* - T)} \cdot e^{-r\cdot(T^* - t)} \cdot \left[1 - N(d_1) \right] \cdot \frac{K_1}{K_2} = 76.77$$

Note that we have assumed that the portfolio mirrors the index. The hedging scheme can be adjusted to deal with other situations (i.e. $\beta \neq 1$):

- the strike price for the options used should be the expected level of the market index when the portfolio's value reaches its insured value;
- the number of index options used should be β times the number of options that would be required if the portfolio had a beta of 1.0.

2.3.3 The rebalancing frequency and other problems

An important issue while creating put options synthetically for portfolio insurance is the frequency with which the portfolio manager's position should be adjusted or rebalanced. With no transaction costs, continuous rebalancing is optimal. However as transactions costs increase, the optimal frequency of rebalancing declines because transaction costs involved in frequent portfolio adjustments erode the benefits.

This issue is discussed in papers dealing with option pricing with transaction costs and optimal solutions are still in the process of being derived.

Etzioni[6] studied three non-continuous rebalancing disciplines for portfolio insurance: rebalancing at regular time intervals of various lengths (the “time” discipline), rebalancing every time the market moves by a pre-specified percentage since the last adjustment (the “market move” discipline) and rebalancing each day when the actual stock and bond mix lags the required mix by more than a pre-specified lag factor in either direction (the “lag” discipline[7]).

Using Monte-Carlo simulation, a standard deviation of 13%, a constant interest rate of 8%, a dividend yield of 4%, and assuming transaction costs of 0.15% and a bid-ask spread of USD 0.10, his results for insuring a hundred million USD portfolio over one year are shown below.

Time discipline					
Number of rebalance intervals	12 (monthly)	52 (weekly)	104 (semi-weekly)	250 (daily)	2000 (hourly)
Average annual transaction costs (‘000 USD)	148.8	323.6	463.4	745.5	2044.7
Average annual replication error (‘000 USD)	893.4	441.1	342.7	199.1	92.4

Lag discipline					
Maximum permissible lag (%)	7	5	3	1	0
Average annual transaction costs (‘000 USD)	182.2	217.5	251.6	439.3	745.5
Average annual replication error (‘000 USD)	733.6	657.3	351.8	241.9	199.1

Market move discipline						
Minimal move to trigger a rebalance (%)	5	4	3	2	1	0
Average annual transaction costs (‘000 USD)	267.8	336.4	377.0	475.5	524.3	745.5
Average annual replication error (‘000 USD)	1170.8	1027.0	583.4	432.4	255.2	199.1

Table 2-3: Impact of various rebalancing strategies

6 See Ethan Etzioni, “Rebalance disciplines for portfolio insurance”, Journal of Portfolio Management, Fall 1986.

7 Note that the “lag” discipline requires adjustment to only the extent of the “over-lag” rather than a full adjustment to the current required mix. If, for instance, we define a lag of ±2% and the current position is at 100 while the required position is at 104, the rebalancing process just needs to build a position at 101.92 (that is, 104 minus 2%) and not 104.

According to these results, it seems that the “lag” strategy turned out to be the most effective, with a permissible lag of 3% proving the most desirable cost/error trade-off.

There exist a few other problems with dynamic portfolio insurance:

- it requires (in theory) continuous trading;
- it cannot take jumps[8] (models with jumps exist, but their estimation is not very satisfactory);
- it requires a constant volatility (alternative models exist);
- interest rates should be both borrowing and lending rates;
- it breaks down with transaction cost (alternative models are still being developed).

Those problems are acceptable, but we have to balance costs versus quality of replication and constant time interval for replication is not optimal. This implies that insurance is not always undertaken.

2.3.4 Portfolio insurance and market crash

A disputed issue is whether volatility is caused solely by the arrival of new information or whether trading itself generates volatility. In other words, we would like to know whether portfolio insurance increases the volatility of stock price movements.

Portfolio insurance schemes such as those described above have the potential to increase volatility. When the market declines, portfolio insurers have to sell stocks to readjust their positions, and that may accentuate the decline; those who have used futures contracts have to reduce their futures positions, driving down futures prices and also creating a selling pressure on stocks via the mechanism of index arbitrage. When the market rises, a similar conclusion applies for an accentuation of the rise.

Whether portfolio insurance schemes affect market volatility depends on how easily the market can absorb the trades that are generated by portfolio insurance. If portfolio insurance trades represent a very small fraction of all trades, it is unlikely to have any effect. But as portfolio insurance becomes more widespread, it is likely to have a destabilising effect on the market[9].

8 Creating put options on the index synthetically does not work well if the volatility of the index changes rapidly or if the index exhibits large jumps. For instance on October 19, 1987, the Dow Jones Industrial Average dropped by over 500 points. Portfolio managers who had insured themselves by buying traded put options survived this crash well. Those who had instead chosen to create put options synthetically found that they were unable to sell either stocks or index futures fast enough to protect their position.

9 For instance, the report of the Brady commission on the October 19, 1987 crash estimated that on Monday, October 19, selling by three portfolio insurers accounted for almost 10 percent of the sales on the New York Stock Exchange, and that portfolio insurance sales amounted to 21.3 percent of all sales in index futures markets.

2.4 Constant proportion portfolio insurance

Fischer Black and Robert Jones introduced this method in 1986 in a research paper from the Goldman Sachs Bank[10], with the express goal of "simplifying portfolio insurance". It requires the choice of a floor (Φ) and a multiple (m), which will remain constant over the insurance period.

A cushion (c) is defined as the value of the portfolio (V_t) minus the floor.

$$c = V_t - \Phi$$

Throughout the insurance period, we maintain the following relationship in this strategy:

$$A_t = N_{S,t} \cdot S_t = m \cdot c$$

where A_t is the total amount invested in risky assets at time t. The amount to be invested in the risk-free asset is computed by the difference $B_t = V_t - A_t$.

At time 1, we start with $A_{0,1}$ invested in the risky asset, and $B_{0,1}$ in the risk-free bond. The total portfolio value is:

$$V_1 = A_{0,1} + B_{0,1}$$

We then determine the new cushion:

$$c_1 = V_1 - \Phi$$

As m is constant, we can compute the new value to be invested in the risky asset as $A_1 = m \cdot c_1$ and the amount to be invested in the risk-free asset as $B_1 = V_1 - m \cdot c_1$.

Let us illustrate this.

Example:
An investor has 1 share with an initial value of EUR 100. He wants to insure his portfolio at EUR 80 for two years. The risk-free rate is 5% (zero-coupon bond with 2 years to maturity). Let us assume a value of m = 5 for the multiple. We have: S_0 = EUR 100, r = 5%, T = 2, Φ = EUR 80, V_0 = EUR 100.

The following table illustrates constant proportion portfolio insurance over 2 years in the case of a bear market.

10 A revised version of the paper was published. See Black and Jones, "Simplifying portfolio insurance", Journal of Portfolio Management, Fall 1987.

t	$P_{stock,t}$	$P_{bond,t}$	$A_{t-1,t}$	$B_{t-1,t}$	V_t	c_t	A_t	B_t
0	100.00	90.48			**100.00**	20.00	100.00	0.00
1	99.29	90.86	99.29	0.00	**99.29**	19.29	96.44	2.85
2	96.48	91.24	93.71	2.86	**96.58**	16.58	82.88	13.70
3	91.64	91.62	78.72	13.75	**92.47**	12.47	62.35	30.12
4	86.82	92.00	59.07	30.25	**89.32**	9.32	46.58	42.74
5	85.89	92.39	46.08	42.91	**89.00**	9.00	44.98	44.02
6	83.12	92.77	43.53	44.20	**87.73**	7.73	38.65	49.08
7	79.25	93.16	36.86	49.28	**86.14**	6.14	30.68	55.45
8	75.15	93.55	29.09	55.68	**84.78**	4.78	23.90	60.88
9	73.37	93.94	23.33	61.14	**84.47**	4.47	22.35	62.12
10	68.52	94.33	20.87	62.38	**83.25**	3.25	16.25	67.00
11	68.08	94.73	16.14	67.28	**83.43**	3.43	17.13	66.30
12	66.34	95.12	16.69	66.58	**83.26**	3.26	16.32	66.94
13	61.86	95.52	15.22	67.22	**82.44**	2.44	12.21	70.23
14	59.22	95.92	11.69	70.52	**82.21**	2.21	11.07	71.15
15	54.22	96.32	10.13	71.44	**81.58**	1.58	7.88	73.69
16	50.37	96.72	7.32	74.00	**81.32**	1.32	6.62	74.70
17	50.17	97.13	6.60	75.01	**81.61**	1.61	8.05	73.56
18	45.26	97.53	7.26	73.86	**81.13**	1.13	5.65	75.48
19	43.47	97.94	5.42	75.80	**81.22**	1.22	6.10	75.12
20	40.63	98.35	5.70	75.43	**81.14**	1.14	5.68	75.45
21	39.14	98.76	5.47	75.77	**81.24**	1.24	6.22	75.03
22	34.69	99.17	5.51	75.34	**80.85**	0.85	4.25	76.60
23	33.81	99.58	4.14	76.92	**81.06**	1.06	5.31	75.75
24	31.75	100.00	4.98	76.07	**81.05**			

The following conventions were adopted:

- $P_{stock,t}$ denotes the stock price at time t
- $P_{bond,t}$ denotes the bond price at time t
- $A_{t-1,t}$ denotes the value of the stock portfolio at the beginning of time t, before rebalancing
- $B_{t-1,t}$ denotes the value of the bond portfolio at the beginning of time t, before rebalancing
- A_t denotes the value of the stock portfolio at time t, after rebalancing
- B_t denotes the value of the bond portfolio at time t, after rebalancing
- c_t is the cushion at time t

The initial cushion is $c_0 = V_0 - \Phi = 100 - 80 =$ EUR 20. As we know $m = 5$, we have $A_0 = m \cdot c_0 = 5 \cdot 20 =$ EUR 100, and $B_0 = V_0 - A_0 = 100 - 100 =$ EUR 0. The portfolio is a pure stock portfolio.

After one period, the value of the risky asset has fallen by 0.71 %. We have $A_{01} = 100 \cdot (1 - 0.0071) =$ EUR 99.29, $B_{01} =$ EUR 0, and $V_1 = A_{01} + B_{01} =$ EUR 99.29. The new cushion can be computed as

$$c1 = V1 - \Phi = 99.29 - 80 = \text{EUR } 19.29$$

As we know $m = 5$, we have $A_1 = m \cdot c_1 = 5 \cdot 19.29 =$ EUR 96.44, and $B_1 = V_1 - A_1 = 99.29 - 96.44 =$ EUR 2.85. The portfolio should be readjusted.

The process continues till we reach T. At each period, the portfolio is rebalanced. Note that when the price of the risky portfolio falls, the position in the risky portfolio is reduced.

The following table illustrates constant proportion portfolio insurance over 2 years in the case of a bull market.

t	$P_{stock,t}$	$P_{bond,t}$	$A_{t-1,t}$	$B_{t-1,t}$	V_t	c_t	A_t	B_t
0	100.00	90.48			**100.00**	20.00	100.00	0.00
1	103.59	90.86	103.59	0.00	**103.59**	23.59	117.94	-14.35
2	106.62	91.24	121.39	-14.41	**106.98**	26.98	134.90	-27.92
3	110.26	91.62	139.51	-28.04	**111.47**	31.47	157.33	-45.87
4	113.67	92.00	162.20	-46.06	**116.14**	36.14	180.72	-64.57
5	117.52	92.39	186.84	-64.84	**122.00**	42.00	209.98	-87.98
6	120.37	92.77	215.07	-88.35	**126.72**	46.72	233.59	-106.87
7	125.35	93.16	243.25	-107.32	**135.93**	55.93	279.65	-143.72
8	128.79	93.55	287.33	-144.32	**143.01**	63.01	315.06	-172.05
9	132.30	93.94	323.65	-172.76	**150.88**	70.88	354.42	-203.54
10	137.20	94.33	367.54	-204.39	**163.15**	83.15	415.76	-252.60
11	141.77	94.73	429.62	-253.66	**175.96**	95.96	479.82	-303.86
12	146.53	95.12	495.91	-305.13	**190.79**	110.79	553.94	-363.15
13	150.69	95.52	569.69	-364.67	**205.02**	125.02	625.08	-420.06
14	152.75	95.92	633.63	-421.82	**211.81**	131.81	659.07	-447.25
15	154.94	96.32	668.50	-449.12	**219.38**	139.38	696.89	-477.51
16	159.19	96.72	716.00	-479.50	**236.50**	156.50	782.48	-545.98
17	161.26	97.13	792.66	-548.26	**244.39**	164.39	821.97	-577.58
18	164.77	97.53	839.87	-579.99	**259.88**	179.88	899.39	-639.52
19	166.64	97.94	909.57	-642.19	**267.39**	187.39	936.94	-669.55
20	168.15	98.35	945.47	-672.35	**273.12**	193.12	965.62	-692.49
21	170.38	98.76	978.41	-695.39	**283.02**	203.02	1015.11	-732.08
22	173.53	99.17	1033.85	-735.14	**298.71**	218.71	1093.54	-794.83
23	177.36	99.58	1117.68	-798.15	**319.53**	239.53	1197.66	-878.13
24	181.48	100.00	1225.48	-881.79	**343.69**			

At each period, the portfolio is rebalanced. Note that when the price of the risky asset rises, the position in the risky asset is increased.

Thus when the price of the risky asset falls (rises), the position in the risky portfolio is reduced (increased). Like the SL2 strategy once the floor is reached, the investment is exclusively made in risk-free bonds. Because of this, the payoff profile of the CPPI strategy is convex.

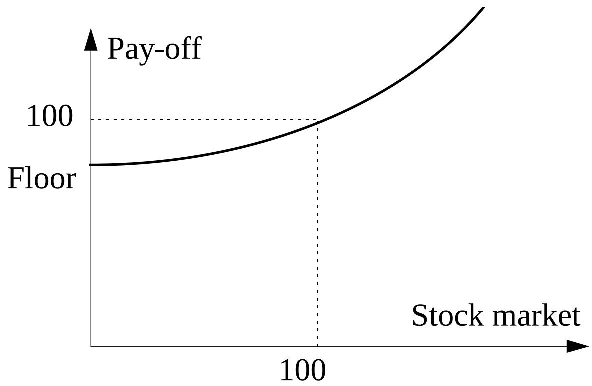


Figure 2-3: CPPI strategy payoff

This strategy is very flexible and can be adapted to a specific investor's preferences. It is easy, for example, to change the multiple to make the strategy more or less aggressive or to let the floor increase at the risk-free rate. Unlike a put, the strategy can continue indefinitely and has no time-horizon.

Note that a well-marketed variant of constant proportion portfolio insurance is called time invariant portfolio protection (TIPP)[11]. This method insures a higher floor each time the value of the risky portfolio increases significantly (so that we do not go back to the original low floor, but to a new, higher floor).

11 See Estep and Kritzman, "TIPP insurance without complexity", Journal of Portfolio Management, Summer 1988.

3. Hedging with stock index futures

Note that the risk of a stock portfolio (σ_P) can be decomposed in two components: a market or systematic risk, which depends on the beta of the portfolio (β_P) and on the market risk (σ_M) and a residual or specific risk (σ_ε). Both are linked by the following relationship:

$$\sigma_P^2 = \beta_P^2\sigma_M^2 + \sigma_\varepsilon^2$$

The specific risk can be reduced by diversification. In fact, in a well diversified portfolio, we should have no specific risk. We have to deal only with the systematic risk. Stock index futures may be used to **hedge the systematic risk** in (well-diversified) portfolios of stocks.

As we know, under appropriate assumptions hedge ratios can be obtained by first running a regression of spot returns against futures returns:

$$\frac{\Delta S_t}{S_t} = \alpha + \beta \cdot \frac{\Delta F_t}{F_{t,T}} + \varepsilon_t$$

and then using the formula:

$$HR = \beta \cdot \frac{S_t}{F_{t,T}}$$

The number of futures to include in the portfolio is given by:

$$\text{Number of futures} = \text{-HR} \cdot \frac{\text{Market value of spot position}}{\text{Futures contract size} \cdot \text{Spot price}}$$

that is,

$$\begin{aligned}\text{Number of futures} &= -\beta \cdot \frac{S_t}{F_{t,T}} \cdot \frac{\text{Market value of spot position}}{\text{Futures contract size} \cdot \text{Spot price}} \\ &= -\beta \cdot \frac{\text{Market value of spot position}}{\text{Futures contract size} \cdot \text{Futures price}}\end{aligned}$$

where, the β coefficient minimises the variance of **hedged returns.** It can be seen as a measure of the sensitivity of the asset returns to the market returns. Thus a common practice is to set the β coefficient equal to the well-known portfolio beta calculated in the CAPM and then to obtain the hedge ratio by adjusting this CAPM-beta by $S_t/F_{t,T}$. But note that we did not use the CAPM (and its assumptions) to derive the optimal hedging ratio.[12]

12 In some practical situations the given beta of the stock portfolio is not the one obtained by regressing the portfolio spot returns against the futures returns, but it is simply the beta β_S of the stock portfolio measured relative to some broad based market Index (like e.g. MSCI World). If the futures CONTRACT used to hedge the portfolio is based on an index different from this broad based market Index (since the Index underlying the futures CONTRACT should mirror at best the portfolio composition, it could happen e.g. to be a futures on the NASDAQ 100 Index) then also the beta β_f of the futures against the broad based market Index is needed, and the formula ABOVE becomes:

$$\text{Number of futures} = -\frac{\beta_S}{\beta_f} \cdot \frac{\text{Market value of spot position}}{\text{Futures contract size} \cdot \text{Futures price}}$$

3.1 Long hedge (Hedging rising share prices)

A short position in shares can be hedged by a long index futures position.

Example:
An investor intends to build up a French shares portfolio, with a beta (regression of portfolio returns against CAC-40 returns) of 1.1 for a total value of EUR 500'000 (as of September), but the necessary means are tied up in a fixed cash investment expiring in three months' time. The CAC-40 is quoted 4'710 and the December futures contract is quoted at 4'735. The investor expects an increase of the CAC-40 value over the next few months and is afraid of buying at considerably higher prices in three months. To safeguard against rising prices, he decides to buy CAC-40 futures.

$$N_F = -\text{Beta} \cdot \frac{\text{Market value of spot position}}{\text{Futures contract size} \cdot \text{Futures price}} = -1.1 \cdot \frac{-500'000}{10 \cdot 4'735} = 11.62 \approx 12 \text{ contracts}$$

Therefore, the investor will buy 12 December CAC-40 futures at 4'735 to cover his « short » position.
The shares prices have indeed risen and in December the CAC 40 is quoted at 4'770. To invest in the shares portfolio, assume that the investor has to pay 506'000 EUR. Shortly before maturity, he has closed his futures position by selling 12 December CAC-40 futures at a price of EUR 4'792.

The result of the strategy are the following:

- the value of the portfolio is EUR 506'000, which means an increased investment of EUR 6'000;
- the futures were purchased in September for EUR 568'200 and sold in December for EUR 575'040. This gives a profit of EUR 6'840.

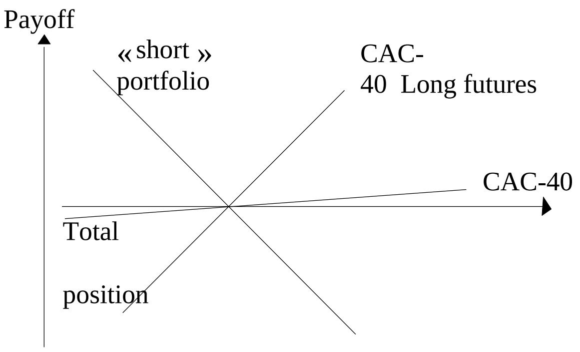


Figure 3-1: Long hedge payoff

On the total position, the investor has gained EUR 840, whereas without hedging the investment in the stock portfolio would have been more expensive than intended by EUR 6'000. As shown in the above figure, the hedge is not perfect (the profit should be zero, whatever the final CAC-40 price and the total position line should be horizontal).

3.2 Short hedge (Hedging falling share prices)

The **short hedge** involves selling index futures to hedge a long position in stocks. The idea is to create a portfolio in which any variation in the value of stock is compensated by an opposite index futures variation.

Example:
An investor manages a French shares portfolio, with a beta (regression of portfolio returns against CAC-40 returns) of 1.06 for a total value of EUR 560’000 (as of July). The CAC-40 is quoted 5’215 and the September futures contract is quoted at 5’225. The investor expects a decrease of the stock's value over the next few months. To safeguard his portfolio from a yield loss, he decides to sell CAC-40 futures.

$$N_F = -\text{Beta} \cdot \frac{\text{Market value of spot position}}{\text{Futures contract size} \cdot \text{Futures price}} = -1.06 \cdot \frac{560'000}{10 \cdot 5'225} = -11.36 \approx 11 \text{ contracts}$$

Therefore, the investor will sell 11 September CAC-40 futures at 5’225.

The shares prices have fallen and in September the CAC-40 is quoted at only 5’065. The investor has closed his futures position before maturity by repurchasing 11 September CAC-40 futures at a price of 5’080.

The result of the strategy is as follows:

- the value of the portfolio is EUR 544’000, which means a loss of EUR 16’000.
- the futures were sold in July for EUR 574’750 and purchased in October for EUR 558’800. This gives a profit of EUR 15’950.

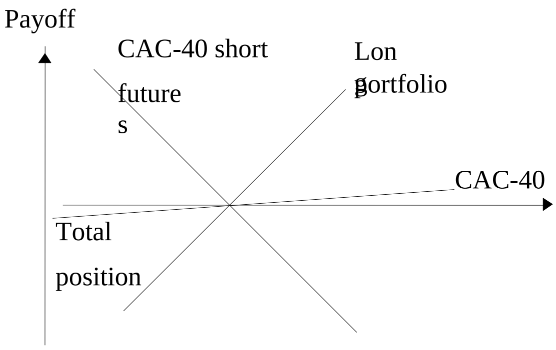


Figure 3-2: Short hedge payoff

On the total position, the investor has lost EUR 50, whereas a similar non-hedged position would have lost EUR 16’000. Again, as shown in the above figure, the hedge is not perfect (the profit/loss should be zero whatever the final CAC-40 price and the total position line should be horizontal).

3.3 A complete hedging analysis

Let us illustrate in detail how a stock index futures strategy is implemented and explain in detail why the hedging is not perfect, that is, decompose the origin of residual gains/losses.

Example:
A French Investor owns a portfolio of European stocks worth $V_{P,0}$ = EUR 4'000'000 with a beta (regression of portfolio returns against CAC-40 returns) of 1.036. On August 7, he decides to cover his portfolio by selling CAC-40 Futures contracts, knowing he will have to liquidate his position in the middle of December. The size of the CAC-40 contract is 10. The spot CAC-40 is at I_0=3’210.05 and a December CAC-40 futures at $F_{0,T}$ = 3’234.15. How many CAC-40 December Futures should he short?

The number of contracts to hold is given by

$$\text{Number of futures} = -\text{Beta} \cdot \frac{\text{Market value of the portfolio}}{\text{Futures contract size} \cdot \text{Futures price}}$$

$$= -1.036 \cdot \frac{4'000'000}{10 \cdot 3'234.15} = -128.1326 \text{ contracts}$$

that is, short 128.1326 contracts.

We know that the hedge will not be perfect, since

- he will have to choose between shorting 128 or 129 Futures contracts
- the futures matures after the Investor's horizon
- there is imperfect correlation between the investor's portfolio and CAC 40

Suppose the investor decides to sell 128 Futures on CAC-40 at $F_{0,T}$ = 3234.15. On December 15, when he liquidates, we observe that the portfolio value is $V_{P,1}$ = EUR 3'756'000 (-6.1%), the CAC-40 futures contract price is $F_{1,T}$ = 3'024.25 (-6.5%), the CAC-40 spot price is I_1=3'010.46 (-6.22%).What is his final payoff?
On December 15, we have:

Profit on portfolio: $V_{P,1}$-$V_{P,0}$ = 3'756'000-4'000'000	-244'000
Profit on Futures position: $N_F \cdot k \cdot (F_{0,T}-F_{1,T})$ = 10·128·(3'234.15-3'024.25)	+268'672
Dividends on portfolio between August 7 and December 15	+ 3'000
	EUR 27'672

Thus the total profit is **EUR 27'672**. But the profit should be zero if the hedge were perfect. How can we explain the origin of the residual?

- First, there is a problem with the non stability over time of the systematic risk coefficient β: the portfolio variation was $V_1 - V_0$ = -244'000 (-6.1%) while the hedging instruments variation was

$$128.1326 \cdot 10 \cdot (3010.46 - 3210.05) = -255'740$$

Thus there is an additional profit of 255'740-244'000 = **EUR 11'740.**

- Second, we have rounding problems, as we have sold 128 futures contracts rather than the exact value of 128.1326. The consequence is a difference of

$$(128 - 128.1326) \cdot 10 \cdot (3'234.15 - 3'024.25) = \textbf{EUR -278}$$

- Third, there is a basis risk, as the futures do not mature at the end of the hedging time-horizon. Initially, on August 7, the basis was $B_0 = F_{0,T} - I_0$ = 24.1 while at the end of the time horizon, on December 15, the basis is $B_1 = F_{1,T} - I_1$ = 13.79. Therefore, there is a gain from the change in basis of

$$128.1326 \cdot 10 \cdot (24.1 - 13.79) = \textbf{EUR 13'210}$$

- The dividends were received for an extra amount of **EUR 3'000.**

The total gain (**EUR 27'672**) is equal to the sum of these four errors. Thus the investor made a profit ex post, but of a "random nature".

3.4 Adjusting the beta of a stock portfolio

To change the systematic risk exposure of his portfolio, a manager has to change his beta. This can be done by varying the respective proportions of high-beta and low-beta shares, or by selling some stocks and holding some cash). However, this is costly in terms of transactions costs and can also impact the quality of the diversification.

An alternative is to use stock index futures. Assuming a contract size of 1, one can show that:

- to reduce the beta of a stock portfolio from β^{actual} to β^{target} ($\beta^{actual} > \beta^{target}$), a short position in $\left(\beta^{target} - \beta^{actual}\right) \cdot \frac{N_S \cdot S_t}{k \cdot F_{t,T}}$ contracts is required.
- to increase the beta of a stock portfolio from β^{actual} to β^{target} ($\beta^{actual} < \beta^{target}$), a long position in $\left(\beta^{target} - \beta^{actual}\right) \cdot \frac{N_S \cdot S_t}{k \cdot F_{t,T}}$ contracts is required.[13]

Let us consider the following example.

Example:
A German portfolio has a beta of 1.4 (with respect to the DAX) for a total value of EUR 9 million. The portfolio manager would like to decrease the beta of his portfolio to 1.
A possibility is to sell short index-futures contract. If the DAX futures contract is quoted at 1'600 and has a contract size of 50 times the index, the manager has to sell short

$$(1 - 1.4) \cdot \frac{9'000'000}{50 \cdot 1'600} = -45 \text{ futures contracts.}$$

13 Similarly to Footnote 82, if the given beta of the stock portfolio is not the one obtained by regressing the portfolio spot returns against the futures returns, but it is the beta β_S of the stock portfolio measured relative to some broad based market Index, then the previous formulas have to be replaced by $\left(\frac{\beta_S^{target} - \beta_S^{actual}}{\beta_f}\right) \cdot \frac{N_S \cdot S_t}{k \cdot F_{t,T}}$, where β_f is the beta of the futures against the broad based market Index.

4. Hedging with foreign exchange futures

We will discuss in this section the different ways of hedging foreign exchange risk.

4.1 Hedging against a rise of the foreign currency

From a USD point of view, a long hedge (i.e. buying foreign currency futures contracts) protects against a rise of the foreign currency value.

Example:
On August 9, an American client signs a contract to purchase a new house in Germany for a total cost of EUR 700'000. The payment date is scheduled to be December 17. Your client is facing the risk that EUR may appreciate between August and December. On August 9, the spot EUR exchange rate is EUR/USD 0.9799, while the December EUR futures price is 0.9745. *What do you suggest?*

In order to protect against an adverse movement in exchange rates, your client should go long 7 December EUR futures contracts (each futures contract calls for delivery of EUR 100'000 at the current futures price) at the price of 0.9745. On December 17 (the last trading day of the December EUR futures contract), he will offset the futures position and purchase EUR 700'000 in the spot market. This allows him to fix the purchase cost of his house at USD 682'150 (700'000·0.9745) on August 9, regardless of the market conditions that prevail on December 17.

The following demonstrates the outcome of this hedging strategy under alternative scenarios.

Case 1: EUR appreciates to 1.0087 on December 17

- Purchase cost: EUR 700'000·1.0087 = USD 706'090
- Gain on futures position: (1.0087-0.9745)·100'000·7 = USD 23'940
- Total cost: USD 682'150

Case 2: EUR depreciates to 0.9661 on December 17

- Purchase cost: EUR 700'000·0.9661 = USD 676'270
- Loss on futures position: (0.9661-0.9745)·100'000·7 = USD 5'880
- Total cost: USD 682'150

In both cases, the price will be the same.

4.2 Hedging against a drop of the foreign currency

From a USD point of view, a short hedge (i.e. selling foreign currency futures contracts) protects against a fall of the foreign currency value.

Example:
On August 9, an American client contacts you. He owns a bond portfolio that matures on September 17, which coincides with the last trading day of the September GBP futures contract. At that future date, he will receive GBP 1'000'000 (in principal and interests). He wants to invest this money in USD, and he fears that the GBP will depreciate against the USD between now and September 17. The September GBP futures price on August 9 is GBP/USD 1.8550. The spot exchange rate on August 9 is GBP/USD 1.8680. *Can you help him?*

Your client could protect himself by going short 16 September GBP futures contracts (each contract calls for delivery of GBP 62'500). He will offset the futures position on September 17 and the GBP 1'000'000 he receives from the bonds sale in the spot market. By hedging this way, your client will be guaranteed a sales revenue of USD 1'855'000 (1'000000·1.8550), independent of the market conditions that exist on September 17.

The following demonstrates the outcome of this hedging strategy under alternative scenarios.

Case 1: GBP appreciates to 1.8893 on September 17:

- Sales revenue: 1'000'000·1.8893 = USD 1'889'300
- Less: Futures loss of (1.8550 - 1.8893)·62'500·16 = USD 34'300
- Net sales revenue: USD 1'855'000

Case 2: GBP depreciates to 1.8000 on September 17:

- Sales revenue: 1'000'000·1.8000 = USD 1'800'000
- Plus: Futures gain of (1.8550 - 1.8000)·62'500·16 = USD 55'000
- Net sales revenue: USD 1'855'000

4.3 Hedging with cross-currency rates

Most of the futures contracts are based on the exchange rate of a foreign currency against the USD (assuming the USD as the local currency). Historically, this made sense as most commercial transactions were performed in USD. But now, foreign currency transactions are increasingly performed in other currencies than the USD.

Fortunately, by combining long and short positions in USD denominated currency futures contracts, it is always possible to create a synthetic futures contract hedging exactly the desired exchange rates.

Example:
Suppose that on August 9, a Japanese client agrees to purchase a new yacht from a German company for EUR 5'000'000. The payment date is December 17 (the last trading day of all December foreign currency futures contracts on the CME). Your client fears that the EUR will appreciate relative to the JPY prior to December. If that happens, the cost (in JPY) of the yacht will increase. On August 9, the price of the December JPY futures contract is JPY/USD 0.008322, and the price of the December EUR futures contract is EUR/USD 0.9745. *What do you suggest?*

Your Japanese client could hedge against a possible EUR appreciation by using a synthetic JPY-denominated EUR futures contract to lock in the exchange rate between the JPY and the EUR on August 9. He goes long the December EUR futures contract and goes short the December JPY futures contract, **both of which are denominated in USD**.

To obtain the required quantity of EUR for the yacht purchase, your client takes a long position of 50 EUR December futures contracts (5'000'000/100'000 = 50). These contracts lock in a dollar cost of USD 4'872'500 on December 17, as EUR 5'000'000·0.9745 = USD 4'872'500.

But your client must also be sure that he will be able to obtain USD 4'872'500 in December with the JPY he has available. To guarantee this, he goes short the December JPY futures contracts, locking in the future JPY/USD exchange rate for December 17. Based on the December JPY futures contract price on August 9, he needs to sell JPY 585'496'275 in order to receive USD 4'872'500 (i.e., USD 4'872'500/0.008322). He therefore has to go short approximately 47 JPY futures contracts on August 9, as JPY 585'496'275 / JPY 12'500'000 = 46.84

Thus, by taking a long position of 50 EUR futures contracts and a short position of 47 JPY futures contracts, the Japanese manufacturer creates a synthetic JPY-dominated EUR futures contract which locks in the rate at which he will exchange JPY for EUR on December 17, the date of his purchase of the German equipment. By doing so, he avoids the risk associated with fluctuations in the EUR/JPY exchange rate. Specifically the manufacturer locks in a EUR/JPY cross exchange rate of 117.10 = 0.9745 / 0.008322.

5. Hedging with interest rate futures

Interest rate futures are mainly used for **asset allocation** (as substitutes for underlying bonds that are risk free) and for **interest rate management**:

- to reduce the interest rate change risk for an existing position;
- to reduce the interest rate change risk of a future spot position;
- to gain from absolute price movements with a large leverage (a pure commitment in an interest rate futures allows the investor to profit from interest rate movements without tying up capital on the spot market);
- to profit from changes in the term structure of interest rates (spread trading).

We will hereafter illustrate some of the basic strategies that can be performed using interest rate futures. We will show that by freezing the current price level, an investor has the possibility to hedge existing or planned commitments on the interest rate market against upwards or downwards price movements.

5.1 Hedging using short term interest rates futures

In the case of short term investments, hedging with interest rate futures is easy:

- by entering today a long position in an interest rate futures and holding it until maturity, an investor engages in buying a short term bond at a future date, but at a given price that depends on current interest rates. This is called a **long-hedge**, and it is often used to « freeze » the higher interest rate today for a future fixed income investment (i.e. buying of an existing short term fixed income portfolio);
- by entering today into a short position in an interest rate futures contract (i.e. by selling a futures contract) and holding it until maturity, an investor engages in delivering a short term bond at a future date, but at a given price that depends on current interest rates. This is called a **short-hedge**, and it is often used to « freeze » the low interest rate today for a future fixed-income borrowing (i.e. selling of an existing short term fixed income portfolio).

As we always deal with short-term investments, there is no necessity for using duration or regression techniques to find the optimal hedge ratio.

Example:
Your client will receive EUR 10 million in 3 months time which he will then invest for three months. On September 14, the three-month Libor for EUR is at 2.875%, and the LIFFE December futures contract price is at 96.87. Your client expects a fall of interest rates and wishes to protect himself against it. *Propose a solution.*

One solution is to buy 10 three month Euro EUR futures contract at 96.87. The implied lending rate (implied by the futures contract price) is 3.13% (= 100 - 96.87).

On 14th of December, the three month Libor EUR is at 2.65%. The December futures contract is quoted at 97.35. The hedge is liquidated upon final settlement of the December futures contract on December 14. The futures P&L is 48 ticks (= 97.35 - 96.87), or 0.48%. The effective lending rate is 3.13% (= 2.65 + 0.48).

Note that in reality, the prevailing cash market bid/offer spread should be taken in consideration. For instance, futures settle at Libor (i.e. on an offered rate), lending on the cash market can only be achieved at Libid (i.e. a bid rate).

If the client wishes to hedge for a longer term, this can be implemented by purchasing several futures contracts with different maturities. In the previous example, this would involve buying a set of futures contracts for the first three month period, another set for the second three month period, etc. In the calculations, do not forget that the interest of the first period has to be reinvested in the second period, and so on...

5.2 Hedging using long term interest rates futures

As the hedging period lengthens, hedging with short-term futures becomes problematic, the number of contracts to be held or sold augments proportionally with time. Thus, it is better to use long-term interest rates futures contracts, which are generally based on long-term government bonds. The calculations are more complex, as they involve the computation of a hedge ratio. We will now discuss two methodologies to compute the hedge ratio: the duration approach and the regression approach.

Consider the situation where a position in an interest rate dependent asset (such as a bond portfolio or a money market security) is being hedged using a long-term interest rate futures contract.

Let

$F_{0,T}$ be the quoted price for the interest rate futures contract. To simplify things, we will generally assume that it is expressed in decimal form[14];

S_0 be the value of the asset being hedged (bond portfolio or money market security) expressed in decimal form;

MD_F be the modified duration of the asset underlying the futures contract, that is, modified duration of the cheapest to deliver bond; and

MD_S be the modified duration of the asset being hedged;

We assume that the change in the yield, Δy, is the same for all maturities (which means that only parallel shifts in the yield curve can occur). From the modified duration definition,

$$\Delta S = -S_0 \cdot MD_S \cdot \Delta y$$

To a reasonable approximation, it is also true that:

$$\Delta F = -F_{0,T} \cdot MD_F \cdot \Delta y$$

Since the Δy's are assumed to be the same, we have $\rho_{\Delta S, \Delta F}=1$. Combining the two previous equations, we can write:

$$\Delta S = \frac{S_0 \cdot MD_S}{F_{0,T} \cdot MD_F} \cdot \Delta F$$

14 By "In decimal form" we mean per one unit (CHF, USD, etc.) of face value. If the quoted futures price is 90.30, its decimal form expression is 0.9030.

It follows that[15]:

$$\frac{\sigma_{\Delta S}}{\sigma_{\Delta F}} = \frac{S_0 \cdot MD_S}{F_{0,T} \cdot MD_F}$$

and the optimal hedge ratio to use for hedging is:

$$HR{=}\rho_{\Delta S,\Delta F} \cdot \frac{\sigma_{\Delta S}}{\sigma_{\Delta F}} = \frac{S_0 \cdot MD_S}{F_{0,T} \cdot MD_F}$$

Using this, results in equating the duration of the whole position equal to zero.

However, the hedge obtained is by no means perfect:

- it assumes that Δy is the same for all yields whereas in practice short-term yields are usually more volatile than and are not closely correlated with long-term yields; as a result, the hedge performance can be disappointing, particularly if there is a big difference between MD_S and MD_F;
- it omits convexity: if the convexity of the asset underlying the futures contract is significantly different from the convexity of the asset being hedged and there is a large change in interest rates, the hedge performance may be worse than expected;
- in order to calculate MD_F, an assumption on what will be the cheapest-to-deliver bond is necessary. If the cheapest-to-deliver bond changes, MD_F changes and the optimal number of contracts also changes.

Knowing the hedge ratio, what is the number of futures contracts to be purchased? As we have seen already, the number of futures contracts required is given by:

$$N_F = -HR \cdot \frac{N_S}{k} = -\frac{S_0 \cdot MD_S}{F_{0,T} \cdot MD_F} \cdot \frac{N_S}{k} = -\frac{N_S \cdot S_0 \cdot MD_S}{k \cdot F_{0,T} \cdot MD_F}$$

where N_S is the number of units of the spot asset to be hedged, and k is the contract size. In fact, what we did is just found another way of estimating the hedge ratio using regression. This can be interpreted as:

$$\begin{matrix}\text{Number of}\\ \text{futures}\\ \text{contracts}\end{matrix} = -\frac{\begin{pmatrix}\text{Market value}\\ \text{of portfolio to hedge}\end{pmatrix}}{\begin{pmatrix}\text{Market value of}\\ \text{one futures contract}\end{pmatrix}} \cdot \frac{\begin{pmatrix}\text{Modified duration}\\ \text{of portfolio}\end{pmatrix}}{\begin{pmatrix}\text{Modified duration}\\ \text{of CTD}\end{pmatrix}}.$$

15 Using the fact that

$$\Delta S = \frac{S_0 \cdot MD_S}{F_{0,T} \cdot MD_F} \cdot \Delta F$$

we have

$$\sigma^2_{\Delta S} = \left(\frac{S_0 \cdot MD_S}{F_{0,T} \cdot MD_F}\right)^2 \cdot \sigma^2_{\Delta F} \qquad \Rightarrow \qquad \sigma_{\Delta S} = \left(\frac{S_0 \cdot MD_S}{F_{0,T} \cdot MD_F}\right) \cdot \sigma_{\Delta F}$$

As we know[16] that

$$F_{0,T} \cdot CF_{CTD,0} = S_{CTD,0},$$

where $CF_{CTD,0}$ is the conversion factor of the cheapest to deliver bond at time 0 and $S_{CTD,0}$ its price, we can replace $F_{0,T}$ in the above formula. This gives:

$$N_F = -\frac{N_S \cdot S_0 \cdot MD_S}{k \cdot F_{0,T} \cdot MD_F} = -\frac{N_S \cdot S_0 \cdot MD_S}{k \cdot S_{CTD,0} \cdot MD_F} \cdot CF_{CTD,0}$$

which can be interpreted as:

$$\begin{matrix}\text{Number of}\\ \text{futures}\\ \text{contracts}\end{matrix} = -\frac{\begin{pmatrix}\text{Market value}\\ \text{of portfolio to hedge}\end{pmatrix}}{\begin{pmatrix}\text{Futures}\\ \text{size}\end{pmatrix} \cdot \begin{pmatrix}\text{CTD spot}\\ \text{price}\end{pmatrix}} \cdot \frac{\begin{pmatrix}\text{Modified duration}\\ \text{of portfolio}\end{pmatrix}}{\begin{pmatrix}\text{Modified duration}\\ \text{of CTD}\end{pmatrix}} \cdot \begin{pmatrix}\text{Conversion}\\ \text{factor}\\ \text{of CTD}\end{pmatrix}$$

We will see examples of this formula in the next section.

An alternative method to compute the hedge ratio is based on the regression method. It assumes that past relationships are stable and will continue to hold in the future, but unlike the duration method, it does not use known information about the maturity, coupon or price characteristics of the relevant bonds.

To estimate a minimum variance hedge ratio HR for interest rate futures contract, one should compute a simple linear regression of ΔS_t on ΔF_t, that is,

$$\Delta S_t = \alpha^* + \beta^* \cdot \Delta F_t + \varepsilon_t$$

where ΔS_t are changes in the price of the security to be hedged, ΔF_t are changes in the futures price, ε_t is a residual with zero expected mean and α^* and β^* are the estimated coefficients. This is intuitively reasonable since we require HR to correspond to the ratio of changes in ΔS to changes in ΔF. Then, set

$$HR = \beta^* = \frac{\Delta S}{\Delta F}$$

The following example illustrates how to hedge a T-Bond portfolio with the regression method.

16 Recall that the invoice price (price for the bonds delivered on an interest rate futures contract) is calculated as the decimal futures settlement price (i.e. the price per one CHF face value) times the conversion factor of the delivered bond times the contract size plus the accrued interest on the delivered bond. If the shorts deliver the cheapest to deliver bond,

$$\text{Invoice price} = F_{t,T}^{(\text{decimal})} \cdot k \cdot CF_{CTD,t} + AI_{CTD,t}$$

In theory, this invoice price should be equal to the purchase price of the delivered bond. Assuming again that the shorts deliver the cheapest to deliver bond, we have:

$$\text{Purchase bond price} = S_{CTD,t}^{(\text{decimal})} \cdot k + AI_{CTD,t}$$

The equality of the purchase price and the invoice price gives:

$$F_{t,T}^{(\text{decimal})} \cdot CF_{CTD,t} = S_{CTD,t}^{(\text{decimal})}$$

which is also valid in a non decimal form.

Example:

On May 5, 2003 we wish to hedge a cash portfolio containing USD 5 million face value of each of the following four Treasury bonds: 9 1/8 of May 2018, 7 7/8 of Feb 2021, 6 1/4 of Aug 2023, and 6 1/8 of Nov 2027.

The average maturity of this portfolio is about 19.3 years, which is 6 months shorter than the maturity of the cheapest-to-deliver T-bond acceptable for delivery against a T-bond futures contract. The portfolio accrues coupon interest of USD 28'245 per week and has an average annual coupon rate of 7.34 percent. Our objective is to hedge this portfolio against all interest rate (or price) risks. Specifically, we want to minimise the standard deviation of week-to-week changes in the total return on the hedged portfolio for the period May 2003 through September 2003.

Date	9 1/8 of May 2018	7 7/8 of Feb 2021	6 1/4 of Aug 2023	6 1/8 of nov.27	Value of bond portfolio	Average portfolio price	Change in portfolio price	Futures price	Changes in futures price
01.11.2002	146.66	133.81	114.22	113.11	25'418'088.75	126.95		106.72	
08.11.2002	149.83	136.89	117.23	116.66	26'058'713.75	130.15	3.20	109.31	2.59
15.11.2002	147.56	134.86	115.36	114.80	25'657'151.25	128.14	-2.01	107.50	-1.81
22.11.2002	145.58	133.06	113.67	113.13	25'300'120.00	126.36	-1.79	106.34	-1.16
29.11.2002	144.86	132.33	113.03	112.44	25'161'057.50	125.66	-0.70	105.56	-0.78
06.12.2002	146.34	133.73	114.28	113.70	25'431'370.00	127.02	1.35	106.75	1.19
13.12.2002	146.83	134.27	114.77	114.05	25'523'557.50	127.48	0.46	107.50	0.75
20.12.2002	148.09	135.50	115.80	114.94	25'744'651.25	128.58	1.11	108.38	0.88
27.12.2002	150.22	137.61	117.78	117.02	26'159'495.00	130.66	2.07	110.25	1.88
03.01.2003	147.27	134.55	115.05	114.20	25'581'370.00	127.77	-2.89	107.72	-2.53
10.01.2003	145.83	133.11	113.66	112.70	25'293'088.75	126.32	-1.44	106.41	-1.31
17.01.2003	147.67	135.14	115.50	114.70	25'679'026.25	128.25	1.93	108.44	2.03
24.01.2003	148.81	136.16	116.41	115.75	25'884'495.00	129.28	1.03	109.75	1.31
31.01.2003	148.72	136.16	116.48	115.92	25'892'307.50	129.32	0.04	109.56	-0.19
07.02.2003	149.30	136.80	117.06	116.55	26'013'401.25	129.93	0.61	110.41	0.84
14.02.2003	148.66	136.03	116.38	115.47	25'854'807.50	129.13	-0.79	109.75	-0.66
21.02.2003	149.66	136.94	117.14	116.25	26'027'463.75	130.00	0.86	110.50	0.75
28.02.2003	152.52	139.80	119.81	119.25	26'596'995.00	132.84	2.85	113.16	2.66
07.03.2003	152.91	140.03	119.98	119.36	26'642'307.50	133.07	0.23	113.63	0.47
14.03.2003	152.23	139.41	119.48	118.77	26'522'776.25	132.47	-0.60	112.88	-0.75
21.03.2003	146.64	133.80	114.34	113.27	25'430'588.75	127.01	-5.46	108.34	-4.53
28.03.2003	149.09	136.17	116.47	115.44	25'886'838.75	129.29	2.28	110.38	2.03
04.04.2003	148.38	135.47	115.75	114.69	25'742'307.50	128.57	-0.72	109.91	-0.47
11.04.2003	148.53	135.75	116.02	114.95	25'790'745.00	128.81	0.24	110.13	0.22
18.04.2003	149.05	136.42	116.75	115.91	25'934'495.00	129.53	0.72	111.00	0.88
25.04.2003	150.16	137.61	117.86	116.98	26'158'713.75	130.65	1.12	112.09	1.09
02.05.2003	149.91	137.48	117.70	116.84	26'125'120.00	130.48	-0.17	112.19	0.09

To estimate the minimum-variance hedge ratio, we compare weekly price changes on the individual bonds and changes in the value of the portfolio as a whole during the prior 6 months, from November 2002 through April 2003, to price movements in September 2003 T-bond futures over the same period. The **value of bond portfolio** column is the sum of the market values of the four bond sub-portfolios plus accrued interest during the week. The last two columns give the **average portfolio price** and the **weekly change in the portfolio price**.

Regressing the average price changes of the bond portfolio on the T-bond futures price changes gives the following estimates ($R^2 = 0.98$):

$$\Delta S_t = -0.1023 + 1.1329 \cdot \Delta F_{t,T}$$

The estimate β^*=1.1329 is the minimum-variance hedge ratio. The high R^2 of 0.98 also indicates that T-bond futures prices tracked changes in cash portfolio values extremely well during the estimation period. Thus a short position of 296 T-bond futures contracts would be appropriate to hedge the bond portfolio against interest rate risk during the indicated period. Indeed,

$$N_F = -1.1329 \cdot \frac{26'125'120}{100'000} = -295.97$$

5.3 Hedging against decreasing rates (long hedge)

By holding a long position in interest rate futures, an investor can « freeze » the higher interest rate today for a future fixed-income investment.

Example:
A portfolio manager expects redemption of a fixed-interest investment of EUR 10 million in mid-December. He would like to invest the money on the German bond market with a modified duration MD_S=6.00 years. Since he expects falling interest rates, he would like to hedge the present lower cost price for the planned investment in bonds as bond prices rise when interest rates fall.

By purchasing the December Euro-Bund futures and later selling the futures position at a higher price, a profit is realised on the futures position that compensates the loss from the price increase for the planned bond purchase.

Let us assume that the futures price is $F_{0,T}$=112.90 and that the cheapest to deliver bond has a modified duration of MD_F=6.54 years and a conversion factor of 1.031156. The contract size is EUR 100'000.

The number of contracts to hold (computed by the duration method) is:

$$\begin{array}{c}\text{Number of}\\ \text{futures}\\ \text{contracts}\end{array} = -\frac{\begin{pmatrix}\text{Market value}\\ \text{of portfolio to hedge}\end{pmatrix}}{\begin{pmatrix}\text{Market value of}\\ \text{one futures contract}\end{pmatrix}} \cdot \frac{\begin{pmatrix}\text{Modified duration}\\ \text{of portfolio}\end{pmatrix}}{\begin{pmatrix}\text{Modified duration}\\ \text{of CTD}\end{pmatrix}}$$

$$= -\frac{-10'000'000}{112'900}\frac{6.00}{6.54} = 81.26 \text{ contracts}$$

Thus the manager decides to purchase 81 December Euro-Bund futures contracts at a price of 112.90.

In December, the yield in the long term sector has fallen by 0.30 percentage points, the futures contract is quoted at 115.13, and the increase in value of the planned portfolio amounts to EUR 182'000. The manager closes his futures position.

The result of the strategy is

- a "loss" (higher bond purchase price) of EUR 182'000, as the value of the portfolio in December is now EUR 10'182'000.
- a profit on the long Euro-Bund futures position of EUR 180'630 (that is, 2.23·81·1'000, where 2.23 is the difference between the purchase price of 112.90 and the selling price of 115.13).

which gives a total loss of EUR 1'370. Without any hedge, the extra-cost for the portfolio would have been EUR 182'000.

5.4 Hedging against increasing rates (short hedge)

By holding a short position in interest rate futures, an investor can « freeze » the lower interest rate today for a future selling of a fixed-income portfolio.

Example:
In October, an institutional investor manages a bond portfolio with a market value of EUR 40’000’000 and a modified duration of 8.20 years. The investor expects rising interest rates in the near future and he would like to hedge his portfolio against expected price losses. The December futures is quoted at 112.90, the cheapest to deliver bond has a modified duration of 6.54 years and a conversion factor of 1.031156.

By selling the December Euro-Bund futures contract at 112.90, and later, buying the futures position back at a lower price, a profit is realised on the futures position that compensates for the price decline of the bond portfolio.

The number of contracts to hold (computed by the duration method) is:

$$\begin{array}{c}\text{Number of}\\ \text{futures}\\ \text{contracts}\end{array} = -\frac{\begin{pmatrix}\text{Market value}\\ \text{of portfolio to hedge}\end{pmatrix}}{\begin{pmatrix}\text{Market value of}\\ \text{one futures contract}\end{pmatrix}} \cdot \frac{\begin{pmatrix}\text{Modified duration}\\ \text{of portfolio}\end{pmatrix}}{\begin{pmatrix}\text{Modified duration}\\ \text{of CTD}\end{pmatrix}}$$

$$= -\frac{40'000'000}{112'900}\frac{8.20}{6.54} = \text{-444.22} \quad \text{contracts}$$

Thus the investor decides to sell 444 December Euro-Bund futures contract at 112.90.

In December, the yields in the long term sector have risen by 0.30 percentage points, the futures contract is quoted at 110.74, and the loss on the portfolio is EUR 970’000. The investor closes his futures position by buying back December Euro-Bund futures at 110.74.

The result of the strategy is

- a «loss» (lower bond price) of 970’000, as the value of the portfolio in December is now EUR 39’030’000.
- a profit on the short Euro-Bund futures position of EUR 959’040 (that is, 2.16·444·1’000, where 2.16 is the difference between the selling price of 112.90 and the buy-back price of 110.74).

which gives a total loss of EUR 10’960. Without any hedge, the loss of the portfolio would have been EUR 970’000.

5.5 Moving to a preferred duration

Let us recall that the price sensitivity to yield changes of a bond portfolio can be calculated from the duration: the longer the duration, the higher the price change. Thus if an investor expects falling interest rates, he will increase the duration of his portfolio (that is, interest rate sensitivity) while he will decrease the duration if he expects increasing interest rates. How can we implement that?

The first solution is called “bond swapping”. If the manager wants to increase the duration of his portfolio, it involves selling bonds with a low duration and in buying bonds with a longer duration. If the manager wants to reduce the duration of his portfolio, it involves selling bonds and holding the proceeds in cash (or in lower duration bonds). The problem of bond swapping is the high level of transaction costs on the bond market.

A better solution is to use interest rates futures contracts as an alternative to restructuring the portfolio in the cash markets. Asset managers can lengthen the effective maturity of short-term investment assets by buying futures contracts, and shorten the effective maturity of those assets by selling futures contracts. Liability managers can achieve the same effects by doing the opposite. The use of futures may be attractive when physical restructuring is not possible (e.g., term deposits cannot be bought back prior to their maturity dates). It may also be cheaper to use futures because transaction costs in the futures market may be lower than those in cash markets, or because liquidity conditions in the cash market would result in substantial market penalties.

When adjusting the duration of a portfolio from D_S^{actual} to D_S^{target}, the hedge ratio can be computed as

$$HR = \frac{S_0 \cdot (D_S^{target} - D_S^{actual})}{F_{0,T} \cdot D_F}$$

and the number of contracts to use:

$$\begin{matrix}\text{Number of}\\ \text{futures}\\ \text{contracts}\end{matrix} = \frac{\begin{matrix}\text{Market value}\\ \text{of portfolio}\end{matrix}}{\begin{matrix}\text{Market value of}\\ \text{futures}\end{matrix}} \cdot \frac{\begin{pmatrix}\text{Target}\\ \text{duration}\end{pmatrix} - \begin{pmatrix}\text{Portfolio}\\ \text{duration}\end{pmatrix}}{\begin{pmatrix}\text{Duration}\\ \text{of CTD}\end{pmatrix}}$$

These two results can be calculated by the same method, which we used for complete hedging. Note that in the complete hedging formulae, we used the modified duration (where we considered a target modified duration of zero). But as a ratio of modified duration is equal to a ratio of duration, we do not need to use modified duration here.

Let us illustrate this by a numerical example.

Example:
In October, an institutional investor manages a bond portfolio with a market value of EUR 40’000’000 and duration of 8.55 years (modified duration: 8.20). The investor expects rising interest rates in the near future and he would like to reduce his interest rate sensitivity and thereby the expected book loss of his portfolio by reducing the portfolio duration to 2 years. The December futures contract is quoted at 112.90, the cheapest to deliver bond has duration of 6.54 years and a conversion factor of 1.031156.

By selling December Euro-Bund futures at 112.90, the investor can decrease the portfolio duration to the desired level. Later, by buying back the futures position, the duration can be increased (almost) to the original level of 8.55 years.

The number of contracts to hold is:

$$\begin{matrix}\text{Number of}\\ \text{futures}\\ \text{contracts}\end{matrix} = \frac{\begin{matrix}\text{Market value}\\ \text{of portfolio}\end{matrix}}{\begin{matrix}\text{Market value of}\\ \text{futures}\end{matrix}} \cdot \frac{\begin{matrix}\text{Target}\\ \text{duration}\end{matrix} - \begin{matrix}\text{Portfolio}\\ \text{duration}\end{matrix}}{\begin{matrix}\text{Duration}\\ \text{of CTD}\end{matrix}}$$

$$= \frac{40'000'000}{112'900} \cdot \frac{(2.0 - 8.55)}{6.54} = -354.84 \text{ contracts}$$

Thus the investor sells 355 December Euro-Bund futures at 112.90.

In December, the expected interest rate increase has occurred and the yields in the long term sector have risen by 0.30 percentage points. The investor is of the opinion that the rise has reached its peak and buys back the Euro-Bund contracts at 110.74.

The result of the strategy is:

- a «loss» (lower bond price) of the portfolio of 970'000, as the value of the portfolio in December is now EUR 39'030'000.
- a profit on the short Euro-Bund futures position of EUR 766'800 (that is, 2.16·355·1'000, where 2.16 is the difference between the selling price of 112.90 and the buy-back price of 110.74).

which gives a limited total loss of EUR 203'200 (but note that this is not a fully hedged position!). Without any duration adjustment, the loss of the portfolio would have been EUR 970'000.

6. Use of swaps in portfolio management

The use of swaps in **existing** assets and liabilities management is predicted on the capacity of such instruments to convert fixed rate exposures to floating rate and vice-versa, assets and liabilities from one currency to another, and more generally, a stream of cash flows into another.

Using swaps, a firm can model its assets or liabilities for a very low cost, while using futures or options, it would have to pay expensive premium. We will briefly quote a few risk-management strategies using swaps.

Swap transactions can also be used to **manage investment portfolios**. For example, in a positively sloped yield curve environment, a portfolio manager may enter into a swap transaction to transform a fixed-rate investment into a floating rate one.

Swaps can also be used to **lock in gains and stop losses** on fixed rate investments. For example, a fixed-rate portfolio manager, after a strong decrease of the interest rates, can lock-in the capital gain from his portfolio by swapping it against a floating rate investment.

Swaps can also be used in order to **improve portfolio performance**, on both fixed-rate and floating-rate investments. Consider the following example: an investor has a portfolio of floating rate securities (USD 6M LIBOR). To increase the yield on this portfolio, he can enter simultaneously into two swaps: in the first one (the “original”), he will swap his floating rate (USD 6M LIBOR) against a fixed rate, let's say 10%; in the second one (the “reverse”), he will swap a lower fixed rate (let's say 9.50%) against the same floating rate (USD 6M LIBOR). After the two transactions, the new rate of return on the portfolio is USD 6M LIBOR, plus 50 basis points. The situation can be illustrated as follows:

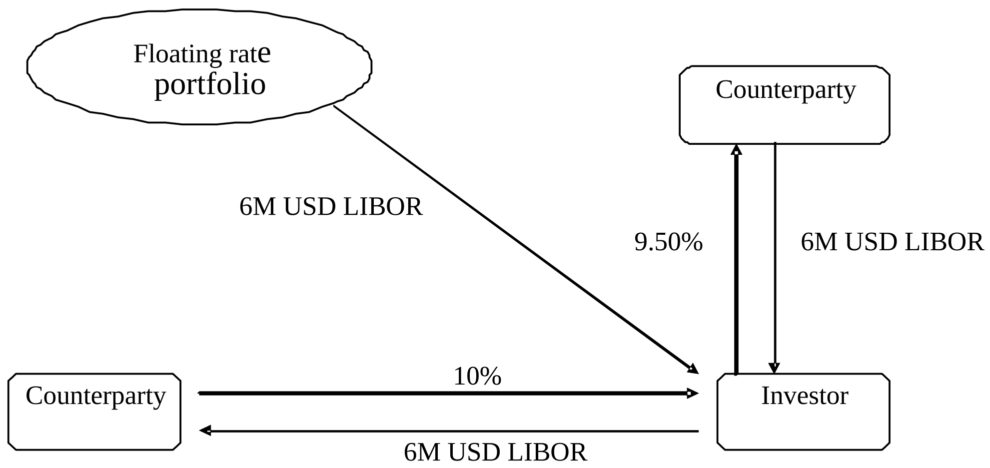


Figure 6-1: Enhancing a portfolio’s performance

Through swaps, it is also possible to create **synthetic foreign currency assets**, or to lock in gains or stop losses on foreign currency investments.

7. Asset allocation with futures

At its most basic level, portfolio management involves a decision concerning what types of assets should be purchased. For example, a fund manager might choose to invest 40 percent of the funds portfolio in stocks, 40 percent in bonds and 20 percent in real estate. Deciding what proportion of fund wealth to place in each category of assets is called the **asset allocation decision**.

Once the asset allocation decision is made and fund wealth is invested, important changes to the allocation are usually avoided because the transaction costs of liquidating assets in one category and buying assets in another are excessive. Instead, fund managers can use futures contracts to change the asset allocation indirectly.

To demonstrate this, assume that a fund consists of V_S in stocks and V_B in bonds for a total value of $V = V_S + V_B$. Now, suppose the fund manager wants to change the amount invested in long-term bonds from V_B to V_B^*. The bond portfolio has a modified duration of MD_B. Rather than selling (buying) stocks to buy (sell) bonds, the portfolio manager can effect the change by buying (selling) T-bond futures contracts. If he wants his bond portfolio to have an income of $MD_B \cdot V_B^*$ if interest rates fall 1 percent, he can generate that amount with the income from the current bond portfolio, $MD_B \cdot B$ and the income from a T-bond futures position, $N_F \cdot MD_F \cdot F$, that is

$$MD_B \cdot V_B^* = MD_B \cdot V_B + N_F \cdot MD_F \cdot F$$

Rearranging to isolate N_F, we get:

$$N_F = \frac{MD_B \cdot \left(V_B^* - V_B\right)}{MD_F \cdot F}$$

Note that if the investment in bonds is to be reduced (i.e. $V_B^* < V_B$), T-bonds futures contracts are sold and if the investment in bonds is to be increased (i.e., $V_B^* > V_B$), T-bond futures contracts are purchased. The reduction (increase) in bond investment is thus transferred to stocks through buying (selling) stock index futures contracts.

Example:
A fund manager currently has USD 50'000'000 in a stock portfolio whose composition matches the S&P 500 and USD 50'000'000 in a bond portfolio whose modified duration is 12.00. Believing that stocks are going to do extraordinarily well over the next three months, the fund manager wants to take advantage of the impending stock market rise and to eliminate his interest rate risk exposure. Unfortunately liquidating bonds and buying stocks is expensive, particularly if at the end of the three months the manager wants to return to his fifty-fifty portfolio mix. How can the fund manager use T-bond and S&P 500 futures to carry out his plans? Assume that the cheapest-to-deliver bond (and hence the T-bond futures) has a duration of 9.00 and that the price of a three-month T-bond futures contract is 96.00. Also, assume that the three-month S&P 500 futures contract is priced at 940.
First, with respect to eliminating the interest rate exposure, the number of T-bond futures to sell is 694.44.

$$N_F = \frac{12.00 \cdot (0 - 50'000'000)}{9 \cdot 96 \cdot 1'000} = -694.44$$

This action is equivalent to liquidating the bond investment. Second, to take a long position of USD 50'000'000 in stocks using the S&P 500 futures, the number of contracts to buy is

$$N_S = \frac{50'000'000}{940 \cdot 250} = 212.76$$