FIXED INCOME

INTEREST RATES - TERM STRUCTURE AND APPLICATIONS

FIXED INCOME

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1. Term structure of interest rates

1.1 Yield curves and shapes

The economics of interest rates deals with the pure **price of time** (time value of money). Awareness and appreciation of the interest rate-maturity relationship is essential in bond management.

The simplest way to compare the yields between bonds of the same credit is to draw a graph depicting the various yields of the similar bonds against the maturity of each bond. Such a simplistic view is called a market **yield curve**.

A more accurate relationship between the yields on otherwise comparable bonds with different maturities is called the **term structure of interest rates**; its graphical depiction is also known as a yield curve.

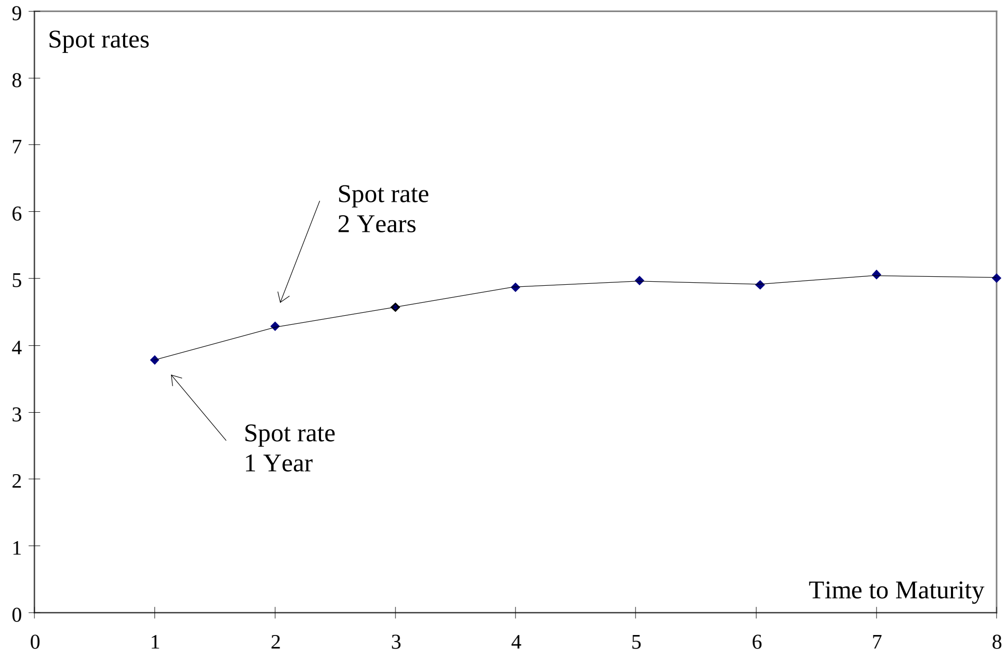


Figure 1-1: The term structure of interest rates

The problems in building the term structure of interest rates are that

- to avoid coupon effects and reinvestment risk, the term structure of interest rates should be built using only zero-coupon bonds.
- some rates are not available: one usually knows the 1, 2, 3, 5, and 10 year rates, but how about a 7.5 year rate?
- there are very few spot rates published for non-government bonds, as there are very few corporate zero-coupon bonds.

Thus, most people will instead use the market **yield curve**, which plots the yield to maturity of various bonds against their respective maturity, assuming all other factors to be equal.

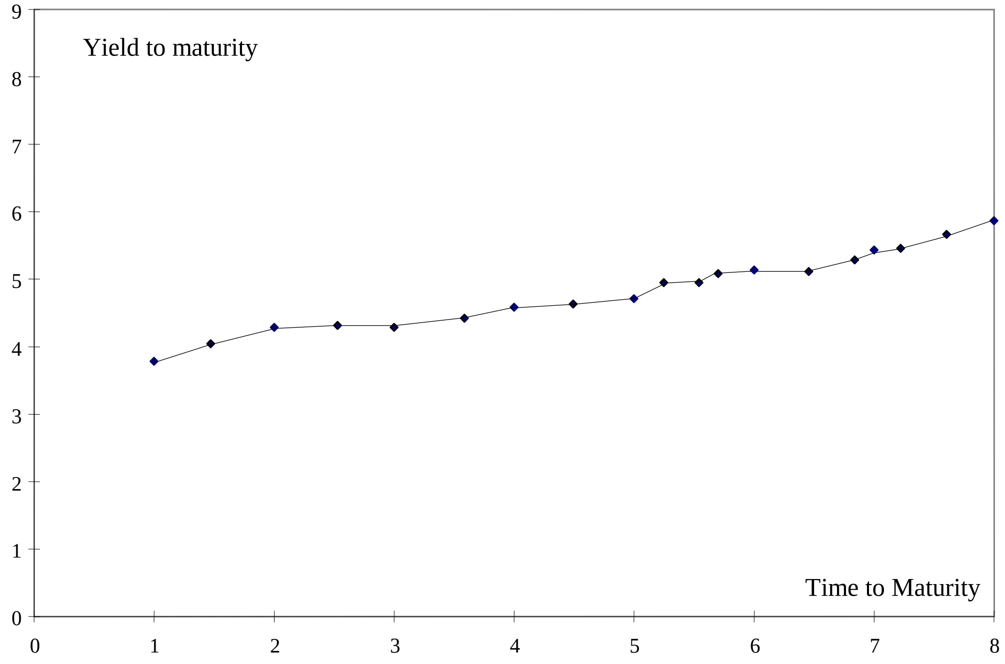


Figure 1-2: The yield curve

Formally, the term structure deals with the relationship between spot rates and time to maturity, whereas the yield curve deals with yield to maturity and time to maturity. Generally, both are similar. But in the analysis of maturity-return relationship, it is better to work with spot rates rather than yields to maturity, as they are (among other things) not contaminated by the coupon effect.

A nominal interest rate can be decomposed into three basic components:

$$\text{nominal rate} = \text{real interest rate} + \text{inflation premium} + \text{risk premium}$$

The **real interest rate** is the compensation for the investor for deferring consumption to a future period (time value of money).

The **inflation premium** is intended to preserve the investor's purchasing power over time, and reflects the expected future inflation level over the life span of the investment.

The **risk premium** compensates the investor against all other potential negatives, including default risk, redemption risk, market risk, etc.[1]

The yield level of all bonds will reflect these three components. Consequently, different issuer sectors will be plotted on different yield curves. Low quality sectors (lower ratings) will be traded at higher yields.

[1] Note that some of these risks can be diversified.

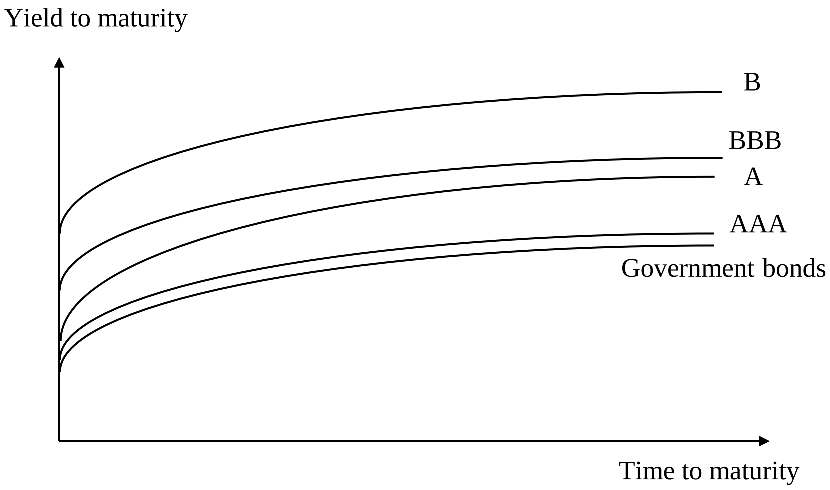


Figure 1-3: The yield/time to maturity relationship of various ratings

Because of this, the liquidity risk, credit risk, call risk, coupon rate, and degree of premium/discount as well as any other risk should be sufficiently similar between the issues in order to build a useful yield curve.

The term structure of interest rates can exhibit four basic shapes: positively sloped (a gently upward slope is the term structure’s usual form), negatively sloped, flat, and humped. The following figures show these four configurations for illustrative purposes only.

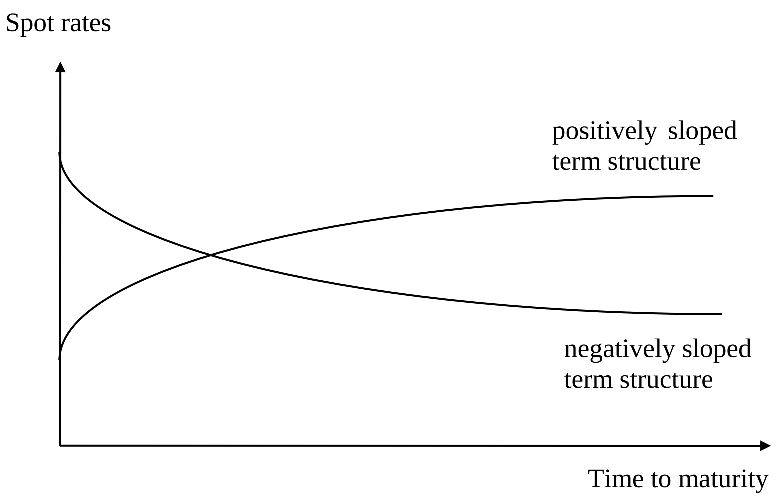


Figure 1-4: Basic shapes of the term structure: positively and negatively sloped

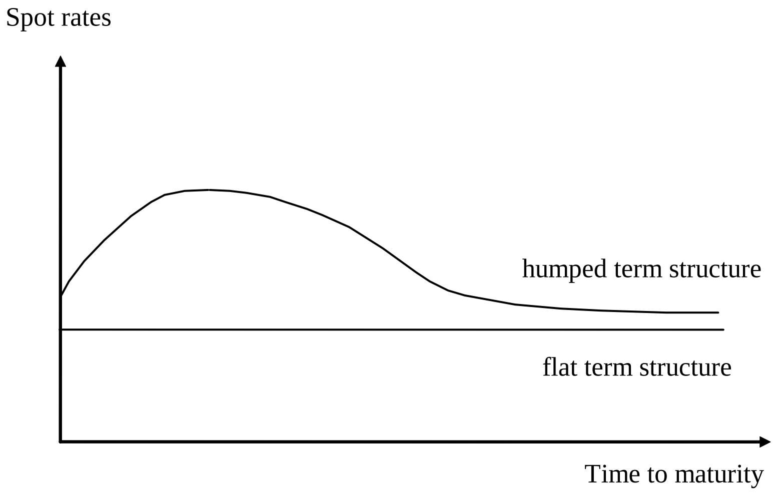


Figure 1-5: Basic shapes of the term structure: flat and humped

The short maturity section is mainly influenced by monetary policy, while the long maturity segment is more sensitive to inflationary expectations.

1.2 Theories of term structures

There are three primary theories that try to explain the shape of the term structure of interest rates: the expectations hypothesis, the liquidity preference, and the market segmentation theory.

Before moving ahead with these theories, we need to define a number of related concepts.

Spot rate is the current interest rate for a given maturity. For example, the one-year spot rate is the yield to maturity of a one-year zero-coupon bond issued today.

Forward rate is the interest rate for a given maturity that begins in the future. For example, the one-year forward rate starting in one year is the yield to maturity of a one-year zero-coupon bond that will be issued in one year.

Forward rates can be calculated using spot rates. A rate calculated this way is called the **implicit forward rate**.

Example:
The one-year spot rate is 4% and the two-year spot rate is 5%. What is the one-year implicit forward rate starting at the end of year 1?

We denote the forward rate as $F_{1,1}$.

To solve this problem, we need to compare two strategies:
Strategy 1: Invest $1 for two years.
Strategy 2: Invest $1 for one year and then reinvest it for another year.
The implicit one-year forward rate will be the one that equates the values of both strategies, so that:

$$(1+0.05)^2 = (1+0.04)\cdot(1+F_{1,1})$$

$$F_{1,1} = \frac{(1+0.05)^2}{1+0.04} - 1 = 6\%$$

Expected future spot rate is the spot rate that is expected by the market to be in effect on some specified date in the future. The main difference between a forward rate and a future spot rate is that a forward rate can be calculated with the information observable in the market today. In contrast, a future spot rate is only observable in the future.

1.2.1 Expectations hypothesis[2]

The expectations theory contends that the shape of the term structure **only** reflects the market consensus forecast on future interest rates levels. Therefore, **the implicit forward rate is an unbiased estimate of the expected future spot rate**.

$$F_{t,t+1} = E\left(\tilde{R}_{t,t+1}\right) \text{(for } t > 0\text{)}$$

A good way to understand this theory is to assume that investors are risk neutral, and that they will select the securities that give them the highest expected return, whatever their time horizon is.

Example:
The yield to maturity on a one-year pure zero-coupon bond is 10%, and 12% for a two-year pure zero-coupon bond. New information makes investors expect the one-year spot rate to be 16% in one year. What should investors do if they consider a one or two-year investment horizon? How would the yield to maturity of the two-year zero-coupon bond evolve?

The investor with a two-year investment horizon can invest 1 EUR in a two-year bond, with a final value of

$$1 \cdot (1.12) \cdot (1.12) = 1.254 \text{ EUR}$$

or hold two one-year bonds, with an expected final value of

$$1 \cdot (1.10) \cdot (1.16) = 1.276 \text{ EUR.}$$

All investors with a two-year investment horizon will want to hold two one-year bonds.
The investor with a one-year investment horizon can invest 1 EUR in a one-year bond, with a final value of

$$1 \cdot (1.10) = 1.10 \text{ EUR}$$

or hold a two-year bond, that he will sell in one year, with an expected final value of

$$1 \cdot (1.12) \cdot \frac{(1.12)}{(1.16)} = 1.081 \text{ EUR}$$

So, all investors with a one-year investment horizon will also want to hold the one-year bonds.

Given this universal preference, all investors will choose a **rollover strategy** and hold only the one-year bonds; thus, prices (i.e. interest rates) of the two-year bonds should adjust until the expected return from holding a two-years bond is exactly the same as the expected return from holding two consecutive one-year bonds.

So,

$$1 \cdot (1 + R_{0,2})^2 = 1 \cdot (1.10) \cdot (1.16)$$

2 There are several different interpretations of the expectation theory. For more information see Fabozzi (2012), chapter 8.

$$R_{0,2} = 12.96\%$$

After this adjustment of the yield, all investors will be indifferent between the two bonds.

The statement that implicit forward rates are unbiased estimates of future spot rates is also based on the following assumptions:

- investors have homogenous expectations.
- investors choose between short or long-term bonds in order to maximise their final expected wealth for a given investment period.
- there are no transaction costs.
- bond markets are efficient, and new information is instantaneously reflected in bond prices.
- future spot rates are statistically independent

In reality, all of these assumptions are subject to criticism. If we assume that the implicit forward rates are an unbiased estimate of the future spot rates, then future spot rates can be derived from spot rates, which implies that, if there are no transaction costs, **each bond is a perfect substitute for any other bond, whatever its maturity** because the expected return will be the same, whatever the bond combination selected by the investor. Indeed, an investor who has a determined investment horizon could do any of the following strategies and obtain the same return:

- buy a pure zero-coupon bond that matures at the end of the investment period, and hold it to maturity ("**buy and hold strategy**").
- buy a short-term maturity bond, and reinvest regularly the proceeds ("**rollover strategy**").
- buy a long-term bond, and sell it with a loss or a gain prior to maturity. The loss or the gain is predictable, using the forward rates.

The explanation of the expectations theory for the four different shapes of the term structure of interest rates is as follows. A positively sloped (respectively negatively sloped) term structure implies that interest rates are expected to rise (respectively to decrease) in the future, while a flat term structure represents a market consensus for stable yields. Finally, a humped term structure shows that market participants expect a rising rate environment for the intermediate times to maturity, followed by a long-term decline in yield levels. This theory, though, does not explain why the term structure of interest rates is generally positively sloped, because the reason is that interest rates are as likely to rise as to fall, thus the yield curve should usually be flat.

1.2.2 Liquidity preferences

As opposed to the expectations theory, the **liquidity preference theory** asserts that investors prefer to hold liquid securities, liquidity being defined as the ability to convert a bond rapidly into cash. As the duration risk in long-term securities is higher than for short-term ones, investors will prefer short-term securities.[3] Thus, there is a shortage of longer term investors.

> **Example:**
> Bond A is a one year bond with a 6% coupon. Bond B is a two-year bond with a 6% coupon. The one year and two-year spot rates are equal to 6%. What happens if there is a sudden unexpected rise of all spot rates to 7%?
>
> Both bonds were priced at par (= 100.00). After the rise, the two-year security will drop to 98.19 in price, while the one year bond will drop to 99.07. So, for one percentage point increase in yield, the two-year security has a more important price decrease than the one-year security. Hence, for a risk-averse investor, the two-year security seems riskier.

On the other hand, the borrowers (governments, firms, ...) prefer to issue long-term securities, to avoid the consequences of interest rates fluctuations on their expenses. In order to induce investors to invest in long-term securities, they will offer them a **risk premium** (a **liquidity premium** or **term premium**).

So, in the liquidity preference theory the implicit forward rate incorporates a risk premium, $L_{t-1,t}$, in the way that the expected future spot rate is equal to:

$$E\left(\tilde{R}_{t,t+1}\right) = F_{t,t+1} - L_{t,t+1} \quad (\text{for } t > 0)$$

$$L_{t,t+1} > 0$$

Therefore we should have a positive difference between implicit forward rates and expected future spot rates:

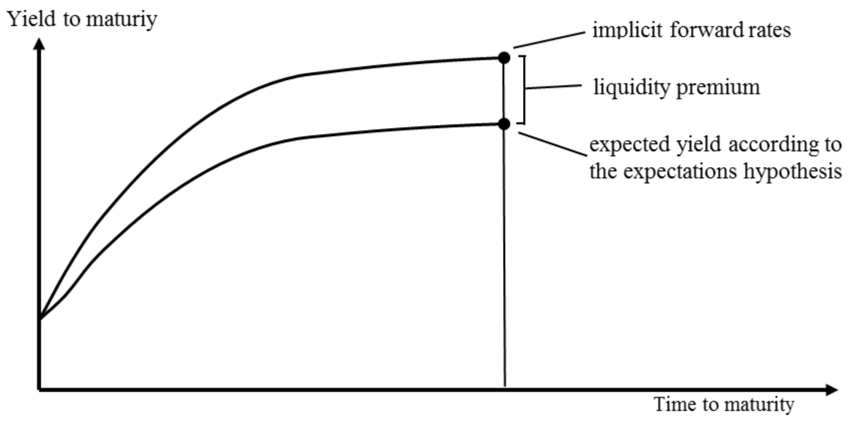


Figure 1-6: The liquidity premium concept

3 Another simple reason to justify this is the unexpected inflation risk: if there is an unexpected rise in the inflation rate, the nominal interest rates should also rise; investors holding short term bonds will be able to reinvest their money at a higher rate, while investors holding long term bonds will have to wait for the final reimbursement before taking advantage of the higher interest rates.

As this theory suggests a higher yield for longer maturity issues caused by their lower degree of “liquidity”, the expected return on a buy and hold strategy has to be higher than the expected return on a rollover strategy. In other words, the return of the roll over strategy is only composed of expectations without risk premium whereas the return of the buy and hold strategy fully incorporates this liquidity premium. So,

$$\left(1+R_{0,t}\right)^t > \left(1+R_{0,1}\right)\left[1+E(\tilde{R}_{1,2})\right]\left[1+E\left(\tilde{R}_{2,3}\right)\right].....\left[1+E\left(\tilde{R}_{t-1,t}\right)\right]$$

Furthermore, as risk increases with time, we should observe:

$$L_{1,2} < L_{2,3} < L_{3,4} < \ldots < L_{n-1,n}$$

Hence, the term structure of interest rates should be (mainly) upward sloping because of the preference of investors for liquidity. Therefore, this theory correctly explains the usual shape of the term structure. Following this theory, the term structure is flat or downward sloping, if the market expects the interest rates to decrease in the future.

1.2.3 *Market segmentation and preferred habitat theories*

Both, the **market segmentation theory** and the **preferred habitat theory** view the bond market as a series of distinct markets that differ by their maturity. In the market segmentation theory, each issuer or investor will have a preferred maturity, and he will have complete risk-aversion so that he operates only in his desired maturity spectrum. Within a given maturity range, the relative supply and demand for funds determines the appropriate clearing price (i.e. the appropriate interest rate).

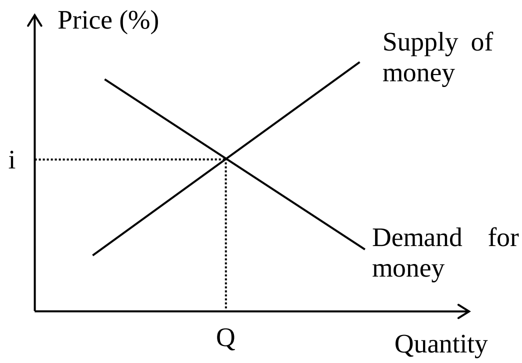


Figure 1-7: Supply and demand of money for a given maturity

The major criticism of the market segmentation theory is that even if investors have a strong maturity preference, the effect of segmentation on interest rates should be offset as soon as some investors start considering relative yields and allocate their funds to another segment which offers a (sufficiently) higher yield. To address this issue, the preferred habitat theory was developed, where any investor will try to reduce risks by staying in its preferred habitat; but he will leave it as soon as he is given a risk premium high enough to cover the assumed risks and the cost of leaving its preferred habitat.

Thus, contrary to the liquidity preference theory, the preferred habitat theory leads to the result that the risk premium attached to bonds can be either positive, negative, or zero.

$$F_{t,t+1} = E\left(\tilde{R}_{t,t+1}\right) + \Pi_{t,t+1}$$

and nothing can be said ex ante about the sign of $\Pi_{t,t+1}$.

In the preferred habitat theory, the money is considered as a commodity, its market clearing price is the interest rate, and the supply and demand of each individual segment connect to create the overall composite term structure of interest rates. By examining flows of funds into the market segments, one could - in theory - predict changes in the term structure of interest rates.

Following this theory, the four basic shapes of the term structure are due to:

- In the case of a positively sloped term structure of interest rates, the investors (buyers) have a preference for the short-term segments of the market, thus prices of bonds with small maturities are high and their yields are low; the reverse is true for the long-term segment.
- In the case of a negatively sloped term structure of interest rates, we have the reverse case of the above scenario.
- If the term structure of interest rates is flat, investors have similar preferences for all segments of the market.
- Finally, the humped term structure of interest rates is due to different preferences, which depend on the maturity segments.

1.2.4 Other theories

The expectation hypothesis, liquidity preference and market segmentation theory are three non-exclusive ways of thinking about interest rates.

But as far as bond pricing is concerned, the most promising theories are the **stochastic process no-arbitrage approaches**. They rely on the following assumptions:

- the term structure and the bond prices are related to some stochastic factors
- these factors evolve over time according to a particular hypothesised stochastic process (i.e. a process with some uncertainty)
- there should be no arbitrage opportunity

Various models have been developed, using single or multiple factors. For example, the Ogden model (1987) assumes that the term structure of interest rates is driven only by the short term interest rate fluctuations, and uses the following process to describe the short term interest rate variations:

$$dr = \underbrace{\beta \cdot (u - r) \cdot dt}_{\text{predictable component}} + \underbrace{\sigma \cdot r \cdot dZ(t)}_{\text{unpredictable component}}$$

where dr is the instantaneous change in the rate r, β is a speed-of-adjustment component, u is the average level of the rate, dt is the passage of time, dZ(t) is a stochastic process, and $\sigma \cdot r$ is the standard deviation of the process.

In other words, such an equation says that the change in the short term rate has two components: one is predictable (the extent to which the current rate differs from its long-term value, multiplied by a coefficient that measures its rate of adjustment to its long-term value), and one unpredictable (the product of the standard deviation of the rate, of the initial level, and of some stochastic process, that acts as a random generator).

Using this specification of the short-term rate, and by solving a partial differential equation, it is possible to find an analytical solution (or a numerical solution) for the bond prices, and therefore for the term structure of interest rates.

Of course, other specifications of the process followed by the short-term rates would lead to another term structure of interest rates. It is also possible to use other factors (such as the long-term rate, the spread between short-term and long-term interest rates, ...), or more than one factors. But each factor stochastic process has to be carefully specified, and the addition of factors complicates the solving of the partial differential equation.[4]

[4] For some interesting specifications, see: Brennan and Schwartz (1982).

2. Risk measurement

The return from holding a bond for a given period can be decomposed in two components: the change in the market value of the security (selling price minus purchase price), and the cash flows received from the security plus any additional income from reinvesting those cash flows. Several market factors impact one or both of these parts of the return.

Hence, we will define the risk of a bond as a **measure of the impact of the market factors on the return characteristics of the bond**.

Hereafter, we will examine the external factors that can affect bond prices.

We have seen that using the yield to maturity concept (denoted YTM), a bond price can be defined as:

$$P = \sum_{t=1}^{T} \frac{CF_t}{(1+YTM)^t} = \frac{CF_1}{(1+YTM)^1} + \frac{CF_2}{(1+YTM)^2} + \ldots + \frac{CF_T}{(1+YTM)^T}$$

where CF_t is the cash flow received at the end of period t (coupons or repayment), and T is the remaining life of the bond (time to maturity).

Hence, it should be clear that the price of a typical fixed income security moves in the opposite direction of the change in interest rates: as interest rates rise (fall), the price of a fixed income security will fall (rise).[5] A bond's **systematic risk** is defined as the volatility in the total return, where the total return takes into account not only price changes, but also coupon payments, due to an instantaneous interest rate fluctuation.

In the past, bonds were considered as safe investments. Interest rates were stable, and investing in bonds was a rather conservative strategy. But increasing interest rates volatility has transformed bonds into an exciting as well as risky investment vehicle.

- **Price risk:**

  For an investor who plans to hold a bond to maturity, the change in the price prior to maturity is of no concern. But if the investor plans to sell the bond prior to the maturity date, an increase in the interest rate will result in a capital loss. This is referred to as the **price risk**, which is by far the major risk faced by an investor in the fixed income market.

- **Reinvestment risk:**

  The reinvestment risk is defined as the variability of the reinvestment income from a given strategy due to changes in interest rates. For example, if interest rates fall, interim cash flows will be reinvested at a lower rate.

It should be noted that price risk and reinvestment risk act in opposite directions. If interest rates rise, the market price of a bond decreases. But, at the same time, the income received by reinvesting the coupons increases. **A strategy based on equalising and thereby nullifying these two offsetting risks is called "immunisation"** and will be examined later.

[5] There are some exceptions, with price changes in the same direction as interest rates, like some putable bonds under certain circumstances.

What happens if there is an instantaneous change in the bond's yield? Empirical investigations show that:

- **Long maturity bonds are more price sensitive than short maturity bonds.**

Example:
The following table lists various bonds differing only by maturity. All bonds have the same face value of EUR 1,000. If the market yield changes from 5% to 5.5%, the bond prices adjust to reflect the new yield. Long-term bonds are clearly more volatile than short-term bonds, and have a larger depreciation.

	Bond 1	**Bond 2**	**Bond 3**	**Bond 4**	**Bond 5**
Maturity (in years)	20	10	5	3	1
Market yield	5.00%	5.00%	5.00%	5.00%	5.00%
Coupon	7.00%	7.00%	7.00%	7.00%	7.00%
Market price (EUR)	1,249.24	1,154.43	1,086.59	1,054.46	1,019.05
New market yield	5.50%	5.50%	5.50%	5.50%	5.50%
New market price (EUR)	1,179.26	1,113.06	1,064.05	1,040.47	1,014.22
ΔP (EUR)	–69.99	–41.37	–22.54	–14.00	–4.83
ΔP / P	**–5.60%**	**–3.58%**	**–2.07%**	**–1.33%**	**–0.47%**

We should note that the relationship is the same in the case of an interest rate decrease. In the case of a decrease in the interest rate, long-term bonds will have the largest price appreciation.

Example:
The following table lists various bonds differing only by maturity. All bonds have the same face value of EUR 1,000. If the market yield changes from 5% to 4.5%, the bond prices adjust to reflect the new yield. Long-term bonds are clearly more volatile than short-term bonds, and have a larger appreciation.

	Bond 1	**Bond 2**	**Bond 3**	**Bond 4**	**Bond 5**
Maturity (in years)	20	10	5	3	1
Market yield	5.00%	5.00%	5.00%	5.00%	5.00%
Coupon	7.00%	7.00%	7.00%	7.00%	7.00%
Market price (EUR)	1,249.24	1,154.43	1,086.59	1,054.46	1,019.05
New market yield	4.50%	4.50%	4.50%	4.50%	4.50%
New market price (EUR)	1,325.20	1,197.82	1,109.75	1,068.72	1,023.92
ΔP (EUR)	+75.95	+43.38	+23.16	+14.26	+4.88
ΔP /P	**+6.08%**	**+3.76%**	**+2.13%**	**+1.35%**	**+0.48%**

Thus, the price/yield curve is steeper for longer maturity issues than for shorter maturity issues. Therefore, a small change in the yield will have a greater impact on the price of a long term bond compared to a short term bond.

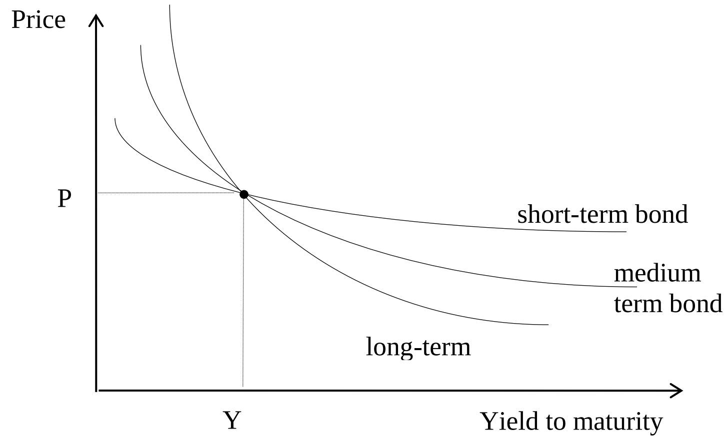


Figure 2-1: Price/yield relationship for various maturities

- **Price volatility is not a symmetric phenomenon**; as shown in previous examples, at a given price, a decrease in the market yield does not have the same effect on the bond price as an identical increase in the market yield.

- For a given maturity**, low coupon bonds are more volatile than high coupon bonds**. Clearly, zero-coupon bonds have the greatest volatility.

Example:
The following table lists various bonds differing only by coupon. All bonds have the same face value of EUR 1,000. If the market yield changes from 5% to 5.5%, the bond prices adjust to reflect the new yield. Low coupons bonds have a greater volatility.

	Bond 1	**Bond 2**	**Bond 3**	**Bond 4**	**Bond 5**
Maturity (in years)	10	10	10	10	10
Market yield	5.00%	5.00%	5.00%	5.00%	5.00%
Coupon	10.00%	7.00%	5.00%	3.00%	0.00%
Market price (EUR)	1,386.09	1,154.43	1,000.00	845.57	613.91
New market yield	5.50%	5.50%	5.50%	5.50%	5.50%
New market price (EUR)	1,339.19	1,113.06	962.31	811.56	585.43
ΔP (EUR)	–46.89	–41.37	–37.69	–34.01	–28.48
ΔP / P	**–3.38%**	**–3.58%**	**–3.77%**	**–4.02%**	**–4.64%**

- For a given maturity**, low yield bonds are more price volatile than high yield bonds**. Thus, the price volatility should be greater in a low interest rate environment.

Example:
The following table lists the effect of a market yield increase on a bond price, with different initial yield levels. All bonds have the same face value of 1,000 EUR. The higher the initial yield, the lower the price decrease. Hence, the bond price is much more volatile when the yield is low, for a same variation of the market yield.

	Bond 1	**Bond 2**	**Bond 3**	**Bond 4**	**Bond 5**
Maturity (in years)	10	10	10	10	10
Market yield	10.00%	7.00%	6.00%	5.00%	3.00%
Coupon	6.00%	6.00%	6.00%	6.00%	6.00%
Market price (EUR)	754.22	929.76	1,000.00	1,077.22	1,255.91
New market yield	10.50%	7.50%	6.50%	5.50%	3.50%
New market price (EUR)	729.34	897.04	964.06	1,037.69	1,207.92
ΔP (EUR)	–24.88	–32.73	–35.94	–39.53	–47.99
ΔP / P	**–3.30%**	**–3.52%**	**–3.59%**	**–3.67%**	**–3.82%**

A bond with a sinking fund provision is less volatile than a similar maturity bullet bond. The existence of a sinking fund provision reduces the effective time to maturity of the bond issue and therefore such bonds are less volatile.

But all these observations are not sufficient for us to derive an adequate bond risk measure. In particular, we cannot compare the risk of two bonds differing in coupon as well as maturity.

Example:
Bonds A and B are described in the following table.

	Bond A	Bond B
Coupon	10%	2%
Time to maturity	12 years	8 years
Market rate	8%	8%
Actual market price	115.07	65.52

What happens if the new market rate is 8.5%? Which bond will be more volatile?

Without any calculation, it is impossible to say anything: bond A has a longer maturity and should have a greater volatility. But bond B's coupon is lower, which should give her the larger volatility.

A quick calculation would lead to the following results:

	Bond A	Bond B
New market rate	8.5%	8.5%
New market price	111.02	63.34
ΔP	–4.06	–2.18
ΔP / P	**–3.52%**	**–3.32%**

Bond A is more volatile. But the results are very close.

Given the various factors affecting bond price volatility, one wonders whether it is possible to derive an adequate bond risk measure to capture the volatility characteristics of a particular bond.

2.1 Risk measurement tools

The most basic bond risk proxies are time to maturity, weighted average maturity, and weighted average cash flow.

The **time to maturity** is the number of years remaining until the bond's final maturity date. It assesses a bond's risk from a final maturity date perspective. Long maturity bonds are riskier than short maturity issues, as the investors have to wait longer to recover the principal, and also because long term bonds are more sensitive to interest rate fluctuations.

But it is a weak proxy for a bond's inherent risk, because:

- it does not consider the cash flows received prior to final maturity, which leads to errors in the risk assessment process.

Example:
Consider a 10% bond and a zero-coupon bond maturing in 10 years. Both should have the same risk, because they have the same maturity.

But after 5 years, supposing the purchase price was equal to the face value, the owner of the coupon bond will have recovered half of its initial investment (which can be reinvested at a higher rate in the case of an interest rate increase), while the owner of the zero-coupon bond will have cashed nothing (all of its proceeds are in terms of principal appreciation).

- There is no linear relationship between time to maturity and price volatility. A 30-year bond is not three times as risky as a 10-year bond.

Example:
The following table lists the effect of a market yield increase on bonds with different maturities. All bonds have a face value of 1,000 EUR.

	Bond 1	**Bond 2**	**Bond 3**	**Bond 4**
Maturity (in years)	5	10	20	40
Market yield	6.00%	6.00%	6.00%	6.00%
Coupon	6.00%	6.00%	6.00%	6.00%
Market price (EUR)	1,000.00	1,000.00	1,000.00	1,000.00
New market yield	6.50%	6.50%	6.50%	6.50%
New market price (EUR)	979.22	964.06	944.91	929.27
ΔP (EUR)	–20.78	–35.94	–55.09	–70.73
ΔP / P	**–2.08%**	**–3.59%**	**–5.51%**	**–7.07%**

It is clear that doubling the time to maturity does not result in twice the initial volatility.

The **weighted average maturity**, or **average life**, is the weighted average maturity of the principal repayment (note that the coupon rate plays no role in the average life, as it only considers principal repayments):

$$\text{Weighted average maturity} = \sum_{t=1}^{T} \frac{\text{Principal paid at time t}}{\text{Total principal to be repaid}} \cdot t$$

It is identical to the time to maturity for bullet bonds, but for sinking fund bonds and mortgage backed securities, it offers some improvements.

Example:
What is the weighted average maturity of a 6% coupon, 10 year sinking fund debenture priced at par (1,000 USD) to yield 6% (discounted cash flow)? The sinking fund retires 20% of the bonds annually, commencing at the end of the sixth year. The interest are paid semi-annual.

The principal cash flows are the following (we are not concerned with the interest payments):

Time	**Principal Cash Flows**
6	200 USD
7	200 USD
8	200 USD
9	200 USD
10	200 USD

Thus, the weighted average maturity of our bond is:

$$\text{Weighted average maturity} = \frac{200 \cdot 6}{1000} + \frac{200 \cdot 7}{1000} + \frac{200 \cdot 8}{1000} + \frac{200 \cdot 9}{1000} + \frac{200 \cdot 10}{1000} = 8 \text{ years}$$

to be compared with the 10 years time to maturity.

But weighted average maturity is still a weak proxy for a bond's inherent risk. It is better than the term to maturity, as it considers the principal repayment cash flows. But it does not consider the full impact of the distribution of cash flows on the bond's risk, as it ignores the coupons. Thus, the weighted average maturity is insensitive to the coupon differentials. For example, an 8% and a 2% sinking-fund debentures could have the same average life, and hence, the same risk.

The **weighted average cash flow** is calculated similar to the weighted average maturity, except that it considers all the cash flows from a bond:

$$\text{Weighted average cash flow} = \sum_{t=1}^{T} \frac{\text{Cash flow paid at time t}}{\text{Total cash flows to be repaid}} \cdot t$$

It assesses a bond's risk by finding the average maturity of a bond's cash flows, considering coupons as well as principal repayments.

Example:
A bond has a face value of EUR 1,000, expires in 4 years, and offers a 6% coupon rate. What is its weighted average cash flow?

The bond's weighted average cash flow is

$$\text{Weighted average cash flow} = \frac{60 \cdot 1}{1240} + \frac{60 \cdot 2}{1240} + \frac{60 \cdot 3}{1240} + \frac{60 \cdot 4}{1240} = 3.71 \text{ years}$$

to be compared with the 4 years time to maturity.

The main drawback of the weighted average cash flow is that repayments are considered on a nominal basis rather than on a present value basis; and we all know that a EUR tomorrow does not have the same value as a EUR today.

Hence, these three average maturities do not provide an adequate bond price volatility measure.[6]

6 Note that the relationships between the three basis measures are the following:

Bond type	Relationship
Coupon bearing bullet bond	WACF < WAM = TTM
Sinking fund bond	WACF < WAM < TTM
Zero-coupon bond	WACF = WAM = TTM

2.2 Duration and modified duration

The duration as a measure of bond risk was initially proposed by Frederick R. Macaulay in 1938.

2.2.1 Definition

The concept of **duration** can be interpreted as an advanced version of the weighted average cash flow. The duration of a series of cash flows is equal to the average time at which the cash flows occur; the weight of each cash flow is calculated using the present value of the cash flow (instead of using the nominal value). The formula for duration is:

$$\text{Duration} = D = \sum_{t=1}^{T} \frac{PV(CF_t)}{P} \cdot t = \sum_{t=1}^{T} w_t \cdot t$$

If we discount all cash flows at the bond's yield to maturity k (as we did to calculate the price), the weight of each cash flow is $w_t = \frac{CF_t / (1+k)^t}{P}$ and the complete formula for duration is:

$$\begin{aligned}\text{Duration} = D &= \sum_{t=1}^{T} \frac{PV(CF_t)}{P} \cdot t \\ &= \frac{1}{P} \cdot \sum_{t=1}^{T} \frac{CF_t}{(1+k)^t} \cdot t \\ &= \frac{1}{P} \cdot \left[\frac{CF_1}{(1+k)^1} \cdot 1 + \frac{CF_2}{(1+k)^2} \cdot 2 + \frac{CF_3}{(1+k)^3} \cdot 3 + \ldots + \frac{CF_T}{(1+k)^T} \cdot T \right]\end{aligned}$$

where:

CF_t	amount of the cash flow (coupon or principal) received at date t
P	the market price of the bond[7] (or present value of all future payments)
T	time to maturity
k	discount rate (market yield)

Similar to the weighted average cash flows, **duration is measured in years**. This formulation of duration is often called **Macaulay duration**.

[7] Note that if the price of the bond is not known, we can use the following formula.

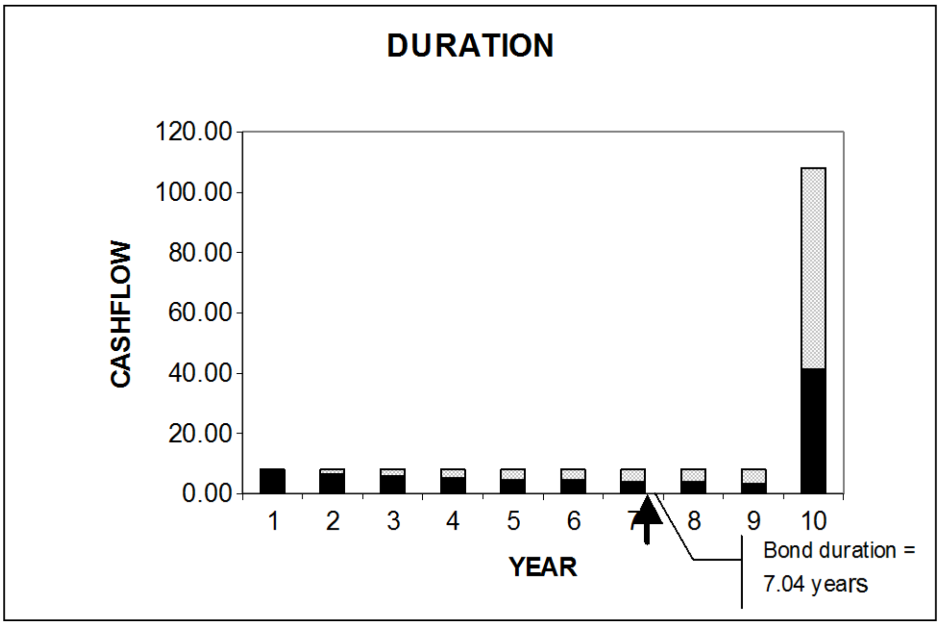
$$\text{Duration} = \frac{\sum_{t=1}^{T} \frac{t \cdot CF_t}{(1+k)^t}}{P} = \frac{\sum_{t=1}^{T} \frac{t \cdot CF_t}{(1+k)^t}}{\sum_{t=1}^{T} \frac{CF_t}{(1+k)^t}}$$

Example:
A bond with a 10-year maturity pays a 8% annual coupon. Its yield to maturity is k=10%. What is its Macaulay duration?

T (Years) [1]	Cash flow CF [2]	PV (CF) [3]=[2]/(1+k)^t	CF weight [4] = [3] / Price	Time weighted by CF weight [5] = [1] · [4]
1	8	7.27	0.0829	0.083
2	8	6.61	0.0754	0.151
3	8	6.01	0.0685	0.206
4	8	5.46	0.0623	0.249
5	8	4.97	0.0566	0.283
6	8	4.52	0.0515	0.309
7	8	4.11	0.0468	0.328
8	8	3.73	0.0425	0.340
9	8	3.39	0.0387	0.348
10	108	41.64	0.4747	4.747
	Price:	**87.71**	**Duration:**	**7.04**

The bond's duration is 7.04 years.

We can graphically represent the duration by plotting the cash flows as a function of time. The height of each bar is the cash flow received [column 2 in the above table]; the lower portion of each bar (in black) is the present value of the cash flow [column 3]. If we think of these values as physical weights placed on a horizontal bar, the duration (marked with an arrow) is the fulcrum point of these weights.



In the case of a zero-coupon bond, as there are no intermediate payments, duration is simply the present value of the final cash flow multiplied by the maturity, divided by the price. But as the price is itself the present value of the final cash flow, **the Macaulay duration of a zero-coupon bond is equal to its maturity**.

Example:
A 10-year zero-coupon bond with a face value of 1,000 EUR is traded at 558.39 EUR. Its yield to maturity is 6%. What is its **Macaulay** duration?

The bond's **Macaulay** duration is:

$$\text{Macaulay Duration} = \frac{\frac{10 \cdot 1000}{1.06^{10}}}{558.39} = \frac{10 \cdot 558.39}{558.39} = 10 \text{ years}$$

which is exactly its time to maturity.

Even if the bond has features which modify its cash flows, such as semi-annual coupon payments or sinking fund requirements, the methodology remains the same.

Example:
What is the duration of a 6% coupon, 10 year sinking fund debenture priced at par (1,000 USD) with a current market yield of 6%? The sinking fund retires 20% of the bonds annually, commencing at the end of the sixth year. The interest is paid semi-annual.

The following table lists the various steps that are necessary to calculate the duration.

T (Years) [1]	**total cash flow** [2]	**PV factor** [3]	**PV (CF)** [4] = [2] · [3]	**CF weight** [5] = [4] / Price	**PV weighted by time t** [6] = [1] · [5]
0.5	30.00	0.9709	29.13	0.0291	0.01
1	30.00	0.9426	28.28	0.0283	0.03
1.5	30.00	0.9151	27.45	0.0275	0.04
2	30.00	0.8885	26.65	0.0267	0.05
2.5	30.00	0.8626	25.88	0.0259	0.06
3	30.00	0.8375	25.12	0.0251	0.08
3.5	30.00	0.8131	24.39	0.0244	0.09
4	30.00	0.7894	23.68	0.0237	0.09
4.5	30.00	0.7664	22.99	0.0230	0.10
5	30.00	0.7441	22.32	0.0223	0.11
5.5	30.00	0.7224	21.67	0.0217	0.12
6	230.00	0.7014	161.32	0.1613	0.97
6.5	24.00	0.6810	16.34	0.0163	0.11
7	224.00	0.6611	148.09	0.1481	1.04
7.5	18.00	0.6419	11.55	0.0116	0.09
8	218.00	0.6232	135.85	0.1359	1.09
8.5	12.00	0.6050	7.26	0.0073	0.06
9	212.00	0.5874	124.53	0.1245	1.12
9.5	6.00	0.5703	3.42	0.0034	0.03
10	206.00	0.5537	114.06	0.1141	1.14
		Price	**1,000.00**	**Duration**	**6.43**

The Macaulay duration of our bond is 6.43 years.

2.2.2 *Interpretations and implicit assumptions*

Macaulay duration takes into account all of the following variables affecting the bond price volatility:

- all the cash flows
- the yield to maturity
- the current market price of the bond

But what does Macaulay duration really mean? In fact, it is more than just a sophisticated average maturity, and there is one basic property that helps in understanding the concept of duration:

- **In interest-rate risk terms, an investor is indifferent between a coupon-bearing bond investment and a zero-coupon instrument maturing on the duration date of the coupon bearing issue.**

Using Macaulay duration, we implicitly assume that all cash flows are discounted (or reinvested) at the same discount rate k, equal to the bond's yield to maturity. But in fact, each cash flow should be discounted at the appropriate rate $R_{0,t}$, and we do not have one single yield, but a part of the term structure of interest rates.

Thus, the assumption made using the Macaulay duration is that **the term structure of interest rates is flat** (that is, **the yields for all maturities are equal** to a single value, called the **market yield**).

Note that if the term structure of interest rates (or the yield curve) is not flat, the implied spot rate curve supplies a series of discount rates (rather than a single one) applicable to the bond's future cash flows, generating a duration that differs from Macaulay duration. For example, the Fisher and Weil's duration is defined as:

$$\text{Fisher and Weil's Duration} = D_{FW} = \sum_{t=1}^{T} \frac{PV(CF_t)}{P} \cdot t = \frac{1}{P} \cdot \sum_{t=1}^{T} \frac{t \cdot CF_t}{(1+R_{0,t})^t}$$

$$= \frac{1}{P} \cdot \left[\frac{1 \cdot CF_1}{(1+R_{0,1})^1} + \frac{2 \cdot CF_2}{(1+R_{0,2})^2} + \frac{3 \cdot CF_3}{(1+R_{0,3})^3} + \ldots + \frac{T \cdot CF_T}{(1+R_{0,T})^T} \right]$$

2.2.3 An example to illustrate the calculation of duration

Consider a 10-year bond with a face value of 100 CHF and a 10% coupon. The current market yield (for all maturities) is 8%.

Let us calculate the current bond

$$P_{k=8\%} = \frac{10}{1.08^1} + \frac{10}{1.08^2} + \frac{10}{1.08^3} + \ldots + \frac{110}{1.08^{10}} = 113.42 \text{ CHF}$$

and the Macaulay duration for the bond

$$\text{Macaulay Duration} = \frac{\frac{1 \cdot 10}{1.08^1} + \frac{2 \cdot 10}{1.08^2} + \frac{3 \cdot 10}{1.08^3} + \ldots + \frac{10 \cdot 110}{1.08^{10}}}{113.42} = 6.97 \text{ years}$$

Now let us determine what happens if immediately after we bought this bond for 113.42 CHF, the market yield decreases from 8% to 4%. The new price of this bond is then

$$P_{k=4\%} = \frac{10}{1.04^1} + \frac{10}{1.04^2} + \frac{10}{1.04^3} + \ldots + \frac{110}{1.04^{10}} = 148.67 \text{ CHF}$$

We can see that with this (extreme) interest rate variation the bondholder has a capital gain of 35.25 CHF. To illustrate the use of duration, we calculate the bond's price for each year remaining until the bond's maturity for a market yield of 8% (before) and a market rate of 4% (after). We also calculate the present value for each year and each market yield of the reinvested coupons C.

$$\text{Future value of reinvested coupons in year t } = \sum_{i=1}^{t} C_i \cdot (1+k)^{t-i}$$

The total value of this bond (calculated as the sum of the bond price and the value of the reinvested coupons) is also given in the following table:

	YTM = 8%			**YTM = 4%**		
Year	**Bond price** [1]	**Value of reinvested coupons** [2]	**Total value** [1] + [2]	**Bond price** [3]	**Value of reinvested coupons** [4]	**Total value** [3] + [4]
0	113.42	0.00	**113.42**	148.67	0.00	**148.67**
1	112.49	10.00	**122.49**	144.61	10.00	**154.61**
2	111.49	20.80	**132.29**	140.40	20.40	**160.80**
3	110.41	32.46	**142.88**	136.01	31.22	**167.23**
4	109.25	45.06	**154.31**	131.45	42.46	**173.92**
5	107.99	58.67	**166.65**	126.71	54.16	**180.87**
6	106.62	73.36	**179.98**	121.78	66.33	**188.11**
7	***105.15***	***89.23***	***194.38***	***116.65***	***78.98***	***195.63***
8	103.57	106.37	**209.93**	111.32	92.14	**203.46**
9	101.85	124.88	**226.73**	105.77	105.83	**211.60**
10	100.00	144.87	**244.87**	100.00	120.06	**220.06**

Table 2-1: Time to maturity and value of a bond

The following figure illustrates the results obtained in the table above:

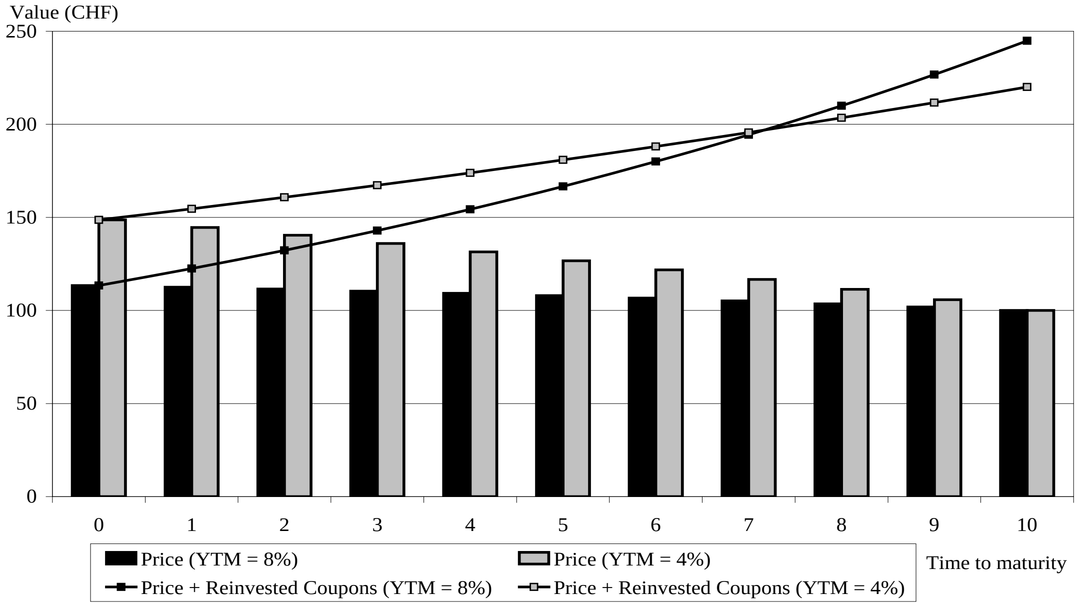


Figure 2-2: Time to maturity and value of a bond

It's interesting to see that in year 7, the total value (= price + reinvested coupons) is almost equal for the two scenarios (194.38 CHF versus 195.63 CHF). From above we know that the duration of this bond is 6.97 years. This is the interpretation of the Macaulay duration. Macaulay duration is equal to the time (in years) at which the total value of the bond is not sensitive to interest rate variations. The total value is the amount we get from this investment.

Another way to show that the total value of the bond is insensitive to interest rate variations in year 7 is to calculate the holding period returns. Remember that we bought the bond in year 0 for 113.42 CHF at a market yield of 8%. What are the holding period returns if the market yield has decreased to 4%? The holding period return is defined as

$$\mathrm{HPR}_{0,t} = \sqrt[t]{\frac{\text{Total value in t}}{\text{Total value in 0}}} - 1$$

Year	**Holding period return**
1	36.31%
2	19.07%
3	13.82%
4	11.28%
5	9.78%
6	8.78%
7	*8.10%*
8	7.6%
9	7.17%
10	6.85%

We see that in year 7 (=Macaulay duration), the holding period return is almost equal to 8% which was the current market yield at which we bought the bond. If you sell a bond at the time of its duration, you have a holding period return equal to the current market yield.

In this example, we calculated the effect on the bond of an extreme interest rate variation (for the purpose of illustration). Now we calculate the total value of this bond in year 7 for other interest rate variations:

Market yield	**Change in market yield**	**Bond price** [1]	**Value of reinvested coupons** [2]	**Total value** [1] + [2]
12%	+4%	95.20	100.89	**196.09**
11%	+3%	97.56	97.83	**195.39**
10%	+2%	100.00	94.87	**194.87**
9%	+1%	102.53	92.00	**194.54**
8%	***0%***	***105.15***	***89.23***	***194.38***
7%	–1%	107.87	86.54	**194.41**
6%	–2%	110.69	83.94	**194.63**
5%	–3%	113.62	81.42	**195.04**
4%	–4%	116.65	78.98	**195.63**

Table 2-2: Value of a bond in its duration year for different interest rate changes

Again we see that the total value of the bond is in year 7 (= Macaulay duration) almost insensitive to interest rate variations. We also see that with bigger interest rate variations (positive and negative) the variation in the total value is bigger. We will come back to this effect in section 2.3 which discusses convexity.

2.2.4 Determinants of Macaulay duration

Macaulay duration of a bond is a function of the bond's time to maturity, coupon rate, accrued interest, market yield, its sinking fund and call features, if any.
Macaulay duration is generally positively related to a bond's **time to maturity**: longer maturity bonds have longer durations.[8] But as maturity increases, duration increases at a decreasing rate: thus, duration cannot grow infinitely, whereas maturity can, and the maximum value of duration is (with an annual coupon payment[9]):

$$\text{Macaulay Duration of a perpetual bond } = \frac{1}{\text{Bond's yield}} + 1$$

We should also note that a zero-coupon bond has a Macaulay duration that exactly matches its time to maturity, while other bonds have durations shorter than their time to maturity (because of the coupon effect).

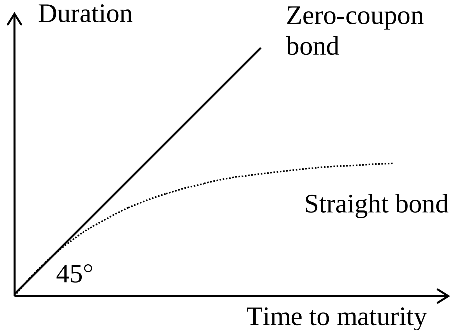


Figure 2-3: Relationship between duration and time to maturity

Macaulay duration is inversely related to the **coupon rate of interest**. Lower coupon bonds have longer Macaulay durations than higher coupon bonds of similar maturity (compare with a zero-coupon bond). Progressively higher coupons lead to a decline in Macaulay duration, but at a diminishing rate.

8 In fact, if the bond is sold at par or over the par duration always increases with maturity. If the bond is sold under par (with a discount), duration also increases with maturity, but starts decreasing at a certain level. It can be shown that the Macaulay duration of a bond paying an annual coupon C, with yield to maturity y and time to maturity T years is given by: $D = \frac{(1+y)}{y} - \frac{(1+y) + T \cdot (C-y)}{C \cdot \left[(1+y)^T - 1\right] + y}$. We can note that, when the coupon C is smaller than the yield y, for large enough T the expression (1+y)+T·(C-y) becomes negative. This means that such a bond has a duration which is higher than the one of a perpetuity!

9 For a semi-annual payment, we have:

$$\text{Maximum Duration} = \frac{1}{\text{Bond's yield}} + 0.5$$

Example:
The following table lists the duration of various bonds differing only by their coupon rate. Higher coupons rates lead to a decline in duration.

	Bond 1	Bond 2	Bond 3	Bond 4	Bond 5
Maturity (years)	10	10	10	10	10
Market yield	6.00%	6.00%	6.00%	6.00%	6.00%
Coupon	0.00%	3.00%	6.00%	9.00%	12.00%
Market price (EUR)	558.39	779.20	1,000.00	1,220.80	1,441.61
Duration (years)	**10.00**	**8.59**	**7.80**	**7.30**	**6.95**

Coupon changes have more impact on duration the lower the initial coupon rate, and the longer the time to maturity.

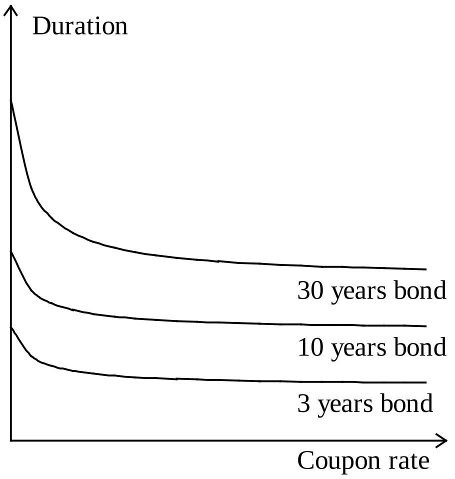


Figure 2-4: Relationship between duration and coupon rate (various maturities)

Duration is, of course, inversely related to the **accumulation of accrued interest**. A bond's duration naturally increases on coupon payment date, as the accrued interest drops off. These effects are especially pronounced for high coupon issues and for long maturity bonds

We should also note that Macaulay duration is inversely related to the **general level of interest rates** (**yield level**). As the concept of duration is based on the discounting process, a higher discount rate will lead to lower duration.

Example:
The following table lists the duration of the same bond using various yields levels. Lower yields lead to a duration increase.

	Case 1	Case 2	Case 3	Case 4	Case 5
Maturity (years)	10	10	10	10	10
Market yield	8.00%	7.00%	6.00%	5.00%	4.00%
Coupon	6.00%	6.00%	6.00%	6.00%	6.00%
Market price (EUR)	865.80	929.76	1,000.00	1,077.22	1,162.22
Duration (years)	**7.62**	**7.71**	**7.80**	**7.89**	**7.98**

2.2.5 Using duration to approximate price changes

From the mathematical derivation of the formula[10] for Macaulay duration, we know that for small changes of the market yield:

$$\frac{\Delta P}{P} = -\frac{D}{(1+k)} \cdot \Delta k$$

This very important formula says that the percentage change in the price of a bond due to an interest rate change is, in first approximation, proportional to its Macaulay duration.[11]

Note that the previous equation is often expressed as:

$$\frac{\Delta P}{P} = -D^{mod} \cdot \Delta k$$

where $D^{mod} = \frac{D}{1+k}$ is called the **modified duration** (or **sensitivity**) of the bond, or as

$$\Delta P = -D^{p} \cdot \Delta k$$

where $D^{p} = \frac{D}{1+k} \cdot P = -\frac{\Delta P}{\Delta k}$ is called the **price duration of the bond.**[12]

Using the modified duration or the price duration of a bond, one can approximate the percentage price change for a given change in the required yield.

Example:
A bond has a face value of EUR 1,000, expires in 4 years and offers a 6% coupon rate. The market yield is 7%. The bond's duration is 3.67 years. What happens if the yield changes by plus 50 basis points (+0.5%, and goes to 7.5%)? How about a 200 basis points change (+2%, to 9%)?

Using duration, we can write:

$$\frac{\Delta P}{P} = -D \cdot \frac{\Delta k}{1+k} = -3.67 \cdot \frac{+0.005}{(1+0.07)} = -1.71\%$$

The duration approach predicts a decrease of 1.71% of the bond price. As the price at a 7% market rate was:

$$P = \frac{\text{EUR } 60}{(1.07)} + \frac{\text{EUR } 60}{(1.07)^2} + \frac{\text{EUR } 60}{(1.07)^3} + \frac{\text{EUR } 1060}{(1.07)^4} = \text{EUR } 966.13$$

the new price should be 966.13 · (1 – 0.0171) = 949.61 EUR.

The effective price with a 7.5% yield is:

$$P = \frac{\text{EUR } 60}{(1.075)} + \frac{\text{EUR } 60}{(1.075)^2} + \frac{\text{EUR } 60}{(1.075)^3} + \frac{\text{EUR } 1060}{(1.075)^4} = \text{EUR } 949.76$$

which is very close to EUR 949.61.

---

10 For the mathematical derivation of the duration see Appendix of this chapter.

11 This formula can be used to *define* duration. In fact the definition of duration as the weighted average life of a bond does not work with particular instruments such as some classes of CMO's (Collateralised Mortgage Obligations) and Inverse Floaters.

12 Or **dollar duration** of the bond in the United States.

The same computation for $\Delta k = +2\%$ would predict a decrease of 6.85% of the price, i.e. a price of EUR 899.95. The effective price with a 9% yield is EUR 902.81.

From the above example, it seems that the duration approach does a good job in estimating the change in the price of a bond **for a small change in the yield**, but not for a large change. We will explain the reason behind this in the next section.

2.2.6 Using the modified duration to approximate the yield to maturity of a bond portfolio

Assume that we have a portfolio containing N different bonds. The yield to maturity of the portfolio (YTM_P) can be approximately calculated from the yields to maturity of the bonds contained in the portfolio using the following formula:

$$YTM_P \cong \sum_{j=1}^{N} \left(\frac{PV_j \cdot D_j^{mod}}{\sum_{i=1}^{N} PV_i \cdot D_i^{mod}} \right) \cdot YTM_j$$

where PV_j is the present value of bond *j*.

Example:
Assume our portfolio contains two different bonds: 50'000 EUR notional of bond 1 with annual coupon C_1 = 2% maturing in 2 years, and 50'000 EUR notional of bond 2 with annual coupon C_2 = 4% maturing in 10 years. Assume that bond 1 has a yield to maturity y_1 = 2%, and bond 2 has a yield to maturity y_2 = 4%, so that the price of bond 1 is $P_{1,cum}$ = 100% and the price of bond 2 is $P_{2,cum}$ = 100%.

The Macaulay durations can be calculated and are equal to 1.98 years for bond 1 and 8.44 years for bond 2. The modified durations are D_1^{mod} = 1.94 for bond 1 and D_2^{mod} = 8.11 for bond 2.

The portfolio yield to maturity can be approximated by:

$$YTM_P \cong \frac{50'000 \cdot 1.94}{50'000 \cdot 1.94 + 50'000 \cdot 8.11} \cdot 2\% + \frac{50'000 \cdot 8.11}{50'000 \cdot 1.94 + 50'000 \cdot 8.11} \cdot 4\% = 0.193 \cdot 2\% + 0.807 \cdot 4\% = 3.614\%$$

If we perform exact calculations [i.e. we list the cashflows CF_t occurring at time t (t=1,…,10) as CF_1 = 50'000 · 2% + 50'000 · 4% = 3'000, CF_2 = 50'000 · 102% + 50'000 · 4% = 53'000, CF_3 = 50'000 · 4% = 2'000, …, CF_{10} = 50'000 · 104% = 52'000, and ask which interest rate has to be used to discount these cashflows in order to get the portfolio present value equal to 100'000 EUR] we get y = 3.627%, which is close to the 3.614% found above.

The yield of each bond is weighted according to the bond value *and* the bond's modified duration. All other things equal, the longer the modified duration of the bond, the higher is its weight to calculate the portfolio yield to maturity.

To approximate the yield to maturity of the portfolio, we should not use the 'naïve' weighted average of the single bonds yields to maturity, where the weight of each bond YTM is equal to its present value divided by the total present value:

Example (continued):
Using the 'naïve' weighted average of the single bonds yields to maturity, we would obtain for the portfolio yield:

$$\text{YTM}_P \cong \frac{50'000}{100'000} \cdot 2\% + \frac{50'000}{100'000} \cdot 4\% = 3\%$$

From the above example we see that the 'naïve' calculation can give very bad approximations of the true portfolio yield to maturity.

2.3 Convexity

How accurately does duration allow us to calculate approximate bond price changes?

Example:
Let us start from the following situation: a 6% 10-year bond is priced at par. Therefore, the market yield is 6%. Its duration is 7.8 years. What are the differences between the effective market price and the price estimated with duration, if the market yield increases?

New market yield	New market price	Estimated price	ΔP / P	Estimated ΔP / P	Difference
0.0625	981.82	981.60	–1.82%	–1.84%	**0.02%**
0.0650	964.06	963.20	–3.59%	–3.68%	**0.09%**
0.0675	946.71	944.80	–5.33%	–5.52%	**0.19%**
0.0700	929.76	926.40	–7.02%	–7.36%	**0.34%**
0.0725	913.21	908.00	–8.68%	–9.20%	**0.52%**
0.0750	897.04	889.60	–10.30%	–11.04%	**0.74%**
0.0775	881.24	871.20	–11.88%	–12.88%	**1.00%**
0.0800	865.80	852.80	–13.42%	–14.72%	**1.30%**
0.0825	850.71	834.40	–14.93%	–16.56%	**1.63%**
0.0850	835.97	816.00	–16.40%	–18.40%	**2.00%**
0.0875	821.56	797.60	–17.84%	–20.24%	**2.40%**
0.0900	807.47	779.20	–19.25%	–22.08%	**2.83%**
0.0925	793.70	760.80	–20.63%	–23.92%	**3.29%**
0.0950	780.24	742.40	–21.98%	–25.76%	**3.78%**
0.0975	767.08	724.00	–23.29%	–27.60%	**4.31%**
0.1000	754.22	705.60	–24.58%	–29.44%	**4.86%**
0.1025	741.64	687.20	–25.84%	–31.28%	**5.44%**
0.1050	729.34	668.80	–27.07%	–33.12%	**6.05%**
0.1075	717.30	650.40	–28.27%	–34.96%	**6.69%**
0.1100	705.54	632.00	–29.45%	–36.80%	**7.35%**
0.1125	694.03	613.60	–30.60%	–38.64%	**8.04%**
0.1150	682.77	595.20	–31.72%	–40.48%	**8.76%**
0.1175	671.76	576.79	–32.82%	–42.32%	**9.50%**

The approximation is accurate for small changes in the market yield, but the error increases for large changes.

The following figure represents the price/yield relationship. We can draw a tangent line at yield k^*, which shows the rate of change of price with respect to a change in interest rate at that point (yield level). The slope of this line is the **price duration**. Mathematically speaking, the price duration is the first derivative of the curvilinear price/yield function.

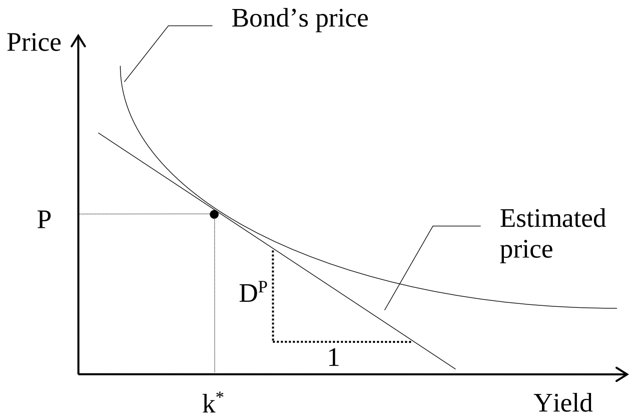


Figure 2-5: Bond's price and market yield

Modified or price duration consider a bond's price/yield relationship as a linear function. In reality, the price/yield function is a convex curve. Thus, duration attempts to estimate a convex relationship with a straight line. Consequently, error terms become large as prices and yields move away from current levels. The further away the new yield is from the initial yield k*, the greater the errors.

Hence,

- duration is an instantaneous value that is continuously modified: even **time has an effect on duration**.
- duration will not exhibit the **asymmetry in price volatility**.
- it should be clear that the approximation will **always underestimate the new price**.
- the **accuracy** of the approximation **depends on the convexity** of the price/yield relationship for the bond.
- we should **not use duration** to approximate a price change if there is a **large variation** in the market yield.

We can better estimate (bigger) price changes if we use another approximation called convexity in addition to duration. The convexity is a proxy for the convexity of the price/yield relationship.[13] The convexity is defined as:

$$\text{Convexity} = C = \frac{1}{2} \cdot \frac{1}{P} \cdot \frac{1}{(1+k)^2} \cdot \sum_{t=1}^{T} \frac{(t)\cdot(t+1)\cdot CF_t}{(1+k)^t}$$

Note that there exist different convexity definitions. Convexity is often defined without the term ½ as:

$$\text{Convexity} = C^* = \frac{1}{P} \cdot \frac{1}{(1+k)^2} \cdot \sum_{t=1}^{T} \frac{(t)\cdot(t+1)\cdot CF_t}{(1+k)^t}$$

Example:
A 10-year bond has a face value of EUR 100, pays a 6% annual coupon rate. The required market yield is 6.5%. What is its convexity?

The following table represents the cash flows from the bond:

13 For a mathematical derivation of the convexity see Appendix of this chapter.

Time	Cash Flow	Present Value (PV)	PV · t · (t + 1)
1	6	5.63	11.27
2	6	5.29	31.74
3	6	4.97	59.61
4	6	4.66	93.28
5	6	4.38	131.38
6	6	4.11	172.70
7	6	3.86	216.22
8	6	3.63	261.03
9	6	3.40	306.37
10	106	56.47	6211.59
Total		**96.41**	**7495.18**

The price P of this bond is 96.41.

The convexity of this bond is:

$$\text{Convexity} = C = \frac{1}{2}\cdot\frac{1}{P}\cdot\frac{1}{(1+k)^2}\cdot\sum_{t=1}^{T}\frac{(t)\cdot(t+1)\cdot CF_t}{(1+k)^t} = \frac{1}{2}\cdot\frac{1}{96.41}\cdot\frac{1}{1.065^2}\cdot 7,495.18 = 34.27$$

With convexity, we can calculate price changes as follows:

$$\Delta P = -D\cdot P\cdot\frac{\Delta k}{1+k} + C\cdot P\cdot(\Delta k)^2$$

or in relative terms:

$$\frac{\Delta P}{P} = -D\cdot\frac{\Delta k}{1+k} + C\cdot(\Delta k)^2$$

If we use the other definition of convexity, we have to modify the above equations with the term ½.

$$\Delta P = -D\cdot P\cdot\frac{\Delta k}{1+k} + \frac{1}{2}\cdot C^*\cdot P\cdot(\Delta k)^2$$

With the above equations one can show that an option-free bond always has a positive convexity for every kind of yield changes. For positive and negative changes in the market yield k, the effect of the convexity term for the price change ΔP is always positive.

We also see from the above equations that the first term is the approximation based on duration and the second term is a proxy for the convexity of the price/yield relationship.

We can define the **price convexity**[14] as the convexity multiplied by the price of the bond.

$$\text{Price convexity} = C^P = C\cdot P$$

Using the price duration and the price convexity, one can estimate the price change of a bond in value in EUR rather than as a percentage:

$$\Delta P = -D^P\cdot\Delta k + C^P\cdot(\Delta k)^2$$

Using both duration and convexity, we should have a more accurate approximation of the bond's price changes for a small variation in the market yield.

14 In the United States: the **dollar convexity**.

Example:
A 10-year bond has a face value of 1,000 EUR, pays a 6% annual coupon rate and is traded at 102%. The market yield is 5.73%. What are its duration and convexity? What happens if the required yield changes by +200 basis points?

The bond's cash flows are as follows:

t	CF	PV(CF)	PV·t	PV · t · (t + 1)
1	60	56.75	56.75	113.50
2	60	53.67	107.34	322.03
3	60	50.76	152.29	609.17
4	60	48.01	192.05	960.26
5	60	45.41	227.05	1,362.33
6	60	42.95	257.70	1,803.90
7	60	40.62	284.35	2,274.84
8	60	38.42	307.35	2,766.29
9	60	36.34	327.05	3,270.47
10	1,060	607.19	6,071.89	66,790.74
		1,020.13	**7,983.85**	**80,273.52**

The bond price P is EUR 1,020.13 EUR.

The duration is:

$$\text{Duration} = D = \sum_{t=1}^{T} \frac{PV(CF_t)}{P} \cdot t = \frac{7,983.85}{1,020.13} = 7.83$$

and the convexity is:

$$\text{Convexity} = C = \frac{1}{2} \cdot \frac{1}{P} \cdot \frac{1}{(1+k)^2} \cdot \sum_{t=1}^{T} \frac{(t) \cdot (t+1) \cdot CF_t}{(1+k)^t} = \frac{1}{2} \cdot \frac{1}{(1.0573)^2} \cdot \frac{80,273.52}{1,020.13} = 35.20$$

If there is a yield increase of 200 basis points, the new market yield is 7.73%. The duration predicts a price change of

$$\frac{\Delta P}{P} = -D \cdot \frac{\Delta k}{1+k} = -7.83 \cdot \frac{+0.02}{(1+0.0573)} = -14.81\%$$

and the new bond price should be 1,020.13 · (1 – 0.1481) = EUR 869.05.

The convexity predicts an additional price change of:

$$\frac{\Delta P}{P} = C \cdot (\Delta k)^2 = 35.20 \cdot (0.02)^2 = +1.41\%$$

Thus, the total price change should be –14.81% + 1.41%= –13.4 %, and the new price should be 1,020.13 · (1 – 0.1340) = EUR 883.43.

The actual price, using a yield of 7.73%, is EUR 882.43.

What does convexity exactly measure? The convexity measures the rate of change of the slope of the price-yield curve with respect to yield changes. Just as duration, convexity changes with time.

Graphically, duration and convexity can be shown as follows:

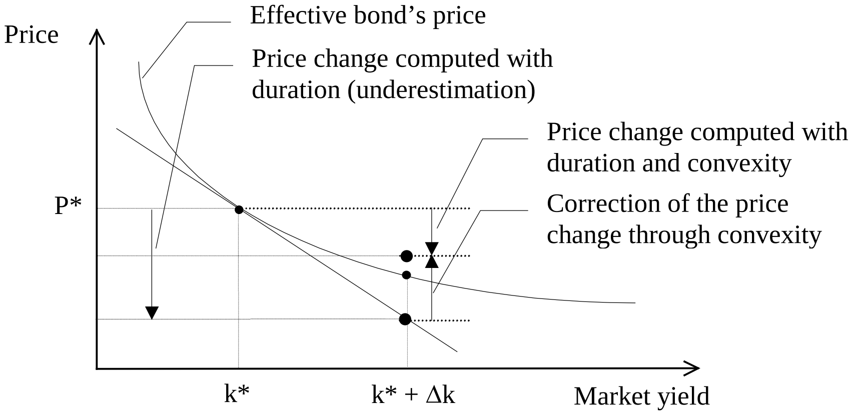


Figure 2-6: Estimating bond price changes using duration and convexity

From the graph above it should be clear that convexity is beneficial to the investor: it has a positive price effect for both increasing and decreasing rates. Thus, all other things being equal, bonds with a larger convexity should be preferred to those with a smaller convexity. Mathematically, we can see this with the already known price change formula:

$$\Delta P = \underbrace{-\text{Duration} \cdot P \cdot \frac{\Delta k}{1+k}}_{\text{negative or positive}} + \underbrace{\text{Convexity} \cdot P \cdot (\Delta k)^2}_{\text{always positive}}$$

In the following figure, we have two bonds A and B with the same duration. But bond B has a smaller convexity than bond A.

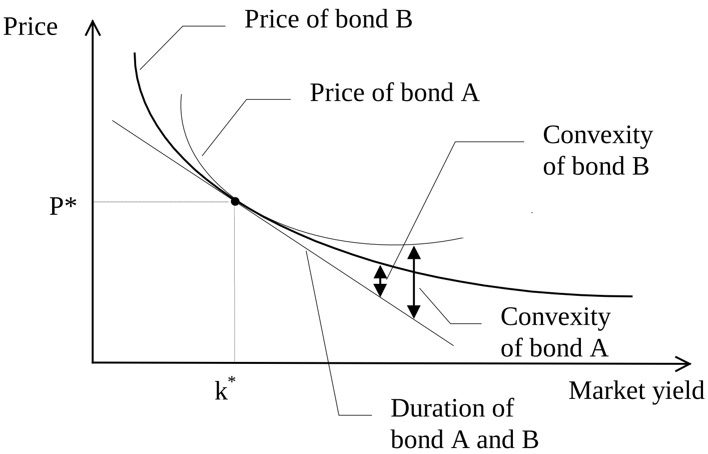


Figure 2-7: Two bonds with different convexities

Now, consider the special case of a zero coupon bond maturing at time T. In this case (there is only one cash flow at T), its convexity simplifies to:

$$C^* = \frac{1}{P} \cdot \frac{CF_T}{(1+k)^T} \cdot \frac{(T)\cdot(T+1)}{(1+k)^2} = \frac{T \cdot (T+1)}{(1+k)^2} \approx T^2$$

Since $\frac{CF_T}{(1+k)^T} = P$

Example:
A 30-year zero coupon bond, with YTM = 3%.

The exact convexity value is $C = \frac{30 \cdot 31}{(1.03)^2} = 876.61$.

The approximated value is $T^2 = 30^2 = 900$.

The convexity of a portfolio[15] consisting of bonds with short and long maturities (barbell portfolio) is generally going to be greater than the convexity of a portfolio concentrated on a single middle length maturity (bullet portfolio).[16] The reason is that the convexity of the barbell portfolio is equal to the weighted average of the convexity of the individual bonds, and the bonds with a long maturity have a dominating contribution. This is because the convexity increases with the square of the maturity.

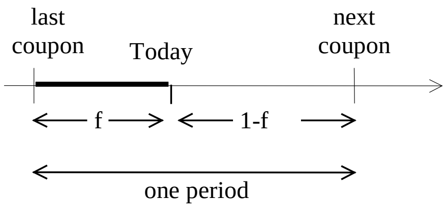
Example:
Consider now a barbell portfolio with 50% each invested in a 1-year and a 30-year zero coupon bond. Using our approximation, the convexity of the barbell portfolio is:

$$50\% \cdot 1^2 + 50\% \cdot 30^2 = 450.5$$

The bullet portfolio with the same duration; i.e., containing a zero coupon bond with a maturity of 15.5 years, has an approximated convexity of $15.5^2 = 240.25$, which is smaller.

2.4 Duration and convexity between coupon payment dates

Determining the duration and convexity of a bond between two coupon payments is very simple. Consider the following situation



P_{ex} denotes the quoted price of the bond (without accrued interests) and C the coupon payment. Starting with the yield to maturity definition formula from “General Principles”:

$$P_{cum} = P_{ex} + f \cdot C = \sum_{t=1}^{T} \frac{CF_t}{(1+k)^{t-f}}$$

15 See section 2.7 for the formulas of the convexity and duration of a portfolio.

16 See chapter “Fixed income portfolio management strategies” for more information about barbell and bullet portfolios.

If we differentiate the right hand side once with respect to the yield k, we get

$$\frac{dP_{cum}}{dk} = \frac{dP_{ex}}{dk} = -\frac{(1+k)^f}{(1+k)} \cdot \left[\sum_{t=1}^{T} \frac{(t-f) \cdot CF_t}{(1+k)^t} \right]$$

The second derivative with respect to the yield is:

$$\frac{d^2P_{cum}}{dk^2} = \frac{(1+k)^f}{(1+k)^2} \cdot \left[\sum_{t=1}^{T} \frac{(t-f) \cdot (t-f+1) \cdot CF_t}{(1+k)^t} \right]$$

From the definition of duration, that is

$$D = -\frac{dP_{cum}}{dk} \cdot \frac{1+k}{P_{cum}}$$

we get by replacing $\frac{dP_{cum}}{dk}$ by its value:

$$D = \frac{(1+k)^f}{P_{cum}} \cdot \left[\sum_{t=1}^{T} \frac{(t-f) \cdot CF_t}{(1+k)^t} \right]$$

From the definition of convexity, that is

$$C = -\frac{d^2P_{cum}}{dk^2} \cdot \frac{1}{P_{cum}}$$

we get by replacing $\frac{d^2P_{cum}}{dk^2}$ by its value:

$$C = \frac{(1+k)^f}{(1+k)^2} \cdot \frac{1}{P} \cdot \left[\sum_{t=1}^{T} \frac{(t-f) \cdot (t-f+1) \cdot CF_t}{(1+k)^t} \right]$$

Thus, we have derived the formulas for the duration and the convexity of a bond between two coupon payments.

2.5 Impact of coupon payments and time lapse on duration

It can be proved that a coupon payment has a positive effect on duration, that is, **duration will suddenly increase at the coupon payment**.

Just before the coupon payment, the duration of the bond is

$$D_{cum} = -\frac{dP_{cum}}{dk} \cdot \frac{1+k}{P_{cum}} \quad \Leftrightarrow \quad D_{cum} \cdot P_{cum} = -\frac{dP_{cum}}{dk} \cdot (1+k)$$

and just after the coupon payment

$$D_{ex} = -\frac{dP_{ex}}{dk} \cdot \frac{1+k}{P_{ex}} \quad \Leftrightarrow \quad D_{ex} \cdot P_{ex} = -\frac{dP_{ex}}{dk} \cdot (1+k)$$

But as

$$\frac{dP_{cum}}{dk} = \frac{dP_{ex}}{dk}$$

one may write

$$D_{cum} \cdot P_{cum} = D_{ex} \cdot P_{ex}$$

that is

$$D_{cum} = D_{ex} \cdot \frac{P_{ex}}{P_{cum}}$$

As $P_{cum} = P_{ex}$ + Coupon, we have $P_{cum} > P_{ex}$. Thus, $D_{cum} < D_{ex}$, which implies that duration will increase just after a coupon payment by an amount of

$$\Delta D = D_{ex} - D_{cum} = D_{ex} - D_{ex} \cdot \frac{P_{ex}}{P_{cum}}$$

$$= \frac{D_{ex} \cdot (P_{cum} - P_{ex})}{P_{cum}} = \frac{D_{ex} \cdot \text{Coupon}}{P_{cum}} > 0$$

One can also prove that **duration will decrease linearly with time** between two coupon payments. Starting from the following equation

$$P_{cum} = P_{ex} + f \cdot I = (1+k)^f \cdot \sum_{t=1}^{T} \frac{CF_t}{(1+k)^t}$$

which gives the yield k of a bond between two coupon payment dates (when a fraction f of a year is elapsed since the last coupon payment date), and replacing P_{cum} in the following formula:

$$D = \frac{(1+k)^f}{P_{cum}} \cdot \sum_{t=1}^{T} \frac{(t-f) \cdot CF_t}{(1+k)^t}$$

which gives the duration of a bond between two coupon payment dates, we get:

$$D = \frac{\sum_{t=1}^{T} \frac{(t-f) \cdot CF_t}{(1+k)^t}}{\sum_{t=1}^{T} \frac{CF_t}{(1+k)^t}}$$

If we derive D with respect to f, we get:

$$\frac{dD}{df} = \frac{\frac{d}{df}\sum_{t=1}^{T} \frac{t \cdot CF_t}{(1+k)^t} - \frac{d}{df}\sum_{t=1}^{T} \frac{f \cdot CF_t}{(1+k)^t}}{\sum_{t=1}^{T} \frac{CF_t}{(1+k)^t}} = \frac{0 - \sum_{t=1}^{T} \frac{CF_t}{(1+k)^t}}{\sum_{t=1}^{T} \frac{CF_t}{(1+k)^t}} = -1$$

that is, duration decreases linearly with f between two coupon payment dates, while f itself increases linearly with time.

In conclusion, one should remember that all other things being equal, the duration of a portfolio will vary linearly over time with an upward adjustment every coupon payment.

2.6 Key rate duration

The usefulness of the (modified) duration as a bond risk proxy is predicated on three assumptions:

- **a small change in the yield**
- **a parallel shift in the yield, whatever the maturity**
- **and instantaneous change in the yield**

Furthermore, as we have already seen, it assumes a **flat yield curve**.

But in reality, the yield curve can make many types of movements, not only parallel shifts. To address this issue, several methods have been suggested to measure the exposure of a bond to a particular rate change. **Partial durations** measure value changes when only the yield for a single maturity is changed leaving the others unchanged. The most popular of these approaches, **key rate durations**, was proposed by Ho (1992).

Following this approach, a finite number of key rates are selected to represent the yield curve. At each key rate, we shift its yield by x basis points, for example 100, declining this shift linearly on both sides of this key rate, reaching 0 at the adjacent key rates.

Key rate durations are not a single measure as is duration; they are a set of measures, each one reflecting the price sensitivity of a security to changes in one key rate and its surroundings, showing where the bond risk concentrates. For example, the ith key rate duration, $KRD(i)$, is the proportional change in the bond price in response to a shift of x basis points in the ith key rate.

$$\Delta \mathrm{P} = -\mathrm{P} \cdot \mathrm{KRD(i)} \cdot \Delta \mathrm{k(i)},$$

where $\Delta \mathrm{k(i)}$ represents the shift of the ith key rate, linearly decreasing to the $(i-1)\,th$ key rate and the $(i+1)\,th$ key rate (as shown in the example below).

Rearranging this equation, we obtain that the ith key rate as:

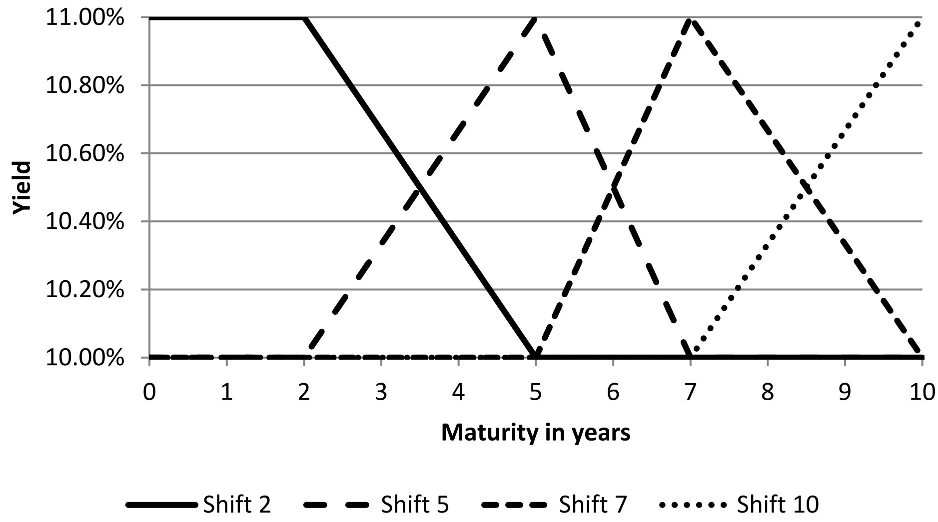
$$\mathrm{KRD(i)} = -\frac{\Delta \mathrm{P}}{\mathrm{P} \cdot \Delta \mathrm{k(i)}}.$$

The idea is that any actual change can be modeled as a combination of key rate shifts, and therefore key rate durations define the price sensitivity of a security over the possible movements of the yield curve. If for example, we combine equal shifts to all the key rates, then its sum is the respond in the bond price to a parallel shift. Hence, key rate durations can be interpreted as a decomposition of the duration.

Due to its definition, in the case of a zero-coupon bond, key rate durations are only different from 0 for the key rates that affect its maturity.

Example:
A bond with a 10-year maturity pays an 8% annual coupon. Its yield to maturity is k=10%. Consider that key rates are the 2, 5, 7 and 10-year rates. What are the key rate durations and their total?

From a previous example, we know that the price of the bond is 87.71. Then we calculate its value after a shift of 100 basis points in each of its key rates, with the shift being linearly declining to the adjacent key rates, as shown in the figure below.



So, for example, for the shift of the 5-year key rate, the above figure shows that due to the linear decline to the adjacent key rates, only the yield on the 5-year rate increases by 100bp, from 10% to 10%+1%=11%, while the adjacent rates increase by less. In particular, the yield on the 4-year rate increases only by 0.66%, to a new rate of 10.66%. The increase reaches 0 at the adjacent key rates of 2 and 7-years. Hence, the new bond price after a shift of 100 basis points in the 5-year key rate is obtained as:

Years T	**Cash flow** CF	**Yield** k	**PV (CF)** $CF/(1+k)^t$
1	8	10%	7.27
2	8	10%	6.61
3	8	**10.33%**	5.96
4	8	**10.66%**	5.33
5	8	**11%**	4.75
6	8	**10.5%**	4.39
7	8	10%	4.11
8	8	10%	3.73
9	8	10%	3.39
10	108	10%	41.64
		Price:	**87.19**

Substituting the price change ΔP= 87.19-87.71 in the equation for the key rate duration, we obtain the value of the 5th key rate duration as:

$$KRD(5) = -\frac{87.19 - 87.71}{87.71 \cdot 0.01} = 0.60 \text{ years} .$$

Following the same procedure, we obtain the remaining key rate durations, which are shown in the table below.

	Value	Key Rate Duration
Initial Curve	87.71	
2-year Shift	87.35	0.41
5-year Shift	87.19	0.60
7-year Shift	87.07	0.73
10-year Shift	83.84	4.41
	Total	**6.15**

The sum of the key rate durations represents the bond's sensitivity to a parallel shift in the yield curve. The above table shows that for a vanilla coupon-bearing bond, the interest rate risk concentrates near its maturity due to the principal payment.

2.7 Portfolio duration, convexity and key rate duration

The **duration of a bond portfolio** is simply the weighted average of the durations of the individual bonds.

$$\text{Portfolio duration} = \sum_{i=1}^{n} w_i \cdot D_i$$

where:

w_i weight (in market value terms) of security i in the portfolio
D_i duration of security i
n number of securities in the portfolio

The **convexity of a bond portfolio** is simply the weighted average of the convexities of the individual bonds.

$$\text{Portfolio convexity} = \sum_{i=1}^{n} w_i \cdot C_i$$

where:

w_i weight (in market value terms) of security i in the portfolio
C_i convexity of security i
n number of securities in the portfolio

The **key rate duration of a bond portfolio** is simply the weighted average of the key rate durations of the individual bonds.

$$\text{Portfolio key rate duration} = \sum_{i=1}^{n} w_i \cdot KRD_i$$

where:

w_i weight (in market value terms) of security i in the portfolio
KRD_i key rate duration of security i
n number of securities in the portfolio

3. Usage

3.1 Bond Yield Curves

3.1.1 Zero (Spot), Coupon and Par curves

As explained in greater detail below, the relationship between par curves and spot or zero-coupon curves is relatively straightforward, at least theoretically. It is important to realise that the relationship is between the whole curves and not between individual points on those curves: we cannot simply "convert" a par bond yield into an equivalent spot yield the way that we can with, for example, a yield expressed in annual terms which can simply be converted into the same yield expressed semi-annually, or a yield expressed in continuous time. This dependence on the whole curve (or at least all of the curve for all maturities shorter than the one being examined) is because the yield of a bond at par (or indeed at any price) is made up of all the yields of all the cash flows included in that bond.

It should be noted that a par curve may have any number of different coupons: in most cases the par coupon will be different for each maturity observed. In fact if the yield curve is positive (i.e. is sloping upwards), then the longer the maturity, the higher the par coupon; and precisely the opposite holds when a curve is inverted: then the longer the maturity of the par bond, the lower the coupon. However, given the spot or zero-coupon curve, the price of any bond (of the same creditworthiness) can be directly constructed, and thus the yield curve that would apply to bonds with any particular coupon rate is relatively trivial to construct.

Let *P* be the (gross) price of the bond, *C* be the coupon of the bond, d_t be the discount factor applying to time *t*, where t_n is the time of the *n*th coupon payment, and *T* is the maturity of the bond, at which time the nominal *N* of the bond is repaid. Assuming, without loss of generality, that *N=1,* then:

$$P = Cd_1 + Cd_2 + ... + Cd_{m-1} + (C+1)d_T$$

$$P = C\sum_{t=1}^{T} d_t + d_T$$

Obviously we then derive the yield of the bond from the resulting *P*. Repeating the exercise we can construct a yield curve for bonds with the coupon *C*, by simply changing the value of *T*.

Calculating the yields involved in the par curve is simply a special case of the equation for *P* above: it is a case where we know *P* (=1) and we need to solve for *C* (and we will obviously NOT need to calculate the yield: it will be the coupon). Thus, using the same notation for the variables described above where P=1 (i.e. par) we have:

$$1 = C\sum_{t=1}^{T} d_t + d_T$$

so for a par curve the coupon, $C = \dfrac{1 - d_T}{\sum_{t=1}^{T} d_t}$

Using the formula above, we can derive a par curve from the spot or zero-coupon curve.

Example:
A bond with, for simplicity, a notional of 1 and a maturity of 3 years, pays an 8% annual coupon. The 1, 2 and 3-year spot rates are 4%, 5%, and 6%, respectively. Calculate the point of the yield and par curves.

First, we calculate the discount factors:

$$d_1 = \frac{1}{(1+1.04)^1} = 0.9616$$
$$d_2 = \frac{1}{(1+1.05)^2} = 0.9070$$
$$d_3 = \frac{1}{(1+1.06)^3} = 0.8396$$

So, the price of the bond is:

$$P = 0.08 \cdot 0.9616 + 0.08 \cdot 0.9070 + 1.08 \cdot 0.8396 = 1.056$$

Therefore, the yield to maturity for this bond, *Y*, is the one that equates:

$$1.056 = \frac{0.08}{(1+Y)^1} + \frac{0.08}{(1+Y)^2} + \frac{1.08}{(1+Y)^3}$$

Using numerical procedures or by testing numbers, it can be determined that the yield is approximately 5.9%.

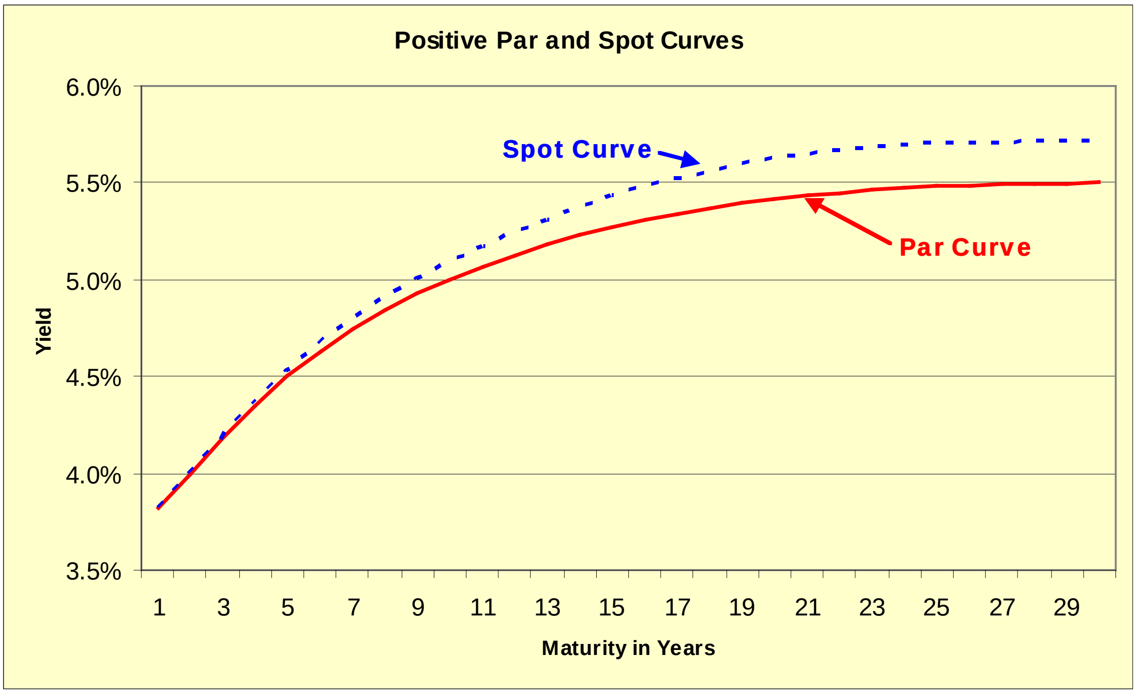
To calculate the par curve point, we equalise the price of the bond with the notional and calculate the coupon needed, so:

$$1 = C \cdot (0.9616 + 0.9070 + 0.8396) + 0.8396$$

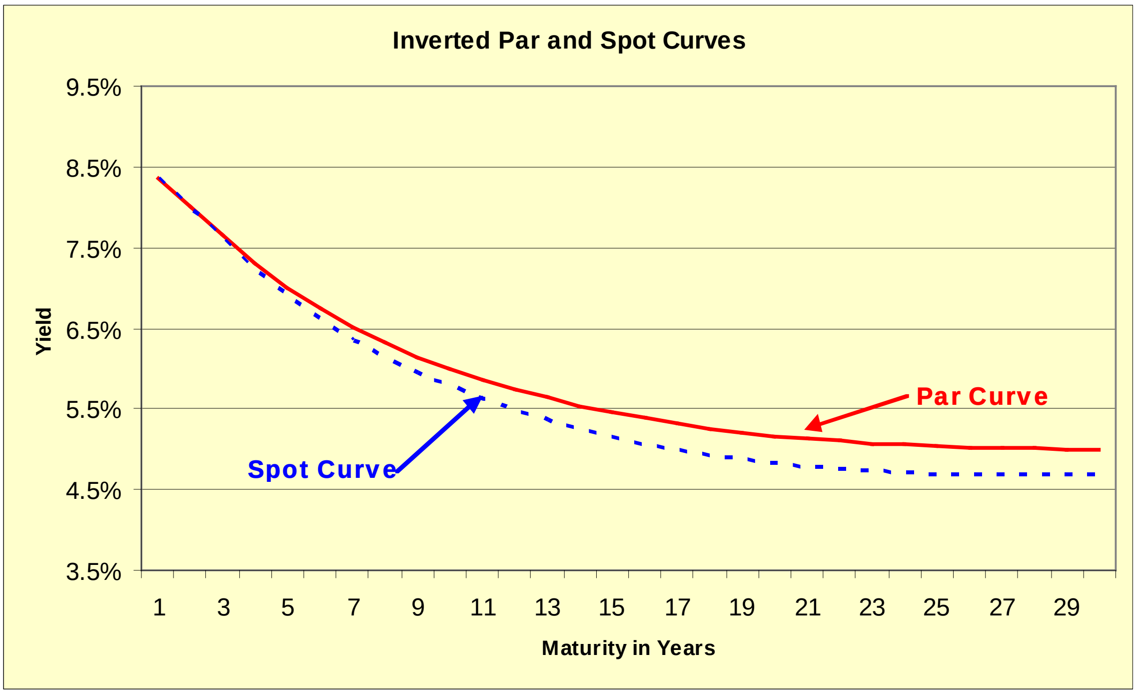
Hence,

$$C = \frac{1 - 0.8396}{0.9616 + 0.9070 + 0.8396} = 5.92\%$$

There are a number of stylised facts relating the shape of par yield curves to that of spot or zero curves. Simply stated, these are essentially that an upward sloping (or positive) yield curve will imply an even steeper spot or zero curve, thus:



and that an inverted par curve implies an even more inverted spot or zero curve, as shown below.



A perfectly flat par curve implies a perfectly flat spot or zero curve. Thus we can imagine the spot or zero curve to have the same direction of slope as the par curve, but always in an amplified way, i.e. with a steeper gradient.

3.2 Bond Curves in Market Usage

3.2.1 *Structure and Smoothness*

From the earliest days of yield curve construction, researchers have recognised that there is a trade-off in the real world between structure and smoothness. One of the earliest papers on yield curve construction puts it thus: "Concern with improving the statistical fit… may, however, complicate and even obscure the true relationship between yield and maturity. Thus, the normal process of curve fitting involves a compromise between some low order curve which is simple and informative and a higher order, more flexible curve, which fits better".[17] By strong implication, better fitting curves are less informative. The practical market trade-off between the two is explained in more detail immediately below.

In one sense, and as illustrated in the examples below, the required granularity to identify aberrations likely to revert to mean depends to a large degree on the focus and reversal horizon of the trade (i.e. the time it is expected to take for the trade to be unwound): the longer the horizon, the less granularity is needed (i.e. the smaller the requirement for the curve "to fit better").

3.2.1.1 Trade Horizon: Yield, Duration & Convexity

One way of understanding the trade-off between structure and smoothness is to consider the horizon and focus of three different types of yield curve trade. The trades described assume that seeming aberrations revert to mean. For the purposes of these examples we will assume that we are trading in the US Treasury (UST) market.

1) A straightforward yield trade would involve the sale of a UST that is overpriced relative to the yield curve (i.e. whose yield is below, or through, the yield curve), against the purchase of a similar maturity UST that is underpriced relative to the yield curve (i.e. whose yield lies above the yield curve). The trade will generally involve roughly equal amounts of each bond. The expectation is that the trade will be "reversed" when the two USTs fall back into line (that is when their yield spreads to the curve have narrowed). In an efficient market, this should not take long, so the trade will have a short horizon, but the granularity of the curve must be high in order to spot small aberrations. In other words, the focus of the trade is on yield, the horizon of the trade is short, and the fitted yield curve should not be overly smoothed.

[17] From Burman and White (1972), p. 467.

2) A straightforward duration trade is likely to involve an expectation that the yield curve slope will change: that is, that the curve will either "flatten" or "steepen". If one expects the yield curve to steepen, one would want to sell a UST with a long maturity, in order to buy one with a short maturity, in the expectations that the spread between the two bonds (which is effectively based on the slope of the curve between the two maturities) will widen.[18] The trade will generally be duration-weighted (i.e. more of the shorter maturity UST will be bought than will be sold of the longer maturity UST, in inverse proportion to their durations). The expectation is that the trade will be reversed when the yield curve steepens, in other words when the spread widens. The horizon of the duration trade is likely to be longer than that of the yield trade, and since we are essentially dealing with the slope of the yield curve, the granularity of the curve fit need not be as high as for that of the yield trade. In other words, the focus of the trade is on duration, the horizon of the trade is longer, and the fitted yield curve can be quite smooth.

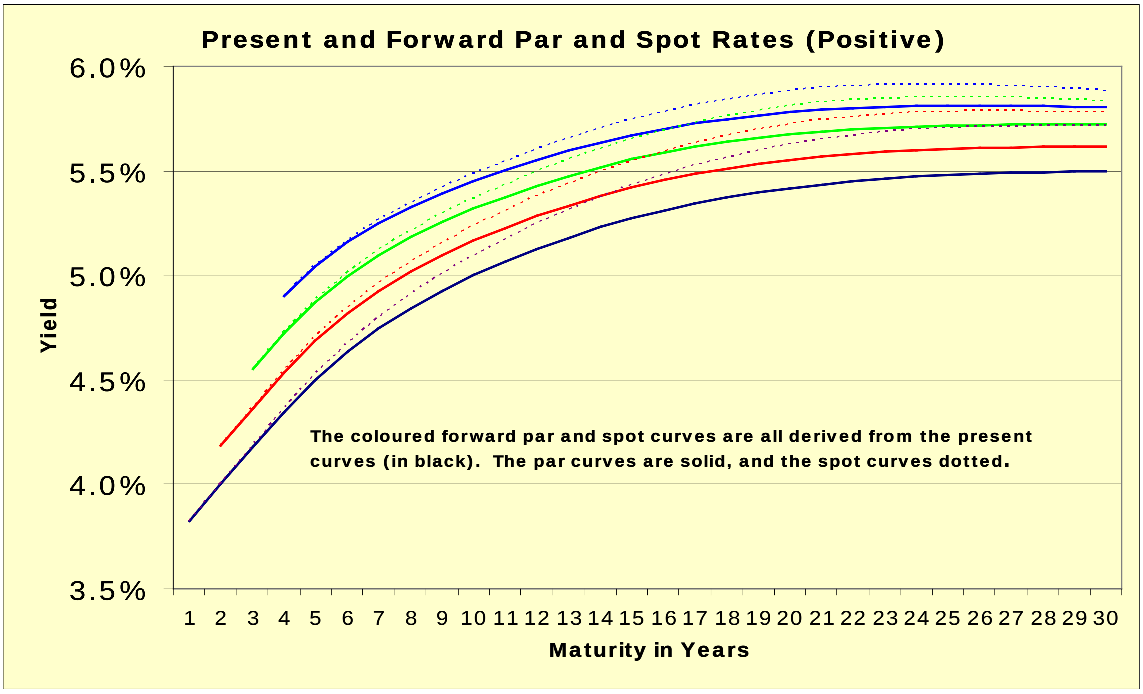
3) A straightforward convexity trade involves an expectation that the curvature of the curve will change. Typical convexity trades are known as "bullet-to-barbell" or vice versa. A bullet to barbell trade would involve selling a bond with a maturity in the middle of the range of those covered by the yield curve, and buying a shorter maturity UST and a longer maturity one.[19] The expectation reflected in this trade is that the yield curve will increase in curvature: i.e. that the yield of the UST being sold will increase relative to that of the shorter and longer one being bought. If the trade is duration-weighted, then although the bond being sold may have a higher yield than the combination of the other two, the convexity of these should be higher, providing protection in parallel yield shifts. The expectation is that the trade will be reversed when the yield curve increases in curvature (i.e. when the spread has widened). The horizon of a convexity trade is likely to be the longest of the three types of yield curve trade examined here, and again, as in the case of the duration trade, the granularity of the curve need not be high. In other words, the focus of the trade is on convexity, and the fitted curve can be quite smooth.

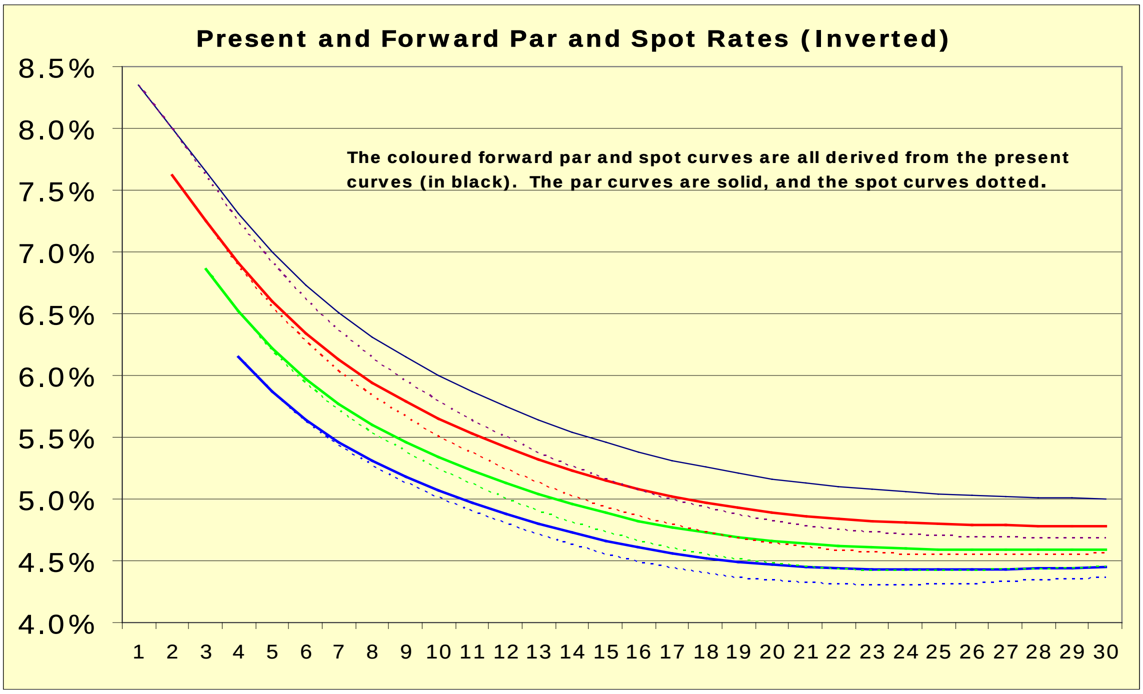
3.3 Curve Shapes and Forward Rates

The general implications about rates in the future drawn from yield curve shapes are straightforward: a positive yield curve is an expectation that yields in the future will be higher, and an inverse yield curve implies that they will be lower. This is illustrated in the attached graphs:

[18] One would, of course, do the opposite if one expected the curve to flatten: i.e. one would sell a short-maturity UST in order to buy a long-maturity UST, expecting the spread between them to narrow.

[19] And conversely, a barbell to bullet trade involves selling two USTs one with a short maturity and the other with a long one, to buy one UST with a maturity in between the two, in the expectation that the yield curve will become less curved: i.e. that the yield on the UST purchased will decrease in relation to that of the other two.





One of the key uses for spot or zero-coupon curves is that it is easy to extract the shape and level of future yield curves from them. Namely, if $d_{t,T}$ is the discount factor at time *t* for maturity *T*, and $z_{t,T}$ is the zero-coupon yield or spot rate at time *t* for maturity *T*, then

$$d_{t,T} = \frac{1}{(1+z_{t,T})^T}$$

so that in *f* time (*f*<*T*)

$$z_{t+f,T-f} = \left[\frac{(1+z_{t,T})^T}{(1+z_{t,f})^f}\right]^{\frac{1}{T-f}} - 1$$

or

$$z_{t+f,T-f} = \left(\frac{d_{t,f}}{d_{t,T}}\right)^{\frac{1}{T-f}} - 1$$

For example, the five year spot rate in two years time implied by the zero-coupon yield curve today, would be today's discount price for a two-year zero-coupon bond, divided by today's discount price for a seven year zero-coupon bond, taken to the fifth root. So the five year rate in two years time depends on the seven year spot rate and the two year spot rate.

3.3.1 Constraints: Absolute and Relative (Slope)

One of the other implications of the fact that we can ascertain future rates from the present spot curve is that the graph of the discount prices corresponding to that curve must always be declining monotonically. In other words the discount price for a given maturity must be higher than the discount price for any longer maturity. Were it smaller than the price for a longer maturity, the future spot rate implied would be negative, which is generally accepted to be impossible. Just as the discount price curve must be monotonically downward sloping, its instantaneous value is obviously 1 (since the immediate present value of 1 is 1!), and the discount price can never reach 0, let alone be negative.

In addition there is a mathematical constraint. Using the **bootstrapping** derivation of the spot curve from the par curve where:

C_T is the coupon for an *T*-year par bond (and must be positive)

d_T is the discount price of the *T*-year zero-coupon bond (and must be positive),

then

$$d_T = \frac{1 - C_T \sum_{t=1}^{T-1} d_t}{1 + C_T}, \text{ and } d_T > 0$$

Because

$$1 - C_T \sum_{t=1}^{T-1} d_t > 0$$

therefore $C_T \sum_{t=1}^{T-1} d_t < 1$ and $C_T < \frac{1}{\sum_{t=1}^{T-1} d_t}$

And since forward rates cannot be negative and nor can the discount factor, then $0 < d_T < d_{T-1}$

3.4 Curves, Economic Activity and Monetary Policy

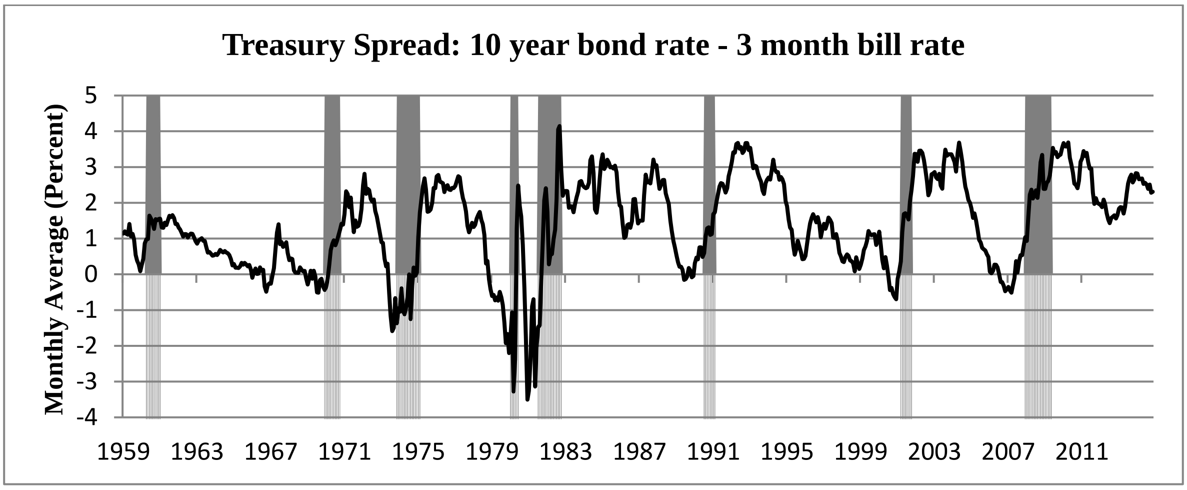
From very early days, it has been assumed that the term structure bears at least some relationship to economic activity. In particular, it seems that the shape of the yield curve can anticipate the evolution of the economic cycle.

So, when a yield curve slopes gently upward, what is known as a **normal yield curve**, the economy is expected to continue to grow at a normal pace. Investors do not perceive a risk of inflation and with a steady economic growth and no inflation pressures, no monetary intervention is expected.

A **steep yield curve**, with long-term rates much higher than short-term rates, appears near the end of a recession. During a recession, short-term rates are low as the central bank has reduced them to stimulate the economy and inflation is lower than normal. When the economy recovers, rates and inflation will increase so investors expecting this recovery begin to demand higher rates for long-term bonds, anticipating these higher short-term rates in the future.

A **flat yield curve**, with similar rates for all maturities, tends to appear at the end of a boom. Investors anticipate a slower economic growth and falling inflation, therefore monetary policy is expected to reduce interest rates to stimulate the economy, so investors demand lower rates for long-term bonds, as future short-term bonds are expected to be lower. This shape may change quickly back to normal or change to the next shape, the inverted yield curve, depending on the (expected) evolution of the economy.

An **inverted yield curve**, with higher short-term rates than long-term rates, provides an early indication that a recession may follow. In the U.S., this has been the case since the 1970s, where inverted curves (defined here as periods when the 10-year UST yield is lower than the 3-month rate) have in fact preceded a recession. This was the case for all observed recessions, including the most recent. The graph below shows the spread between the interest rate on the ten-year Treasury note and the three-month Treasury bill since 1959, where shaded areas indicate recessions.



Source: Federal Reserve Board

Even if inverted curves have indeed preceded all recent U.S. recessions, their lead time is somewhat volatile, ranging from five to thirteen months. A good summary of the situation in the U.S. can be found in Ang, Piazzesi and Wei (2006).[20] Furthermore, an inversion has not preceded all recent recessions in other countries. For example in Japan, short-term rates have been near 0% for the past 15 years, and no inversion could occur under these conditions. However, Japan has suffered five recessions in this period.

Although the history of the Eurozone is shorter, the subject has been explored in several studies such as Chionis, Gogas and Pragidis (2010). Briefly, the paper concludes that the yield curve has forecasting power in terms of the EU15 real output according to data from 1994 to 2008.

As the above discussion shows, expectations on monetary policy affect the shape of the yield curve. However, this influence works in both ways. That is, central banks place a particular emphasis on the information content of the yield curve. Typically, the information is used (i) to predict economic activity, especially recessions and inflation, and (ii) to measure expectations regarding short-term interest rate movements (see above). These findings can be combined to estimate movements in real interest rates, and thus to permit monetary authorities to form a view about the relative tightness of monetary conditions.

3.5 Portfolio Valuation and Mark-to-Market with Unobserved Prices

Any form of portfolio valuation, including valuation on a mark-to-market basis has to follow the same structure. Observed prices (such as those at which an actual trade takes place) are seldom the basis for a valuation price, except for a minority of liquid issues, where observed prices are not affected by a liquidity premium. In most cases, the valuation of bonds tends to be valued in two stages: in the first stage, a benchmark term structure is estimated, and in the second stage the expected spread at which the bond is valued to reflect its default risk and any options embedded in it. The bond specific spread is synthesised from various factors involving comparisons to bonds whose prices have indeed been observed that share comparable characteristics with the valuated bond.

3.6 Financial Engineering

The use of term structures to value cash flows, together with option valuation techniques, is central to much of financial engineering. The ease with which future yield curves can be derived from present term structures greatly facilitates the construction and valuation of all sorts of products, from standard new issues to structured products.

[20] The Federal Reserve Bank of New York maintains a website with information and data about the yield curve as an indicator for recessions, and shows results for a model that calculates the probability of a recession in the United States. http://www.newyorkfed.org/research/capital_markets/ycfaq.html

3.6.1 Structured Product

Structured product generally refers to synthetic investment instruments specifically created to meet explicit needs that cannot be met from standard financial market instruments. Structured products can be used as an alternative to direct investment; as part of the asset allocation process to reduce the risk exposure of a portfolio, or simply opportunistically to express a particular market projection. In addition, structured products can be used to circumvent constraints that may be imposed on the investor, either by regulation, investment mandate, or simply by local investor preferences, or even to arbitrage between different legal, regulatory or tax regimes.

Since these products are essentially synthetic, they depend in large part on the valuation of instruments that do not actually exist, and whose price cannot therefore be directly observed in the market place. Therefore, term structure models are central to the creation and subsequent valuation of almost all structured product.

3.7 Risk Management

An essential part of financial risk management (in, for example a bank, or an insurance company) is constantly to assess the exposure of the bank's or company's position to various changes in the term structure. This process involves the evaluation of their positions to various scenarios of changes in the term structure. Regulators increasingly demand risk measurements of the exposures of financial institutions. In particular, the implementation of the Basel 2 agreements is accelerating the pace of risk management regulation.

Recent events have put an important weight to stress tests. In these tests the term structure is changed in various relatively extreme ways, subject of course to the constraints of the term structure model, to determine the ability of the financial institution to deal with a series of worse case scenarios.

Risk-valuation models commonly have to be approved by competent authorities, and these valuation models themselves incorporate models of the term-structure, which may well differ from institution to institution. In any case, robust term structure models are an essential part of compliance with regulatory reporting requirements.

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