FIXED INCOME

APPENDIX - DERIVATION OF DURATION AND CONVEXITY

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1. Appendix: Derivation of duration and convexity

In this appendix we present the mathematical derivation of the duration and convexity formulae.

1.1 Duration

Generally, risk exposure is defined as the change in value of the assets (liability) in reaction to a factor influencing the return from the asset.

$$\text{Risk exposure} = \frac{\text{Change of the value}}{\text{Value}} = \frac{\Delta\text{Value}}{\text{Value}}$$

The interest rate is the most important factor affecting the market value of a bond. So the risk of the bond can be defined as the change in value of a bond in reaction to a given change in interest rates.

In the case of bonds, the problem is well structured, because:

- Bonds have a well-defined stream of future cash flows without option-like characteristics or prepayable mortgages.
- The term structure of interest rates is assumed to be flat at the level of the bond yield
- Interest rate changes are assumed to lead to parallel shifts in the term structure.
- The interest rate changes are usually small.

To compute the risk exposure and then the duration, we start with the pricing formula for coupon-bearing bonds:

$$P = \frac{CF_1}{(1+k)} + \frac{CF_2}{(1+k)^2} + \frac{CF_3}{(1+k)^3} + \ldots + \frac{CF_T}{(1+k)^T} = \sum_{t=1}^{T} \frac{CF_t}{(1+k)^t}$$

where:

P	Bond price now (at time 0)
CF_t	Cash flow at time t
k	Market yield

To determine the approximate price changes for small changes in the yield, we take the first derivative of the Price P with respect to the market yield k:

$$\frac{\partial P}{\partial k} = \frac{(-1)\cdot CF_1}{(1+k)^2} + \frac{(-2)\cdot CF_2}{(1+k)^3} + \frac{(-3)\cdot CF_3}{(1+k)^4} + \ldots + \frac{(-T)\cdot CF_T}{(1+k)^{T+1}} \quad (1)$$

using the usual rules for differentiation:

$$f(x) = \frac{1}{x^n} = x^{-n} \quad \rightarrow \quad \frac{df}{dx} = -n \cdot x^{-(n+1)} = \frac{-n}{x^{n+1}}$$

$$f(x) = a \cdot g(x) \quad \rightarrow \quad \frac{df}{dx} = a \cdot \frac{dg}{dx}$$

$$f(x) = g(x) + h(x) \quad \rightarrow \quad \frac{df}{dx} = \frac{dg}{dx} + \frac{dh}{dx}$$

We can rearrange equation (1) by extracting $-\frac{1}{1+k}$:

$$\frac{\partial P}{\partial k} = -\frac{1}{1+k} \cdot \left[\frac{1\cdot CF_1}{(1+k)} + \frac{2\cdot CF_2}{(1+k)^2} + \frac{3\cdot CF_3}{(1+k)^3} + \ldots + \frac{T\cdot CF_T}{(1+k)^T}\right] = -\frac{1}{1+k} \cdot \left[\sum_{t=1}^{T} \frac{t\cdot CF_t}{(1+k)^t}\right] \quad (2)$$

If we insert equation (2) in the equation $dP = \frac{\partial P}{\partial k} \cdot dk$, we get:

$$dP = -\frac{1}{1+k} \cdot \left[\sum_{t=1}^{T} \frac{t \cdot CF_t}{(1+k)^t} \right] \cdot dk \qquad (3)$$

and divide equation (3) by the bond price P

$$\frac{dP}{P} = -\frac{1}{1+k} \cdot \frac{\sum_{t=1}^{T} \frac{t \cdot CF_t}{(1+k)^t}}{P} \cdot dk \qquad (4)$$

For small changes, we can rewrite dP as ΔP and dk as Δk and get from equation (4):

$$\frac{\Delta P}{P} \approx -\frac{1}{1+k} \cdot \frac{\sum_{t=1}^{T} \frac{t \cdot CF_t}{(1+k)^t}}{P} \cdot \Delta k \qquad (5)$$

Equation (5) is a formula for the risk exposure we have defined above.

Substituting the formual for Price P in the above expression, we get Macaulay's duration:

$$\text{Macaulay's duration } = D = \frac{\sum_{t=1}^{T} \frac{t \cdot CF_t}{(1+k)^t}}{\sum_{t=1}^{T} \frac{CF_t}{(1+k)^t}} = \frac{\frac{1 \cdot CF_1}{(1+k)} + \frac{2 \cdot CF_2}{(1+k)^2} + \ldots + \frac{T \cdot CF_T}{(1+k)^T}}{\frac{CF_1}{(1+k)} + \frac{CF_2}{(1+k)^2} + \ldots + \frac{CF_T}{(1+k)^T}} \qquad (6)$$

Inserting Macaulay's Duration D (equation (6)) in equation (5), we get:

$$\frac{\Delta P}{P} \approx -\frac{1}{1+k} \cdot D \cdot \Delta k \qquad (7)$$

We define the modified Duration as follows,

$$\text{Modified duration } = D^{mod} = \frac{D}{1+k}$$

Inserting the modified duration D^{mod} in equation (7), we get:

$$\frac{\Delta P}{P} \approx -D^{mod} \cdot \Delta k$$

To calculate the price change ΔP:

$$\Delta P \approx -D^{mod} \cdot \Delta k \cdot P$$

1.2 Convexity

The accuracy of approximation improves if we use the first two terms of a **Taylor series** to approximate the price change, as follows:

$$dP = \frac{\partial P}{\partial k} \cdot dk + \frac{1}{2} \cdot \frac{\partial^2 P}{\partial k^2} \cdot (dk)^2 + \varepsilon \qquad (8)$$

where:

ε Residual term

$\frac{\partial P}{\partial k}$ First derivative of the price with respect to the yield

$\frac{\partial^2 P}{\partial k^2}$ Second derivative of the price with respect to the yield

If we divide the two sides of equation (8) by the price P, we get:

$$\frac{dP}{P} = \frac{\partial P}{\partial k} \cdot \frac{1}{P} \cdot dk + \frac{1}{2} \cdot \frac{\partial^2 P}{\partial k^2} \cdot \frac{1}{P} \cdot (dk)^2 + \xi \qquad (9)$$

where:

$\xi = \frac{\varepsilon}{P}$ Residual term divided by P

$-\frac{\partial P}{\partial k} \cdot \frac{1}{P} = D$ Duration

$\frac{1}{2} \cdot \frac{\partial^2 P}{\partial k^2} \cdot \frac{1}{P} = C$Convexity

From equation (1) we know that:

$$\frac{\partial P}{\partial k} = \frac{-1 \cdot CF_1}{(1+k)^2} + \frac{-2 \cdot CF_2}{(1+k)^3} + \frac{-3 \cdot CF_3}{(1+k)^4} + \ldots + \frac{-T \cdot CF_T}{(1+k)^{T+1}}$$

$\frac{\partial^2 P}{\partial k^2}$ is also the first derivative of $\frac{\partial P}{\partial k}$ with respect to the yield.

Similar to how we obtained equation (1), we can obtain the second derivative:

$$\frac{\partial P^2}{\partial^2 k} = \frac{(-1) \cdot (-2) \cdot CF_1}{(1+k)^3} + \frac{(-2) \cdot (-3) \cdot CF_2}{(1+k)^4} + \frac{(-3) \cdot (-4) \cdot CF_3}{(1+k)^5} + \ldots + \frac{(-T) \cdot [-(T+1)] \cdot CF_T}{(1+k)^{T+2}}$$

and this is equivalent to:

$$\frac{\partial P^2}{\partial^2 k} = \frac{1}{(1+k)^2} \cdot \left[\frac{1 \cdot 2 \cdot CF_1}{(1+k)} + \frac{2 \cdot 3 \cdot CF_2}{(1+k)^2} + \frac{3 \cdot 4 \cdot CF_3}{(1+k)^3} + \ldots + \frac{T \cdot (T+1) \cdot CF_T}{(1+k)^T} \right]$$
$$= \frac{1}{(1+k)^2} \cdot \sum_{t=1}^{T} \frac{t \cdot (t+1) \cdot CF_t}{(1+k)^t} \qquad (10)$$

The convexity is defined as:

$$\text{Convexity} = C = \frac{1}{2} \cdot \frac{d^2 P}{dk^2} \cdot \frac{1}{P} \qquad (11)$$

We know that the bond price P is:

$$P = \frac{CF_1}{(1+k)} + \frac{CF_2}{(1+k)^2} + \frac{CF_3}{(1+k)^3} + \ldots + \frac{CF_T}{(1+k)^T} = \sum_{t=1}^{T} \frac{CF_t}{(1+k)^t} \tag{12}$$

If we insert equations (10) and (12) in equation (11), we get:

$$\text{Convexity} = C = \frac{1}{2} \cdot \frac{1}{(1+k)^2} \cdot \left[\frac{\frac{1 \cdot 2 \cdot CF_1}{(1+k)} + \frac{2 \cdot 3 \cdot CF_2}{(1+k)^2} + \ldots + \frac{T \cdot (T+1) \cdot CF_T}{(1+k)^T}}{\frac{CF_1}{(1+k)} + \frac{CF_2}{(1+k)^2} + \ldots + \frac{CF_T}{(1+k)^T}} \right] \tag{13}$$

$$= \frac{1}{2} \cdot \frac{1}{(1+k)^2} \cdot \frac{\sum_{t=1}^{T} \frac{t \cdot (t+1) \cdot CF_t}{(1+k)^t}}{\sum_{t=1}^{T} \frac{CF_t}{(1+k)^t}}$$

If we insert the convexity (equation (13)) and the duration (equation (6)) in equation (9)), we get:

$$\frac{\partial P}{P} = -\frac{D}{1+k} \cdot dk + C \cdot (dk)^2 \tag{14}$$

For small yield changes (Δk), we can rewrite equation (14) as:

$$\frac{\Delta P}{P} \approx -\frac{D}{1+k} \cdot \Delta k + C \cdot (\Delta k)^2 \tag{15}$$

or as:

$$\Delta P \approx -D \cdot P \cdot \Delta k + C \cdot P \cdot (\Delta k)^2 \tag{16}$$

1.3 Bootstraping

The relationship between spot rates and par yields is a straightforward one and deriving a spot- or zero-curve from a par curve is a simple business, known as bootstrapping. We assume for the purposes of the example that we have the data on yields of bonds priced at par (in which case, coupon and yield will be the same), for every year from exactly one year from now to whatever the longest maturity happens to be, and that coupon payments are annual.

The first step in bootstrapping is to notice that there is one bond in the series of par yields that only makes one payment: it is the one year bond. Therefore it is fair to say that the yield on a one-year zero is the same as the yield on a one year bond. Armed with that first (one-year) value, and given the par yield of the two-year bond it should be apparent that:

c_T is the coupon/yield of a par bond in T years,

z_T is the yield of a zero-coupon bond maturing in T years

d_T is the T-year discount price (i.e. the price of a zero-coupon bond maturing in T years)

Then, $c_1 = z_1$ and the cash flows of the two-year bond consists of two payments, which are c_2 in one year's time and $1+c_2$ in two years' time (where 1 is the principal repayment). The value of the par bond today is, of course, par, or 1 (=100%) and the present value or price of c_1 is d_1 so that we have

$$1 = d_1 c_2 + d_2(c_2 + 1).$$

Solving for d_2, we get: $d_2 = \dfrac{1 - d_1 c_2}{1 + c_2}$ and $z_2 = \left(\dfrac{1}{d_2}\right)^{1/2} - 1$.

Now, the cashflows for the 3rd year are c_3 in one year, c_3 in two years and $(\mathrm{c}_3 + 1)$ in three year, so that we have:

$$1 = d_1 c_3 + d_2 c_3 + d_3(c_3 + 1) = c_3(d_1 + d_2) + d_3(c_3 + 1)$$

Solving the above equation for d_3, we get:

$$d_3 = \frac{1 - c_3(d_1 + d_2)}{1 + c_3} \quad \text{and} \quad z_3 = \left(\frac{1}{d_3}\right)^{1/3} - 1, \text{ etc.}$$

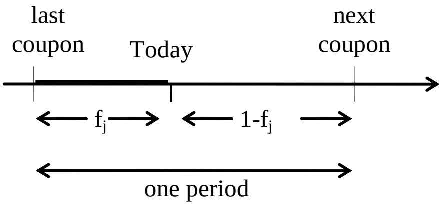
Generalasing, we get

$$d_T = \frac{1 - c_T \sum_{i=1}^{T-1} d_i}{1 + c_T}.$$

Thus, we can continue calculating the value of the discount factor curve, and since $z_T = \left(\dfrac{1}{d_T}\right)^{1/T} - 1$, we can populate the zero-coupon or spot curve accordingly.

1.4 Approximation of the yield to maturity of a bond portfolio

The approximation can be explained as follows:



The present value PV_j of the position held in bond j is simply:

$$PV_j = Q_J \cdot P_{j,cum}$$

where Q_j is the quantity of the bond j held in the portfolio and $P_{j,cum}$ is its dirty price.

$P_{j,ex}$ denotes the quoted price of bond j (without accrued interests) and C_j its coupon. The dirty price $P_{j,cum}$ can be written as a function h_j of the bond's j yield to maturity YTM_j, and is obtained by discounting the cashflows $CF_{j,t}$ paid by the bond j at time t and summing over t, using bond's j yield to maturity:

$$P_{j,cum} = P_{j,ex} + f_j \cdot C_j = h_j(YTM_j) = \sum_{t=1}^{T_j} \frac{CF_{j,t}}{(1+YTM_j)^{t-f_j}}$$

We know that if the yield YTM_j changes, the price change $dP_{j,cum}$ can be approximated using the modified duration approach, according to the formula:

$$\frac{dP_{j,cum}}{dYTM_j} = h_j'(YTM_j) = -P_{j,cum} \cdot D_j^{mod}$$

where h_j' is the first derivative of the function h_j with respect to YTM_j.

For a portfolio of N bonds, the portfolio present value PV is equal to:

$$PV = \sum_{j=1}^{N} Q_j \cdot P_{j,cum} = \sum_{j=1}^{N} Q_j \cdot h_j(YTM_j) .$$

On the other hand, the present value can be obtained using a same yield YTM_P for all the bonds, the portfolio yield to maturity, to discount all the cash-flows:

$$PV = \sum_{j=1}^{N} Q_j \cdot h_j(YTM_P) .$$

Let us define ΔYTM_i as the difference between YTM_i and YTM_P:

$$\Delta YTM_i = YTM_i - YTM_P$$

Example (continued):
We have YTM_P = 3.627%, YTM_1 = 2% and YTM_2 = 4%. Therefore $\Delta YTM_1 = YTM_1 - YTM_P$ = -1.627% and ΔYTM_2 = 0.373%.

Then we have:

$$\begin{aligned}
PV &= \underbrace{\sum_{j=1}^{N} Q_j \cdot h_j(YTM_j)}_{A} \\
&= \sum_{j=1}^{N} Q_j \cdot h_j(YTM_P + \Delta YTM_j) \\
&\cong \sum_{j=1}^{N} Q_j \cdot \left[h_j(YTM_P) + h_j{}'(YTM_P) \cdot \Delta YTM_j\right] \\
&\cong \underbrace{\sum_{j=1}^{N} Q_j \cdot h_j(YTM_P)}_{B} - \underbrace{\sum_{j=1}^{N} Q_j \cdot P_{j,cum} \cdot D_j^{mod} \cdot \Delta YTM_j}_{C}
\end{aligned}$$

Since A = B, it has to hold that C = 0. Therefore:

$$\sum_{j=1}^{N} Q_j \cdot P_{j,cum} \cdot D_j^{mod} \cdot (YTM_j - YTM_P) = \sum_{j=1}^{N} PV_j \cdot D_j^{mod} \cdot (YTM_j - YTM_P) = 0,$$

Hence[1]:

$$YTM_P = \sum_{j=1}^{N} \left(\frac{PV_j \cdot D_j^{mod}}{\sum_{i=1}^{N} PV_i \cdot D_i^{mod}} \right) \cdot YTM_j$$

1 See K. Garbade, Fixed Income Analytics, MIT Press, 1996.