CM1

Actuarial Mathematics

Combined Materials Pack
for exams in 2019

The Actuarial Education Company
on behalf of the Institute and Faculty of Actuaries

Subject CM1

2019 Study Guide

Introduction

This Study Guide has been created to help guide you through Subject CM1. It contains all the information that you will need before starting to study Subject CM1 for the 2019 exams and you may also find it useful to refer to throughout your Subject CM1 journey.

The guide is split into two parts:

- Part 1 contains general information about the Core Principles subjects
- Part 2 contains specific information about Subject CM1.

Please read this Study Guide carefully before reading the Course Notes, even if you have studied for some actuarial exams before.

Contents

1.1 Before you start

When studying for the UK actuarial exams, you will need:

- a copy of the Formulae and Tables for Examinations of the Faculty of Actuaries and the Institute of Actuaries, 2nd Edition (2002) – these are often referred to as simply the Yellow Tables or the Tables
- a 'permitted' scientific calculator – you will find the list of permitted calculators on the profession's website. Please check the list carefully, since it is reviewed each year.

These are both available from the Institute and Faculty of Actuaries' eShop. Please visit **www.actuaries.org.uk**.

1.2 Core study material

This section explains the role of the Syllabus, Core Reading and supplementary ActEd text. It also gives guidance on how to use these materials most effectively in order to pass the exam.

Some of the information below is also contained in the introduction to the Core Reading produced by the Institute and Faculty of Actuaries.

Syllabus

The Syllabus for Subject CM1 has been produced by the Institute and Faculty of Actuaries. The relevant individual Syllabus Objectives are included at the start of each course chapter and a complete copy of the Syllabus is included in Section 2.2 of this Study Guide. We recommend that you use the Syllabus as an important part of your study.

Core Reading

The Core Reading has been produced by the Institute and Faculty of Actuaries. The purpose of the Core Reading is to ensure that tutors, students and examiners understand the requirements of the syllabus for the qualification examinations for Fellowship of the Institute and Faculty of Actuaries.

The Core Reading supports coverage of the syllabus in helping to ensure that both depth and breadth are re-enforced. It is therefore important that students have a good understanding of the concepts covered by the Core Reading.

The examinations require students to demonstrate their understanding of the concepts given in the syllabus and described in the Core Reading; this will be based on the legislation, professional guidance *etc* that are in force when the Core Reading is published, *ie* on 31 May in the year preceding the examinations.

Therefore the exams in April and September 2019 will be based on the Syllabus and Core Reading as at 31 May 2018. We recommend that you always use the up-to-date Core Reading to prepare for the exams.

Examiners will have this Core Reading when setting the papers. In preparing for examinations, students are advised to work through past examination questions and may find additional tuition helpful. The Core Reading will be updated each year to reflect changes in the syllabus and current practice, and in the interest of clarity.

Accreditation

The Institute and Faculty of Actuaries would like to thank the numerous people who have helped in the development of the material contained in this Core Reading.

ActEd text

Core Reading deals with each syllabus objective and covers what is needed to pass the exam. However, the tuition material that has been written by ActEd enhances it by giving examples and further explanation of key points. Here is an excerpt from some ActEd Course Notes to show you how to identify Core Reading and the ActEd material. **Core Reading is shown in this bold font.**

Note that in the example given above, the index *will* fall if the actual share price goes below the theoretical ex-rights share price. Again, this is consistent with what would happen to an underlying portfolio.

This is ActEd text

After allowing for chain-linking, **the formula for the investment index then becomes:**

This is Core Reading

$$I(t) = \frac{\sum_i N_{i,t} P_{i,t}}{B(t)}$$

where $N_{i,t}$ **is the number of shares issued for the** ***i*****th constituent at time** ***t*****;**

$B(t)$ **is the base value, or divisor, at time** ***t*****.**

Copyright

1.3 ActEd study support

This section gives a description of the products offered by ActEd.

Successful students tend to undertake three main study activities:

1. *Learning* – initial study and understanding of subject material
2. *Revision* – learning subject material and preparing to tackle exam-style questions
3. *Rehearsal* – answering exam-style questions, culminating in answering questions at exam speed without notes.

Different approaches suit different people. For example, you may like to learn material gradually over the months running up to the exams or you may do your revision in a shorter period just before the exams. Also, these three activities will almost certainly overlap.

We offer a flexible range of products to suit you and let you control your own learning and exam preparation. The following table shows the products that we produce. Note that not all products are available for all subjects.

LEARNING	**LEARNING & REVISION**	**REVISION**	**REVISION & REHEARSAL**	**REHEARSAL**
Course Notes	Assignments	Flashcards	Revision Notes	Mock Exam
	Combined Materials Pack (CMP)		ASET	Mock Marking
	Assignment Marking			
	Tutorials			
	Online Classroom			

The products and services are described in more detail below.

'Learning' products

Course Notes

The Course Notes will help you develop the basic knowledge and understanding of principles needed to pass the exam. They incorporate the complete Core Reading and include full explanation of all the syllabus objectives, with worked examples and questions (including some past exam questions) to test your understanding.

Each chapter includes:

- the relevant syllabus objectives
- a chapter summary
- a page of important formulae or definitions (where appropriate)
- practice questions with full solutions.

Paper B Online Resources (PBOR)

The Paper B Online Resources (PBOR) will help you prepare for the computer-based paper. Delivered through a virtual learning environment (VLE), you will have access to worked examples and practice questions. PBOR will also include the Y Assignments, which are two exam-style assessments.

'Learning & revision' products

X Assignments

The Series X Assignments are written assessments that cover the material in each part of the course in turn. They can be used to both develop and test your understanding of the material.

Combined Materials Pack (CMP)

The Combined Materials Pack (CMP) comprises the Course Notes, PBOR and the Series X Assignments.

The CMP is available in **eBook** format for viewing on a range of electronic devices. eBooks can be ordered separately or as an addition to paper products. Visit **www.ActEd.co.uk** for full details about the eBooks that are available, compatibility with different devices, software requirements and printing restrictions.

X / Y Assignment Marking

We are happy to mark your attempts at the X and/or Y assignments. Marking is not included with the Assignments or the CMP and you need to order both Series X and Series Y Marking separately. You should submit your script as an attachment to an email, in the format detailed in your assignment instructions. You will be able to download your marker's feedback via a secure link on the internet.

Don't underestimate the benefits of doing and submitting assignments:

- Question practice during this phase of your study gives an early focus on the end goal of answering exam-style questions.
- You're incentivised to keep up with your study plan and get a regular, realistic assessment of your progress.
- Objective, personalised feedback from a high quality marker will highlight areas on which to work and help with exam technique.

In a recent study, we found that students who attempt more than half the assignments have significantly higher pass rates.

There are two different types of marking product: Series Marking and Marking Vouchers.

Series Marking

Series Marking applies to a specified subject, session and student. If you purchase Series Marking, you will **not** be able to defer the marking to a future exam sitting or transfer it to a different subject or student.

We typically provide full solutions with the Series Assignments. However, if you order Series Marking at the same time as you order the Series Assignments, you can choose whether or not to receive a copy of the solutions in advance. If you choose not to receive them with the study material, you will be able to download the solutions via a secure link on the internet when your marked script is returned (or following the final deadline date if you do not submit a script).

If you are having your attempts at the assignments marked by ActEd, you should submit your scripts regularly throughout the session, in accordance with the schedule of recommended dates set out in information provided with the assignments. This will help you to pace your study throughout the session and leave an adequate amount of time for revision and question practice.

The recommended submission dates are realistic targets for the majority of students. Your scripts will be returned more quickly if you submit them well before the final deadline dates.

Any script submitted *after* the relevant final deadline date will not be marked. It is your responsibility to ensure that we receive scripts in good time.

Marking Vouchers

Marking Vouchers give the holder the right to submit a script for marking at any time, irrespective of the individual assignment deadlines, study session, subject or person.

Marking Vouchers can be used for any assignment. They are valid for four years from the date of purchase and can be refunded at any time up to the expiry date.

Although you may submit your script with a Marking Voucher at any time, you will need to adhere to the explicit Marking Voucher deadline dates to ensure that your script is returned before the date of the exam. The deadline dates are provided with the assignments.

Tutorials

Our tutorials are specifically designed to develop the knowledge that you will acquire from the course material into the higher-level understanding that is needed to pass the exam.

We run a range of different tutorials including face-to-face tutorials at various locations, and Live Online tutorials. Full details are set out in our *Tuition Bulletin*, which is available on our website at **www.ActEd.co.uk**.

Regular and Block Tutorials

In preparation for these tutorials, we expect you to have read the relevant part(s) of the Course Notes before attending the tutorial so that the group can spend time on exam questions and discussion to develop understanding rather than basic bookwork.

You can choose ***one*** of the following types of tutorial:

- **Regular Tutorials** spread over the session.
- **A Block Tutorial** held two to eight weeks before the exam.

The tutorials outlined above will focus on and develop the skills required for the written Paper A examination. Students wishing for some additional tutor support working through exam-style questions for Paper B may wish to attend a Preparation Day. These will be available Live Online or face-to-face, where students will need to provide their own device capable of running Excel or R as required.

Online Classroom

The Online Classroom acts as either a valuable add-on or a great alternative to a face-to-face or Live Online tutorial, focussing on the written Paper A examination.

At the heart of the Online Classroom in each subject is a comprehensive, easily-searched collection of tutorial units. These are a mix of:

- teaching units, helping you to really get to grips with the course material, and
- guided questions, enabling you to learn the most efficient ways to answer questions and avoid common exam pitfalls.

The best way to discover the Online Classroom is to see it in action. You can watch a sample of the Online Classroom tutorial units on our website at **www.ActEd.co.uk**.

'Revision' products

Flashcards

For most subjects, there is ***a lot of material*** to revise. Finding a way to fit revision into your routine as painlessly as possible has got to be a good strategy. Flashcards are a relatively inexpensive option that can provide a massive boost. They can also provide a variation in activities during a study day, and so help you to maintain concentration and effectiveness.

Flashcards are a set of A6-sized cards that cover the key points of the subject that most students want to commit to memory. Each flashcard has questions on one side and the answers on the reverse. We recommend that you use the cards actively and test yourself as you go.

Flashcards are available in **eBook** format for viewing on a range of electronic devices. eBooks can be ordered separately or as an addition to paper products. Visit **www.ActEd.co.uk** for full details about the eBooks that are available, compatibility with different devices, software requirements and printing restrictions.

The following questions and comments might help you to decide if flashcards are suitable for you:

- Do you have a regular train or bus journey?

  Flashcards are ideal for regular bursts of revision on the move.

- Do you want to fit more study into your routine?

  Flashcards are a good option for 'dead time', eg using flashcards on your phone or sticking them on the wall in your study.

- Do you find yourself cramming for exams (even if that's not your original plan)?

  Flashcards are an extremely efficient way to do your pre-exam memorising.

If you are retaking a subject, then you might consider using flashcards if you didn't use them on a previous attempt.

'Revision & rehearsal' products

Revision Notes

Our Revision Notes have been designed with input from students to help you revise efficiently. They are suitable for first-time sitters who have worked through the ActEd Course Notes or for retakers (who should find them much more useful and challenging than simply reading through the course again).

The Revision Notes are a set of A5 booklets – perfect for revising on the train or tube to work. Each booklet covers one main theme or a set of related topics from the course and includes:

- Core Reading with a set of integrated short questions to develop your bookwork knowledge
- relevant past exam questions with concise solutions from the last ten years
- other useful revision aids.

ActEd Solutions with Exam Technique (ASET)

The ActEd Solutions with Exam Technique (ASET) contains our solutions to eight past exam papers, plus comment and explanation. In particular, it highlights how questions might have been analysed and interpreted so as to produce a good solution with a wide range of relevant points. This will be valuable in approaching questions in subsequent examinations.

'Rehearsal' products

Mock Exam

The Mock Exam consists of two papers. There is a 100-mark mock exam for the written Paper A examination and a separate mock exam for the computer-based Paper B exam. These provide a realistic test of your exam readiness.

Mock Marking

We are happy to mark your attempts at the mock exams. The same general principles apply as for the Assignment Marking. In particular:

- Mock Exam Marking applies to a specified subject, session and student. In this subject it covers the marking of both papers.
- Marking Vouchers can be used for each mock exam paper. Note that you will need two marking vouchers in order to have the two mock papers marked.

Recall that:

- marking is not included with the products themselves and you need to order it separately
- you should submit your script via email in the format detailed in the mock exam instructions
- you will be able to download the feedback on your marked script via a secure link on the internet.

1.4 Skills

Technical skills

The Core Reading and exam papers for these subjects tend to be very technical. The exams themselves have many calculation and manipulation questions. The emphasis in the exam will therefore be on *understanding* the mathematical techniques and applying them to various, frequently unfamiliar, situations. It is important to have a feel for what the numerical answer should be by having a deep understanding of the material and by doing reasonableness checks.

As a high level of pure mathematics and statistics is generally required for the Core Principles subjects, it is important that your mathematical skills are extremely good. If you are a little rusty you may wish to consider purchasing additional material to help you get up to speed. The course 'Pure Maths and Statistics for Actuarial Studies' is available from ActEd and it covers the mathematical techniques that are required for the Core Principles subjects, some of which are beyond A-Level (or Higher) standard. You do not need to work through the whole course in order – you can just refer to it when you need help on a particular topic. An initial assessment to test your mathematical skills and further details regarding the course can be found on our website at www.ActEd.co.uk.

Study skills

Overall study plan

We suggest that you develop a realistic study plan, building in time for relaxation and allowing some time for contingencies. Be aware of busy times at work, when you may not be able to take as much study leave as you would like. Once you have set your plan, be determined to stick to it. You don't have to be too prescriptive at this stage about what precisely you do on each study day. The main thing is to be clear that you will cover all the important activities in an appropriate manner and leave plenty of time for revision and question practice.

Aim to manage your study so as to allow plenty of time for the concepts you meet in these courses to 'bed down' in your mind. Most successful students will probably aim to complete the courses at least a month before the exam, thereby leaving a sufficient amount of time for revision. By finishing the courses as quickly as possible, you will have a much clearer view of the big picture. It will also allow you to structure your revision so that you can concentrate on the important and difficult areas.

You can also try looking at our discussion forum on the internet, which can be accessed at **www.ActEd.co.uk/forums** (or use the link from our home page at **www.ActEd.co.uk**). There are some good suggestions from students on how to study.

Study sessions

Only do activities that will increase your chance of passing. Try to avoid including activities for the sake of it and don't spend time reviewing material that you already understand. You will only improve your chances of passing the exam by getting on top of the material that you currently find difficult.

Ideally, each study session should have a specific purpose and be based on a specific task, *eg* '*Finish reading Chapter 3 and attempt Practice Questions 3.4, 3.7 and 3.12* ', as opposed to a specific amount of time, *eg 'Three hours studying the material in Chapter 3*'.

Try to study somewhere quiet and free from distractions (*eg* a library or a desk at home dedicated to study). Find out when you operate at your peak, and endeavour to study at those times of the day. This might be between 8*am* and 10*am* or could be in the evening. Take short breaks during your study to remain focused – it's definitely time for a short break if you find that your brain is tired and that your concentration has started to drift from the information in front of you.

Order of study

We suggest that you work through each of the chapters in turn. To get the maximum benefit from each chapter you should proceed in the following order:

1. Read the Syllabus Objectives. These are set out in the box at the start of each chapter.

2. Read the Chapter Summary at the end of each chapter. This will give you a useful overview of the material that you are about to study and help you to appreciate the context of the ideas that you meet.

3. Study the Course Notes in detail, annotating them and possibly making your own notes. Try the self-assessment questions as you come to them. As you study, pay particular attention to the listing of the Syllabus Objectives and to the Core Reading.

4. Read the Chapter Summary again carefully. If there are any ideas that you can't remember covering in the Course Notes, read the relevant section of the notes again to refresh your memory.

5. Attempt (at least some of) the Practice Questions that appear at the end of the chapter.

6. Where relevant, work through the relevant Paper B Online Resources for the chapter(s). You will need to have a good understanding of the relevant section of the paper-based course before you attempt the corresponding section of PBOR.

It's a fact that people are more likely to remember something if they review it several times. So, do look over the chapters you have studied so far from time to time. It is useful to re-read the Chapter Summaries or to try the Practice Questions again a few days after reading the chapter itself. It's a good idea to annotate the questions with details of when you attempted each one. This makes it easier to ensure that you try all of the questions as part of your revision without repeating any that you got right first time.

Once you've read the relevant part of the notes and tried a selection of questions from the Practice Questions (and attended a tutorial, if appropriate) you should attempt the corresponding assignment. If you submit your assignment for marking, spend some time looking through it carefully when it is returned. It can seem a bit depressing to analyse the errors you made, but you will increase your chances of passing the exam by learning from your mistakes. The markers will try their best to provide practical comments to help you to improve.

To be really prepared for the exam, you should not only know and understand the Core Reading but also be aware of what the examiners will expect. Your revision programme should include plenty of question practice so that you are aware of the typical style, content and marking structure of exam questions. You should attempt as many past exam questions as you can.

Active study

Here are some techniques that may help you to study actively.

1. Don't believe everything you read. Good students tend to question everything that they read. They will ask 'why, how, what for, when?' when confronted with a new concept, and they will apply their own judgement. This contrasts with those who unquestioningly believe what they are told, learn it thoroughly, and reproduce it (unquestioningly?) in response to exam questions.

2. Another useful technique as you read the Course Notes is to think of possible questions that the examiners could ask. This will help you to understand the examiners' point of view and should mean that there are fewer nasty surprises in the exam room. Use the Syllabus to help you make up questions.

3. Annotate your notes with your own ideas and questions. This will make you study more actively and will help when you come to review and revise the material. Do not simply copy out the notes without thinking about the issues.

4. Attempt the questions in the notes as you work through the course. Write down your answer before you refer to the solution.

5. Attempt other questions and assignments on a similar basis, *ie* write down your answer before looking at the solution provided. Attempting the assignments under exam conditions has some particular benefits:
   - It forces you to think and act in a way that is similar to how you will behave in the exam.
   - When you have your assignments marked it is *much* more useful if the marker's comments can show you how to improve your performance under exam conditions than your performance when you have access to the notes and are under no time pressure.
   - The knowledge that you are going to do an assignment under exam conditions and then submit it (however good or bad) for marking can act as a powerful incentive to make you study each part as well as possible.
   - It is also quicker than trying to write perfect answers.

6. Sit a mock exam four to six weeks before the real exam to identify your weaknesses and work to improve them. You could use a mock exam written by ActEd or a past exam paper.

You can find further information on how to study in the profession's Student Handbook, which you can download from their website at:

www.actuaries.org.uk/studying

Revision and exam skills

Revision skills

You will have sat many exams before and will have mastered the exam and revision techniques that suit you. However it is important to note that due to the high volume of work involved in the Core Principles subjects it is not possible to leave all your revision to the last minute. Students who prepare well in advance have a better chance of passing their exams on the first sitting.

Unprepared students find that they are under time pressure in the exam. Therefore it is important to find ways of maximising your score in the shortest possible time. Part of your preparation should be to practise a large number of exam-style questions under timed exam conditions as soon as possible. This will:

- help you to develop the necessary understanding of the techniques required
- highlight the key topics, which crop up regularly in many different contexts and questions
- help you to practise the specific skills that you will need to pass the exam.

There are many sources of exam-style questions. You can use past exam papers, the Practice Questions at the end of each chapter (which include many past exam questions), assignments, mock exams, the Revision Notes and ASET.

Exam question skill levels

Exam questions are not designed to be of similar difficulty. The Institute and Faculty of Actuaries specifies different skill levels that questions may be set with reference to.

Questions may be set at any skill level:

- Knowledge – demonstration of a detailed knowledge and understanding of the topic
- Application – demonstration of an ability to apply the principles underlying the topic within a given context
- Higher Order – demonstration of an ability to perform deeper analysis and assessment of situations, including forming judgements, taking into account different points of view, comparing and contrasting situations, suggesting possible solutions and actions, and making recommendations.

Command verbs

The Institute and Faculty of Actuaries use command verbs (such as ‘Define’, ‘Discuss’ and ‘Explain’) to help students to identify what the question requires. The profession has produced a document, ‘Command verbs used in the Associate and Fellowship written examinations’, to help students to understand what each command verb is asking them to do.

It also gives the following advice:

- The use of a specific command verb within a syllabus objective does not indicate that this is the only form of question which can be asked on the topic covered by that objective.
- The Examiners may ask a question on any syllabus topic using any of the agreed command verbs, as are defined in the document.

You can find the relevant document on the profession's website at:

https://www.actuaries.org.uk/studying/prepare-your-exams

1.5 The examination

What to take to the exam

IMPORTANT NOTE: The following information was correct at the time of printing, however it is important to keep up-to-date with any changes. See the profession's website for the latest guidance.

For the written exams the examination room will be equipped with:

- the question paper
- an answer booklet
- rough paper
- a copy of the Yellow Tables.

Remember to take with you:

- black pens
- a permitted scientific calculator – please refer to **www.actuaries.org.uk** for the latest advice.

Please also refer to the profession's website and your examination instructions for details about what you will need for the computer-based Paper B exam.

Past exam papers

You can download some past exam papers and Examiners' Reports from the profession's website at **www.actuaries.org.uk**. However, please be aware that these exam papers are for the pre-2019 syllabus and not all questions will be relevant.

1.6 Queries and feedback

Questions and queries

From time to time you may come across something in the study material that is unclear to you. The easiest way to solve such problems is often through discussion with friends, colleagues and peers – they will probably have had similar experiences whilst studying. If there's no-one at work to talk to then use our discussion forum at **www.ActEd.co.uk/forums** (or use the link from our home page at **www.ActEd.co.uk**).

Our online forum is dedicated to actuarial students so that you can get help from fellow students on any aspect of your studies from technical issues to study advice. You could also use it to get ideas for revision or for further reading around the subject that you are studying. ActEd tutors will visit the site from time to time to ensure that you are not being led astray and we also post other frequently asked questions from students on the forum as they arise.

If you are still stuck, then you can send queries by email to the relevant subject email address (see Section 2.6), but we recommend that you try the forum first. We will endeavour to contact you as soon as possible after receiving your query but you should be aware that it may take some time to reply to queries, particularly when tutors are away from the office running tutorials. At the busiest teaching times of year, it may take us more than a week to get back to you.

If you have many queries on the course material, you should raise them at a tutorial or book a personal tuition session with an ActEd tutor. Information about personal tuition is set out in our current brochure. Please email **ActEd@bpp.com** for more details.

Feedback

If you find an error in the course, please check the corrections page of our website (**www.ActEd.co.uk/paper_corrections.html**) to see if the correction has already been dealt with. Otherwise please send details via email to the relevant subject email address (see Section 2.6).

Each year our tutors work hard to improve the quality of the study material and to ensure that the courses are as clear as possible and free from errors. We are always happy to receive feedback from students, particularly details concerning any errors, contradictions or unclear statements in the courses. If you have any comments on this course please email them to the relevant subject email address (see Section 2.6).

Our tutors also work with the profession to suggest developments and improvements to the Syllabus and Core Reading. If you have any comments or concerns about the Syllabus or Core Reading, these can be passed on via ActEd. Alternatively, you can send them directly to the Institute and Faculty of Actuaries' Examination Team by email to **education.services@actuaries.org.uk**.

2.1 Subject CM1 – background

History

The Actuarial Mathematics subjects (Subjects CM1 and CM2) are new subjects in the Institute and Faculty of Actuaries 2019 Curriculum.

Subject CM1 is *Actuarial Mathematics*.

Predecessors

The topics covered in the Actuarial Mathematics subjects (Subjects CM1 and CM2) cover content previously in Subjects CT1, CT5, CT8 and a small amount from Subjects CT4, CT6 and CT7:

- Subject CM1 contains material from Subjects CT1, CT4 and CT5.
- Subject CM2 contains material from Subjects CT8, CT6, CT1 and CT7.

Exemptions

You will need to have passed or been granted an exemption from Subjects CT1 and CT5 to be eligible for a pass in Subject CM1 during the transfer process.

Links to other subjects

Concepts are introduced in:

- Subject CS1 – Actuarial Statistics

Topics in this subject are further built upon in:

- Subject CM2 – Financial Engineering and Loss Reserving
- Subject CB1 – Business Finance
- Subject CP1 – Actuarial Practice
- Subject CP2 – Modelling Practice
- Subject SP1 – Health and Care Principles
- Subject SP2 – Life Insurance Principles
- Subject SP4 – Pensions and Other Benefits Principles.

2.2 Subject CM1 – Syllabus and Core Reading

Syllabus

The Syllabus for Subject CM1 is given here. To the right of each objective are the chapter numbers in which the objective is covered in the ActEd course.

Aim

The aim of the Actuarial Mathematics subject is to provide a grounding in the principles of modelling as applied to actuarial work – focusing particularly on deterministic models which can be used to model and value known cashflows as well as those which are dependent on death, survival or other uncertain risks.

Competences

On successful completion of this subject, the candidate will be able to:

1. describe the basic principles of actuarial modelling
2. describe, interpret and discuss the theories on interest rates
3. describe, interpret and discuss mathematical techniques used to model and value cashflows which are contingent on mortality and morbidity risks.

Syllabus topics

1.	Data and basics of modelling	(10%)
2.	Theory of interest rates	(20%)
3.	Equation of value and its applications	(15%)
4.	Single decrement models	(10%)
5.	Multiple decrement and multiple life models	(10%)
6.	Pricing and reserving	(35%)

The weightings are indicative of the approximate balance of the assessment of this subject between the main syllabus topics, averaged over a number of examination sessions.

The weightings also have a correspondence with the amount of learning material underlying each syllabus topic. However, this will also reflect aspects such as:

- the relative complexity of each topic, and hence the amount of explanation and support required for it
- the need to provide thorough foundation understanding on which to build the other objectives
- the extent of prior knowledge which is expected
- the degree to which each topic area is more knowledge or application based.

Detailed syllabus objectives

1. Data and basics of modelling (10%)

1.1 Data analysis (Chapter 1)

1.1.1 Describe the possible aims of data analysis (*eg* descriptive, inferential and predictive).

1.1.2 Describe the stages of conducting a data analysis to solve real-world problems in a scientific manner and describe tools suitable for each stage.

1.1.3 Describe sources of data and explain the characteristics of different data sources, including extremely large data sets.

1.1.4 Explain the meaning and value of reproducible research and describe the elements required to ensure a data analysis is reproducible.

1.2 Describe the principles of actuarial modelling. (Chapter 2)

1.2.1 Describe why and how models are used including, in general terms, the use of models for pricing, reserving, and capital modelling.

1.2.2 Explain the benefits and limitations of modelling.

1.2.3 Explain the difference between a stochastic and a deterministic model, and identify the advantages/disadvantages of each.

1.2.4 Describe the characteristics of, and explain the use of, scenario-based and proxy models.

1.2.5 Describe, in general terms, how to decide whether a model is suitable for any particular application.

1.2.6 Explain the difference between the short-run and long-run properties of a model, and how this may be relevant in deciding whether a model is suitable for any particular application.

1.2.7 Describe, in general terms, how to analyse the potential output from a model, and explain why this is relevant to the choice of model.

1.2.8 Describe the process of sensitivity testing of assumptions and explain why this forms an important part of the modelling process.

1.2.9 Explain the factors that must be considered when communicating the results following the application of a model.

1.3 Describe how to use a generalised cashflow model to describe financial transactions. (Chapter 3)

1.3.1 State the inflows and outflows in each future time period and discuss whether the amount or the timing (or both) is fixed or uncertain for a given cashflow process.

1.3.2 Describe in the form of a cashflow model the operation of financial instruments like a zero-coupon bond, a fixed-interest security, an index-linked security, cash on deposit, an equity, an interest-only loan, a repayment loan, and an annuity-certain; and insurance contracts like an endowment, a term assurance, a contingent annuity, car insurance and health cash plans.

2. Theory of interest rates (20%)

2.1 Show how interest rates may be expressed in different time periods. (Chapters 4 and 5)

2.1.1 Describe the relationship between the rates of interest and discount over one effective period arithmetically and by general reasoning.

2.1.2 Derive the relationships between the rate of interest payable once per measurement period (effective rate of interest) and the rate of interest payable p (> 1) times per measurement period (nominal rate of interest) and the force of interest.

2.1.3 Calculate the equivalent annual rate of interest implied by the accumulation of a sum of money over a specified period where the force of interest is a function of time.

2.2 Demonstrate a knowledge and understanding of real and money interest rates. (Chapters 6 and 13)

2.3 Describe how to take into account time value of money using the concepts of compound interest and discounting. (Chapter 4)

2.3.1 Accumulate a single investment at a constant rate of interest under the operation of simple and compound interest.

2.3.2 Define the present value of a future payment.

2.3.3 Discount a single investment under the operation of a simple (commercial) discount at a constant rate of discount.

2.4 Calculate the present value and accumulated value for a given stream of cashflows under the following individual or combination of scenarios: (Chapter 7)

2.4.1 Cashflows are equal at each time period.

2.4.2 Cashflows vary with time which may or may not be a continuous function of time.

2.4.3 Some of the cashflows are deferred for a period of time.

2.4.4 Rate of interest or discount is constant.

2.4.5 Rate of interest or discount varies with time which may or may not be a continuous function of time.

2.5 Define and derive the following compound interest functions (where payments can be in advance or in arrears) in terms of i, v, n, d, δ, $i^{(p)}$ and $d^{(p)}$:

(Chapters 8 and 9)

2.5.1 $a_{\overline{n}|}$, $s_{\overline{n}|}$, $a_{\overline{n}|}^{(p)}$, $s_{\overline{n}|}^{(p)}$, $\ddot{a}_{\overline{n}|}$, $\ddot{s}_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}^{(p)}$, $\ddot{s}_{\overline{n}|}^{(p)}$, $\bar{a}_{\overline{n}|}$ and $\bar{s}_{\overline{n}|}$.

2.5.2 ${}_{m|}a_{\overline{n}|}$, ${}_{m|}a_{\overline{n}|}^{(p)}$, ${}_{m|}\ddot{a}_{\overline{n}|}$, ${}_{m|}\ddot{a}_{\overline{n}|}^{(p)}$ and ${}_{m|}\bar{a}_{\overline{n}|}$.

2.5.3 $(Ia)_{\overline{n}|}$, $(I\ddot{a})_{\overline{n}|}$, $(I\bar{a})_{\overline{n}|}$ and $(\bar{I}\bar{a})_{\overline{n}|}$ and the respective deferred annuities.

2.6 Show an understanding of the term structure of interest rates. (Chapter 14)

2.6.1 Describe the main factors influencing the term structure of interest rates.

2.6.2 Explain what is meant by, derive the relationships between and evaluate:

- discrete spot rates and forward rates.
- continuous spot rates and forward rates.

2.6.3 Explain what is meant by the par yield and yield to maturity.

2.7 Understanding duration, convexity and immunisation of cashflows. (Chapter 14)

2.7.1 Define the duration and convexity of a cashflow sequence, and illustrate how these may be used to estimate the sensitivity of the value of the cashflow sequence to a shift in interest rates.

2.7.2 Evaluate the duration and convexity of a cashflow sequence.

2.7.3 Explain how duration and convexity are used in the (Redington) immunisation of a portfolio of liabilities.

3. Equation of value and its applications (15%)

3.1 Define an equation of value. (Chapter 10)

3.1.1 Define an equation of value, where payment or receipt is certain.

3.1.2 Describe how an equation of value can be adjusted to allow for uncertain receipts or payments.

3.1.3 Understand the two conditions required for there to be an exact solution to an equation of value.

3.2 Use the concept of equation of value to solve various practical problems. (Chapters 11 and 13)

3.2.1 Apply the equation of value to loans repaid by regular instalments of interest and capital. Obtain repayments, interest and capital components, the effective interest rate (APR) and construct a schedule of repayments.

3.2.2 Calculate the price of, or yield (nominal or real allowing for inflation) from, a bond (fixed-interest or index-linked) where the investor is subject to deduction of income tax on coupon payments and redemption payments are subject to deduction of capital gains tax.

3.2.3 Calculate the running yield and the redemption yield for the financial instrument as described in 3.2.2.

3.2.4 Calculate the upper and lower bounds for the present value of the financial instrument as described in 3.2.2 when the redemption date can be a single date within a given range at the option of the borrower.

3.2.5 Calculate the present value or yield (nominal or real allowing for inflation) from an ordinary share or property, given constant or variable rate of growth of dividends or rents.

3.3 Show how discounted cashflow and equation of value techniques can be used in project appraisals. (Chapter 12)

3.3.1 Calculate the net present value and accumulated profit of the receipts and payments from an investment project at given rates of interest.

3.3.2 Calculate the internal rate of return, payback period and discounted payback period and discuss their suitability for assessing the suitability of an investment project.

4. Single decrement models (10%)

4.1 Define various assurance and annuity contracts. (Chapters 15, 16, 17, 19 and 26)

4.1.1 Define the following terms:

- whole life assurance
- term assurance
- pure endowment
- endowment assurance
- whole life level annuity
- temporary level annuity
- guaranteed level annuity
- premium
- benefit

including assurance and annuity contracts where the benefits are deferred.

4.1.2 Describe the operation of conventional with-profits contracts, in which profits are distributed by the use of regular reversionary bonuses, and by terminal bonuses. Describe the benefits payable under the above assurance-type contracts.

4.1.3 Describe the operation of conventional unit-linked contracts, in which death benefits are expressed as combination of absolute amount and relative to a unit fund.

4.1.4 Describe the operation of accumulating with-profits contracts, in which benefits take the form of an accumulating fund of premiums, where either:

- the fund is defined in monetary terms, has no explicit charges, and is increased by the addition of regular guaranteed and bonus interest payments plus a terminal bonus; or
- the fund is defined in terms of the value of a unit fund, is subject to explicit charges, and is increased by regular bonus additions plus a terminal bonus (unitised with-profits).

In the case of unitised with-profits, the regular additions can take the form of (a) unit price increases (guaranteed and/or discretionary), or (b) allocations of additional units.

In either case, a guaranteed minimum monetary death benefit may be applied.

4.2 Develop formulae for the means and variances of the payments under various assurance and annuity contracts, assuming constant deterministic interest rate.
(Chapters 15, 16, 17, 18, 19 and 21)

4.2.1 Describe the life table functions l_x and d_x and their select equivalents $l_{[x]+r}$ and $d_{[x]+r}$.

4.2.2 Define the following probabilities: ${}_np_x$, ${}_nq_x$, ${}_{n|m}q_x$, ${}_{n|}q_x$ and their select equivalents ${}_np_{[x]+r}$, ${}_nq_{[x]+r}$, ${}_{n|m}q_{[x]+r}$, ${}_{n|}q_{[x]+r}$.

4.2.3 Express the probabilities defined in 4.2.2 in terms of life table functions defined in 4.2.1.

4.2.4 Define the assurance and annuity factors and their select and continuous equivalents. Extend the annuity factors to allow for the possibility that payments are more frequent than annual but less frequent than continuous.

4.2.5 Understand and use the relations between annuities payable in advance and in arrear, and between temporary, deferred and whole life annuities.

4.2.6 Understand and use the relations between assurance and annuity factors using equation of value, and their select and continuous equivalents.

4.2.7 Obtain expressions in the form of sums/integrals for the mean and variance of the present value of benefit payments under each contract defined in 4.1.1, in terms of the (curtate) random future lifetime, assuming:

- contingent benefits (constant, increasing or decreasing) are payable at the middle or end of the year of contingent event or continuously.
- annuities are paid in advance, in arrear or continuously, and the amount is constant, increases or decreases by a constant monetary amount or by a fixed or time-dependent variable rate.
- premiums are payable in advance, in arrear or continuously; and for the full policy term or for limited period.

Where appropriate, simplify the above expressions into a form suitable for evaluation by table look-up or other means.

4.2.8 Define and evaluate the expected accumulations in terms of expected values for the contracts described in 4.1.1 and contract structures described in 4.2.7.

5. Multiple decrement and multiple life models (10%)

5.1 Define and use assurance and annuity functions involving two lives. (Chapters 22 and 23)

5.1.1 Extend the techniques of objectives 4.2 to deal with cashflows dependent upon the death or survival of either or both of two lives.

5.1.2 Extend the technique of 5.1.1 to deal with functions dependent upon a fixed term as well as age.

5.2 Describe and illustrate methods of valuing cashflows that are contingent upon multiple transition events. (Chapter 25)

5.2.1 Define health insurance, and describe simple health insurance premium and benefit structures.

5.2.2 Explain how a cashflow, contingent upon multiple transition events, may be valued using a multiple-state Markov Model, in terms of the forces and probabilities of transition.

5.2.3 Construct formulae for the expected present values of cashflows that are contingent upon multiple transition events, including simple health insurance premiums and benefits, and calculate these in simple cases. Regular premiums and sickness benefits are payable continuously and assurance benefits are payable immediately on transition.

5.3 Describe and use methods of projecting and valuing expected cashflows that are contingent upon multiple decrement events. (Chapter 25)

5.3.1 Describe the construction and use of multiple decrement tables.

5.3.2 Define a multiple decrement model as a special case of multiple-state Markov model.

5.3.3 Derive dependent probabilities for a multiple decrement model in terms of given forces of transition, assuming forces of transition are constant over single years of age.

5.3.4 Derive forces of transition from given dependent probabilities, assuming forces of transition are constant over single years of age.

6. Pricing and reserving (35%)

6.1 Define the gross random future loss under an insurance contract, and state the principle of equivalence. (Chapter 20)

6.2 Describe and calculate gross premiums and reserves of assurance and annuity contracts. (Chapters 20 and 21)

6.2.1 Define and calculate gross premiums for the insurance contract benefits as defined in objective 4.1 under various scenarios using the equivalence principle or otherwise:

- contracts may accept only single premium;
- regular premiums and annuity benefits may be payable annually, more frequently than annually, or continuously;
- death benefits (which increase or decrease by a constant compound rate or by a constant monetary amount) may be payable at the end of the year of death, or immediately on death;
- survival benefits (other than annuities) may be payable at defined intervals other than at maturity.

6.2.2 State why an insurance company will set up reserves.

6.2.3 Define and calculate gross prospective and retrospective reserves.

6.2.4 State the conditions under which, in general, the prospective reserve is equal to the retrospective reserve allowing for expenses.

6.2.5 Prove that, under the appropriate conditions, the prospective reserve is equal to the retrospective reserve, with or without allowance for expenses, for all fixed benefit and increasing / decreasing benefit contracts.

6.2.6 Obtain recursive relationships between successive periodic gross premium reserves, and use this relationship to calculate the profit earned from a contract during the period.

6.2.7 Outline the concepts of net premiums and net premium valuation and how they relate to gross premiums and gross premium valuation respectively.

6.3 Define and calculate, for a single policy or a portfolio of policies (as appropriate):

- death strain at risk;
- expected death strain;
- actual death strain; and
- mortality profit

for policies with death benefits payable immediately on death or at the end of the year of death; for policies paying annuity benefits at the start of the year or on survival to the end of the year; and for policies where single or non-single premiums are payable. (Chapter 24)

6.4 Project expected future cashflows for whole life, endowment and term assurances, annuities, unit-linked contracts, and conventional/unitised with-profits contracts, incorporating multiple decrement models as appropriate. (Chapters 27 and 28)

6.4.1 Profit test life insurance contracts of the types listed above and determine the profit vector, the profit signature, the net present value, and the profit margin.

6.4.2 Show how a profit test may be used to price a product, and use a profit test to calculate a premium for life insurance contracts of the types listed above.

6.4.3 Show how gross premium reserves can be computed using the above cashflow projection model and included as part of profit testing.

6.5 Show how, for unit-linked contracts, non-unit reserves can be established to eliminate ('zeroise') future negative cashflows, using a profit test model. (Chapter 28)

Core Reading

The Subject CM1 Course Notes include the Core Reading in full, integrated throughout the course.

Accreditation

The Institute and Faculty of Actuaries would like to thank the numerous people who have helped in the development of the material contained in the Core Reading.

Further reading

The exam will be based on the relevant Syllabus and Core Reading and the ActEd course material will be the main source of tuition for students.

2.3 Subject CM1 – the course structure

There are five parts to the Subject CM1 course. The parts cover related topics and have broadly equal marks in the paper-based exam. The parts are broken down into chapters.

The following table shows how the parts, the chapters and the syllabus items relate to each other. The end columns show how the chapters relate to the days of the regular tutorials. We have also given you a broad indication of the length of each chapter. This table should help you plan your progress across the study session.

Part	*Chapter*	*Title*	*No of pages*	*Syllabus objectives*	*5 full days*
1	1	Data analysis	23	1.1	1
	2	Principles of actuarial modelling	25	1.2	
	3	Cashflow models	24	1.3	
	4	The time value of money	27	2.1, 2.3	
	5	Interest rates	40	2.1	
	6	Real and money interest rates	13	2.2	
	7	Discounting and accumulating	30	2.4	
	8	Level annuities	36	2.5	
	9	Increasing annuities	30	2.5	
2	10	Equations of value	20	3.1	2
	11	Loan schedules	31	3.2	
	12	Project appraisal	32	3.3	
	13	Bonds, equity and property	55	2.2, 3.2	
3	14	Term structure of interest rates	50	2.6, 2.7	3
	15	The life table	45	4.1, 4.2	
	16	Life assurance contracts	45	4.1, 4.2	
	17	Life annuity contracts	44	4.1, 4.2	
	18	Evaluation of assurances and annuities	31	4.2	
	19	Variable benefits and conventional with-profits policies	41	4.1, 4.2	
4	20	Gross premiums	41	6.1, 6.2	4
	21	Gross premium reserves	59	4.2, 6.2	
	22	Joint life and last survivor functions	38	5.1	
	23	Contingent and reversionary benefits	61	5.1	

Part	*Chapter*	*Title*	*No of pages*	*Syllabus objectives*	*5 full days*
5	24	Mortality profit	35	6.3	5
	25	Competing risks	61	5.2, 5.3	
	26	Unit-linked and accumulating with-profits contracts	25	4.1	
	27	Profit testing	51	6.4	
	28	Reserving aspects of profit testing	48	6.4, 6.5	

2.4 Subject CM1 – summary of ActEd products

The following products are available for Subject CM1:

- Course Notes
- PBOR (including the Y Assignments)
- X Assignments – five assignments:
  - X1, X2, X3: 80-mark tests (you are allowed 2¾ hours to complete these)
  - X4, X5: 100-mark tests (you are allowed 3¼ hours to complete these)
- Series X Marking
- Series Y Marking
- Online Classroom – over 150 tutorial units
- Flashcards
- Revision Notes
- ASET – four years' exam papers, *ie* eight papers, covering the period April 2014 to September 2017
- Mock Exam
- Mock Exam Marking
- Marking Vouchers.

We will endeavour to release as much material as possible but unfortunately some revision products may not be available until the September 2019 or even April 2020 exam sessions. Please check the ActEd website or email ActEd@bpp.com for more information.

The following tutorials are typically available for Subject CM1:

- Regular Tutorials (five days)
- Block Tutorials (five days)
- a Preparation Day for the computer-based exam.

Full details are set out in our *Tuition Bulletin*, which is available on our website at **www.ActEd.co.uk**.

2.5 Subject CM1 – skills and assessment

Technical skills

The *Actuarial Mathematics* subjects (Subjects CM1 and CM2) are very mathematical and have relatively few questions requiring wordy answers.

Exam skills

Exam question skill levels

In the CM subjects, the approximate split of assessment across the three skill types is:

- Knowledge – 20%
- Application – 65%
- Higher Order skills – 15%.

Assessment

Assessment consists of a combination of a 3¼-hour written examination and a 1¾-hour computer-based modelling examination.

2.6 Subject CM1 – frequently asked questions

Q: ***What knowledge of earlier subjects should I have and what level of mathematics is required?***

A: Subject CM1 does require some knowledge of statistics. In particular, you need to be familiar with random variables, probabilities, expectations and variances. These topics are prerequisites for studying *any* of the IFoA examinations, and are further developed in Subject CS1, although it is not essential to have studied Subject CS1 before Subject CM1.

The level of maths you need for this course is broadly A-level standard. In particular, you will need a knowledge of calculus, *ie* differentiation and integration. You will find the course much easier if you feel comfortable with the mathematical techniques used and you feel confident in applying them yourself.

If your maths or statistics is a little rusty you may wish to consider purchasing additional material to help you get up to speed. The course 'Pure Maths and Statistics for Actuarial Studies' is available from ActEd and it covers the mathematical techniques that are required for the Core Principles subjects, some of which are beyond A-Level (or Higher) standard. You do not need to work through the whole course in order – you can just refer to it when you need help on a particular topic. An initial assessment to test your mathematical skills and further details regarding the course can be found on our website.

Q: ***What should I do if I discover an error in the course?***

A: If you find an error in the course, please check our website at:

www.ActEd.co.uk/paper_corrections.html

to see if the correction has already been dealt with. Otherwise please send details via email to **CM1@bpp.com**.

Q: ***Who should I send feedback to?***

A: We are always happy to receive feedback from students, particularly details concerning any errors, contradictions or unclear statements in the courses.

If you have any comments on this course in general, please email them to **CM1@bpp.com**.

If you have any comments or concerns about the Syllabus or Core Reading, these can be passed on to the profession via ActEd. Alternatively, you can send them directly to the Institute and Faculty of Actuaries' Examination Team by email to **education.services@actuaries.org.uk**.

1

Data analysis

Syllabus objectives

1.1 Data analysis

1.1.1 Describe the possible aims of data analysis (*eg* descriptive, inferential and predictive).

1.1.2 Describe the stages of conducting a data analysis to solve real-world problems in a scientific manner and describe tools suitable for each stage.

1.1.3 Describe sources of data and explain the characteristics of different data sources, including extremely large data sets.

1.1.4 Explain the meaning and value of reproducible research and describe the elements required to ensure a data analysis is reproducible.

0 Introduction

This chapter provides an introduction to the underlying principles of data analysis, in particular within an actuarial context.

Data analysis is the process by which data is gathered in its raw state and analysed or processed into information which can be used for specific purposes. This chapter will describe some of the different forms of data analysis, the steps involved in the process and consider some of the practical problems encountered in data analytics.

Although this chapter looks at the general principles involved in data analysis, it does not deal with the statistical techniques required to perform a data analysis. These are covered elsewhere, in CS1 and CS2.

1 Aims of a data analysis

Three keys forms of data analysis will be covered in this section:

- **descriptive;**
- **inferential; and**
- **predictive.**

1.1 Descriptive analysis

Data presented in its raw state can be difficult to manage and draw meaningful conclusions from, particularly where there is a large volume of data to work with. A descriptive analysis solves this problem by presenting the data in a simpler format, more easily understood and interpreted by the user.

Simply put, this might involve summarising the data or presenting it in a format which highlights any patterns or trends. A descriptive analysis is not intended to enable the user to draw any specific conclusions. Rather, it describes the data actually presented.

For example, it is likely to be easier to understand the trend and variation in the sterling/euro exchange rate over the past year by looking at a graph of the daily exchange rate rather than a list of values. The graph is likely to make the information easier to absorb.

Two key measures, or parameters, used in a descriptive analysis are the *measure of central tendency* and the *dispersion*. The most common measurements of central tendency are the mean, the median and the mode. Typical measurements of the dispersion are the standard deviation and ranges such as the interquartile range. These measurements are described in CS1.

Measures of central tendency tell us about the 'average' value of a data set, whereas measures of dispersion tell us about the 'spread' of the values.

It can also be important to describe other aspects of the shape of the (empirical) distribution of the data, for example by calculating measures of skewness and kurtosis.

Empirical means 'based on observation'. So an empirical distribution relates to the distribution of the actual data points collected, rather than any assumed underlying theoretical distribution.

Skewness is a measure of how symmetrical a data set is, and kurtosis is a measure of how likely extreme values are to appear (*ie* those in the tails of the distribution). Detailed knowledge of these measures is not required for this course.

1.2 Inferential analysis

Often it is not feasible or practical to collect data in respect of the whole population, particularly when that population is very large. For example, when conducting an opinion poll in a large country, it may not be cost effective to survey every citizen. A practical solution to this problem might be to gather data in respect of a sample, which is used to represent the wider population. The analysis of the data from this sample is called inferential analysis.

The sample analysis involves estimating the parameters as described in Section 1.1 above and testing hypotheses. It is generally accepted that if the sample is large and taken at random (selected without prejudice), then it quite accurately represents the statistics of the population, such as distribution, probability, mean, standard deviation, However, this is also contingent upon the user making reasonably correct hypothesis about the population in order to perform the inferential analysis.

Care may need to be taken to ensure that the sample selected is likely to be representative of the whole population. For example, an opinion poll on a national issue conducted in urban locations on weekday afternoons between 2pm and 4pm may not accurately reflect the views of the whole population. This is because those living in rural areas and those who regularly work during that period are unlikely to have been surveyed, and these people might tend to have a different viewpoint to those who have been surveyed.

Sampling, inferential analysis and parameter estimation are covered in more detail in CS1.

1.3 Predictive analysis

Predictive analysis extends the principles behind inferential analysis in order for the user to analyse past data and make predictions about future events.

It achieves this by using an existing set of data with known attributes (also known as features), known as the *training set* in order to discover potentially predictive relationships. Those relationships are tested using a different set of data, known as the *test set*, to assess the strength of those relationships.

A typical example of a predictive analysis is regression analysis, which is covered in more detail in CS1 and CS2. The simplest form of this is linear regression where the relationship between a scalar dependent variable and an explanatory or independent variable is assumed to be linear and the training set is used to determine the slope and intercept of the line. A practical example might be the relationship between a car's braking distance against speed.

In this example, the car's speed is the explanatory (or independent) variable and the braking distance is the dependent variable.

Question

Based on data gathered at a particular weather station on the monthly rainfall in mm (r) and the average number of hours of sunshine per day (s), a researcher has determined the following explanatory relationship:

$$s = 9 - 0.1r$$

Using this model:

(i) Estimate the average number of hours of sunshine per day, if the monthly rainfall is 50mm.

(ii) State the impact on the average number of hours of sunshine per day of each extra millimetre of rainfall in a month.

Solution

(i) When $r = 50$:

$$s = 9 - 0.1 \times 50 = 4$$

ie there are 4 hours of sunshine per day on average.

(ii) For each extra millimetre of rainfall in a month, the average number of hours of sunshine per day falls by 0.1 hours, or 6 minutes.

2 The data analysis process

While the process to analyse data does not follow a set pattern of steps, it is helpful to consider the key stages which might be used by actuaries when collecting and analysing data.

The key steps in a data analysis process can be described as follows:

1. ***Develop* a well-defined set of objectives which need to be met by the results of the data analysis.**

   The objective may be to summarise the claims from a sickness insurance product by age, gender and cause of claim, or to predict the outcome of the next national parliamentary election.

2. ***Identify* the data items required for the analysis.**

3. ***Collection* of the data from appropriate sources.**

   The relevant data may be available internally (*eg* from an insurance company's administration department) or may need to be gathered from external sources (*eg* from a local council office or government statistical service).

4. ***Processing* and formatting data for analysis, *eg* inputting into a spreadsheet, database or other model.**

5. ***Cleaning data*, *eg* addressing unusual, missing or inconsistent values.**

6. ***Exploratory data analysis*, which may include:**

   (a) Descriptive analysis; producing summary statistics on central tendency and spread of the data.

   (b) Inferential analysis; estimating summary parameters of the wider population of data, testing hypotheses.

   (c) Predictive analysis; analysing data to make predictions about future events or other data sets.

7. ***Modelling* the data.**

8. ***Communicating* the results.**

   It will be important when communicating the results to make it clear what data was used, what analyses were performed, what assumptions were made, the conclusion of the analysis, and any limitations of the analysis.

9. ***Monitoring* the process; updating the data and repeating the process if required.**

   A data analysis is not necessarily just a one-off exercise. An insurance company analysing the claims from its sickness policies may wish to do this every few years to allow for the new data gathered and to look for trends. An opinion poll company attempting to predict an election result is likely to repeat the poll a number of times in the weeks before the election to monitor any changes in views during the campaign period.

Throughout the process, the modelling team needs to ensure that any relevant professional guidance has been complied with. For example, the Financial Reporting Council has issued a Technical Actuarial Standard (TAS) on the principles for Technical Actuarial Work (TAS100) which includes principles for the use of data in technical actuarial work. Knowledge of the detail of this TAS is not required for CM1.

Further, the modelling team should also remain aware of any legal requirement to be complied with. Such legal requirement may include aspects around consumer/customer data protection and gender discrimination.

3 Data sources

Step 3 of the process described in Section 2 above refers to collection of the data needed to meet the objectives of the analysis from *appropriate sources*. As consideration of Steps 3, 4, and 5 makes clear, getting data into a form ready for analysis is a process, not a single event. Consequently, what is seen as the source of data can depend on your viewpoint.

Suppose you are conducting an analysis which involves collecting survey data from a sample of people in the hope of drawing inferences about a wider population. If you are in charge of the whole process, including collecting the *primary* data from your selected sample, you would probably view the 'source' of the data as being the people in your sample. Having collected, cleaned and possibly summarised the data you might make it available to other investigators in JavaScript object notation (JSON) format via a web Application programming interface (API). You will then have created a *secondary* 'source' for others to use.

In this section we discuss how the characteristics of the data are determined both by the primary source and the steps carried out to prepare it for analysis – which may include the steps on the journey from primary to secondary source. Details of particular data formats (such as JSON), or of the mechanisms for getting data from an external source into a local data structure suitable for analysis, are not covered in CM1.

Primary data can be gathered as the outcome of a designed *experiment* or from an *observational study* (which could include a *survey* of responses to specific questions). In all cases, knowledge of the details of the collection process is important for a complete understanding of the data, including possible sources of bias or inaccuracy. Issues that the analyst should be aware of include:

- **whether the process was manual or automated;**
- **limitations on the precision of the data recorded;**
- **whether there was any validation at source; and**
- **if data wasn't collected automatically, how was it converted to an electronic form.**

These factors can affect the accuracy and reliability of the data collected. For example:

- in a survey, an individual's salary may be specified as falling into given bands, *eg* £20,000 - £29,999, £30,000 - £39,999 *etc*, rather than the precise value being recorded
- if responses were collected on handwritten forms, and then manually input into a database, there is greater scope for errors to appear.

Where randomisation has been used to reduce the effect of bias or confounding variables it is important to know the sampling scheme used:

- **simple random sampling;**
- **stratified sampling; or**
- **another sampling method.**

Question

A researcher wishes to survey 10% of a company's workforce.

Describe how the sample could be selected using:

(a) simple random sampling

(b) stratified sampling.

Solution

(a) ***Simple random sampling***

Using simple random sampling, each employee would have an equal chance of being selected. This could be achieved by taking a list of the employees, allocating each a number, and then selecting 10% of the numbers at random (either manually, or using a computer-generated process)

(b) ***Stratified sampling***

Using stratified sampling, the workforce would first be split into groups (or *strata*) defined by specific criteria, *eg* level of seniority. Then 10% of each group would be selected using simple random sampling. In this way, the resulting sample would reflect the structure of the company by seniority.

This aims to overcome one of the issues with simple random sampling, *ie* that the sample obtained does not fully reflect the characteristics of the population. With a simple random sample, it would be possible for all those selected to be at the same level of seniority, and so be unrepresentative of the workforce as a whole.

Data may have undergone some form of *pre-processing*. A common example is *grouping* (*eg* by geographical area or age band). In the past, this was often done to reduce the amount of storage required and to make the number of calculations manageable. The scale of computing power available now means that this is less often an issue, but data may still be grouped: perhaps to anonymise it, or to remove the possibility of extracting sensitive (or perhaps commercially sensitive) details.

Other aspects of the data which are determined by the collection process, and which affect the way it is analysed include the following:

- ***Cross-sectional* data involves recording values of the variables of interest for each case in the sample at a single moment in time.**

  For example, recording the amount spent in a supermarket by each member of a loyalty card scheme this week.

- ***Longitudinal* data involves recording values at intervals over time.**

  For example, recording the amount spent in a supermarket by a particular member of a loyalty card scheme each week for a year.

- ***Censored* data occurs when the value of a variable is only *partially known*, for example, if a subject in a survival study withdraws, or survives beyond the end of the study: here a lower bound for the survival period is known but the exact value isn't.**

  Censoring is dealt with in detail in CS2.

- ***Truncated* data occurs when measurements on some variables are not recorded so are completely unknown.**

  For example, if we were collecting data on the periods of time for which a user's internet connection was disrupted, but only recorded the duration of periods of disruption that lasted 5 minutes or longer, we would have a truncated data set.

3.1 Big data

The term *big data* is not well defined but has come to be used to describe data with characteristics that make it impossible to apply traditional methods of analysis (for example, those which rely on a single, well-structured data set which can be manipulated and analysed on a single computer). Typically, this means automatically collected data with characteristics that have to be inferred from the data itself rather than known in advance from the design of an experiment.

Given the description above, the properties that can lead data to be classified as 'big' include:

- ***size*, not only does big data include a very large number of individual cases, but each might include very many variables, a high proportion of which might have empty (or null) values – leading to *sparse* data;**
- ***speed*, the data to be analysed might be arriving in real time at a very fast rate – for example, from an array of sensors taking measurements thousands of time every second;**
- ***variety*, big data is often composed of elements from many different sources which could have very different structures – or is often largely unstructured;**
- ***reliability*, given the above three characteristics we can see that the reliability of individual data elements might be difficult to ascertain and could vary over time (for example, an internet connected sensor could go offline for a period).**

Examples of 'big data' are:

- the information held by large online retailers on items viewed, purchased and recommended by each of its customers
- measurements of atmospheric pressure from sensors monitored by a national meteorological organisation
- the data held by an insurance company received from the personal activity trackers (that monitor daily exercise, food intake and sleep, for example) of its policyholders.

Although the four points above (size, speed, variety, reliability) have been presented in the context of big data, they are characteristics that should be considered for any data source. For example, an actuary may need to decide if it is advisable to increase the volume of data available for a given investigation by combining an internal data set with data available externally. In this case, the extra processing complexity required to handle a variety of data, plus any issues of reliability of the external data, will need to be considered.

3.2 Data security, privacy and regulation

In the design of any investigation, consideration of issues related to data security, privacy and complying with relevant regulations should be paramount. It is especially important to be aware that combining different data from different 'anonymised' sources can mean that individual cases become identifiable.

Another point to be aware of is that just because data has been made available on the internet, doesn't mean that that others are free to use it as they wish. This is a very complex area and laws vary between jurisdictions.

4 Reproducible research

An example reference for this section is in Peng (2016). For the full reference, see the end of this section.

4.1 The meaning of reproducible research

***Reproducibility* refers to the idea that when the results of a statistical analysis are reported, sufficient information is provided so that an independent third party can repeat the analysis and arrive at the same results.**

In science, *reproducibility* is linked to the concept of *replication* which refers to someone repeating an experiment and obtaining the same (or at least consistent) results. Replication can be hard, or expensive or impossible, for example if:

- **the study is big;**
- **the study relies on data collected at great expense or over many years; or**
- **the study is of a unique occurrence (the standards of healthcare in the aftermath of a particular event).**

Due to the possible difficulties of replication, reproducibility of the statistical analysis is often a reasonably alternative standard.

So, rather than the results of the analysis being validated by an independent third party completely *replicating* the study from scratch (including gathering a new data set), the validation is achieved by an independent third party *reproducing* the same results based on the same data set.

4.2 Elements required for reproducibility

Typically, reproducibility requires the original data and the computer code to be made available (or fully specified) so that other people can repeat the analysis and verify the results. In all but the most trivial cases, it will be necessary to include full documentation (*eg* description of each data variable, an audit trail describing the decisions made when cleaning and processing the data, and full documented code). Documentation of models is covered in Subject CP2.

Full documented code can be achieved through *literate statistical programming* (as defined by Knuth, 1992) where the program includes an explanation of the program in plain language, interspersed with code snippets. Within the R environment, a tool which allows this is R-markdown.

A detailed knowledge of the statistical package R is not required for CM1 – R is covered in CS1 and CS2. R-markdown enables documents to be produced that include the code used, an explanation of that code, and, if desired, the output from that code.

As a simpler example, it may be possible to document the work carried out in a spreadsheet by adding comments or annotations to explain the operations performed in particular cells, rows or columns.

Although not strictly required to meet the definition of reproducibility, a good version control process can ensure evolving drafts of code, documentation and reports are kept in alignment between the various stages of development and review, and changes are reversible if necessary. There are many tools that are used for version control. A popular tool used for version control is *git*.

A detailed knowledge of the version control tool 'git' is not required in CM1.

In addition to version control, documenting the software environment, the computing architecture, the operating system, the software toolchain, external dependencies and version numbers can all be important in ensuring reproducibility.

As an example, in the R programming language, the command:

```
sessionInfo()
```

provides information about the operating system, version of R and version of all R packages being used.

Question

Give a reason why documenting the version number of the software used can be important for reproducibility of a data analysis.

Solution

Some functions might be available in one version of a package that are not available in another (older) version. This could prevent someone being able to reproduce the analysis.

Where there is randomness in the statistical or machine learning techniques being used (for example random forests or neural networks) or where simulation is used, replication will require the random seed to be set.

Machine learning is covered in Subject CS2.

Simulation will be dealt with in more detail in the next chapter. At this point, it is sufficient to know that each simulation that is run will be based on a series of pseudo-random numbers. So, for example, one simulation will be based on one particular series of pseudo-random numbers, but unless explicitly coded otherwise, a different simulation will be based on a *different* series of pseudo-random numbers. The second simulation will then produce different results, rather than replicating the original results, which is the desired outcome here.

To ensure the two simulations give the same results, they would both need to be based on the *same* series of pseudo-random numbers. This is known as 'setting the random seed'.

Doing things 'by hand' is very likely to create problems in reproducing the work. Examples of doing things by hand are:

- **manually editing spreadsheets (rather than reading the raw data into a programming environment and making the changes there);**
- **editing tables and figures (rather than ensuring that the programming environment creates them exactly as needed);**
- **downloading data manually from a website (rather than doing it programmatically); and**
- **pointing and clicking (unless the software used creates an audit trail of what has been clicked).**

'Pointing and clicking' relates to choosing a particular operation from an on-screen menu, for example. This action would not ordinarily be recorded electronically.

The main thing to note here is that the more of the analysis that is performed in an automated way, the easier it will be to reproduce by another individual. Manual interventions may be forgotten altogether, and even if they are remembered, can be difficult to document clearly.

4.3 The value of reproducibility

Many actuarial analyses are undertaken for commercial, not scientific, reasons and are not published, but reproducibility is still valuable:

- **reproducibility is necessary for a complete technical work review (which in many cases will be a professional requirement) to ensure the analysis has been correctly carried out and the conclusions are justified by the data and analysis;**
- **reproducibility may be required by external regulators and auditors;**
- **reproducible research is more easily extended to investigate the effect of changes to the analysis, or to incorporate new data;**
- **it is often desirable to compare the results of an investigation with a similar one carried out in the past; if the earlier investigation was reported reproducibly an analysis of the differences between the two can be carried out with confidence;**
- **the discipline of reproducible research, with its emphasis on good documentation of processes and data storage, can lead to fewer errors that need correcting in the original work and, hence, greater efficiency.**

There are some issues that reproducibility does not address:

- **Reproducibility does not mean that the analysis is correct. For example, if an incorrect distribution is assumed, the results may be wrong – even though they can be reproduced by making the same incorrect assumption about the distribution. However, by making clear how the results are achieved, it does allow transparency so that incorrect analysis can be appropriately challenged.**
- **If activities involved in reproducibility happen only at the end of an analysis, this may be too late for resulting challenges to be dealt with. For example, resources may have been moved on to other projects.**

4.4 References

Further information on the material in this section is given in the references:

- **Knuth, Donald E. (1992). *Literate Programming*. California: Stanford University Center for the Study of Language and Information. ISBN 978-0-937073-80-3.**
- **Peng, R. D., 2016, *Report Writing for Data Science in R*, www.Leanpub.com/reportwriting**

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 1 Summary

The three key forms of data analysis are:

- descriptive analysis: producing summary statistics (*eg* measures of central tendency and dispersion) and presenting the data in a simpler format
- inferential analysis: using a data sample to estimate summary parameters for the wider population from which the sample was taken, and testing hypotheses
- predictive analysis: extends the principles of inferential analysis to analyse past data and make predictions about future events.

The key steps in the data analysis process are:

1. Develop a well-defined set of objectives which need to be met by the results of the data analysis.
2. Identify the data items required for the analysis.
3. Collection of the data from appropriate sources.
4. Processing and formatting data for analysis, *eg* inputting into a spreadsheet, database or other model.
5. Cleaning data, *eg* addressing unusual, missing or inconsistent values.
6. Exploratory data analysis, which may include descriptive analysis, inferential analysis or predictive analysis.
7. Modelling the data.
8. Communicating the results.
9. Monitoring the process; updating the data and repeating the process if required.

In the data collection process, the primary source of the data is the population (or population sample) from which the 'raw' data is obtained. If, once the information is collected, cleaned and possibly summarised, it is made available for others to use via a web interface, this is then a secondary source of data.

Other aspects of the data determined by the collection process that may affect the analysis are:

- Cross-sectional data involves recording values of the variables of interest for each case in the sample at a single moment in time.
- Longitudinal data involves recording values at intervals over time.
- Censored data occurs when the value of a variable is only partially known.
- Truncated data occurs when measurements on some variables are not recorded so are completely unknown.

The term 'big data' can be used to describe data with characteristics that make it impossible to apply traditional methods of analysis. Typically, this means automatically collected data with characteristics that have to be inferred from the data itself rather than known in advance from the design of the experiment.

Properties that can lead to data being classified as 'big' include:

- size of the data set
- speed of arrival of the data
- variety of different sources from which the data is drawn
- reliability of the data elements might be difficult to ascertain.

Replication refers to an independent third party repeating an experiment and obtaining the same (or at least consistent) results. Replication of a data analysis can be difficult, expensive or impossible, so reproducibility is often used as a reasonably alternative standard. Reproducibility refers to reporting the results of a statistical analysis in sufficient detail that an independent third party can repeat the analysis on the same data set and arrive at the same results.

Elements required for reproducibility:

- the original data and fully documented computer code need to be made available
- good version control
- documentation of the software used, computing architecture, operating system, external dependencies and version numbers
- where randomness is involved in the process, replication will require the random seed to be set
- limiting the amount of work done 'by hand'.

Chapter 1 Practice Questions

1.1 The data analysis department of a mobile phone messaging app provider has gathered data on the number of messages sent by each user of the app on each day over the past 5 years. The geographical location of each user (by country) is also known.

(i) Describe each of the following terms as it relates to a data set, and give an example of each as it relates to the app provider's data:

(a) cross-sectional

(b) longitudinal

(ii) Give an example of each of the following types of data analysis that could be carried out using the app provider's data:

(a) descriptive

(b) inferential

(c) predictive.

1.2 Explain the regulatory and legal requirements that should be observed when conducting a data analysis exercise.

1.3 A car insurer wishes to investigate whether young drivers (aged 17-25) are more likely to have an accident in a given year than older drivers.

Exam style

Describe the steps that would be followed in the analysis of data for this investigation. [7]

1.4 (i) In the context of data analysis, define the terms 'replication' and 'reproducibility'. [2]

Exam style

(ii) Give three reasons why replication of a data analysis can be difficult to achieve in practice. [3]

[Total 5]

The solutions start on the next page so that you can
separate the questions and solutions.

Chapter 1 Solutions

1.1 (i)(a) ***Cross-sectional***

Cross-sectional data involves recording the values of the variables of interest for each case in the sample at a single moment in time.

In this data set, this relates to the number of messages sent by each user on any particular day.

(i)(b) ***Longitudinal***

Longitudinal data involves recording the values of the variables of interest at intervals over time.

In this data set, this relates to the number of messages sent by a particular user on each day over the 5-year period.

(ii)(a) ***Descriptive analysis***

Examples of descriptive analysis that could be carried out on this data set include:

- calculating the mean and standard deviation of the number of messages sent each day by users in each country
- plotting a graph of the total messages sent each day worldwide, to illustrate the overall trend in the number of messages sent over the 5 years
- calculating what proportion of the total messages sent in each year originate in each country.

(ii)(b) ***Inferential analysis***

Examples of inferential analysis that could be carried out on this data set include:

- testing the hypothesis that more messages are sent at weekends than on weekdays
- assessing whether there is a significant difference in the rate of growth of the number of messages sent each day by users in different countries over the 5-year period.

(ii)(c) ***Predictive analysis***

Examples of predictive analysis that could be carried out on this data set include:

- forecasting which countries will be the major users of the app in 5 years' time, and will therefore need the most technical support staff
- predicting the number of messages sent on the app's busiest day (*eg* New Year's Eve) next year, to ensure that the provider continues to have sufficient capacity.

1.2 Throughout the data analysis process, it is important to ensure that any relevant professional guidance has been complied with. For example, the UK's Financial Reporting Council has issued a Technical Actuarial Standard (TAS) on the principles for Technical Actuarial Work (TAS100). This describes the principles that should be adhered to when using data in technical actuarial work.

The data analysis team must also be aware of any legal requirements to be complied with relating to, for example:

- protection of an individual's personal data and privacy
- discrimination on the grounds of gender, age, or other reasons.

With regard to privacy regulations, it is important to note that combining data from different sources may mean that individuals can be identified, even if they are anonymous in the original data sources.

Finally, data that have been made available on the internet cannot necessarily be used for any purpose. Any legal restrictions should be checked before using the data, noting that laws can vary between jurisdictions.

1.3 The key steps in the data analysis process in this scenario are:

1. Develop a well-defined set of objectives that need to be met by the results of the data analysis. [½]

   Here, the objective is to determine whether young drivers are more likely to have an accident in a given year than older drivers. [½]

2. Identify the data items required for the analysis. [½]

   The data items needed would include the number of drivers of each age during the investigation period and the number of accidents they had. [½]

3. Collection of the data from appropriate sources. [½]

   The insurer will have its own internal data from its administration department on the number of policyholders of each age during the investigation period and which of them had accidents. [½]

   The insurer may also be able to source data externally, *eg* from an industry body that collates information from a number of insurers. [½]

4. Processing and formatting the data for analysis, *eg* inputting into a spreadsheet, database or other model. [½]

   The data will need to be extracted from the administration system and loaded into whichever statistical package is being used for the analysis. [½]

   If different data sets are being combined, they will need to be put into a consistent format and any duplicates (*ie* the same record appearing in different data sets) will need to be removed. [½]

5. Cleaning data, *eg* addressing unusual, missing or inconsistent values. [½]

   For example, the age of the driver might be missing, or be too low or high to be plausible. These cases will need investigation. [½]

6. Exploratory data analysis, which here takes the form of inferential analysis… [½]

   … as we are testing the hypothesis that younger drivers are more likely to have an accident than older drivers. [½]

7. Modelling the data. [½]

   This may involve fitting a distribution to the annual number of accidents arising from the policyholders in each age group. [½]

8. Communicating the results. [½]

   This will involve describing the data sources used, the model and analyses performed, and the conclusion of the analysis (*ie* whether young drivers are indeed more likely to have an accident than older drivers), along with any limitations of the analysis. [½]

9. Monitoring the process – updating the data and repeating the process if required. [½]

   The car insurer may wish to repeat the process again in a few years' time, using the data gathered over that period, to ensure that the conclusions of the original analysis remain valid. [½]

10. Ensuring that any relevant professional guidance and legislation (*eg* on age discrimination) has been complied with. [½]

[Maximum 7]

1.4 (i) ***Definitions***

Replication refers to an independent third party repeating an analysis from scratch (including gathering an independent data sample) and obtaining the same (or at least consistent) results. [1]

Reproducibility refers to reporting the results of a statistical analysis in sufficient detail that an independent third party can repeat the analysis on the same data set and arrive at the same results. [1]

[Total 2]

(ii) ***Three reasons why replication is difficult***

Replication of a data analysis can be difficult if:

- the study is big; [1]
- the study relies on data collected at great expense or over many years; or [1]
- the study is of a unique occurrence (*eg* the standards of healthcare in the aftermath of a particular event). [1]

[Total 3]

2

Principles of actuarial modelling

Syllabus objectives

1.2 Describe the principles of actuarial modelling.

1.2.1 Describe why and how models are used including, in general terms, the use of models for pricing, reserving and capital modelling.

1.2.2 Explain the benefits and limitations of modelling.

1.2.3 Explain the difference between a stochastic and a deterministic model, and identify the advantages/disadvantages of each.

1.2.4 Describe the characteristics of, and explain the use of, scenario-based and proxy models.

1.2.5 Describe, in general terms, how to decide whether a model is suitable for any particular application.

1.2.6 Explain the difference between the short-run and long-run properties of a model, and how this may be relevant in deciding whether a model is suitable for any particular application.

1.2.7 Describe, in general terms, how to analyse the potential output from a model, and explain why this is relevant to the choice of model.

Syllabus objectives continued

1.2.8 Describe the process of sensitivity testing of assumptions and explain why this forms an important part of the modelling process.

1.2.9 Explain the factors that must be considered when communicating the results following the application of a model.

0 Introduction

This chapter provides an introduction to modelling, in particular within an actuarial context. The following general principles are covered:

- why we want to model
- how to model
- the benefits and limitations of modelling
- testing the suitability of the model
- analysing the output
- communicating the results.

1 Models

1.1 Why models are used

A model is an imitation of a real-world system or process. Models of many activities can be developed, for example:

- **the economy of a country**
- **the workings of the human heart, and**
- **the future cashflows of the broker distribution channel of a life insurance company.**

The expression 'broker distribution channel' refers to the fact that in life insurance it is common for payments to go through a broker who acts as an intermediary between the policyholder and the insurance company. This is also true of general insurance.

Suppose we wished to 'predict' the effect that a real-world change would have on these three models. In some cases it might be too risky, or too expensive or too slow, to try a proposed change in the real world even on a sample basis. Trying out the change first without the benefit of a model could have serious consequences. The economy might go into recession costing a government the next election, the patient might die and the life office could suffer a surge in new business but at highly unprofitable premium rates.

Parameters

A model enables the possible consequences to be investigated. The effect of changing certain input parameters can be studied before a decision is made to implement the plans in the real world.

Suppose, for example, we want to buy shares in a particular company. In order to calculate what price we should pay, we would like to have some idea of the future price of these shares, so we could develop a model relating the price of the shares to various factors. In our model we *might* assume (among other things) a fixed value for future interest rates, say 6% *pa*. This value is an input parameter. It is fixed within the model, but we can vary it to obtain different models. Using the figure 6% will give us one value for the price we should pay, but we cannot be certain that 6% is the correct input. What if interest rates rise to 10% *pa*, or fall to 3% *pa*, for example? By varying the parameters in the model we can answer questions such as these.

If we assume that the interest rate is a lognormal random variable with parameters μ and σ, then although the interest rate itself is not fixed, its distribution is. The input parameters are then μ and σ.

To build a model of a system or process, a set of mathematical or logical assumptions about how it works needs to be developed. The complexity of a model is determined by the complexity of the relationships between the various model parameters. For example, in modelling a life office, consideration must be given to issues such as regulations, taxation and cancellation terms. Future events affecting investment returns, inflation, new business, lapses, mortality and expenses also affect these relationships.

In insurance terms, a cancellation occurs when a policyholder terminates their insurance cover, *eg* when a driver changes their car insurance to a different insurer mid-year. A lapse occurs when a policyholder doesn't pay a premium that is due, *eg* when a policyholder changes to a new insurer when their existing policy is up for renewal.

Data

In order to produce the model and determine suitable parameters, data need to be considered and judgements need to be made as to the relevance of the observed data to the future environment. Such data may result from past observations, from current observations (such as the rate of inflation) or from expectations of future changes.

An example of a future change in an insurance context would be if the government decided to increase the rate of insurance tax.

Where observed data are considered to be suitable for producing the parameters for a chosen model, statistical methods can be used to fit the data.

For example, if we were to model the daily changes in the price of shares in a particular company using a normal distribution, then the parameters would be the mean and the variance. We could estimate these by looking at past data and calculating the sample mean and sample variance. These would be sensible estimates.

Objectives

Before finalising the choice of model and parameters, it is important to consider the objectives for creation and use of the model. For example, in many cases there may not be a desire to create the most accurate model, but instead to create a model that will not understate costs or other risks that may be involved.

Moreover, we need to decide what the 'best' model is, within the context of its intended use. For example, we might wish to model the number of claims made to an insurance company for various amounts. To simplify matters, suppose there are only two possible probability distributions that we are choosing between. If most claims are for small amounts and one distribution models these small amounts better than the other, then this might seem a natural choice. However, if this same distribution models the large claims relatively poorly, then although there are not many of these large claims, the fact that they are for large amounts is significant. The decisions resulting from a poor model of the higher claims may be more costly to the company than those based on a model that is slightly inaccurate on a lot of small claims.

1.2 How models are used

While in reality a modelling process does not follow a rigid pattern of prescribed steps, it is helpful in introducing the topic to imagine a set of key steps. In practice, actuaries who build and use models move back and forth between these key steps continuously to improve the model.

The key steps in a modelling process can be described as follows:

1. **Develop a well-defined set of objectives that need to be met by the modelling process.**

   Continuing our last example on modelling the size of insurance claims, in addition to the basic purpose of being able to predict the number of claims of different sizes, one objective might be to give as accurate a prediction as possible for 95% of the claims.

2. **Plan the modelling process and how the model will be validated.**

   The validation of the model will involve a series of diagnostic tests to ensure that the model meets the objectives we want.

3. **Collect and analyse the necessary data for the model.**

4. **Define the parameters for the model and consider appropriate parameter values.**

5. **Define the model initially by capturing the essence of the real-world system. Refining the level of detail in the model can come at a later stage.**

6. **Involve experts on the real-world system you are trying to imitate to get feedback on the validity of the conceptual model.**

7. **Decide on whether a simulation package or a general-purpose language is appropriate for the implementation of the model. Choose a statistically reliable random number generator that will perform adequately in the context of the complexity of the model.**

   Models based on a deterministic approach would not need this.

8. **Write the computer program for the model.**

   After this stage we can run the model.

9. **Debug the program to make sure it performs the intended operations in the model definition.**

10. **Test the reasonableness of the output from the model.**

11. **Review and carefully consider the appropriateness of the model in the light of small changes in input parameters.**

    Suppose, for example, that small changes in the input parameters lead to large changes in the output. If these parameters cannot be known with great accuracy, then we cannot be sure that our predictions will be valid. However, we could still use the model to come up with a range of possible outputs by assuming a range of values for the input parameters. This procedure is called sensitivity testing and is discussed later in this chapter.

12. **Analyse the output from the model.**

13. **Ensure that any relevant professional guidance has been complied with. For example, the Financial Reporting Council has issued a Technical Actuarial Standard on the principles for Technical Actuarial Work (TAS100), which includes principles for models used in technical actuarial work. (Knowledge of the detail of this TAS is not required for CM1.)**

14. Communicate and document the results and the model.

After the model has been developed, if it continues to be used over time, it should be reviewed periodically to ensure that it continues to meet the user's objectives and to investigate the possibility of making improvements by repeating Steps 3 to 12 in the light of data obtained since the original predictions were made.

2 Modelling – the benefits and limitations

2.1 Advantages of models

In actuarial work, one of the most important benefits of modelling is that systems with long time frames – such as the operation of a pension fund – can be studied in compressed time.

Other benefits include:

- **Complex systems with stochastic elements, such as the operation of a life insurance company, cannot be properly described by a mathematical or logical model that is capable of producing results that are easy to interpret. Simulation modelling is a way of studying the operation of a life insurance company.**

  In broad terms, 'stochastic' means 'random'. So, where randomness is an essential part of the process being modelled, this randomness must be included in the model in order to get meaningful results.

- **Different future policies or possible actions can be compared to see which best suits the requirements or constraints of a user.**
- **In a model of a complex system we can usually get control over the experimental conditions so that we can reduce the variance of the results output from the model without upsetting their mean values.**

2.2 Disadvantages

However, models are not the simple solution to all actuarial problems – they have drawbacks that must be understood when interpreting the output from a model and communicating the results to clients.

The drawbacks include:

- **Model development requires a considerable investment of time, and expertise. The financial costs of development can be quite large given the need to check the validity of the model's assumptions, the computer code, the reasonableness of results and the way in which results can be interpreted in plain language by the target audience.**

  In an actuarial context, the target audience could be, for example:

  – a life office policyholder, who wants an idea of how much will be received from a particular policy when it matures

  – a pension fund client who needs to know how much to pay into the company's pension fund next year

  – the finance director of a general insurance company who wants an estimate of the end-of-year profit figures.

- **In a stochastic model, for any given set of inputs each run gives only estimates of a model's outputs. So, to study the outputs for any given set of inputs, several independent runs of the model are needed.**

  A stochastic model allows for randomness and each computer 'run' would give the figures for one possible outcome.

- **As a rule, models are more useful for comparing the results of input variations than for optimising outputs.**

  In other words, it's easier to use a model to predict what *might* happen than it is to determine what inputs would be required to obtain a particular outcome.

- **Models can look impressive when run on a computer so that there is a danger that one gets lulled into a false sense of confidence. If a model has not passed the tests of validity and verification, its impressive output is a poor substitute for its ability to imitate its corresponding real-world system.**

- **Models rely heavily on the data input. If the data quality is poor or lacks credibility, then the output from the model is likely to be flawed.**

  In actuarial terminology, 'credibility' refers to the extent to which data can be relied on.

- **It is important that the users of the model understand the model and the uses to which it can be safely put. There is a danger of using a model as a 'black box' from which it is assumed that all results are valid without considering the appropriateness of using that model for the data input and the output expected.**

- **It is not possible to include all future events in a model. For example, a change in legislation could invalidate the results of a model, but may be impossible to predict when the model is constructed.**

- **It may be difficult to interpret some of the outputs of the model. They may only be valid in relative rather than absolute terms, as when, for example, comparing the level of risk of the outputs associated with different inputs.**

  'Risk' refers to the level of uncertainty associated with an outcome.

3 Stochastic and deterministic models

If it is desired to represent reality as accurately as possible, the model needs to imitate the random nature of the variables. A *stochastic* model is one that recognises the random nature of the input components. A model that does not contain any random component is *deterministic* in nature.

In a deterministic model, the output is determined once the set of fixed inputs and the relationships between them have been defined. By contrast, in a stochastic model the output is random in nature – like the inputs, which are random variables. The output is only a snapshot or an estimate of the characteristics of the model for a given set of inputs. Several independent runs are required for each set of inputs so that statistical theory can be used to help in the study of the implications of the set of inputs.

A deterministic model is really just a special (simplified) case of a stochastic model.

The following example illustrates the difference between the two approaches.

Question

An investor has bought shares worth £5,000 and wants to estimate how much they will be worth in a year's time. Describe both a deterministic and stochastic model based on an expected growth rate of 7% over the year.

Solution

Deterministic model

The outcome from a deterministic model is the prediction that the value in a year's time would be:

$$5{,}000 \times 1.07 = £5{,}350$$

Stochastic model

A stochastic model allows for randomness in the growth rate. For example, it might be decided (based on past performance of the company and prospects for the company, the investment sector and the economy in general) that the probabilities of different growth rates for the shares are:

$$\text{growth} = \begin{cases} 20\% & \text{with probability } 0.1 \\ 10\% & \text{with probability } 0.6 \\ 0\% & \text{with probability } 0.2 \\ -10\% & \text{with probability } 0.1 \end{cases}$$

These probabilities add up to 1 and the expected growth rate is 7% for this model since:

$$0.1 \times 20\% + 0.6 \times 10\% + 0.2 \times 0\% + 0.1 \times (-10\%) = 7\%$$

The outcome from this model, if it is run 100 times, is a list of 100 predicted values, which might look like this:

£5500, £5000, £5000, £6000, £5000, … , £4500, £6000

Whether one wishes to use a deterministic or a stochastic model depends on whether one is interested in the results of a single 'scenario' or in the distribution of results of possible 'scenarios'. A deterministic model will give one the results of the relevant calculations for a single scenario; a stochastic model gives distributions of the relevant results for a distribution of scenarios. If the stochastic model is investigated by using 'Monte Carlo' simulation, then this provides a collection of a suitably large number of different deterministic models, each of which is considered equally likely.

'Monte Carlo' simulation is where a computer is set up to run a stochastic model a number of times, *eg* 10,000 times, using pseudo-random numbers generated by the computer. The results look like the list of numbers in the stochastic model in the example above. The numbers generated by the computer are pseudo-random rather than truly random because they are generated using a prescribed method.

As stochastic models provide a distribution of outputs, they can be used to estimate the probability that a particular event occurs, or to calculate approximate confidence intervals. This wouldn't be possible using a deterministic model.

The results for a deterministic model can often be obtained by direct calculation, but sometimes it is necessary to use numerical approximations, either to integrate functions or to solve differential equations.

If a stochastic model is sufficiently tractable, it may be possible to derive the results one wishes by analytical methods. If this can be done it is often preferable to, and also often much quicker than, Monte Carlo simulation. One gets precise results and can analyse the effect of changes in the assumptions more readily. Monte Carlo simulation is covered in CS1.

Many practical problems, however, are too complicated for analytical methods to be used easily, and Monte Carlo simulation is an extremely powerful method for solving complicated problems. But if even part of a model can be treated analytically, it may provide a check on any simulation method used. It may be possible to use a deterministic method to calculate the expected values, or possibly the median values, for a complicated problem, where the distributions around these central values are estimated by simulation.

One also needs to recognise that a simulation method generally provides 'what if?' answers; what the results are on the basis of the assumptions that have been made. It is much harder to use simulation to provide the optimum solution; in other words to find the set of assumptions that maximises or minimises some desired result.

A further limitation is that the precision of a simulated result depends on the number of simulations that are carried out. This is covered in more detail in CS1.

4 Discrete and continuous state spaces and time sets

The state of a model is the set of variables that describe the system at a particular point in time taking into account the goals of the study. It is possible to represent any future scenarios as states, as will be developed in later chapters.

Consider an insurance policy that will provide an income if the policyholder is sick and unable to work. We might model the future state of the policyholder as:

- healthy (*ie* not receiving a benefit)
- sick (*ie* receiving a benefit)
- dead.

Discrete states are where the variables exhibit step function changes in time. For example, from a state of alive to dead, or an increase in the number of policies for an insurer. Continuous states are where the variables change continuously with respect to time. For example, real time changes in values of investments.

The decision to use a discrete or continuous state model for a particular system is driven by the objectives of the study, rather than whether or not the system itself is of a discrete or continuous nature.

A model may also consider time in a discrete or a continuous way. This may reflect the fact that outputs from the model are only required at discrete points in time, or may be to satisfy the objectives of the modelling.

For example, an insurance company may wish to model the running total of the number of claims it has received since the start of the year (*ie* in continuous time), or it may simply model the number of claims it has received at the end of each month (*ie* in discrete time).

One cannot in practice use Monte Carlo simulation for a continuous time problem; one has to discretise the time step. This can be done with whatever precision one likes, but the higher the precision the longer the time taken to process any particular model. This may or may not be a limitation in practice. And it should be remembered that some results for continuous time, continuous space models cannot be obtained by discrete simulation at all.

Processes with continuous state spaces are discussed in Subjects CS2 and CM2.

5 Scenario-based and proxy models

Most models depend on many input parameters. A *scenario-based* model would take into consideration a particular scenario; that is a series of input parameters based on this scenario. Different scenarios would be useful in decision analysis as one can evaluate the expected impact of a course of action.

For example, we could model the financial performance of a company under different future scenarios, such as 'recession' or 'economic boom'. The values of any input parameters (*eg* rate of inflation, interest rate, level of taxation) would be selected to be appropriate to the specific scenario under consideration.

The projected valuation of assets and liabilities for an insurer can be very complex, requiring significant runs of Monte Carlo simulation. Therefore, *proxy* models are used to replace Monte Carlo simulations. These are expected to be faster but less accurate.

For example, a Monte Carlo simulation evaluating the total claims paid each year of a car insurer may depend on the number of cars insured and written premium (this is too simplistic for a real-life example). A Monte-Carlo simulation would provide a range of results, including the mean for different input scenarios. However, given a range of inputs and outputs, a regression function can be set up based on these two parameters and their derivatives (such as squared value). This would then act as a proxy model for the actual expected total amount of claims paid for other intermediate values.

Performing a Monte Carlo simulation is a time-consuming process. If it is possible to develop a simplified formula, based on the same inputs, that we believe predicts the result with reasonable accuracy, we may use this instead to save time. The simplified formula is then used as a substitute (or proxy) for running the full model.

So, using the example given above, if N denotes the number of cars insured and P denotes the written premium, then a proxy model may take the form:

$$\text{Total claims paid each year} = a + bN + cP + dP^2$$

where a, b, c and d are constants.

6 Suitability of a model

In assessing the suitability of a model for a particular exercise it is important to consider the following:

- **The objectives of the modelling exercise.**
- **The validity of the model for the purpose to which it is to be put.**
- **The validity of the data to be used.**
- **The validity of the assumptions.**
- **The possible errors associated with the model or parameters used not being a perfect representation of the real-world situation being modelled.**
- **The impact of correlations between the random variables that 'drive' the model.**
- **The extent of correlations between the various results produced from the model.**
- **The current relevance of models written and used in the past.**
- **The credibility of the data input.**
- **The credibility of the results output.**
- **The dangers of spurious accuracy.**
- **The ease with which the model and its results can be communicated.**
- **Regulatory requirements.**

The important actuarial/investment concept of 'matching' of assets and liabilities relies on the fact that the value of the matched assets and liabilities will tend to move together, *ie* they are positively correlated. In models of such situations, it would be essential to incorporate this correlation.

Similarly, when modelling the value of a portfolio invested in a variety of different shares, it would be important to allow for the fact that the different share prices are likely to be positively correlated. This is because there is a tendency for the prices of different shares to move together in response to general factors affecting the whole market, *eg* changes in interest rates.

An example of 'spurious accuracy' is the statement: 'The value of our company pension fund's assets is £46,279,312.86.' This is spurious accuracy because the market value of the investments will change by the minute and this level of accuracy cannot be justified. A more appropriate figure to give is £46.3 million.

7 Short-term and long-term properties of a model

The stability of the relationships incorporated in the model may not be realistic in the longer term. For example, exponential growth can appear linear if surveyed over a short period of time. If changes can be predicted, they can be incorporated in the model, but often it must be accepted that longer-term models are suspect.

Models are, by definition, simplified versions of the real world. They may therefore ignore 'higher order' relationships which are of little importance in the short term, but which may accumulate in the longer term.

There is an analogy here with series. For small values of time t, the exponential function e^t appears linear, since:

$$e^t = 1 + t + o(t)$$

But if t is greater, we need to include the higher order terms:

$$e^t = 1 + t + \tfrac{1}{2}t^2 + \cdots$$

8 Analysing the output of a model

Statistical sampling techniques are needed to analyse the output of a model, as a simulation is just a computer-aided statistical sampling project. The actuary must exercise great care and judgement at this stage of the modelling process as the observations in the process are correlated with each other and the distributions of the successive observations change over time. The useless and fatally attractive temptation of assuming that the observations are independent and identically distributed is to be avoided.

If there is a real-world system against which results can be compared, a 'Turing' test should be used. In a Turing test, experts on the real-world system are asked to compare several sets of real-world and model data without being told which are which. If these experts can differentiate between the real-world and model data, their techniques for doing so could be used to improve the model.

This is an extension of the original meaning of a Turing test named after the British mathematician and early computer pioneer, Alan Turing. The original Turing test relates to one objective of artificial intelligence researchers, which is to write a computer program that cannot be distinguished from a real person by a user asking questions over a computer link.

9 Sensitivity testing

Where possible, it is important to test the reasonableness of the output from the model against the real world. To do this, an examination of the sensitivity of the outputs to small changes in the inputs or their statistical distributions should be carried out. The appropriateness of the model should then be reviewed, particularly if small changes in inputs or their statistical distributions give rise to large changes in the outputs. In this way, the key inputs and relationships to which particular attention should be given in designing and using the model can be determined.

If small changes in the inputs give rise to large changes in the outputs, then our initial choices are more crucial. How confident are we in our choices of input? If the resulting changes in output are small, then our initial choices are less important in this respect.

Sensitivity testing also refers to the approach of using a deterministic model with changes in one or more of the assumptions to see the range of possible outcomes that might occur. For example, an insurance company providing personal pension plans might give illustrations to policyholders showing how much pension they would get if growth rates were 2%, 5% and 8% *pa* in the period before retirement. This allows the policyholder to gauge the extent to which the resulting pension will be affected by changes in the growth rate.

The model should be tested by designing appropriate simulation experiments. It is through this process that the model can be refined.

An approach that has been traditionally used by actuaries in the fields of insurance, pensions and investment is to carry out a set of deterministic calculations using different actuarial bases, *ie* under different sets of assumptions. By varying the assumptions, the actuary could use the model to arrive at figures that are 'best estimates' (the most likely, or median, result) or 'optimistic' (if things work out favourably) or 'cautious' (if things work out badly). This is an example of a scenario-based approach to modelling.

10 Communication of the results

The final step in the modelling process is the communication and documentation of the results and the model itself to others. The communication must be such that it takes account of the knowledge of the target audience and their viewpoint. A key issue here is to make sure that the client accepts the model as being valid and a useful tool in decision making. It is important to ensure that any limitations on the use and validity of the model are fully appreciated.

The following example illustrates one possible limitation of a model that would need to be explained to a client.

Example

An actuarial student working at a consultancy is asked by a client how expensive it would be for the client's company pension scheme to offer more generous benefits to members who leave the company before reaching retirement age.

As the company has historically had a very stable workforce with very little turnover of staff, the current model assumes that everyone stays with the company until retirement age. So the client's question could not be answered using the existing model.

To answer the client's question, the model could be extended to allow for an appropriate rate of withdrawal of staff before retirement age. This amended model could then be used to compare the costs of the scheme with the existing withdrawal benefit with the costs of the 'improved' one.

Chapter 2 Summary

A model is an imitation of a real-world system or process.

It enables possible consequences to be investigated without carrying out the actions themselves.

There are benefits to modelling, such as the possibility of looking at long-term phenomena in an accelerated time frame, but there are also limitations that must be considered such as the time and expertise required to develop and run a model.

The following 14 key steps can be considered in the construction and use of a model:

1. Develop a well-defined set of objectives that need to be met by the modelling process.
2. Plan the modelling process and how the model will be validated.
3. Collect and analyse the necessary data for the model.
4. Define the parameters for the model and consider appropriate parameter values.
5. Define the model initially by capturing the essence of the real-world system. Refining the level of detail in the model can come at a later stage.
6. Involve the experts on the real-world system you are trying to model in order to get feedback on the validity of the conceptual model.
7. Decide on whether a simulation package or general-purpose language is appropriate for the implementation of the model. If necessary, choose a statistically reliable random number generator that will perform adequately in the context and complexity of the model.
8. Write the computer program for the model.
9. Debug the program to make sure it performs the intended operations in the model definition.
10. Test the reasonableness of the output from the model.
11. Review and carefully consider the appropriateness of the model in the light of small changes to the input parameters.
12. Analyse the output from the model.
13. Ensure that any relevant professional guidance has been complied with. For example, the Financial Reporting Council has issued a Technical Actuarial Standard on the principles for Technical Actuarial Work (TAS100), which includes principles for models used in technical actuarial work.
14. Communicate and document the results and the model.

When building a model, the suitability of the model to the objectives should be borne in mind. Some mathematical or logical assumptions about the workings of the real-world system must be made. Input parameters need to be chosen, possibly by statistical analysis of past data. Sensitivity analysis of the dependence of the output on these parameters can be carried out.

A model may be stochastic or deterministic. For stochastic models it is better to use direct calculation if possible, but in complex situations it may be necessary to use Monte Carlo simulation on a computer.

Models may also be constructed in discrete or continuous time and with discrete or continuous state spaces. Monte Carlo simulations have to be run in discrete time.

A scenario-based model would take into consideration a particular scenario, with the values of any input parameters being selected to be appropriate to that specific scenario.

A proxy model may be used to replace Monte Carlo simulation, providing faster but less accurate results. A simplified formula is developed that we believe predicts the result with reasonable accuracy, and this is then used as a substitute for running the full model.

The output from any model needs to be analysed. This can be done using the idea of a Turing test – can an expert tell the difference between a set of simulated outputs and actual occurrences?

The results need to be communicated to other people. The key question when framing the level of communication is 'to whom?'

Chapter 2 Practice Questions

2.1 Explain what is meant by a 'stochastic model' and state two advantages these have over deterministic models.

2.2 Describe the role of simulation in sensitivity analysis.

2.3 The government of a small island state intends to set up a model to analyse the mortality of the island's population over the past 50 years.

Exam style

Describe the process that would be followed to carry out the analysis. [6]

2.4 (i) List the advantages and disadvantages of using models in actuarial work. [4]

Exam style

A new town is planned in a currently rural area. A model is to be developed to recommend the number and size of schools required in the new town. The proposed modelling approach is as follows:

- The current age distribution of the population in the area is multiplied by the planned population of the new town to produce an initial population distribution.
- Current national fertility and mortality rates by age are used to estimate births and deaths.
- The births and deaths are applied to the initial population distribution to generate a projected distribution of the town's population by age for each future year, and hence the number of school age children.

(ii) Discuss the appropriateness of the proposed modelling approach. [5]
[Total 9]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 2 Solutions

2.1 A stochastic model is one that recognises the random nature of the input components.

Two advantages of stochastic models over deterministic models are:

1. To reflect reality as accurately as possible, the model should imitate the random nature of the variables involved.

2. A stochastic model can provide information about the distribution of the results (*eg* probabilities, variances *etc*), not just a single best estimate figure.

2.2 Sensitivity analysis involves testing the robustness of the model by making small changes to the input parameters. This should result in small changes to the output from the model that are consistent with the real-world behaviour of the situation we are modelling.

The usual method of carrying out a sensitivity analysis is to run a large number of computer simulations based on the original parameter values, then to repeat these using several sets of slightly different parameter values.

For consistency, the same set of pseudo-random numbers should be used for each set of simulations. This removes the effect of the additional source of randomness that would be introduced by using different pseudo-random numbers.

2.3 *This question is Subject CT4, October 2010, Question 3.*

The process that would be followed to carry out the analysis is as follows:

1. Develop a well-defined set of objectives that need to be met by the modelling process, *ie* state the point of the exercise. In this case we should think about what aspects of mortality are to be analysed, *eg* average mortality rates, split of males/females, trends in mortality over the last 50 years.

2. Plan the modelling process and how the model will be validated.

3. Collect and analyse the necessary data for the model. In this case we would need the numbers of deaths over the last 50 years and any available census data. Problems may arise as some of the data may be missing or inaccurate, and recording practices may have changed over the last 50 years.

4. Define the parameters for the model and consider appropriate parameter values.

5. Define the model firstly by capturing the essence of the real-world system. Refining the level of detail in the model can come at a later stage. For this model, this means we should identify the main features of mortality.

6. Involve people with expert knowledge of the real-world system you are trying to imitate so as to get feedback on the validity of the conceptual model. For example, there may be a national census office or government department that can help.

7. Decide on whether a simulation package or a general-purpose language is appropriate for the implementation of the model.

8. Write the computer program for the model.

9. Debug the program to make sure it performs the intended operations defined in the initial modelling process.

10. Test the reasonableness of the output from the model. For example, we could check that it is a good fit to the actual mortality experience of the island over subsets of the last 50 years.

11. Review and carefully consider the appropriateness of the model in the light of small changes in input parameters.

12. Analyse the output from the model.

13. Ensure that any relevant professional guidance has been complied with. This may include standards on data, modelling and reporting.

14. Document the results of the model and communicate these to the government.

[½ for each point, up to a maximum of 6]

2.4 *This question is Subject CT4, April 2012, Question 6.*

(i) ***Advantages / disadvantages of using models in actuarial work***

Advantages

Models allow us to investigate the future behaviour of a process in compressed time.

Models allow us to study the stochastic nature of the results, *eg* using Monte Carlo simulation.

Models allow us to compare a large number of different scenarios (in a short space of time) to determine the best strategy.

Models allow us to consider scenarios that would not be feasible in practice, *eg* because of cost or other business considerations.

Models allow control over experimental conditions, so that we can reduce the variance of the results output, without affecting the mean values.

[½ for each advantage, up to a maximum of 2]

Disadvantages

Models can be time-consuming and expensive to set up.

Stochastic models require a large number of simulations to be carried out to get accurate results.

In general, models are more useful for comparing the results of input variations than for optimising outputs.

Models can give an impression of greater accuracy and reliability than they actually have, and so may create a sense of false security.

The results from models are dependent on the data used, which may be inaccurate or unreliable.

Users of the model may not understand fully how it works and its limitations.

Models may not capture sufficiently accurately the real-world situation.

Models require simplifying assumptions that might turn out to be wrong, *eg* they may ignore certain features (*eg* inflation) or certain types of events that could occur (*eg* catastrophes).

Some models may be difficult to explain to clients.

[½ for each disadvantage, up to a maximum of 2]
[Total 4]

(ii) ***Appropriateness of the proposed modelling approach***

The model uses an established method that is straightforward to apply, understand and explain to the town planners. [1]

If the rural area is in a developed country, mortality will probably have little effect (since the mortality of children and those of child-bearing age will be relatively low) and so the model could be simplified by ignoring this factor. [1]

The results will probably be accurate over the short term (*ie* for the next few years), but may become less reliable if applied over longer periods. [½]

The model makes no allowance for migration, *ie* people moving in or out of the town. This could be an important factor that could increase or reduce the size of the population. [½]

It should be possible to obtain sufficiently accurate data about the current age distribution. [½]

However, the model assumes that the initial population profile of the town will be the same as for the rural area in which it is located, which may not be true for a new town. [1]

The initial population profile will depend on the type of housing (*eg* family homes, bungalows for the elderly) and the type of employment opportunities available in the town. [1]

It should be possible to obtain reliable estimates for fertility rates and mortality rates, but these may change in the future, *eg* people may have fewer children during an economic recession. [½]

Also the fertility rates may be different in different areas, so that the national rates would not be appropriate. [½]

There could be changes to government education policy that would affect the number of school places required. For example, the ages when school attendance is compulsory could be changed or new types of school could be introduced (*eg* boarding schools, where children are schooled outside the area). [½]

[Maximum 5]

3

Cashflow models

Syllabus objectives

1.3 Describe how to use a generalised cashflow model to describe financial transactions.

1.3.1 State the inflows and outflows in each future time period and discuss whether the amount or the timing (or both) is fixed or uncertain for a given cashflow process.

1.3.2 Describe in the form of a cashflow model the operation of financial instruments like a zero-coupon bond, a fixed-interest security, an index-linked security, cash on deposit, an equity, an interest-only loan, a repayment loan, and an annuity-certain; and insurance contracts like an endowment, a term assurance, a contingent annuity, car insurance and health cash plans.

0 Introduction

A cashflow model is a mathematical projection of the payments arising from a financial transaction, *eg* a loan, a share or a capital project. Payments received are referred to as *income* and are shown as *positive* cashflows. Payments made are referred to as *outgo* and are shown as *negative* cashflows. The difference at a single point in time (income less outgo) is called the *net cashflow* at that point in time.

This chapter considers the cashflows that emerge in a number of practical situations that you will come across in the actuarial field.

1 Cashflow process

The practical work of the actuary often involves the management of various *cashflows.* These are simply sums of money, which are paid or received at different times. The timing of the cashflows may be known or uncertain. The amount of the individual cashflows may also be known or unknown in advance.

For example, a company operating a privately owned bridge, road or tunnel will receive toll payments. The company will pay out money for maintenance, debt repayment and for other management expenses. From the company's viewpoint the toll payments are positive cashflows (*ie* money received) while the maintenance, debt repayments and other expenses are negative cashflows (*ie* money paid out). Similar cashflows arise in all businesses.

From a theoretical viewpoint one may also consider a *continuously* payable cashflow.

Continuously payable cashflows are often used when payments are made very frequently, *eg* daily or weekly. This is because the mathematics is sometimes easier if we assume that the payments are made continuously rather that at regular intervals. This will become clearer when this mathematics is introduced to you later in the course.

Question

Outline some cashflows, both positive and negative, that will occur in the next month where one of the parties involved in the cashflow is (i) you, (ii) your employer.

Solution

(i) ***Cashflows involving you***

Positive: receiving your salary, borrowing some money from a friend.

Negative: repaying some borrowed money, buying something from a shop using cash.

(ii) ***Cashflows involving your employer***

Positive: receiving payments for supplying products or services, *eg* receiving premiums.

Negative: paying salaries, paying property expenses, *eg* electricity bills.

In some businesses, such as insurance companies, investment income will be received in relation to positive cashflows (premiums) received before the negative cashflows (claims and expenses).

For example, consider a premium received by an insurance company from a policyholder. Some of the premium might be used to cover the costs associated with setting up the insurance policy. The remainder of the premium could be put into a bank account. Investment income, in this case interest, will be earned on the money in the bank account until the money is needed for further expenses or payments back to the policyholder.

Where there is uncertainty about the amount or timing of cashflows, an actuary can assign probabilities to both the amount and the existence of a cashflow.

The amount and timing of some cashflows will be known with great certainty. For example, an employed person who gets paid on the last Friday of every month will be almost certain to receive a payment on the last Friday of this month. The amount of the payment is also likely to be known.

Question

List reasons why this person is not *completely certain* of receiving a payment on the last Friday of this month.

Solution

The payment might not be made if:

- the employer makes an error or goes bankrupt
- the bank makes an error and doesn't make the payment on the due date
- the person leaves the company before the end of the month
- the last Friday is a public holiday and other arrangements are in place.

Other cashflows are not so certain. If you buy a lottery ticket every week, you don't know when, or if, you will win or how much you may win.

The probability that the payment will take place could be estimated by looking at past results. If there is no past information relating to the event being considered, then data from similar events would be used.

2 Examples of cashflow scenarios

In this section, we provide examples of practical situations with cashflows that are assumed to be certain. In reality, this may not be the case, as the counterparty of a particular cashflow may not be able to pay out. For example, a company may fail and not be able to pay out interest on issued bonds.

If a company fails to pay the interest it owes on its bonds, this is called a default event.

The first few examples considered below are types of security or investment. A security is a tradeable financial instrument, *ie* a financial contract that can be bought and sold.

2.1 A zero-coupon bond

The term *zero-coupon bond* is used to describe a security that is simply a contract to provide a specified lump sum at some specified future date. For the investor there is a negative cashflow at the point of investment and a single known positive cashflow on the specified future date.

For example, the investor may give the issuer of the zero-coupon bond £400,000, and in return the investor will receive £500,000 from the issuer in exactly 5 years' time. The issuer may be a government or a large company.

The positive cashflow is paid on a set future date and is of a set amount, but, as mentioned above, it is not certain that the payment will be made. There is a chance that the issuing organisation will not make the payment, *ie* that it will default. This risk is usually negligible for bonds issued by governments of developed countries, since the government can always raise taxes. The risk of default is greater for issuing organisations that may go bust, *eg* companies.

A zero-coupon bond is a form of loan from the investor to the issuer. The loan is repaid by one single payment of a fixed amount at a fixed date in the future. It is a special case of a fixed-interest security with no interest payments before redemption. We will study fixed-interest securities in the next section.

We can plot the cashflows of the investor on a timeline:

For the issuing organisation, there is a positive cashflow at the point of investment and a single known negative cashflow on a specified future date. These cashflows are the opposite of those experienced by the investor, and can be shown on a timeline as:

The investor may also be referred to as the lender, and the issuer may be referred to as the borrower.

2.2 A fixed-interest security

A body such as an industrial company, a local authority, or the government of a country may raise money by floating a loan on the stock exchange.

This means that the organisation borrows money by issuing a loan to investors. The loan is simultaneously listed on the stock exchange so that after issue the securities can be traded on the stock exchange. This means that investors can sell their right to receive the future cashflows.

In many instances such a loan takes the form of a *fixed-interest security*, which is issued in bonds of a stated nominal amount. The characteristic feature of such a security in its simplest form is that the holder of a bond will receive a lump sum of specified amount at some specified future time together with a series of regular level interest payments until the repayment (or 'redemption') of the lump sum.

The regular level interest payments are referred to as coupons. Thus a zero-coupon bond has no interest payments.

The investor has an initial negative cashflow, a single known positive cashflow on the specified future date, and a series of smaller known positive cashflows on a regular set of specified future dates.

An investor might buy a 20-year fixed-interest security of nominal amount £10,000. This means that the face value of the loan is £10,000. The investor is unlikely to pay exactly £10,000 for this security but will pay a price that is acceptable to both parties. This may be higher or lower than £10,000. The investor will then receive a lump sum payment in 20 years' time. This lump sum is most commonly equal to the nominal amount, in this case £10,000, but could be a pre-specified amount higher or lower than this. The investor will also receive regular payments throughout the 20 years of, say, £500 *pa*. These regular payments could be made at the end of each year or half-year or at different intervals.

In the case where the regular payments are made at the end of each year, the cashflows of the investor can be represented on a timeline, as follows:

– £price	+£500	+£500	+£500	…	+£500	+£10,500
0	1	2	3	…	19	20

The last payment is made up of the final regular payment (£500) and the lump sum payment (£10,000).

2.3 An index-linked security

Inflation is a measure of the rate of change of the price of goods and services, including salaries. High inflation implies that prices are rising quickly and low inflation implies that prices are rising slowly.

If a pair of socks costs £1, then £5 could be used to buy 5 pairs of socks. However, if inflation is high, then the cost of socks in 1 year's time might be £1.25. £5 would then only buy 4 pairs of socks.

This simple example shows how the 'purchasing power' of a given sum of money, *ie* the quantity of goods that can be bought with the money, can diminish if inflation is high. In the socks example, the annual rate of inflation is 25%.

With a conventional fixed-interest security, the interest payments are all of the same amount. If inflationary pressures in the economy are not kept under control, the purchasing power of a given sum of money diminishes with the passage of time, significantly so when the rate of inflation is high. For this reason some investors are attracted by a security for which the actual cash amount of interest payments and of the final capital repayment are linked to an 'index' which reflects the effects of inflation.

Here the initial negative cashflow is followed by a series of unknown positive cashflows and a single larger unknown positive cashflow, all on specified dates. However, it is known that the amounts of the future cashflows relate to the inflation index. Hence these cashflows are said to be known in 'real' terms.

Real terms means taking into account inflation, whereas *nominal* (or *money*) *terms* means ignoring inflation. For example, if your salary is rising at 5% *pa* and inflation is 7% *pa*, your salary is *falling* in real terms (as you will be able to buy less with your 'higher salary'), even though your salary is *rising* in nominal (or money) terms.

So for an index-linked bond, the cashflows are known in real terms, but are unknown in nominal (or money) terms. For a fixed-interest bond, on the other hand, the cashflows are known in nominal (or money) terms, but are unknown in real terms.

Both the regular interest payments and the final capital payment from an index-linked security are linked to the inflation index. If inflation is high, then the regular payments will rise by larger amounts than if inflation is low.

If inflation is 10% per time period and the regular interest payment after one time period is £500, then the payment after two time periods will be £550 ($=500\times 1.1$), and the payment after three time periods will be £605 ($=500\times 1.1^2$) *etc*.

Inflation is often measured by reference to an index. For example an inflation index might take values as set out in the table below.

Date	1.1.2015	1.1.2016	1.1.2017	1.1.2018
Index	100.00	105.00	108.00	113.00

Based on this, the rate of inflation during 2016 is 2.86% *pa* (*ie* $108/105-1$).

Question

An investor purchased a three-year index-linked security on 1.1.2015. In return, the investor received payments at the end of each year plus a final redemption payment, all of which were increased in line with the index given in the table above. The payments would have been £600 each year and £11,000 on redemption if there had been no inflation.

Calculate the payments actually received by the investor.

Solution

The payments received by the investor are calculated in the table below:

Payment date	Payment amount
1.1.2016	$600\times\frac{105}{100}=£630$
1.1.2017	$600\times\frac{108}{100}=£648$
1.1.2018	$(11{,}000+600)\times\frac{113}{100}=£13{,}108$

Note that in practice the operation of an index-linked security will be such that the cashflows do not relate to the inflation index at the time of payment, due to delays in calculating the index. It is also possible that the need of the borrower (or perhaps the investors) to know the amounts of the payments in advance may lead to the use of an index from an earlier period.

Question

An investor purchased a two-year index-linked security on 1.1.2016. In return, the investor received payments at the end of each year plus a final redemption payment, all of which were increased in line with the index given in the table above, with a one-year indexation lag, *ie* the index value one year before each payment is used. The payments would have been £600 each year and £11,000 on redemption if there had been no inflation.

Calculate the payments actually received by the investor.

Solution

The payments received by the investor are calculated in the table below:

Payment date	Payment amount
1.1.2017	$600\times\frac{105}{100}=£630$
1.1.2018	$(11{,}000+600)\times\frac{108}{100}=£12{,}528$

The one-year indexation lag means that the payment on 1 January 2017 is calculated using the index values on 1 January 2015 (one year before the date of issue of the bond) and on 1 January 2016 (one year before the date of payment).

2.4 Cash on deposit

If cash is placed on deposit, the investor can choose when to disinvest and will receive interest additions during the period of investment. The interest additions will be subject to regular change as determined by the investment provider. These additions may only be known on a day-to-day basis. The amounts and timing of cashflows will therefore be unknown.

This is describing a bank account that pays interest and allows instant access. Consider your own bank account. You can choose when to invest money, *ie* pay money in, and when to disinvest money, *ie* withdraw money. The interest you receive on your money will depend on the current interest rate and this may change with little or no notice.

2.5 An equity

Equity shares (also known as *shares* or *equities* in the UK and as *common stock* in the USA) are securities that are held by the owners of an organisation. Equity shareholders own the company that issued the shares. For example, if a company issues 4,000 shares and an investor buys 1,000, the investor owns 25% of the company. In a small company all the equity shares may be held by a few individuals or institutions. In a large organisation there may be many thousands of shareholders.

Equity shares do not earn a fixed rate of interest as fixed-interest securities do. Instead the shareholders are entitled to a share in the company's profits, in proportion to the number of shares owned.

The distribution of profits to shareholders takes the form of regular payments of *dividends*. Since they are related to the company profits that are not known in advance, dividend rates are variable. It is expected that company profits will increase over time, and also, therefore, expected that dividends per share will increase – though there are likely to be fluctuations. This means that, in order to construct a cashflow schedule for an equity, it is necessary first to make an assumption about the growth of future dividends. It also means that the entries in the cashflow schedule are uncertain – they are estimates rather than known quantities.

In practice the relationship between dividends and profits is not a simple one. Companies will, from time to time, need to hold back some profits to provide funds for new projects or expansion. They may also hold back profits in good years to subsidise dividends in years with poorer profits. Additionally, companies may be able to distribute profits in a manner other than dividends, such as by buying back the shares issued to some investors.

Share buy-backs will result in some investors having to sell their shares back to the company. The remaining shareholders will subsequently own a greater percentage of the company and should expect greater future profits.

The following table shows the projected future cashflows for a shareholder who has just purchased a block of shares for £6,000 and expects the dividends in each year to be 5% higher than the corresponding amounts in the previous year. Dividends are paid twice yearly. This shareholder expects the two dividends in the first year to be £100 each and intends to sell all the shares after 2 years.

In the table, time is measured from the date of purchase.

Time (years)	Purchase price (£)	Dividends (£)	Sale proceeds (£)
0	– 6,000		
½		+100	
1		+100	
1½		+105	
2		+105	+6,615

In this example we have assumed that the price also grows at 5% *pa* to calculate the sale proceeds.

Since equities do not have a fixed redemption date, but can be held in perpetuity, we may assume that dividends continue indefinitely (unless the investor sells the shares or the company buys them back), but it is important to bear in mind the risk that the company will fail, in which case the dividend income will cease and the shareholders would only be entitled to any assets which remain after creditors are paid. The future positive cashflows for the investor are therefore uncertain in amount and may even be lower, in total, than the initial negative cashflow.

'In perpetuity' means that the payments continue forever.

2.6 An annuity-certain

An *annuity-certain* provides a series of regular payments in return for a single premium (*ie* a lump sum) paid at the outset. The precise conditions under which the annuity payments will be made will be clearly specified. In particular, the number of years for which the annuity is payable, and the frequency of payment, will be specified. Also, the payment amounts may be level or might be specified to vary – for example in line with an inflation index, or at a constant rate.

The cashflows for the investor will be an initial negative cashflow followed by a series of smaller regular positive cashflows throughout the specified term of payment. In the case of level annuity payments, the cashflows are similar to those for a fixed-interest security.

However there will not be a redemption payment as there normally is for a fixed-interest security.

From the perspective of the annuity provider, there is an initial positive cashflow followed by a known number of regular negative cashflows.

A key characteristic of an annuity-certain is that a known number of payments will be made, *eg* payments of £1,000 at the end of each of the next 10 years.

The theory can be extended to deal with annuities where the payment term is uncertain, that is, for which payments are made only so long as the annuity policyholder survives – see Section 3.4 below.

An example of this is a *whole life annuity* where a policyholder receives a fixed amount of money per month until they die. In more common language, this type of policy is called a pension.

2.7 An 'interest-only' loan

An 'interest-only' loan is a loan that is repayable by a series of interest payments followed by a return of the initial loan amount.

The regular repayments only cover the interest owed, so the full capital amount borrowed remains outstanding throughout the term of the loan.

In the simplest of cases, the cashflows are the reverse of those for a fixed-interest security. The provider of the loan effectively buys a fixed-interest security from the borrower.

In practice, however, the interest rate need not be fixed in advance. The regular cashflows may therefore be of unknown amounts.

It may also be possible for the loan to be repaid early. The number of cashflows and the timing of the final cashflows may therefore be uncertain.

2.8 A repayment loan (or mortgage)

A repayment loan is a loan that is repayable by a series of payments that include partial repayment of the loan capital in addition to the interest payments.

In its simplest form, the interest rate will be fixed and the payments will be of fixed equal amounts, paid at regular known times.

The cashflows are similar to those for an annuity-certain.

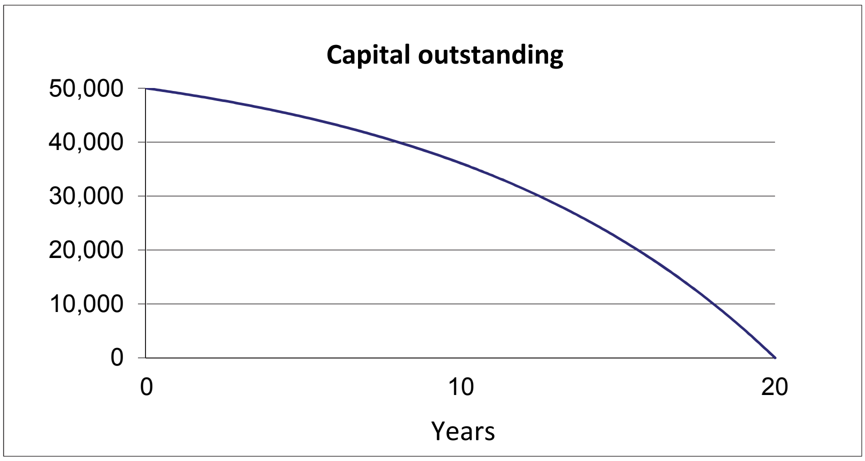
As for the 'interest-only' loan, complications may be added by allowing the interest rate to vary or the loan to be repaid early. Additionally, it is possible that the regular repayments could be specified to increase (or decrease) with time. Such changes could be smooth or discrete.

Each payment can be considered to consist of an interest payment and a capital repayment. The interest payment covers the interest that will be charged over the period since the previous payment. The interest payment will be calculated by reference to the amount of the loan outstanding just after the last payment. The remainder of each payment is the capital repayment, which is used to reduce the amount of the loan outstanding.

It is important to appreciate that with a repayment loan the breakdown of each payment into 'interest' and 'capital' changes significantly over the period of the loan. The first repayment will consist almost entirely of interest and will provide only a very small capital repayment. In contrast, the final repayment will consist almost entirely of capital and will have a small interest content.

This is because the amount of the loan outstanding will reduce throughout the term of the loan. At the start of the contract, the entire loan will be outstanding and so the interest portion of each payment will be large. The remainder, the capital portion, will therefore be relatively small. At the end of the contract, the amount of the loan outstanding will be small and so the interest due will also be small. The capital repayment will then be much larger.

The diagram below shows how the capital outstanding reduces for a repayment loan – relatively slowly at first, but then more quickly towards the end of the term of the loan. The graph is based on a loan of £50,000 being repaid by monthly instalments over 20 years.



We'll look at numerical examples of repayment loans in Chapter 11.

3 Insurance contracts

In the previous section, we looked at the cashflows arising from different financial securities. We now turn our attention to the cashflows arising from different policies sold by insurance companies.

The cashflows for the examples covered in this section differ from those in the previous section in that the frequency, severity, and/or timing of the cashflows may be unknown. For example, a typical life cover may have a specified date on which a pre-agreed amount is paid on survival (Section 3.1) – but the benefit payment may not be paid if the individual does not survive. Similarly a pension pays out a known amount at a specified time per month, but only if the individual is alive (Section 3.4). Typically the severity is known and pre-specified in life insurance contracts.

Here, 'severity' is referring to the benefit or claim amount paid out by the insurer.

On the other hand, a non-life (general) insurance cover tends not to have known severities. For example, the cost of a car accident may range from a few pounds in the case of a small collision to millions in the case of a major accident that caused death.

A life insurer sells policies related to events that might be experienced by an individual in the course of their future life, *eg* policies may provide a payment if an individual survives, dies or falls sick. A general insurer sells policies related to specific items, *eg* car insurance, home insurance.

3.1 A pure endowment

A pure endowment is an insurance policy which provides a lump sum benefit on survival to the end of a specified term usually in return for a series of regular premiums.

For example, a payment of £50,000 is made if a life now aged 30 survives to age 60, but no payment is made if this life dies before age 60.

The cashflows for the policyholder will be a series of negative cashflows throughout the specified term or until death, if earlier. A large, positive cashflow occurs at the end of the term, only if the policyholder has survived. If the policyholder dies before the end of the term there is no positive cashflow.

From the perspective of the insurer, there is a stream of regular positive cashflows which cease at a specified point (or earlier, if the policyholder dies) followed by a large negative cashflow, contingent on policyholder survival.

The cashflows experienced by the insurer are the exact opposite to those experienced by the policyholder.

In some cases a lump sum premium is paid. In this case, the cashflows for the policyholder will be a negative cashflow at inception and a positive cashflow at the end of the term, only if the policyholder has survived.

In general, insurance policies can be paid for by either:

- a one-off, lump sum premium payable at outset, or
- regular premiums payable during the term of the contract, while the policyholder is alive.

3.2 An endowment assurance

An endowment assurance is similar to a pure endowment **in that it provides a survival benefit at the end of the term, but it also provides a lump sum benefit on death before the end of the term. The benefits are provided in return for a series of regular premiums.**

So, under an endowment assurance, a payment is made whether the policyholder dies during the term or survives to the end of the term.

The cashflows for the policyholder will be a series of negative cashflows throughout the specified term or until death, if earlier, followed by a large positive cashflow at the end of the term (or death, if earlier). Depending on the terms of the policy, the amount payable on death may not be the same as that payable on survival.

For example, the payment may be £10,000 if the policyholder dies during the term, and £20,000 if the policyholder survives to the end of the term.

From the perspective of the insurer, there is a stream of regular positive cashflows which cease at a specified point (or earlier, if the policyholder dies) followed by a large negative cashflow. The negative cashflow is certain to be paid, but the timing of that payment depends on whether/when the policyholder dies.

The proceeds from an endowment assurance could be used to repay the loan amount under an 'interest-only' loan, by ensuring that the payment made from the endowment assurance matches the loan amount. In this way, the loan will be repaid whether the policyholder dies during the term or survives to the end of the term.

3.3 A term assurance

A term assurance is an insurance policy which provides a lump sum benefit on death before the end of a specified term usually in return for a series of regular premiums.

The cashflows for the policyholder will be a series of negative cashflows throughout the specified term or until death, if earlier, (or one negative cashflow at inception if paid on a lump-sum basis), followed by a large positive cashflow payable on death, if death occurs before the end of the term. If the policyholder survives to the end of the term there is no positive cashflow.

If a policyholder purchases a term assurance and a pure endowment, this is equivalent to purchasing an endowment assurance (as a payment will be made from the term assurance if the policyholder dies, or from the pure endowment if the policyholder survives).

From the perspective of the insurer, there is a stream of regular positive cashflows which cease at a specified point (or earlier, if the policyholder dies) followed by a large negative cashflow, contingent on policyholder death during the term.

Generally, the negative cashflow (death benefit), if it occurs, is significantly higher than the positive cashflow (premiums), when compared to, say, a pure endowment. This is because, for each individual policy, the probability of the benefit being paid is generally lower than for endowments, because it is contingent on death, rather than on survival.

Question

Consider a 10-year term assurance and a 10-year pure endowment, both providing a lump sum benefit of £50,000.

Explain which policy would have higher premiums, if both policies are sold to an individual currently aged:

(a) 40

(b) 90.

Solution

The premiums charged under any insurance policy reflect the risk covered. So, the more likely the insurance company is to have to make a payment, the higher the premium will be.

(a) A 40-year-old is more likely to survive for the next 10 years than to die during the next 10 years. So the premiums charged for the pure endowment will be higher than those charged for the term assurance, as the payment on survival is more likely to be made than the payment on death.

(b) On the other hand, a 90-year-old is more likely to die during the next 10 years than to survive to age 100. So, in this case, the premiums charged for the term assurance will be higher than those charged for the pure endowment.

3.4 A contingent annuity

This is a similar contract to the annuity-certain (see Section 2.6 above) but the payments are contingent upon certain events, such as survival, hence the payment term for the regular cashflows (which will be negative from the perspective of the annuity provider) is uncertain.

As mentioned before, a pension payable after retirement is an example of a contingent annuity.

Typical examples of contingent annuities include:

- **a single life annuity – where the regular payments made to the annuitant are contingent on the survival of that annuitant**
- **a joint life annuity – which covers two lives, where the regular payments are contingent on the survival of one or both of those lives**
- **a reversionary annuity – which is based on two lives, where the regular payments start on the death of the first life if, and only if, the second life is alive at the time. Payments then continue until the death of the second life.**

A reversionary annuity is one which would be paid, for example, to a wife after her husband has died, or to a husband after his wife has died. It provides the surviving partner with an income, and hence some financial security, for their remaining lifetime. In the context of pensions, a reversionary annuity is sometimes called a spouse's pension or a dependant's pension.

3.5 A car insurance policy

A typical car insurance contract lasts for one year. In return for a premium which can be paid as a single lump sum or at monthly intervals, the insurer will provide cover to pay for damage to the insured vehicle or fire or theft of the vehicle, known as 'property cover'. In many countries, such as the UK, the contract also provides cover for compensation payable to third parties for death, injury or damage to their property, known as 'liability cover'.

So, property cover relates to damaged caused *to* the car, and liability cover relates to damaged caused *by* the car.

Depending on the terms of the policy, the insurance company may settle claims directly with the policyholder or with another party. For example, in the case of theft or total loss, the insurance company may pay a lump sum to the policyholder in lieu of that loss. In the case of damage to the insured vehicle, the insurance company may settle the claim directly with the party undertaking the repairs without involving the policyholder. In the case of third party liability claims, the insurance company may settle the claims directly with the third party.

The 'third party' here is the person whose car or property has been damaged by the insured vehicle.

In some cases, the policyholder may be required to cover the cost of damage or repairs first before the insurance company settles the claim, in which case the insurance company will pay the policyholder directly.

The cashflows for the policyholder will usually be a single negative cashflow at the beginning of the year. Further cashflows only take place in the event of a claim. If the policyholder has to pay for repairs or compensation, this will incur a further negative cashflow, followed by a positive cashflow when the insurance company settles the claim. If the insurance company settles the claim directly with the repair company or third party, the policyholder may not experience further cashflows.

From the insurer's perspective, there will be a positive cashflow at the beginning of the policy, followed by a negative cashflow when the claim is settled.

The timing of the cashflows will depend on how long the claim takes to be reported and settled. Typically property claims take less time to settle than liability claims. Where liability claims involve disputes, for example necessitating court judgements, they can take years to settle and the amounts are less certain.

For example, it may be difficult to establish precisely which vehicle was responsible for a major car accident.

Cashflows tend to be short-term and are payable within the year.

3.6 A health cash plan

A typical health insurance contract lasts for one year. In return for a premium, the policyholder is entitled to benefits which may include hospital treatment either paid for in full or in part, and/or cash benefits in lieu of treatment, such as a fixed sum per day spent in hospital as an in-patient.

Typical benefits provided by a health cash plan might include:

- £150 towards any dentist or optician fees
- £75 per day or overnight stay in hospital up to a maximum of 25 stays
- £350 towards the cost of a specialist consultation or procedure, such as an MRI scan.

From the policyholder's perspective, the cashflows will include a negative cashflow at the beginning of the year followed by positive cashflows in the event of a claim in the case of a cash benefit. Where the insurance company pays for hospital treatment directly, the policyholder may experience no more cashflows after paying the initial premium.

From the perspective of the insurer, there will be an initial positive cashflow at the start of the policy followed by negative cashflows in the event of a claim, when those claims are settled.

Cashflows tend to be short-term and are payable within the year.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 3 Summary

Cashflows are sums of money that are paid or received at particular times.

Where there is uncertainty about the amount or timing of cashflows, an actuary can assign probabilities to both the amount and the existence of a cashflow.

A *zero-coupon bond* is a security that provides a specified lump sum at a specified future date and no other positive cashflows.

The holder of a *fixed-interest security* will receive a lump sum of specified amount at a specified future time, together with a series of regular level interest payments until the repayment of the lump sum.

With *cash on deposit* the amounts and timing of cashflows will usually be unknown.

Equity shares are securities that are held by the owners of an organisation. Shareholders are entitled to a share in the company's profits in proportion to the number of shares owned.

An *annuity-certain* provides a series of regular payments for a fixed term in return for a single premium.

An *interest-only loan* is a loan that is repayable by a series of interest payments followed by a return of the initial loan amount.

A *repayment loan* is a loan that is repayable by a series of payments, each including a partial repayment of the loan capital in addition to the interest payments.

A *pure endowment* is an insurance contract that provides a lump sum benefit on survival to the end of a specified term.

A *term assurance* is an insurance contract that provides a lump sum benefit on death during a specified term.

An *endowment assurance* is an insurance contract that provides a lump sum benefit on death during a specified term, or on survival to the end of that term.

A *contingent annuity* provides a series of regular payments, where the payments are contingent (*ie* dependent) upon certain events, such as survival.

A *car insurance policy* usually lasts for one year, and provides payments to cover any damage to the insured vehicle (property cover) or any damage caused by the insured vehicle (liability cover).

A *health cash plan* usually lasts for one year, and provides payments to cover medical expenses, such as hospital treatment.

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 3 Practice Questions

3.1 Complete the table below using the symbols: ✓ (= yes) or × (= no).

Security	Absolute amount of payments known in advance?	Timing of payments known in advance?
Zero-coupon bond		
Fixed-interest security		
Index-linked security		
Cash on deposit		
Equity		

3.2 Describe the characteristics of:

Exam style

(a) an interest-only loan (or mortgage); and

(b) a repayment loan (or mortgage). [4]

3.3 Complete the table below using the symbols: ✓ (= yes) or × (= no).

Insurance contract	Absolute amount of payments made by insurer known in advance?	Timing of payments made by insurer known in advance?
Pure endowment		
Term assurance		
Endowment assurance		
Car insurance		
Health cash plan		

3.4 Outline the similarities and differences between an annuity-certain and a contingent annuity. [3]

Exam style

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 3 Solutions

3.1 The completed table is as follows:

Contract	Absolute amount of payments known in advance?	Timing of payments known in advance?
Zero-coupon bond	✓	✓
Fixed-interest security	✓	✓
Index-linked security	×	✓
Cash on deposit	×	×
Equity	×	×

3.2 *This question is Subject CT1, April 2009, Question 2.*

(a) ***Interest-only loan***

An interest-only loan is a loan repayable by a series of interest payments during the term of the loan, followed by repayment of the full capital amount at the end of the loan. The amount of capital outstanding therefore remains fixed over the term of the loan. [1]

If the interest rate is fixed, the amount of each interest cashflow is known in advance. If the interest rate is variable, the interest payments will be unknown at outset. [1]

(b) ***Repayment loan***

A repayment loan is a loan repayable by a series of payments, each of which includes partial repayment of the loan capital in addition to interest. This means the amount of capital outstanding reduces over the term of the loan. [1]

If the interest rate is fixed over the term of the loan, the repayments will all be for fixed amounts. If the interest rate varies, so will the repayment amounts. [1]

[Total 4]

3.3 The completed table is as follows:

Insurance contract	Absolute amount of payments made by insurer known in advance?	Timing of payments made by insurer known in advance?
Pure endowment	✓	✓
Term assurance	✓	×
Endowment assurance	✓	×
Car insurance	×	×
Health cash plan	×	×

3.4 Both an annuity-certain and a contingent annuity provide a regular series of payments in return for a single premium (*ie* a lump sum) paid at the outset. [1]

For both contracts, the frequency of payments will be specified, as will the payment amount, *eg* payments could be level, increasing at a constant rate, or increasing in line with an inflation index. [1]

The difference between these contracts is that under an annuity-certain the number of annuity payments is fixed in advance, whereas for a contingent annuity the payments are dependent upon certain events, usually the survival of the policyholder. So for a contingent annuity the number of payments made will not be known in advance. [1]

[Total 3]

4

The time value of money

Syllabus objectives

2.1 Show how interest rates may be expressed in different time periods.

2.1.1 Describe the relationship between the rates of interest and discount over one effective period arithmetically and by general reasoning.

2.3 Describe how to take into account the time value of money using the concepts of compound interest and discounting.

2.3.1 Accumulate a single investment at a constant rate of interest under the operation of simple and compound interest.

2.3.2 Define the present value of a future payment.

2.3.3 Discount a single investment under the operation of a simple (commercial) discount at a constant rate of discount.

0 Introduction

***Interest* may be regarded as a reward paid by one person or organisation (the *borrower*) for the use of an asset, referred to as *capital*, belonging to another person or organisation (the *lender*).**

In return for the use of the investor's capital, the borrower will be expected to pay interest to the lender. For example:

- a bank will be expected to pay interest to its customers on money held in their savings accounts
- a company will be expected to pay interest to a bank on money lent to it for a business project.

In this chapter we will look at the basic ideas underlying the theory of interest rates. Interest rates are a fundamental part of actuarial work.

Question

List some common situations in which a bank will act as:

(i) a lender

(ii) a borrower.

Solution

(i) A bank acts as a lender when:

- it offers a mortgage to a person wanting to buy a house
- it makes a business loan to a company
- it buys fixed-interest securities.

(ii) A bank acts as a borrower when:

- it accepts money from savers
- it issues shares (*ie* the bank's own shares) to investors
- it sells fixed interest securities.

When the capital and interest are expressed in monetary terms, capital is also referred to as *principal*. The total received by the lender after a period of time is called the *accumulated value*. The difference between the principal and the accumulated value is called the *interest*. Note that we are assuming here that no other payments are made or incurred (*eg* charges, expenses).

For example, consider the situation where a bank lends an individual £1,000, and in return the individual has to pay £1,200 to the bank in 1 year's time. The *capital* (or *principal*) is the £1,000 lent to the individual. The *accumulated value* is the £1,200 paid back. Since the individual has to pay back £200 more than was borrowed, this is the *interest*.

The above example is a very simple case where there is just one payment. Things will get messier when payments are more frequent.

If there is some risk of default (*ie* loss of capital or non-payment of interest) a lender would expect to be paid a higher rate of interest than would otherwise be the case.

If an investor lends money both to the US government (by purchasing a fixed-interest security) and to a property developer, the investor will probably demand a higher rate of interest from the property developer. This is because the property developer is more likely not to repay the loan or not to pay the interest due on the loan.

Another factor which may influence the rate of interest on any transaction is an allowance for the possible depreciation or appreciation in the value of the currency in which the transaction is carried out. This factor is very important in times of high inflation.

If the rate of inflation over the period of a loan is expected to be high, then the lender may demand a higher rate of interest to compensate for the lower *real* value of the capital when it is returned at the end of the loan period. We will consider real interest rates in more detail later in the course.

We will now consider two types of interest within the framework of a savings account.

1 Interest

1.1 Simple interest

The essential feature of *simple interest* is that interest, once credited to an account, does not itself earn further interest.

Suppose an amount C is deposited in an account that pays simple interest at the rate of $i \times 100\%$ per annum. Then after n years the deposit will have accumulated to:

$$C(1+ni) \tag{1.1}$$

Question

An investor deposits £10,000 in a bank account that pays simple interest at a rate of 5% *pa*. Calculate the accumulated value of the deposit after 3 years.

Solution

After 1 year, the investor will have earned interest of $0.05 \times £10,000 = £500$.

Whilst the investor's bank balance is now £10,500, we know that interest is not earned on interest. So in the second year, the investor will again earn interest of $0.05 \times £10,000 = £500$.

There is now £11,000 in the bank, and in the third year the investor will again earn interest of $0.05 \times £10,000 = £500$. So at the end of the third year the investor's deposit will have accumulated to £11,500.

Essentially the investor is only earning interest on the original capital of £10,000. Since this is the same every year, the total interest after 3 years is $3 \times 0.05 \times £10,000 = £1,500$. So the final accumulated value is the original capital plus the interest earned:

$$£10,000 + 3 \times 0.05 \times £10,000 = £11,500$$

Taking out the common factor of £10,000, we have:

$$£10,000(1+3 \times 0.05) = £11,500$$

This figure can also be obtained using the formula $C(1+ni)$.

When n is not an integer, interest is paid on a pro-rata basis.

This is normal commercial practice so equation (1.1) could be applied to an accumulation over 3.6 years, say.

However, the situation where further interest is not earned on earlier interest can lead to problems, as illustrated in the following question.

Question

An investor deposits £5,000 into a savings account that pays 10% simple interest at the end of each year. Compare how much the investor would have after 6 years if the money were:

(i) invested for 6 years

(ii) invested for 3 years, then immediately reinvested for a further 3 years.

Solution

(i) At the end of 6 years, the investor will have $5{,}000(1+6\times 0.1)=£8{,}000$.

(ii) At the end of 3 years, the investor will have $5{,}000(1+3\times 0.1)=£6{,}500$.

At the end of 6 years, the investor will have $6{,}500(1+3\times 0.1)=£8{,}450$.

So under a simple interest arrangement, an investor can earn more interest by withdrawing the capital and then immediately reinvesting it.

The situation illustrated above is undesirable, since it encourages unnecessary additional transactions. Hence we will now consider the case where interest does itself earn interest. This is called *compound interest*.

1.2 Compound (effective) interest

The essential feature of compound interest is that interest itself earns interest.

Suppose an amount C is deposited in an account that pays compound interest at the rate of $i\times 100\%$ per annum. Then after n years the deposit will have accumulated to:

$$C(1+i)^n \qquad \textbf{(1.2)}$$

Question

An investor deposits £10,000 in a bank account that pays compound interest at a rate of 5% *pa*. Calculate the accumulated value of the deposit after 3 years.

Solution

After 1 year, the investor will earn interest of $0.05\times £10{,}000=£500$. So there will be £10,500 in the bank.

In the second year, the investor will earn interest of $0.05\times £10{,}500=£525$. So there will then be £11,025 in the bank.

In the third year, the investor will earn interest of $0.05\times £11{,}025=£551.25$. So at the end of the third year the investor's deposit will have accumulated to £11,576.25.

Every year interest of 5% is added to the balance – so the new balance is 105% of the old balance, which we can calculate by multiplying by 1.05. So after 3 years the accumulation is:

$$£10,000 \times 1.05 \times 1.05 \times 1.05 = £11,576.25$$

This is the same as:

$$£10,000 \times 1.05^3 = £11,576.25$$

which is given directly by the formula $C(1+i)^n$.

Unlike simple interest, with compound interest, an investor cannot earn more interest by withdrawing the capital and then immediately reinvesting it.

Question

An investor deposits £5,000 into a savings account that pays 10% compound interest at the end of each year. Compare how much the investor would have after 6 years if the money were:

(i) invested for 6 years

(ii) invested for 3 years, then immediately reinvested for a further 3 years.

Solution

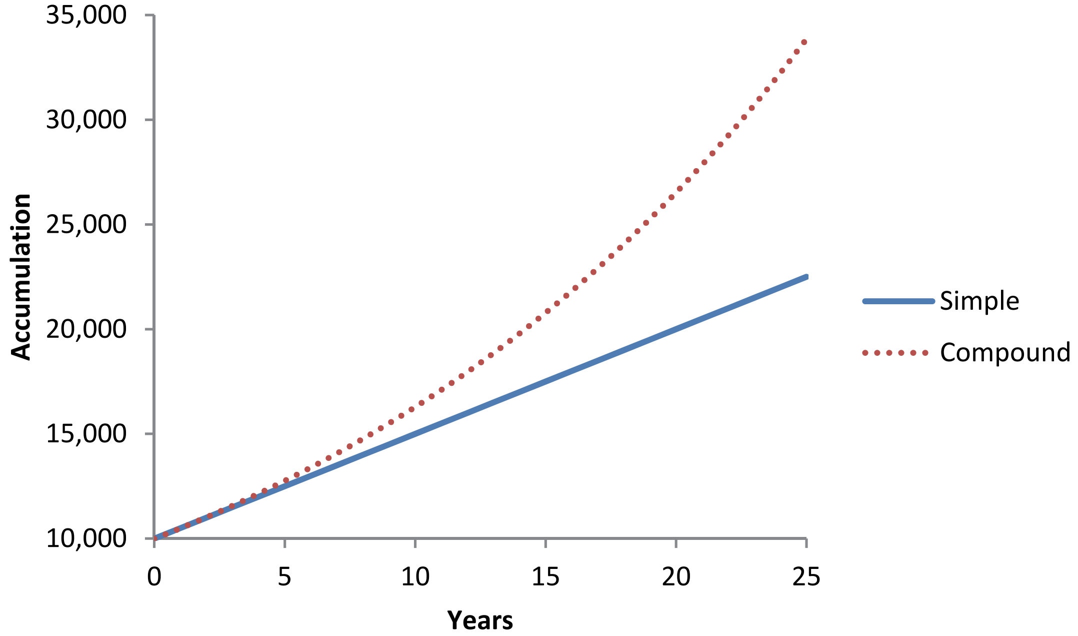
(i) At the end of 6 years the investor will have $5,000 \times 1.1^6 = £8,858$.

(ii) At the end of 3 years the investor has $5,000 \times 1.1^3 = £6,655$.

At the end of 6 years the investor will have $6,655 \times 1.1^3 = £8,858$.

The accumulated value is the same with and without the reinvestment.

The following graph shows how £10,000 invested in a savings account would grow over the next few years if the account paid either 5% *pa* simple interest (solid line) or 5% *pa* compound interest (dotted line):



We see that the simple interest account produces a straight-line graph, whereas the graph of the compound interest account is exponentially shaped.

1.3 Accumulation factors

For $t_1 < t_2$ we define $A(t_1,t_2)$ to be the accumulation at time t_2 of an investment of 1 at time t_1.

The number $A(t_1,t_2)$ is often called an *accumulation factor*, since the accumulation at time t_2 of an investment of C at time t_1 is, by proportion:

$$CA(t_1,t_2) \tag{1.3}$$

$A(n)$ is often used as an abbreviation for the accumulation factor $A(0,n)$.

For example, if an investor deposits £10,000 in a bank account that pays simple interest at a rate of 5% *pa*, after 3 years the accumulated value of the deposit will be £11,500. So the accumulation factor from time 0 to time 3 is:

$$A(0,3) = \frac{11,500}{10,000} = 1.15$$

So here the accumulation factor is of the form $1+ni$, as in equation (1.1).

In summary, using equation (1.1) the accumulation factor for simple interest is:

$$A(n) = A(0,n) = 1 + ni$$

and using equation (1.2) the accumulation factor for compound interest is:

$$A(n) = A(0,n) = (1+i)^n$$

Question

An investment of £1,000 accumulates to £2,500 after 5 years.

(i) Calculate the accumulation factor $A(0,5)$.

(ii) (a) Calculate the simple annual interest rate that would give the accumulation factor in part (i).

(b) Calculate the annual compound interest rate that would give the accumulation factor in part (i).

Solution

(i) $A(0,5) = \dfrac{2,500}{1,000} = 2.5$

(ii)(a) $A(0,5) = (1+5i) = 2.5 \quad \Rightarrow \quad i = \dfrac{2.5-1}{5} = 30\%$

(ii)(b) $A(0,5) = (1+i)^5 = 2.5 \quad \Rightarrow \quad i = (2.5)^{1/5} - 1 = 20.1\%$

1.4 The principle of consistency

Now let $t_0 \le t_1 \le t_2$ and consider an investment of 1 at time t_0. The proceeds at time t_2 will be $A(t_0,t_2)$ if one invests at time t_0 for term $t_2 - t_0$, or $A(t_0,t_1)A(t_1,t_2)$ if one invests at time t_0 for term $t_1 - t_0$ and then, at time t_1, reinvests the proceeds for term $t_2 - t_1$. In a consistent market these proceeds should not depend on the course of action taken by the investor. Accordingly, we say that under the *principle of consistency*:

$$A(t_0,t_n) = A(t_0,t_1)A(t_1,t_2)\ldots A(t_{n-1},t_n) \qquad (1.4)$$

If we invest £1,000 for 8 years in a bank account that pays compound interest of 10% *pa*, the balance at the end of this period would be $£1,000A(0,8) = £1,000 \times 1.1^8 = £2,143.59$.

Similarly, the balance after 3 years would be $£1,000A(0,3) = £1,000 \times 1.1^3 = £1,331.00$. If we then reinvest this in the same account for a further 5 years, the balance would be $£1,331.00A(3,8) = £1,331.00 \times 1.1^5 = £2,143.59$, as before.

Hence $A(0,8) = A(0,3)A(3,8)$.

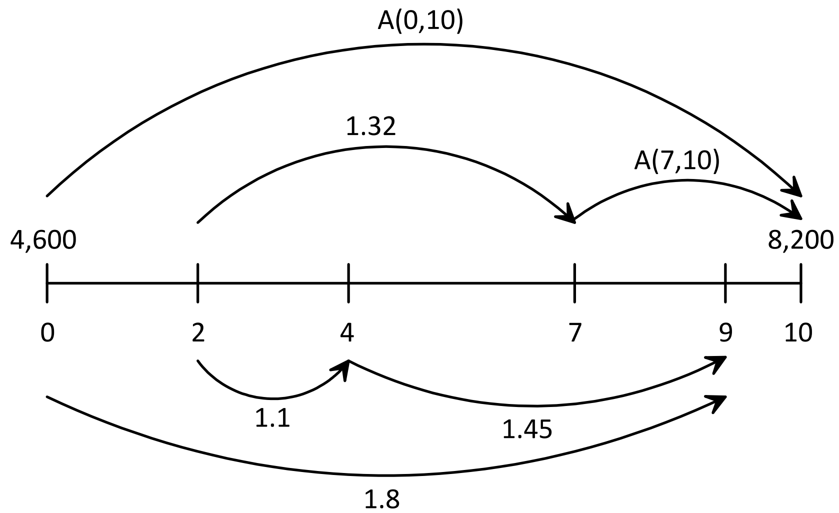
Question

£4,600 is invested at time 0 and the proceeds at time 10 are £8,200.

Calculate $A(7,10)$ if $A(0,9)=1.8$, $A(2,4)=1.1$, $A(2,7)=1.32$, $A(4,9)=1.45$.

Solution

Representing the initial investment, proceeds, and accumulation factors on a diagram, we have:



Since £4,600 accumulates to £8,200 in 10 years, we know:

$$A(0,10)=\frac{8,200}{4,600}=1.7826$$

Using the principle of consistency:

$$A(0,10)=1.7826=A(0,2)A(2,7)A(7,10) \quad \Rightarrow \quad A(7,10)=\frac{1.7826}{A(0,2)\times 1.32}$$

where $A(2,7)=1.32$. We can find $A(0,2)$ using the principle of consistency again:

$$A(0,9)=A(0,2)A(2,4)A(4,9) \quad \Rightarrow \quad A(0,2)=\frac{A(0,9)}{A(2,4)A(4,9)}=\frac{1.8}{1.1\times 1.45}=1.1285$$

So:

$$A(7,10)=\frac{1.7826}{1.1285\times 1.32}=1.1967$$

2 Present values

In the previous section we saw how to answer the question: how much will a single payment accumulate to at a later time? In actuarial work, we are usually aiming to make payments at certain future dates, *eg* making pension payments to a worker after retirement or making a life assurance payment when an individual dies. So actuaries are usually more interested in answering the question: how much do we need to invest now to provide payments at a later time? This amount is called the *present value* (PV) or *discounted value* of the payments.

Question

An investor must make a payment of £5,000 in 5 years' time. The investor wishes to make provision for this payment by investing a single sum now in a deposit account that pays 10% per annum compound interest.

Calculate the initial investment required to meet the payment of £5,000 in 5 years' time.

Solution

By the end of 5 years an initial payment of $£X$ will accumulate to $£X \times 1.1^5$.

So:

$$X \times 1.1^5 = 5{,}000 \quad \Rightarrow \quad X = \frac{5{,}000}{1.1^5} = £3{,}104.61$$

It follows by formula (1.2) that an investment of:

$$C/(1+i)^n \qquad \textbf{(2.1)}$$

at time 0 (the present time) will give C at time $n \geq 0$.

This is called the *discounted present value* (or, more briefly, the *present value*) of C due at time $n \geq 0$.

It is the amount that needs to be invested at time 0 at compound interest rate i to accumulate to C at time n.

We can define the function:

$$v = \frac{1}{1+i} \qquad \textbf{(2.2)}$$

It follows by formulae (2.1) and (2.2) that the discounted present value of C due at time $n \geq 0$ is:

$$Cv^n \qquad \textbf{(2.3)}$$

Using this notation, we could have found the answer to the previous question by first calculating:

$$v = \frac{1}{1+i} = \frac{1}{1.1} = 0.9090909$$

and then the amount of the initial investment required, *ie* the present value of £5,000 due at time 5 years, is:

$$\Rightarrow \quad PV = 5{,}000v^5 = 5{,}000 \times 0.9090909^5 = £3{,}104.61$$

Although the above approach is correct, it's quicker to cut out the middle step of calculating the value of v , and instead work with negative powers:

$$PV = 5{,}000v^5 = 5{,}000 \times 1.1^{-5} = £3{,}104.61$$

Formulae and tables for actuarial examinations

Values of v^n at various interest rates are tabulated in 'Formulae and Tables for Examinations', which are provided in the exams. They are available to purchase from the Institute and Faculty of Actuaries. From now on we will refer to this book as simply the *Tables*.

Question

Calculate v, assuming an annual compound rate of interest of 4%.

Solution

Using the definition of v :

$$v = \frac{1}{1+i} = \frac{1}{1.04} = 0.96154$$

Alternatively, we can find this on page 56 of the *Tables*, either at the top of the v^n column or among the 'constants' listed on the left-hand side.

Note on rounding

How to round your answers is an important factor that you need to consider in your calculations. The rounding used should be appropriate for the level of accuracy used in your calculations. For example, if you are using numbers from the *Tables* then you shouldn't quote an answer to more significant figures than given in the *Tables*. The number of significant figures that you quote is usually more important than the number of decimal places. When using your calculator you can keep the accurate values of intermediate calculations in your memories. Your final answer will then be more accurate. In an exam you need not write down the fully accurate intermediate figures.

Remember that 104.27 is rounded to 5 significant figures, but 2 decimal places!

3 Discount rates

In Section 1, we were given an interest rate, which we used to obtain the appropriate accumulation factor. Multiplying the amount invested by the accumulation factor gives the accumulated value of the investment. For example, with compound interest rate i the *accumulation factor* is $(1+i)^n$.

In Section 2, we looked at discounting – that is, given the accumulated value, how much was the initial investment? We also developed the compound *discount factor* v^n. Multiplying the accumulated value at time n by the discount factor gives the present value of the investment.

We now want to calculate the corresponding compound *discount rate* that goes with this discount factor. In addition, we'll also need to obtain *simple* discount rates and factors.

An alternative way of obtaining the discounted value of a payment is to use discount rates.

3.1 Simple discount

As has been seen with simple interest, the interest earned is not itself subject to further interest. The same is true of simple discount, which is defined below.

Suppose an amount C is due after n years and a rate of *simple discount* of d per annum applies. Then the sum of money required to be invested now to amount to C after n years (*ie* the present value of C) is:

$$C(1-nd) \qquad (3.1)$$

Question

Calculate the present value of £10,000 due at time 3 years, using a simple discount rate of 5% *pa*.

Solution

After 1 year the discount will be $0.05\times £10,000 = £500$. This amount is deducted from the payment of £10,000, leaving £9,500.

Whilst we've now got £9,500 left, we know that simple discount does not discount the discount. So in the second year the discount will again be $0.05\times £10,000 = £500$.

We've now got £9,000 left. In the third year the discount will again be $0.05\times £10,000 = £500$. So at the end of the third year we will have £8,500.

Essentially we are only discounting the original £10,000. Since this is the same every year, the total discount after 3 years is $3\times 0.05\times £10,000 = £1,500$. So the final discounted value is the original amount minus the discount:

$$£10,000 - 3\times 0.05\times £10,000 = £8,500$$

Taking out the common factor of £10,000, we have:

$$£10,000(1-3\times 0.05)=£8,500$$

This is the present value of £10,000 due in 3 years and can be obtained directly using the formula $C(1-nd)$.

In normal commercial practice, d is usually encountered only for periods of less than a year. If a lender bases his short-term transactions on a simple rate of discount d then, in return for a repayment of X after a period t $(t<1)$ he will lend $X(1-td)$ at the start of the period. In this situation, d is also known as a rate of *commercial discount*.

Simple discount is the rate quoted for treasury bills, which are short-term loans made by the government. Rather than quoting the amount it wishes to borrow, the government quotes the amount of the repayment. The rate of simple discount is then used to calculate the purchase price of the treasury bill, *ie* the amount actually lent to the government.

Question

An 8-month loan is repayable by a single payment of £100,000. If the loan is issued at a rate of commercial discount of 15% *pa*, calculate how much is initially lent to the borrower.

Solution

The amount lent is:

$$£100,000\left(1-\frac{8}{12}\times 0.15\right)=£90,000$$

Since 15% is the annual rate of discount, we use $\frac{8}{12}$ as the length of time, ie 8 months expressed in years.

3.2 Compound (effective) discount

As has been seen with compound interest, the interest earned is subject to further interest. The same is true of compound discount, which is defined below.

Suppose an amount C is due after n years and a rate of compound (or effective) discount of d per annum applies. Then the sum of money required to be invested now to accumulate to C after n years (*ie* the present value of C) is:

$$C(1-d)^n \tag{3.2}$$

Question

Calculate the present value of £10,000 due at time 3 years, using a compound discount rate of 5% *pa*.

Solution

After 1 year, the discount will be $0.05 \times £10,000 = £500$. This amount is deducted from the payment of £10,000, leaving £9,500.

In the second year, we discount the discounted amount, so the discount will be $0.05 \times £9,500 = £475$. So we will now have £9,025 left.

In the third year, the discount will be $0.05 \times £9,025 = £451.25$. So after discounting for the third year, there will be £8,573.75 left.

Every year a 5% discount is subtracted from the amount – so the new amount is 95% of the old amount, which we can calculate by multiplying by 0.95. So after 3 years the final discounted value is:

$$£10,000 \times 0.95 \times 0.95 \times 0.95 = £8,573.75$$

This can be written as:

$$£10,000 \times 0.95^3 = £8,573.75$$

This is the present value of £10,000 due in 3 years and can be obtained directly using the formula $C(1-d)^n$.

3.3 Discount factors

In the same way that the accumulation factor $A(n)$ gives the accumulation at time n of an investment of 1 at time 0, we define $v(n)$ to be the present value of a payment of 1 due at time n. Hence:

$$\boldsymbol{v(n) = \frac{1}{A(n)}} \qquad \textbf{(3.3)}$$

So using equation (3.2) the discount factor for compound discount is:

$$v(n) = (1-d)^n$$

However, using equation (3.3) and equation (1.2), we could also give a discount factor in terms of compound interest:

$$v(n) = \frac{1}{A(n)} = \frac{1}{(1+i)^n} \qquad (3.4)$$

Using the definition of v from (2.2) in (3.4) we get:

$$v(n) = \frac{1}{A(n)} = \frac{1}{(1+i)^n} = v^n$$

This is the discount factor that we used in (2.3) to calculate the present value.

We could also use (3.3) and (3.2) to give an *accumulation* factor in terms of compound discount:

$$A(n)=\frac{1}{v(n)}=\frac{1}{(1-d)^n}=(1-d)^{-n}$$

So regardless of whether we're given interest or discount rates, we can calculate both accumulations and present values. Of course this does mean that we have to read questions carefully to see what rates we are given and what we are asked to do with them!

Question

(i) Given an investment of €1,000, calculate the accumulation after 5 years using:

(a) simple discount of 8% *pa*

(b) compound discount of 8% *pa*

(c) compound interest of 8% *pa.*

(ii) A payment of €2,000 is due in 4 years' time. Calculate the present value using:

(a) simple interest of 3% *pa*

(b) simple discount of 3% *pa*

(c) compound interest of 3% *pa*.

Solution

(i)(a) Using (3.3) and (3.1), the accumulation is:

$$1{,}000A(5)=1{,}000\times\frac{1}{v(5)}=1{,}000\times\frac{1}{1-5\times 0.08}=€1{,}666.67$$

(i)(b) Using (3.3) and (3.2), the accumulation is:

$$1{,}000A(5)=1{,}000\times\frac{1}{v(5)}=1{,}000\times\frac{1}{(1-0.08)^5}=€1{,}517.26$$

(i)(c) Just using (1.2), the accumulation is:

$$1{,}000A(5)=1{,}000\times 1.08^5=€1{,}469.33$$

(ii)(a) Using (3.3) and (1.1), the present value is:

$$2{,}000v(4)=2{,}000\times\frac{1}{A(4)}=2{,}000\times\frac{1}{1+4\times 0.03}=€1{,}785.71$$

(ii)(b) Just using (3.1), the present value is:

$$2{,}000v(4)=2{,}000\times(1-4\times 0.03)=€1{,}760$$

(ii)(c) Using (3.3) and (1.2) the present value is:

$$2,000v(4) = 2,000 \times \frac{1}{A(4)} = 2,000 \times \frac{1}{(1.03)^4} = €1,776.97$$

4 Effective rates of interest and discount

Effective rates are compound rates that have interest paid *once* per unit time either at the end of the period (effective interest) or at the beginning of the period (effective discount). This distinguishes them from nominal rates where interest is paid more frequently than once per unit time.

Bank accounts sometimes use nominal rates of interest. They might quote the annual interest rate (and so the unit time is one year) but interest is actually added at the end of each month (so interest is paid more frequently than once per unit year). We will meet nominal rates in the next chapter.

We can demonstrate the equivalence of compound and effective rates by an alternative way of considering effective rates.

4.1 Effective rate of interest

An investor will lend an amount 1 at time 0 in return for a repayment of $(1+i)$ at time 1. Hence we can consider i to be the interest paid at the *end* of the year. Accordingly i is called the *rate of interest* (or the *effective rate of interest*) per unit time.

Now we'll consider the general case. Investing 1 at time 0, the accumulation at time $n-1$ will be $A(n-1)$ and the accumulation at time n will be $A(n)$.

So denoting the effective rate of interest during the nth period (*ie* between time $n-1$ and time n) **by i_n, we have:**

$$A(n) = (1+i_n)A(n-1)$$

Rearranging gives:

$$\boldsymbol{i_n = \frac{A(n) - A(n-1)}{A(n-1)}} \qquad \textbf{(4.1)}$$

This formula should be intuitive – subtracting the accumulations in the numerator gives the interest earned during the nth period. By dividing by the amount we started with, we will obtain the interest rate.

Question

An investor's bank balance at various times is as follows:

1 Jan 2017	1 Jul 2017	1 Jan 2018
£3,000	£3,100	£3,300

Calculate the:

(i) effective six-monthly rate between 1 January 2017 and 1 July 2017

(ii) effective annual rate between 1 January 2017 and 1 January 2018.

Solution

(i) $i = \dfrac{£3{,}100 - £3{,}000}{£3{,}000} = 3.33\%$ per six months

(ii) $i = \dfrac{£3{,}300 - £3{,}000}{£3{,}000} = 10\%\ pa$

If *i* is the compound rate of interest, we have:

$$A(n) = (1+i)^n \text{ and } A(n-1) = (1+i)^{n-1}$$

Hence substituting in (4.1) we have:

$$\boldsymbol{i_n = \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^{n-1}} = (1+i) - 1 = i} \qquad \textbf{(4.2)}$$

Since this is independent of n, we see that the effective rate of interest is identical to the compound rate of interest we met earlier.

Hence, hereafter, we shall use the terms compound interest and effective interest interchangeably.

Question

Show that the effective rate of interest, when accumulating using a constant simple interest rate, decreases over time.

Solution

For simple interest i, the accumulations are:

$$A(n) = (1+ni) \text{ and } A(n-1) = (1+(n-1)i)$$

Hence substituting in (4.1) we have:

$$i_n = \frac{(1+ni) - (1+(n-1)i)}{(1+(n-1)i)} = \frac{i}{1+(n-1)i}$$

So the effective interest rate decreases as n gets larger. This makes sense as simple interest pays a constant amount each year (as interest doesn't earn interest) but the accumulation is getting larger each year.

4.2 Effective rate of discount

We can think of compound discount as an investor lending an amount $(1-d)$ at time 0 in return for a repayment of 1 at time 1. The sum of $(1-d)$ may be considered as a loan of 1 (to be repaid after 1 unit of time) on which interest of amount d is payable in *advance*. Accordingly d is called the *rate of discount* (or the *effective rate of discount*) per unit time.

Consider the situation where an individual borrows a sum of £5,000 and agrees to pay this back at the end of 1 year with interest calculated at an effective rate of 10% per annum. The total amount repayable will therefore be £5,500.

An alternative way of looking at this arrangement would be to say that the individual has borrowed £5,500 (the amount to be repaid) but the lender has deducted the interest payment of £500 at the time the money was lent. Presented in this way, it would seem logical to express the interest rate as 9.09% (*ie* $500/5{,}500$) of the amount borrowed, where the interest is payable at the *beginning* of the year.

Therefore the effective rate of discount over a given time period is the amount of interest payable at the beginning of the time period, expressed as a proportion of the total amount paid at the end of the period.

In symbols, denoting the effective rate of interest during the nth period (*ie* between time $n-1$ and time n) by d_n, we have:

$$d_n = \frac{A(n) - A(n-1)}{A(n)}$$

We can also show that the effective rate of discount is identical to the compound rate of discount we met earlier.

For compound discount, d, we have:

$$A(n) = \frac{1}{v(n)} = \frac{1}{(1-d)^n} = (1-d)^{-n} \quad \text{and} \quad A(n-1) = \frac{1}{v(n-1)} = \frac{1}{(1-d)^{n-1}} = (1-d)^{-(n-1)}$$

Hence:

$$d_n = \frac{A(n) - A(n-1)}{A(n)} = \frac{(1-d)^{-n} - (1-d)^{-(n-1)}}{(1-d)^{-n}} = 1 - (1-d) = d$$

Hence, hereafter, we shall use the terms compound discount and effective discount interchangeably.

5 Equivalent rates

Two rates of interest and/or discount are equivalent if a given amount of principal invested for the same length of time produces the same accumulated value under each of the rates.

Converting between interest rates efficiently is an important skill. The key is to equate the accumulation factors or to equate the discount factors over the same time period.

Question

Calculate the effective annual interest rate that is equivalent to a simple interest rate of 3% *pa* over 4 years.

Solution

The accumulation factor for 3% *pa* simple interest over 4 years is:

$$A(4) = 1 + 4 \times 0.03 = 1.12$$

The accumulation factor for effective interest over 4 years is:

$$A(4) = (1+i)^4$$

If two rates are equivalent, then they will result in the same accumulation factors:

$$(1+i)^4 = 1.12 \quad \Rightarrow \quad i = 2.87\% \ pa$$

Comparing formulae (2.3) and (3.2), we see that the present value of C due at time n can be expressed as Cv^n or as $C(1-d)^n$. So equating the discount factors we see that:

$$\boldsymbol{v = 1 - d} \qquad \textbf{(5.1)}$$

And from (2.2) and (5.1) we obtain the rearrangements:

$$\boldsymbol{d} = 1 - v = 1 - \frac{1}{1+i} = \frac{\boldsymbol{i}}{\boldsymbol{1+i}} \qquad \textbf{(5.2)}$$

or: $\boldsymbol{d = iv}$ **(5.3)**

Recall that d is the interest paid at time 0 on a loan of 1, whereas i is the interest paid at time 1 on the same loan. If the rates are equivalent then if we discount i from time 1 to time 0 we will obtain d. This is the interpretation of equations (5.2) and (5.3).

Note that formulae (5.1), (5.2) and (5.3) apply for effective rates. They *cannot* be used to convert between simple interest and discount.

Chapter 4 Summary

Many financial arrangements involve a borrower and a lender. Borrowers reward lenders by paying interest to them.

Two factors that might influence the level of interest rates are the likelihood of default on payments and the possible appreciation or depreciation of currency.

The calculation of the amount of interest payable under a financial arrangement can be expressed in terms of compound (effective) interest or simple interest.

The essential feature of *simple interest* is that interest, once credited to an account, does not itself earn further interest. In the case of simple interest, the formula for the n-year accumulation factor is:

$$A(n) = 1 + ni$$

The essential feature of *compound (effective) interest* is that interest itself earns interest. In the case of compound interest, the formula for the n-year accumulation factor is:

$$A(n) = (1+i)^n$$

The principle of consistency says that the accumulated proceeds of an investment in a consistent market will not depend on the action of the investor.

The *present value* of a series of payments is the amount that is invested now in order to meet those payments. We can obtain present values using discount factors. In terms of effective interest the n-year discount factor is:

$$v(n) = v^n = \frac{1}{(1+i)^n}$$

The n-year discount factor in terms of the effective discount rate is: $v(n) = (1-d)^n$

The n-year discount factor in terms of the simple discount rate is: $v(n) = 1 - nd$

The relationship between discount and accumulation factors is: $v(n) = \dfrac{1}{A(n)}$

The *effective rate of interest* over a given time period is the amount of interest a single initial investment will earn at the end of the time period, expressed as a proportion of the initial amount.

The *effective rate of discount* over a given time period is the amount of interest a single initial investment will earn at the start of the time period, expressed as a proportion of the final amount.

Useful relationships between effective rates are: $v = 1 - d$ $\quad d = \dfrac{i}{1+i}$ $\quad d = iv$

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 4 Practice Questions

4.1 An investor pays £100 into an account today. The account pays simple interest at a rate of 4% *pa*. Calculate the amount in the account in five years' time.

4.2 A company is due to receive a payment of £500,000 from a customer in 6 months' time. To smooth its cashflows, the company would prefer to receive the payment immediately, and has agreed to transfer its entitlement to this payment to a third party (a discount house) in return for an immediate payment calculated using a rate of commercial discount of 16% per annum.

Calculate the amount of the immediate payment made by the discount house.

4.3 A bank account pays an effective annual interest rate of 10% over 5 years. Calculate the equivalent:

(i) simple annual interest rate

(ii) effective monthly interest rate

(iii) effective two-yearly interest rate

(iv) effective annual discount rate

(v) simple annual discount rate.

4.4 A woman who has won a prize is offered either:

- a lump sum of £100,000 to invest now, or
- £55,000 to invest in one year's time and another £55,000 to invest in two years' time.

If all investments are assumed to earn interest at a rate of 7% *pa* effective, determine which option she should choose if she intends to withdraw the money after:

(i) 4 years

(ii) 2 years.

4.5 Calculate the effective annual rate of interest for:

(i) a transaction in which £200 is invested for 18 months to give £350.

(ii) a transaction in which £100 is invested for 24 months and another £100 is invested for 12 months (starting 12 months after the first investment) to give a total of £350.

4.6 A 182-day treasury bill, redeemable at $100, was purchased for $96.50 at the time of issue and later sold to another investor for $98 who held the bill to maturity. The rate of return received by the initial purchaser was 4% per annum effective.

Exam style

(i) Calculate the length of time in days for which the initial purchaser held the bill. [2]

(ii) Calculate the annual simple rate of return achieved by the second investor. [2]

(iii) Calculate the annual effective rate of return achieved by the second investor. [2]
[Total 6]

Chapter 4 Solutions

4.1 The amount in the account will be $100(1+0.04\times 5)=£120$.

4.2 The amount of the immediate payment will be:

$$500,000\,(1-\tfrac{6}{12}\times 0.16)=£460,000$$

4.3 (i) Equating the accumulation factors for simple and effective interest over 5 years:

$$A(5)=(1+5i)=1.10^5 \quad\Rightarrow\quad i=12.2\%\ \ pa$$

(ii) Equating the accumulation factors for monthly and annual effective interest over 5 years gives:

$$A(5)=(1+i)^{5\times 12}=1.1^5 \quad\Rightarrow\quad (1+i)^{12}=1.1 \quad\Rightarrow\quad i=0.797\%\ \text{ per month}$$

Note that because we are changing between two effective rates the 5 years is not actually needed. (The same would be true if we were changing between two simple rates.)

(iii) Equating the accumulation factors for two-yearly and annual effective interest over 5 years gives:

$$A(5)=(1+i)^{2.5}=1.1^5 \quad\Rightarrow\quad (1+i)^{½}=1.1 \quad\Rightarrow\quad i=21\%\ \text{ per two-years}$$

Again because we are changing between two effective rates the 5 years is not actually needed.

(iv) Equating discount factors for effective annual interest and effective annual discount over the 5 years gives:

$$v(5)=v^5=1.1^{-5}=(1-d)^5 \quad\Rightarrow\quad 1.1^{-1}=1-d \quad\Rightarrow\quad d=9.09\%\ \ pa$$

Again because we are changing between two effective rates the 5 years is not actually needed.

(v) Equating discount factors for effective annual interest and simple annual discount over the 5 years gives:

$$v(5)=v^5=1.1^{-5}=(1-5d) \quad\Rightarrow\quad d=7.58\%\ \ pa$$

4.4 The time at which she intends to withdraw the money is irrelevant – she should choose the option that maximises the present value of the payments. This is because the higher the present value, the higher the accumulated value will be, irrespective of the time of withdrawal.

The PV of the lump sum option is £100,000. The PV of the two-payment option is:

$$55,000(v+v^2)=55,000(1.07^{-1}+1.07^{-2})=£99,441\,.$$

So she should choose the lump sum in either case.

4.5 (i) The effective annual interest rate i satisfies:

$$200(1+i)^{\frac{18}{12}}=350$$

Solving for i gives:

$$i=\left(\frac{350}{200}\right)^{\frac{12}{18}}-1=0.4522 \quad ie\ \ 45.22\%$$

(ii) The effective annual interest rate i satisfies:

$$100(1+i)^2+100(1+i)=350$$

Dividing by 100 and rearranging:

$$(1+i)^2+(1+i)-3.5=0$$

This is a quadratic equation in $(1+i)$. Applying the quadratic formula:

$$1+i=\frac{-1\pm\sqrt{1-4(-3.5)}}{2}=1.4365 \ \text{(or } -2.4365\text{)}$$

Therefore $i=43.65\%$.

4.6 *This question is Subject CT1, April 2015, Question 3.*

(i) ***Length of time held***

We have:

$$96.5\times 1.04^t=98$$ [1]

Taking logs of both sides of this equation gives:

$$\log 96.5+t\log 1.04=\log 98$$

Rearranging this, we find that:

$$t=\frac{\log 98-\log 96.5}{\log 1.04}=0.39327$$

Converting this to days, we find that the length of time is:

$$t=0.39327\times 365=143.545$$

or about 144 days. [1]

[Total 2]

(ii) ***Annual simple rate of return***

Since the original investor held the bill for 144 days, the bill was held by the second investor for $182-144=38$ days. So the simple rate of return experienced by the second investor is the solution of the equation:

$$98\left(1+\frac{38}{365}\times i\right)=100$$ [1]

Solving this equation, we find that:

$$i=\left(\frac{100}{98}-1\right)\times\frac{365}{38}=0.19603 \quad \textit{ie} \;\; 19.60\% \; \textit{pa}$$ [1]

[Total 2]

(iii) ***Annual effective rate of return***

The equation of value is now:

$$98(1+i)^{38/365}=100$$ [1]

Solving this equation, we find that:

$$i=\left(\frac{100}{98}\right)^{365/38}-1=0.21416 \quad \textit{ie} \;\; 21.42\% \; \textit{pa}$$ [1]

[Total 2]

5

Interest rates

Syllabus objectives

2.1 Show how interest rates may be expressed in different time periods.

2.1.2 Derive the relationships between the rate of interest payable once per measurement period (effective rate of interest) and the rate of interest payable p (>1) times per measurement period (nominal rate of interest) and the force of interest.

2.1.3 Calculate the equivalent annual rate of interest implied by the accumulation of a sum of money over a specified period where the force of interest is a function of time.

0 Introduction

The previous chapter considered the accumulation and present value of a single payment and introduced the ideas of effective interest rates and effective discount rates. This chapter describes the alternative ways of expressing interest rates and shows the relationships between them.

1 Nominal rates of interest and discount

Recall from Chapter 4 that 'effective' rates of interest and discount have interest paid once per measurement period, either at the end of the period or at the beginning of the period.

The effective annual rate of interest tells us the amount of interest to be paid at the *end* of each year. The effective annual rate of discount tells us the amount of interest to be paid at the *start* of each year.

'Nominal' is used where interest is paid more (or less) frequently than once per measurement period.

The example we gave in the previous chapter was that bank accounts sometimes use nominal rates. They might quote the annual interest rate (and so the time unit or measurement period is one year), but interest might actually be added at the end of each month (so interest is paid more frequently than once per unit year).

1.1 Nominal rates of interest

We denote the nominal rate of interest *payable p* times per period by $i^{(p)}$. This is also referred to as the rate of interest *convertible p*thly or *compounded p*thly.

Therefore, working in years, $i^{(12)}$ is referred to as a nominal interest rate convertible monthly and $i^{(4)}$ as a nominal interest rate convertible quarterly, *etc.*

A nominal rate of interest per period, payable *p*thly, $i^{(p)}$, is defined to be a rate of interest of $i^{(p)}/p$ applied for each *p*th of a period. For example, a nominal rate of interest of 6% *pa* convertible quarterly means an interest rate of $6/4 = 1.5\%$ per quarter.

Essentially what we are doing is 'annualising' a *p*thly effective interest rate. That is, we are converting a non-annual rate to an annual rate by multiplying. For example, suppose interest is 3% effective per half-year. We could annualise this rate by doubling it, which would give 6% *pa*. However, this is clearly not the correct annual effective rate as it has ignored the effect of compounding. The true effective annual rate is $1.03^2 - 1 = 6.09\%$ *pa*. We call 6% a nominal rate and give the period it actually refers to. Dividing by p gives us the correct *p*thly effective rate. So, in this case, we would say that the nominal rate of interest is 6% *pa* convertible half-yearly and we denote this by $i^{(2)}$.

Hence, by definition, $i^{(p)}$ is equivalent to a *p*thly *effective* rate of interest of $i^{(p)}/p$.

Question

(i) Express a monthly effective interest rate of 2% as a nominal annual interest rate convertible monthly.

(ii) (a) State the two-monthly effective interest rate that corresponds to a nominal interest rate of 3% *pa* convertible two-monthly.

(b) Hence, calculate the equivalent annual effective interest rate.

Solution

(i) A monthly effective interest rate of 2% is equivalent to a nominal annual interest rate of $2\% \times 12 = 24\%$ *pa* convertible monthly. In symbols, this is $i^{(12)} = 24\%$.

(ii) (a) There are six two-monthly periods in a year, so we are given $i^{(6)} = 3\%$. Therefore the two-monthly effective interest rate is $\frac{3\%}{6} = 0.5\%$.

(b) Using the two-monthly effective interest rate of 0.5%, the accumulation factor for one year is $1.005^6 = 1.030378$. Hence, the annual effective rate is 3.0378% *pa*.

(The effective rate is greater than the nominal rate as the effective rate takes account of the effect of compounding.)

In part (ii) of the above question, we have a nominal rate of interest of 3% *pa* convertible two-monthly (*ie* $i^{(6)} = 3\%$), which is equivalent to an effective interest rate of $\frac{i^{(6)}}{6} = 0.5\%$ per two-months. So, an initial investment of 1 unit will amount to 1.005^6 *ie* $\left(1+\frac{i^{(6)}}{6}\right)^6$ by the end of one year. If the equivalent effective annual rate of interest is i, an initial investment of 1 unit will amount to $1+i$. So we must have $1.005^6 = 1+i$, *ie* $\left(1+\frac{i^{(6)}}{6}\right)^6 = 1+i$. Hence, in the question, the equivalent effective annual rate of interest is $i = 1.005^6 - 1 = 0.030378$, *ie* 3.0378%.

In general, if we are given $i^{(p)}$, then the *p*thly effective interest rate is $\frac{i^{(p)}}{p}$, *ie* we have an effective rate of interest of $\frac{i^{(p)}}{p}$ for each period of length $\frac{1}{p}$. Hence, the accumulation factor over one time period is $\left(1+\frac{i^{(p)}}{p}\right)^p$.

Therefore the effective interest rate i is obtained from:

$$\mathbf{1+i = \left(1+\frac{i^{(p)}}{p}\right)^p} \qquad \textbf{(1.1)}$$

Note that $i^{(1)} = i$.

This can be seen by substituting $p=1$ into equation (1.1). If $p=1$ we don't bother to write the superscript.

Rearranging equation (1.1) gives:

$$i^{(p)}=p\left[(1+i)^{1/p}-1\right] \tag{1.2}$$

This equation appears on page 120 of the *Tables*.

We can use equations (1.1) and (1.2) to convert between nominal and effective interest rates.

Question

(i) Calculate the nominal annual interest rate convertible quarterly that is equivalent to an interest rate of 5% *pa* effective.

(ii) Calculate the annual effective interest rate that is equivalent to a nominal interest rate of 12% *pa* convertible four-monthly.

Solution

(i) Using equation (1.2) we have:

$$i^{(4)}=4(1.05^{\frac{1}{4}}-1)=4.90889\%$$

(ii) There are three four-month periods in a year, so we are given $i^{(3)}=12\%$. Using equation (1.1) we have:

$$1+i=\left(1+\frac{0.12}{3}\right)^3=1.124864 \quad\Rightarrow\quad i=12.4864\% \ \textit{pa} \text{ effective}$$

1.2 Accumulating and discounting using nominal interest rates

Since a nominal interest rate is a multiple of an effective interest rate for a period, we cannot directly use a nominal interest rate to accumulate or discount sums of money. However, we could use equation (1.1) to convert the nominal rate into an effective rate and then use the formulae from the previous chapter:

$$A(n)=(1+i)^n$$

$$v(n)=v^n=(1+i)^{-n}$$

Question

(i) €500 is invested in an account that pays nominal interest of 8% *pa* convertible half-yearly. Calculate the accumulated amount in the account after 3 years.

(ii) A payment of $800 is due in 5 years' time. Calculate the present value of this payment using an interest rate of 9% *pa* convertible monthly.

Solution

(i) We are given $i^{(2)} = 8\%$. Using equation (1.1), this is equivalent to an annual effective rate of $i = \left(1 + \frac{0.08}{2}\right)^2 - 1 = 8.16\%$. Accumulating the €500 for 3 years at this rate gives:

$$€500 \times 1.0816^3 = €632.66$$

(ii) We are given $i^{(12)} = 9\%$. Using equation (1.1), this is equivalent to an annual effective rate of $i = \left(1 + \frac{0.09}{12}\right)^{12} - 1 = 9.3806898\%$. Discounting the $800 for 5 years at this rate gives:

$$\$800 \times 1.093806898^{-5} = \$510.96$$

Instead of converting the nominal interest rate convertible *p*thly into an annual effective rate, we could convert it to a *p*thly effective rate.

The treatment of problems involving nominal rates of interest (or discount) is almost always considerably simplified by an appropriate choice of the time unit.

By choosing the basic time unit to be the period corresponding to the frequency with which the nominal rate of interest is convertible, we can use $i^{(p)}/p$ as the effective rate of interest per unit time. For example, if we have a nominal rate of interest of 18% per annum convertible monthly, we should take one month as the unit of time and 1½% as the rate of interest per unit time.

We'll now repeat the previous question using this method.

Question

(i) €500 is invested in an account that pays nominal interest of 8% *pa* convertible half-yearly. Calculate the accumulated amount in the account after 3 years.

(ii) A payment of $800 is due in 5 years' time. Calculate the present value of this payment using an interest rate of 9% *pa* convertible monthly.

Solution

(i) We are given $i^{(2)} = 8\%$. This is equivalent to a half-yearly effective rate of $\frac{8\%}{2} = 4\%$. Accumulating the €500 for 3 years (= 6 half-years) at this rate gives:

$$€500 \times 1.04^6 = €632.66$$

(ii) We are given $i^{(12)} = 9\%$. This is equivalent to a monthly effective rate of $\frac{9\%}{12} = 0.75\%$. Discounting the \$800 for 5 years (= 60 months) at this rate gives:

$$\$800 \times 1.0075^{-60} = \$510.96$$

1.3 Nominal rates of discount

We denote the nominal rate of discount *payable p* times per period by $d^{(p)}$. This is also referred to as the rate of discount *convertible* *p*thly or *compounded* *p*thly.

Therefore, working in years, $d^{(12)}$ is referred to as a nominal discount rate convertible monthly and $d^{(4)}$ as a nominal discount rate convertible quarterly, *etc.*

A nominal rate of discount per period payable *p*thly, $d^{(p)}$, is defined as a rate of discount of $d^{(p)}/p$ applied for each *p*th of a period.

Again, we are 'annualising' a *p*thly effective discount rate. For example, to 'annualise' an effective discount rate of 3% per half-year, we would double it to get 6% *pa*. This is not the correct annual effective discount rate (as it has ignored the effect of compounding). The 6% we have obtained is referred to as a nominal discount rate convertible half-yearly and is denoted by $d^{(2)}$.

Hence, by definition, $d^{(p)}$ is equivalent to a *p*thly *effective* rate of discount of $d^{(p)}/p$.

Recall from the previous chapter that given an effective discount *rate*, d, the discount *factor* for n periods is:

$$v(n) = (1-d)^n$$

Question

(i) Express a monthly effective discount rate of 2% as a nominal annual discount rate convertible monthly.

(ii) Calculate the effective annual discount rate that is equivalent to a nominal discount rate of 3% *pa* convertible two-monthly.

Solution

(i) A monthly effective discount rate of 2% is equivalent to a nominal annual discount rate of $2\% \times 12 = 24\%$ *pa* convertible monthly. In symbols this is $d^{(12)} = 24\%$.

(ii) There are six two-monthly periods in a year, so we are given $d^{(6)} = 3\%$. Therefore the two-monthly effective discount rate is $\frac{3\%}{6} = 0.5\%$.

The discount factor for one year is $(1-0.005)^6 = 0.970373$. Hence, the annual effective discount rate is $1-0.970373 = 0.029627$, *ie* 2.9627%.

In this question, we have a nominal rate of discount of 3% *pa* convertible two-monthly (*ie* $d^{(6)} = 3\%$) which is equivalent to an effective discount rate of $\frac{d^{(6)}}{6} = 0.5\%$ per two-months. So, the present value of 1 unit due in one year's time is 0.995^6, *ie* $\left(1-\frac{d^{(6)}}{6}\right)^6$. If the equivalent effective annual rate of discount is d, the present value of 1 unit in one year's time is $1-d$. So we must have $0.995^6 = 1-d$, *ie* $\left(1-\frac{d^{(6)}}{6}\right)^6 = 1-d$. Hence, in the example, the equivalent effective annual rate of discount is $d = 1-0.995^6 = 0.029627$ *ie* 2.9627%.

In general, if we are given $d^{(p)}$, then the *p*thly effective discount rate is $\frac{d^{(p)}}{p}$, *ie* we have an effective rate of discount of $\frac{d^{(p)}}{p}$ for each period of length $\frac{1}{p}$. Hence, the discount factor over one time period is $\left(1-\frac{d^{(p)}}{p}\right)^p$.

Therefore the effective discount rate d is obtained from:

$$\mathbf{1-d = \left(1-\frac{d^{(p)}}{p}\right)^p} \qquad \textbf{(1.3)}$$

Note that $d^{(1)} = d$.

This can be seen by substituting $p=1$ into equation (1.3). If $p=1$ we don't bother to write the superscript.

Rearranging equation (1.3) gives:

$$d^{(p)} = p\left[1-(1-d)^{1/p}\right] \qquad (1.4)$$

We can use equations (1.3) and (1.4) to convert quickly between nominal and effective discount rates.

Question

(i) Calculate the nominal annual discount rate convertible quarterly that is equivalent to an effective discount rate of 5% *pa*.

(ii) Calculate the annual effective discount rate that is equivalent to a nominal discount rate of 12% *pa* convertible four-monthly.

Solution

(i) Using equation (1.4) we have:

$$d^{(4)} = 4(1-(1-0.05)^{1/4}) = 5.09658\%$$

(ii) There are three four-month periods in a year, so we are given $d^{(3)} = 12\%$. Using equation (1.3) we have:

$$1-d = \left(1-\frac{0.12}{3}\right)^3 = 0.884736 \quad \Rightarrow \quad d = 11.5264\%$$

However, it's often far more convenient to convert a nominal discount rate into an effective *interest* rate and *vice versa*.

Since $v = 1-d$, we could rewrite equation (1.4) as:

$$d^{(p)} = p\left[1-v^{1/p}\right]$$

Recalling that $v = \frac{1}{1+i} = (1+i)^{-1}$, we then have:

$$d^{(p)} = p\left[1-(1+i)^{-1/p}\right] \tag{1.5}$$

Rearranging (1.5) gives:

$$1+i = \left(1-\frac{d^{(p)}}{p}\right)^{-p} \tag{1.6}$$

This allows us to calculate the effective interest rate given a nominal discount rate convertible *p*thly.

Question

(i) Calculate the nominal annual discount rate convertible monthly that is equivalent to an effective annual interest rate of 10%.

(ii) Calculate the annual effective interest rate that is equivalent to a discount rate of 8% *pa* convertible quarterly.

Solution

(i) Using equation (1.5) we have:

$$d^{(12)} = 12\left[1 - 1.1^{-1/12}\right] = 9.49327\%$$

(ii) Using equation (1.6) we have:

$$1 + i = \left(1 - \frac{0.08}{4}\right)^{-4} = 1.0841658 \quad \Rightarrow \quad i = 8.41658\% \ pa$$

1.4 Accumulating and discounting using nominal discount rates

Since a nominal discount rate is a multiple of an effective discount rate for a period, we cannot directly use a nominal discount rate to accumulate or discount sums of money. However, we could use equation (1.3) to convert it to an effective discount rate and then use the following formulae from the previous chapter:

$$v(n) = (1-d)^n$$

$$A(n) = \frac{1}{v(n)} = (1-d)^{-n}$$

Alternatively, we could use equation (1.6) to convert it to an effective interest rate and then use the following formulae from the previous chapter:

$$A(n) = (1+i)^n$$

$$v(n) = v^n = (1+i)^{-n}$$

We will use the second method in the next question.

Question

(i) €500 is invested in an account that pays interest equivalent to a nominal discount rate of 8% *pa* convertible half-yearly. Calculate the accumulated amount in the account after 3 years.

(ii) A payment of \$800 is due in 5 years' time. Calculate the present value of this payment using a discount rate of 9% *pa* convertible monthly.

Solution

(i) We are given $d^{(2)} = 8\%$. Using equation (1.6), this is equivalent to an annual effective interest rate of $i = \left(1 - \frac{0.08}{2}\right)^{-2} - 1 = 8.50694\%$. Accumulating the €500 for 3 years at this rate gives:

$$€500 \times 1.0850694^3 = €638.77$$

(ii) We are given $d^{(12)} = 9\%$. Using equation (1.6), this is equivalent to an annual effective interest rate of $i = \left(1 - \frac{0.09}{12}\right)^{-12} - 1 = 9.4545487\%$. Discounting the \$800 for 5 years at this rate gives:

$$\$800 \times 1.094545487^{-5} = \$509.24$$

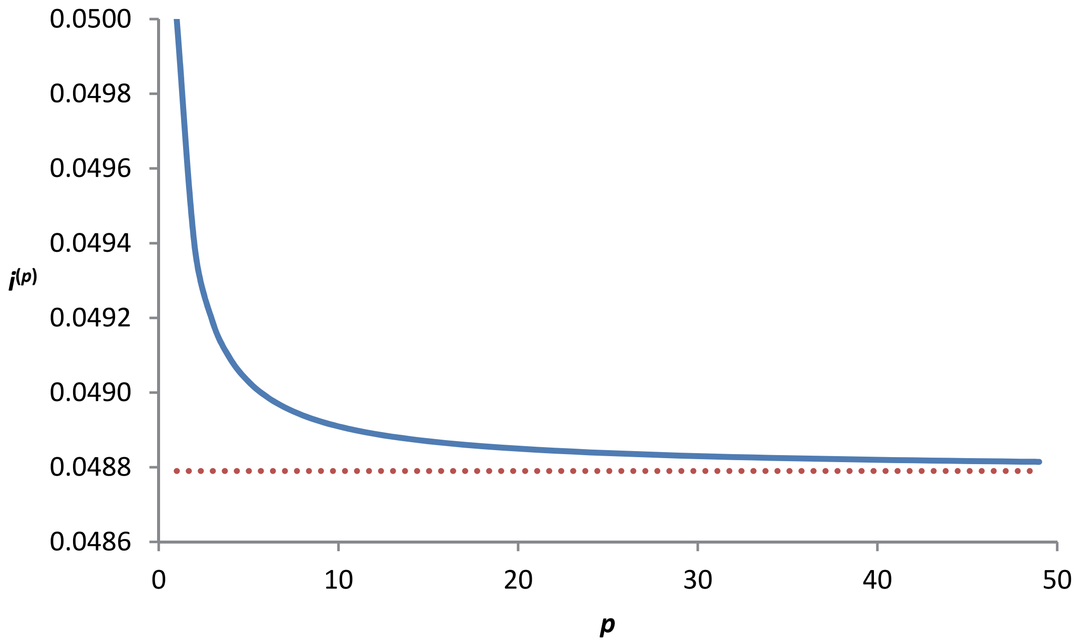
2 The force of interest

An effective rate of interest is the rate of interest a single initial investment will earn at the end of the time period. We now move on to the case where the interest is paid continuously throughout the time period.

If we consider a nominal interest rate convertible very frequently (*eg* every second), we are no longer thinking of a fund that suddenly acquires an interest payment at the end of each interval, but of a fund that steadily accumulates over the period as interest is earned and added. In the limiting case, the amount of the fund can be considered to be subject to a constant 'force' causing it to grow. This leads us to the concept of a *force of interest*, which is the easiest way to model continuously paid interest rates mathematically.

2.1 Derivation from nominal interest convertible *p*thly

Using $i^{(p)} = p\left[(1+i)^{1/p} - 1\right]$ with an effective rate of interest of 5% *pa*, we can obtain the equivalent nominal rates of interest convertible *p*thly (*eg* $i^{(2)} = 0.04949$, $i^{(4)} = 0.04909$, *etc*). If we let the value of *p* increase, we obtain the following graph:



We can see that $i^{(p)}$ is approaching a limit as $p \to \infty$. This is the nominal rate of interest convertible *continuously* (*ie* every instant) and is the called the *force of interest*. We denote the force of interest by δ.

We assume that for each value of *i* there is number, δ, such that:

$$\lim_{p\to\infty} i^{(p)} = \delta$$

δ is the nominal rate of interest per unit time convertible continuously (or momently). This is also referred to as the rate continuously compounded. We call it the *force of interest*.

Whilst this is all very interesting, it gives us no practical way of *calculating* the value of δ. So what we're going to do now is derive a relationship between the force of interest and the effective rate of interest.

Euler's rule states that:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

Applying this to the right-hand-side of (1.1), which states that:

$$1+i=\left(1+\frac{i^{(p)}}{p}\right)^p$$

gives:

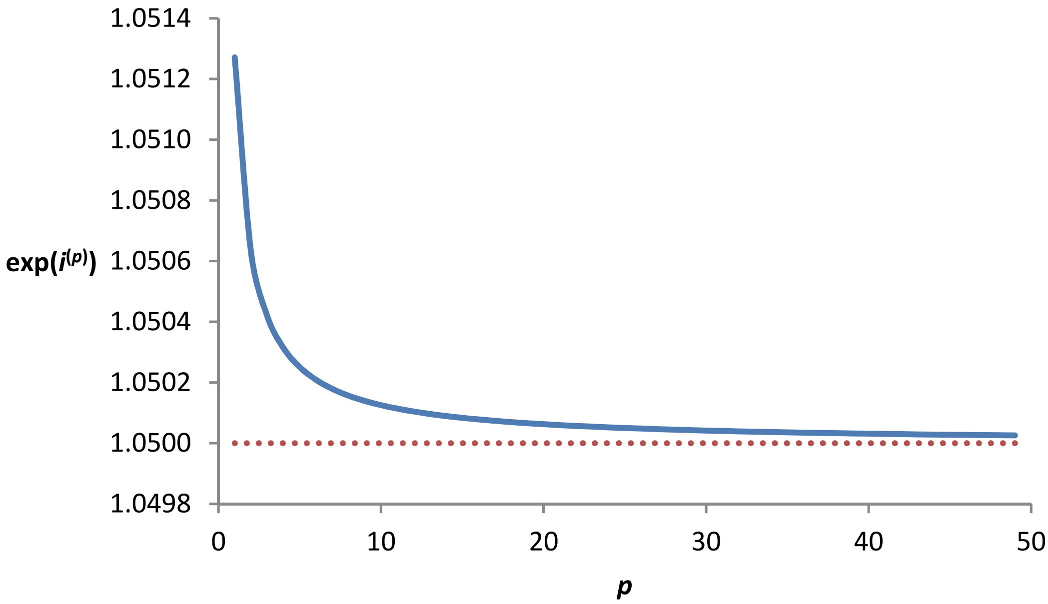
$$1+i=\lim_{p\to\infty}\left(1+\frac{i^{(p)}}{p}\right)^p = e^{i^{(\infty)}}$$

But we defined $i^{(\infty)}$ to be δ. **Hence:**

$$1+i=e^{\delta} \tag{2.1}$$

This gives us our connection between the effective interest rate, i, and the force of interest, δ.

Taking the exponential of the $i^{(p)}$'s that we calculated using an effective rate of interest of 5% *pa* in our previous graph we obtain the following:



We can see that $\lim_{p\to\infty} e^{i^{(p)}} = e^{\delta} = 1.05$. This is the value of $1+i$, as seen in equation (2.1).

Rearranging the equation $e^{\delta} = 1+i$ gives:

$$\delta = \ln(1+i) \tag{2.2}$$

In our first graph the limiting value of $i^{(p)}$ is $\delta = \ln 1.05 \approx 0.048790$.

Since we will only ever use natural log (*ie* $\ln = \log_e$) in this course, the Core Reading, the examiners and ActEd materials use $\log_e$, log and ln interchangeably.

2.2 Accumulating and discounting using the force of interest

Since the force of interest is also an annualised rate, we could convert it to an effective rate before we accumulate or discount.

Question

€500 is invested in an account that pays a force of interest of 8% *pa*. Calculate the accumulated amount in the account after 3 years.

Solution

We are given $\delta = 8\%$. Using equation (2.1), this is equivalent to an annual effective interest rate of $i = e^{0.08} - 1 = 8.3287068\%$. Accumulating the €500 for 3 years at this rate gives:

$$€500 \times 1.083287068^3 = €635.62$$

However, we can actually obtain the accumulation factor without first converting to an effective interest rate. From the previous chapter, the accumulation factor for effective interest is:

$$A(n) = (1+i)^n$$

Now (2.1) tells us that $1+i = e^{\delta}$. Substituting this in gives:

$$A(n) = (e^{\delta})^n = e^{\delta n} \tag{2.3}$$

So, in the previous question, we could have calculated the accumulation over 3 years as:

$$€500 \times e^{0.08\times 3} = €635.62$$

Similarly we can develop a formula for a discount factor using the force of interest.

Since $v = (1+i)^{-1}$, and (2.1) tells us that $1+i = e^{\delta}$, we have:

$$\boldsymbol{v = e^{-\delta}} \tag{2.4}$$

From equation (2.4) we have:

$$\boldsymbol{v^t = (e^{-\delta})^t = e^{-\delta t}}$$

In the previous chapter, we saw that the discount factor over n years is:

$$v(n) = v^n = (1+i)^{-n}$$

Hence, the discount factor for a force of interest δ is:

$$\boldsymbol{v(n) = e^{-\delta n}} \qquad \textbf{(2.5)}$$

Question

A payment of $800 is due in 5 years' time. Calculate the present value of this payment using a force of interest of 9% *pa*.

Solution

We are given $\delta = 9\%$. Using equation (2.5), the present value is:

$$\$800 \times e^{-0.09\times 5} = \$510.10$$

Alternatively, we can first calculate the annual effective interest rate, i, using equation (2.1):

$$i = e^{0.09} - 1 = 9.4174284\% \, .$$

Then we can discount the $800 for 5 years at this rate to give:

$$\$800 \times 1.094174284^{-5} = \$510.10$$

2.3 Derivation from nominal discount convertible *p*thly

In Section 2.1 we defined the force of interest, δ, as $\lim\limits_{p\to\infty} i^{(p)}$.

It can also be shown that:

$$\boldsymbol{\lim_{p\to\infty} d^{(p)} = \delta} \qquad \textbf{(2.6)}$$

The proof is very similar to before. We apply Euler's rule, $\lim\limits_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n = e^x$, to the RHS of (1.6):

$$1+i = \lim_{p\to\infty}\left(1-\frac{d^{(p)}}{p}\right)^{-p}$$

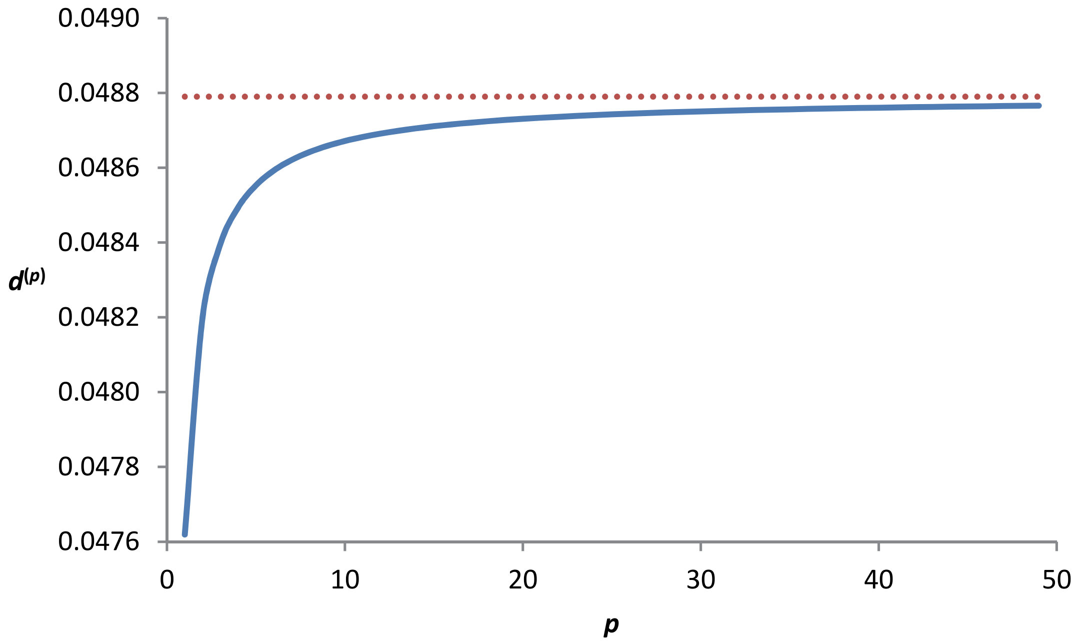
We then use the fact that $\lim \frac{1}{f(x)} = \frac{1}{\lim f(x)}$. Hence:

$$1+i = \frac{1}{\lim\limits_{p\to\infty}\left(1-\frac{d^{(p)}}{p}\right)^p} = \frac{1}{e^{-d^{(\infty)}}} = e^{d^{(\infty)}}$$

We showed that $1+i = e^{\delta}$ in (2.1) and here we have seen that $1+i = e^{d^{(\infty)}}$. Therefore $d^{(\infty)} = \delta$, or more correctly $\lim\limits_{p\to\infty} d^{(p)} = \delta$.

Let's have a look at an example of this.

Using $d^{(p)} = p\left[1-(1+i)^{-1/p}\right]$ with an effective rate of interest of 5% *pa* we can obtain the equivalent nominal rates of discount convertible *p*thly. If we let the value of *p* increase, we obtain the graph below:



We can see that $d^{(p)}$ is approaching the same limit of $\delta = \ln 1.05 = 0.048790$ as before.

However, $d^{(p)}$ tends to this limit from below whereas $i^{(p)}$ tends to this limit from above.

Hence, we have:

$$d < d^{(2)} < d^{(3)} < \cdots < \delta < \cdots < i^{(3)} < i^{(2)} < i$$

This order reflects how late interest is paid. For example, d corresponds to interest paid immediately, and so requires a smaller payment amount than i, which corresponds to interest paid in one year's time.

To demonstrate this further, pick an interest rate (say $i = 8\%$), go to page 60 of the *Tables* and look up some values.

Being able to convert quickly between the different rates of interest and discount is a key skill needed for the exam.

Question

(i) Given $\delta = 8\%$, calculate i, $i^{(4)}$ and $d^{(12)}$.

(ii) Given $i = 7\%$, calculate d, $d^{(4)}$, $i^{(2)}$ and δ.

(iii) Given $d = 9\%$, calculate i, $d^{(2)}$, $i^{(12)}$ and δ.

Solution

(i) $1+i = e^{\delta} = e^{0.08} \quad \Rightarrow \quad i = e^{0.08} - 1 = 8.33\%$

$$i^{(4)} = 4\left[(1+i)^{¼} - 1\right] = 4\left[(e^{0.08})^{¼} - 1\right] = 4\left[e^{0.02} - 1\right] = 8.08\%$$

$$d^{(12)} = 12\left[1 - (1+i)^{-1/12}\right] = 12\left[1 - (e^{0.08})^{-1/12}\right] = 7.97\%$$

(ii) $d = 1 - v = 1 - 1.07^{-1} = 6.54\%$

$$d^{(4)} = 4\left[1 - 1.07^{-¼}\right] = 6.71\%$$

$$i^{(2)} = 2\left[1.07^{½} - 1\right] = 6.88\%$$

$$\delta = \ln(1.07) = 6.77\%$$

(iii) $v = 1 - d = 0.91 \quad \Rightarrow \quad 1+i = \dfrac{1}{0.91} \quad \Rightarrow \quad i = 9.89\%$

$$d^{(2)} = 2\left[1 - (1-d)^{½}\right] = 2\left[1 - (1-0.09)^{½}\right] = 9.21\%$$

$$i^{(12)} = 12\left[(1+i)^{1/12} - 1\right] = 12\left[0.91^{-1/12} - 1\right] = 9.47\%$$

$$\delta = \ln(1+i) = \ln 1.0989 = 9.43\%$$

The following table summarises the formulae we've met that link the effective annual interest rate i, the effective annual discount rate d, the force of interest δ, and the one-year discount factor v.

		Value of..			
		δ	i	v	d
In terms of..	δ		$e^{\delta}-1$	$e^{-\delta}$	$1-e^{-\delta}$
	i	$\ln(1+i)$		$(1+i)^{-1}$	$i/(1+i)$
	v	$-\ln v$	$v^{-1}-1$		$1-v$
	d	$-\ln(1-d)$	$(1-d)^{-1}-1$	$1-d$	

These can all be obtained by manipulating the basic relationships:

- $v=\dfrac{1}{1+i}$
- $\delta=\ln(1+i)$
- $d=iv=1-v$

3 Relationships between effective, nominal and force of interest

3.1 An alternative way of considering nominal interest convertible *p*thly

Recall that effective interest i can be thought of as interest paid at the *end* of the period. Hence, an investor lending an amount 1 at time 0 receives a repayment of $(1+i)$ at time 1.

Similarly, nominal interest convertible *p*thly can be thought of as the total interest per unit of time paid on a loan of amount 1 at time 0, where interest is paid in p equal instalments at the *end* of each *p*th subinterval (*ie* at times $1/p, 2/p, 3/p, \dots, 1$).

Instead of paying one instalment of interest at time 1, we are paying p instalments throughout the time period. Therefore, the accumulated value of the p interest payments, each of amount $i^{(p)}/p$, is equal to i.

Since $i^{(p)}$ is the *total* interest paid and each interest payment is of amount $i^{(p)}/p$, the accumulated value at time 1 of the interest payments is:

$$\frac{i^{(p)}}{p}(1+i)^{(p-1)/p} + \frac{i^{(p)}}{p}(1+i)^{(p-2)/p} + \cdots + \frac{i^{(p)}}{p} = i$$

The terms on the left-hand side of the above equation form a geometric progression, so we can use the formula for the sum of the first n terms of a geometric progression:

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a\left(1-r^n\right)}{1-r}$$

where: $a = \text{first term} = \dfrac{i^{(p)}}{p}(1+i)^{(p-1)/p}$

$r = \text{common ratio} = (1+i)^{-1/p}$

and: $n = \text{number of terms in the sum} = p$.

So, provided that $i \neq 0$, the left-hand side is:

$$\frac{\dfrac{i^{(p)}}{p}(1+i)^{(p-1)/p}\left(1-\left[(1+i)^{-1/p}\right]^p\right)}{1-(1+i)^{-1/p}} = \frac{\dfrac{i^{(p)}}{p}(1+i)^{(p-1)/p}\left(1-(1+i)^{-1}\right)}{1-(1+i)^{-1/p}}$$

Multiplying the numerator and denominator by $(1+i)^{1/p}$ gives:

$$\frac{\dfrac{i^{(p)}}{p}(1+i)^1\left(1-(1+i)^{-1}\right)}{(1+i)^{1/p}-1} = i^{(p)} \times \frac{(1+i)^1 - 1}{p\left[(1+i)^{1/p}-1\right]} = i^{(p)} \times \frac{i}{p\left[(1+i)^{1/p}-1\right]}$$

Now recall that this sum is equal to i, so:

$$i^{(p)} \times \frac{i}{p\left[(1+i)^{1/p}-1\right]} = i$$

Hence, cancelling the i's and rearranging:

$$\boldsymbol{i^{(p)} = p\left[(1+i)^{1/p}-1\right]}$$

This is the result we obtained in equation (1.2), which confirms that our alternative way of thinking about nominal interest is correct. The fact that $i^{(p)}$ can be used to represent *p*thly payments of interest will be useful to us later when we introduce *p*thly annuities.

3.2 An alternative way of considering nominal discount convertible *p*thly

Recall that effective discount d can be thought of as interest paid at the *start* of the period. Hence, an investor lending an amount 1 at time 0 receives a repayment of 1 at time 1, but d is paid at the start so a sum of $(1-d)$ is lent at time 0.

Similarly, $d^{(p)}$ is the total amount of interest per unit of time payable in equal instalments at the *start* of each *p*th subinterval (*ie* at times $0, 1/p, 2/p, \ldots, (p-1)/p$).

Instead of paying one instalment of interest of d at time 0, we are paying p instalments, each of amount $d^{(p)}/p$, throughout the time period.

As a consequence the present value at time 0 of the interest payments is:

$$\boldsymbol{\frac{d^{(p)}}{p} + \frac{d^{(p)}}{p}(1-d)^{1/p} + \cdots + \frac{d^{(p)}}{p}(1-d)^{(p-1)/p} = d}$$

The terms on the left-hand side of the above equation form a geometric progression, with first term $\frac{d^{(p)}}{p}$, common ratio $(1-d)^{1/p}$ and p terms. Summing this (provided that $d \neq 0$) gives:

$$\frac{\frac{d^{(p)}}{p}\left(1-\left[(1-d)^{1/p}\right]^{p}\right)}{1-(1-d)^{1/p}}$$

Simplifying this gives:

$$\frac{d^{(p)}}{p} \times \frac{\left(1-(1-d)\right)}{1-(1-d)^{1/p}} = d^{(p)} \times \frac{d}{p\left[1-(1-d)^{1/p}\right]}$$

Now recall that this sum is equal to d, so:

$$d^{(p)} \times \frac{d}{p\left[1-(1-d)^{1/p}\right]} = d$$

Hence, cancelling the d 's and rearranging:

$$\boldsymbol{d^{(p)} = p\left[1-(1-d)^{1/p}\right]}$$

This is the result we obtained in equation (1.4), which confirms that our alternative way of thinking about nominal discount is correct. The fact that $d^{(p)}$ can be used to represent *p*thly payments of interest in advance will also be useful to us later.

3.3 An alternative way of considering force of interest

Now δ is the total amount of interest payable as a continuous payment stream, *ie* an amount δdt is paid over an infinitesimally small period dt at time t.

So the accumulated value at time 1 of a single payment of δdt at time t is:

$$\delta(1+i)^{1-t}\,dt$$

If t can take any value between 0 and 1, then the total accumulated value (allowing for all values of t) can be calculated by integration. (Integration can be thought of as summation in a continuous sense.)

As a consequence the accumulated value at time 1 of these interest payments is:

$$\boldsymbol{\int_0^1 \delta(1+i)^{1-t}\,dt}$$

The total of all these interest payments is again equal to a single payment of i at time 1, *ie*:

$$\int_0^1 \delta(1+i)^{1-t}\,dt = i$$

Since we are integrating $\delta(1+i)^{1-t}$ from $t=0$ to 1, the integrand ranges from $\delta(1+i)$ to δ. Equally we could have integrated $\delta(1+i)^t$ from $t=1$ to 0 to give the same answer.

So the integral, **by symmetry, is equal to:**

$$\boldsymbol{\int_0^1 \delta(1+i)^t\,dt = i}$$

This can also be seen by making the substitution $u=1-t$ in the original integral above.

To evaluate the integral above, note that:

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{\ln a^x} = \frac{d}{dx}e^{x\ln a} = e^{x\ln a} \times \ln a = a^x \times \ln a$$

So $\int a^x \, dx = \frac{a^x}{\ln a}$ (ignoring the constant of integration). Here, a is $(1+i)$, so:

$$\int_0^1 \delta(1+i)^t \, dt = \delta\left[\frac{(1+i)^t}{\ln(1+i)}\right]_0^1 = \delta\frac{(1+i)-1}{\ln(1+i)} = \delta\frac{i}{\ln(1+i)}$$

We now set this equal to i and rearrange. **Hence:**

$$\boldsymbol{\delta = \ln(1+i) \qquad \text{or} \qquad e^{\delta} = 1+i}$$

This is the result we obtained in equation (2.1), which confirms that our alternative way of thinking about the force of interest is correct. The fact that δ can be used to represent continuous interest payments will also be useful to us later.

We have shown several different but equivalent ways of interest being paid over time. These are summarised below.

It is essential to appreciate that, at force of interest δ per unit time, the five series of payments illustrated in Figure 1 below all have the same value.

	0	$\frac{1}{p}$	$\frac{2}{p}$	$\frac{3}{p}$	**...**	$\frac{p-1}{p}$	**1**	**time**
(1)	d							
(2)	$\frac{d^{(p)}}{p}$	$\frac{d^{(p)}}{p}$	$\frac{d^{(p)}}{p}$	$\frac{d^{(p)}}{p}$	**...**	$\frac{d^{(p)}}{p}$		
(3)		$\frac{i^{(p)}}{p}$	$\frac{i^{(p)}}{p}$	$\frac{i^{(p)}}{p}$	**...**	$\frac{i^{(p)}}{p}$	$\frac{i^{(p)}}{p}$	**equivalent payments**
(4)							i	
(5)	$\longleftarrow$			δ			$\longrightarrow$	

Figure 1 Equivalent payments

4 Force of interest as a function of time

In Section 2, we introduced the constant force of interest to describe the value of payments that are made continuously. In this section, we generalise this idea to consider the case when the force of interest is a function of time.

4.1 Formal definition

The force of interest is the instantaneous change in the fund value, expressed as an annualised percentage of the current fund value.

So the force of interest at time t is defined to be:

$$\boldsymbol{\delta(t) = \frac{V_t'}{V_t}} \qquad \textbf{(4.1)}$$

where V_t is the value of the fund at time t and V_t' is the derivative of V_t with respect to t.

How does this relate back to our previous definition of the (constant) force of interest? Suppose that we have a constant effective rate i and we invest C at time zero. Then the value at the fund at time t is:

$$V_t = C(1+i)^t$$

We now differentiate this with respect to t. Recall that $\frac{d}{dx}a^x = a^x \ln a$. So we have:

$$V_t' = C(1+i)^t \ln(1+i)$$

Substituting these into our definition (4.1) gives a constant force of interest of:

$$\delta(t) = \frac{C(1+i)^t \ln(1+i)}{C(1+i)^t} = \ln(1+i)$$

This is what we saw in equation (2.2). So our previous definition of a constant force of interest is actually a special case of definition (4.1).

Let's now obtain an expression for the accumulation factor using this variable force of interest.

Using the chain rule for differentiation, $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$. Applying this result to our definition $\delta(t) = \frac{V_t'}{V_t}$, we see that:

$$\boldsymbol{\delta(t) = \frac{d}{dt}\ln V_t}$$

Integrating this from t_1 to t_2 gives:

$$\int_{t_1}^{t_2} \delta(t)dt = \left[\ln V_t\right]_{t_1}^{t_2} = \ln V_{t_2} - \ln V_{t_1} = \ln\left(\frac{V_{t_2}}{V_{t_1}}\right)$$

Taking exponentials of both sides gives:

$$\Rightarrow \quad \frac{V_{t_2}}{V_{t_1}} = \exp\left(\int_{t_1}^{t_2} \delta(t)\,dt\right)$$

Hence:

$$A(t_1,t_2) = \exp\left(\int_{t_1}^{t_2} \delta(t)\,dt\right) \qquad \textbf{(4.2)}$$

This formula gives us an accumulation factor between times t_1 and t_2 when we have a variable force of interest, and appears on page 31 of the *Tables*. We can also write:

$$A(n) = A(0,n) = \exp\left(\int_0^n \delta(t)\,dt\right)$$

Let's apply this formula.

Question

The force of interest at time t is $\delta(t) = 0.02 + 0.01t$. Calculate the accumulated value at time 8 of an investment of £1,000 at time:

(i) 0

(ii) 5

Solution

(i) The accumulated value is:

$$V_8 = 1{,}000A(0,8)$$

Using (4.2) we have:

$$V_8 = 1{,}000\exp\left(\int_0^8 0.02 + 0.01t\,dt\right) = 1{,}000\exp\left(\left[0.02t + 0.005t^2\right]_0^8\right)$$

$$= 1{,}000e^{0.48} = £1{,}616.07$$

(ii) The accumulated value is:

$$V_8 = 1{,}000A(5,8)$$

Using (4.2) we have:

$$V_8 = 1{,}000\exp\left(\int_5^8 0.02 + 0.01t\,dt\right) = 1{,}000\exp\left(\left[0.02t + 0.005t^2\right]_5^8\right)$$

$$= 1{,}000e^{0.48-0.225} = £1{,}290.46$$

The force of interest can also be a stepwise function of time. In that case, we can apply (4.2) repeatedly, once for each of the different time periods covered by the accumulation.

Question

The force of interest at time t is given by:

$$\delta(t) = \begin{cases} 0.08 & 0 \le t < 5 \\ 0.13 - 0.01t & 5 \le t \end{cases}$$

Calculate the accumulated value at time 10 of an investment of $500 at time 2.

Solution

The accumulated value is:

$$\begin{aligned} V_{10} &= 500A(2,10) \\ &= 500A(2,5)A(5,10) \\ &= 500\exp\left(\int_2^5 0.08\,dt\right)\exp\left(\int_5^{10} 0.13 - 0.01t\,dt\right) \\ &= 500\exp\left(\left[0.08t\right]_2^5\right)\exp\left(\left[0.13t - 0.005t^2\right]_5^{10}\right) \\ &= 500e^{0.4-0.16}e^{0.8-0.525} \\ &= 500e^{0.515} \\ &= \$836.82 \end{aligned}$$

4.2 Relationship to constant force of interest

For the case when the force of interest is constant, δ, between time 0 and time n, we have:

$$A(n) = \boldsymbol{A(0,n) = e^{\int_0^n \delta\, dt} = e^{\delta n}}$$

This is equation (2.3) that we obtained earlier.

Equating this accumulation factor to the effective interest accumulation factor, we can obtain the relationship between constant force of interest and effective interest.

Hence:

$$\boldsymbol{(1+i)^n = e^{\delta n}}$$

Therefore:

$$\boldsymbol{(1+i) = e^{\delta}}$$

as before.

This is equation (2.1), which we obtained from our definition of the force of interest in terms of nominal rates of interest convertible *p*thly. Here we have reached it from a different starting point.

4.3 Present values

We can also obtain present values when we have a variable force of interest. The accumulation at time t_2 of an investment of C at time t_1 is given by:

$$C\,A(t_1,t_2) = C\exp\left(\int_{t_1}^{t_2} \delta(t)\, dt\right)$$

Therefore an investment of:

$$\frac{C}{A(t_1,t_2)} = \frac{C}{\exp\left(\int_{t_1}^{t_2} \delta(t)\, dt\right)} = C\exp\left(-\int_{t_1}^{t_2} \delta(t)\, dt\right)$$

at time t_1 will give C at time t_2. This is the discounted value at time t_1 of C due at time t_2. We simply have a negative power for discounting, which is similar to using $(1+i)^{-n}$ to discount for n years at a constant effective rate of interest.

In particular, the discounted value at time 0 (the present time) of C due at time $n \geq 0$ is called its discounted *present* value (or, more briefly, its present value). It is equal to:

$$C\exp\left(-\int_0^n \delta(t)\,dt\right)$$

Using the notation of $v(n)$ to represent the present value of a payment of 1 due at time n as defined in the previous chapter, we have:

$$v(n) = \exp\left(-\int_0^n \delta(t)\,dt\right)$$

We'll now look at some examples of calculating present values when the force of interest changes over time. We'll apply the same principles as when we were accumulating, but just put a negative in the power of the exponential.

Question

The force of interest at time t is given by:

$$\delta(t) = \begin{cases} 0.08 & 0 \leq t < 5 \\ 0.13 - 0.01t & 5 \leq t \end{cases}$$

(i) Calculate the present value at time 3 of a payment of \$500 at time 10.

(ii) Calculate the annual effective rate of interest that is equivalent to this variable force of interest from time 3 to time 10.

Solution

(i) The present value is:

$$\begin{aligned} V_3 &= 500\frac{1}{A(3,10)} \\ &= 500\frac{1}{A(3,5)}\frac{1}{A(5,10)} \\ &= 500\exp\left(-\int_3^5 0.08\,dt\right)\exp\left(-\int_5^{10} 0.13 - 0.01t\,dt\right) \\ &= 500\exp\left(-\left[0.08t\right]_3^5\right)\ \exp\left(-\left[0.13t - 0.005t^2\right]_5^{10}\right) \\ &= 500e^{-(0.4-0.24)}e^{-(0.8-0.525)} = 500e^{-0.435} \\ &= \$323.63 \end{aligned}$$

(ii) In terms of an annual effective rate of interest i, the discount factor for 7 years, from time 10 to time 3 is:

$$v^7 = (1+i)^{-7}$$

Setting this equal to the discount factor from time 10 to time 3 based on the variable force of interest calculated in part (i), we find:

$$(1+i)^{-7} = e^{-0.435} \quad \Rightarrow \quad i = \left(e^{-0.435}\right)^{-1/7} - 1 = 0.06411 \quad ie \quad 6.411\%$$

Finally, we'll obtain some general expressions for accumulation factors where there is a variable force of interest. We can use the same approach to obtain general expressions for discount factors.

Question

The force of interest at time t is:

$$\delta(t) = \begin{cases} 0.08 & 0 \le t < 5 \\ 0.13 - 0.01t & 5 \le t \end{cases}$$

Determine expressions for the accumulation factor from time 0 to time t.

Solution

We need separate expressions here for the accumulation factor depending on whether t is less than 5 or greater than or equal to 5. When $0 \le t < 5$ we have:

$$A(0,t) = \exp\left(\int_0^t 0.08\,ds\right) = \exp\left(\left[0.08s\right]_0^t\right) = e^{0.08t}$$

We should not use t as both the variable of integration and the upper limit on the integral. So we change the variable to another letter, say s.

When $5 \le t$, we need to determine the product of two accumulation factors:

$$A(0,t) = A(0,5)A(5,t) = \exp\left(\int_0^5 0.08\,dt\right)\exp\left(\int_5^t 0.13 - 0.01s\,ds\right)$$

$$= \exp\left(\left[0.08t\right]_0^5\right)\exp\left(\left[0.13s - 0.005s^2\right]_5^t\right)$$

$$= e^{0.4}e^{0.13t - 0.005t^2 - 0.525}$$

$$= e^{0.13t - 0.005t^2 - 0.125}$$

We could also have obtained $A(0,5)$ *by substituting* $t=5$ *into our first expression for* $A(0,t)$.

So, in summary:

$$A(0,t)=\begin{cases} e^{0.08t} & 0\le t<5 \\ e^{0.13t-0.005t^2-0.125} & 5\le t \end{cases}$$

4.4 Applications of force of interest

Although the force of interest is a theoretical measure, it is the most fundamental measure of interest (as all other interest rates can be derived from it). However, since the majority of transactions involve discrete processes we tend to use other interest rates in practice.

It still remains a useful conceptual and analytical tool and can be used as an approximation to interest paid very frequently, *eg* daily.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 5 Summary

Effective rates have interest paid *once* per measurement period. Effective *interest*, i, refers to the amount of interest that is paid at the *end* of the year. Effective *discount*, d, refers to the amount of interest that is paid at the *start* of the year.

Nominal rates are paid *more frequently* than once per measurement period. A nominal rate of *interest* convertible *p*thly, $i^{(p)}$, refers to the total amount of interest that is paid in p equal instalments at the *end* of each p th subinterval.

The relationships between nominal interest and effective interest are:

$$1+i=\left(1+\frac{i^{(p)}}{p}\right)^{p} \qquad i^{(p)}=p\left[(1+i)^{1/p}-1\right]$$

$i^{(p)}/p$ is the effective p thly rate of interest (*ie* the effective rate of interest for a time period of length $1/p$).

A nominal rate of *discount* convertible *p*thly, $d^{(p)}$, refers to the total amount of interest that is paid in p equal instalments at the *start* of each p th subinterval.

The relationships between nominal discount and effective interest and discount are:

$$1-d=\left(1-\frac{d^{(p)}}{p}\right)^{p} \qquad d^{(p)}=p\left[1-(1-d)^{1/p}\right]$$

$$1+i=\left(1-\frac{d^{(p)}}{p}\right)^{-p} \qquad d^{(p)}=p\left[1-(1+i)^{-1/p}\right]$$

$d^{(p)}/p$ is the effective p thly rate of discount (*ie* the effective rate of discount for a time period of length $1/p$).

To accumulate or discount with nominal rates requires that they are first converted to effective rates.

The **force of interest**, δ, is the amount of interest that is paid continuously over a time period. It is defined as $\delta=\lim\limits_{p\to\infty} i^{(p)}=\lim\limits_{p\to\infty} d^{(p)}$.

The relationships between the force of interest and effective interest are:

$$1+i=e^{\delta} \qquad \delta=\ln(1+i)$$

To accumulate or discount at a (constant) force of interest δ, we use the following:

$$A(n) = e^{\delta n} \qquad v(n) = e^{-\delta n}$$

More generally, we can define the force of interest as the instantaneous change in the fund value, expressed as an annualised percentage of the current fund value:

$$\delta(t) = \frac{V_t'}{V_t}$$

where V_t is the value of the fund at time t and V_t' is the derivative of V_t with respect to t.

Using this, we obtain the following accumulation factor:

$$A(t_1, t_2) = \exp\left(\int_{t_1}^{t_2} \delta(t)\, dt \right)$$

Chapter 5 Practice Questions

5.1 Given that $i = 0.07$, calculate:

(i) $i^{(6)}$

(ii) $d^{(6)}$

(iii) $i^{(4)}$

(iv) $d^{(2)}$

5.2 Calculate the annual effective rate of discount that is equivalent to a rate of interest of 4% *pa* convertible monthly.

5.3 (i) Calculate the effective annual rate of interest corresponding to:

Exam style

(a) a nominal rate of interest of 11% *pa* convertible half-yearly

(b) a nominal rate of interest of 12% *pa* convertible monthly. [2]

(ii) Calculate the rate of interest convertible monthly corresponding to:

(a) an effective rate of interest of 14.2% *pa*

(b) a nominal rate of interest of 11% *pa* convertible three times a year. [3]

[Total 5]

5.4 The constant nominal rate of interest convertible quarterly is 15% *pa*. Calculate the accumulated value after 7 years of a payment of £300.

5.5 £250 is invested at a discount rate of 18% *pa* convertible monthly for the first 3 months followed by an interest rate of 20% *pa* convertible quarterly for the next 9 months. Calculate the accumulated sum at the end of the year.

5.6 Assuming a force of interest of 9% *pa*, calculate the accumulated value of £6.34 after:

(i) 3 months

(ii) 3 years

(iii) 7 years and 5 days.

5.7 Exam style

(i) Calculate the accumulated value after 6 months of an investment of £100 at time 0 using the following rates of interest:

(a) a force of interest of 5% *pa*

(b) a rate of interest of 5% *pa* convertible monthly

(c) an effective rate of interest of 5% *pa*. [3]

(ii) Explain why the answer obtained in (i)(a) is higher than the answer obtained in (i)(c). [2]
[Total 5]

5.8 The force of interest at time *t* is:

$$\delta(t)=\begin{cases} 0.04 & 0\le t<6 \\ 0.2-0.02t & 6\le t \end{cases}$$

Calculate the accumulated value at time 8 of a payment of \$400 at time 3.

5.9 The force of interest at time t is given by $\delta(t)=0.01t+0.04$. Calculate the corresponding nominal rate of discount convertible half-yearly for the period $t=1$ to $t=2$.

5.10 The force of interest at time t is given by:

Exam style

$$\delta(t)=\begin{cases} 0.08-0.001t & 0\le t<3 \\ 0.025t-0.04 & 3\le t<5 \\ 0.03 & 5\le t \end{cases}$$

(i) Calculate the present value at time 2 of a payment of £1,000 at time 10. [5]

(ii) Calculate the annual effective rate of interest that is equivalent to this variable force of interest from time 2 to time 10. [2]
[Total 7]

5.11 The force of interest at time t is given by:

Exam style

$$\delta(t)=\begin{cases} 0.04+0.002t & 0\le t<10 \\ 0.015t-0.08 & 10\le t<12 \\ 0.07 & 12\le t \end{cases}$$

Determine expressions for the present value at time 0 of 1 unit of money due at time t. [5]

Chapter 5 Solutions

5.1 (i) $i^{(6)} = 6\left((1+i)^{1/6} - 1\right) = 6\left((1.07)^{1/6} - 1\right) = 0.068042 \quad ie \quad 6.8042\%$

(ii) $d^{(6)} = 6\left(1 - (1+i)^{-1/6}\right) = 6\left(1 - (1.07)^{-1/6}\right) = 0.067279 \quad ie \quad 6.7279\%$

The relationship $d = iv$ only holds for effective rates of interest and discount, so we cannot calculate $d^{(6)}$ from $i^{(6)}$ using $d^{(6)} = \dfrac{i^{(6)}}{1+i^{(6)}}$. However, we can calculate it using:

$$\frac{d^{(6)}}{6} = \frac{\frac{i^{(6)}}{6}}{1 + \frac{i^{(6)}}{6}}$$

(iii) $i^{(4)} = 4\left((1+i)^{1/4} - 1\right) = 4\left((1.07)^{1/4} - 1\right) = 0.068234 \quad ie \quad 6.8234\%$

Alternatively, this value can be taken directly from the Tables.

(iv) $d^{(2)} = 2\left(1 - (1+i)^{-1/2}\right) = 2\left(1 - (1.07)^{-1/2}\right) = 0.066527 \quad ie \quad 6.6527\%$

Alternatively, this value can be taken directly from the Tables.

5.2 Given $i^{(12)} = 4\%$, we can calculate the annual effective interest rate as follows:

$$i = \left(1 + \frac{i^{(12)}}{12}\right)^{12} - 1 = \left(1 + \frac{0.04}{12}\right)^{12} - 1 = 0.040742$$

So the annual effective discount rate is:

$$d = 1 - v = 1 - \frac{1}{1+i} = 1 - \frac{1}{1.040742} = 0.039147 \quad ie \quad 3.9147\%$$

Alternatively, we can calculate d directly, using different forms of discount factor:

$$d = 1 - v = 1 - \left(1 + \frac{i^{(12)}}{12}\right)^{-12} = 1 - \left(1 + \frac{0.04}{12}\right)^{-12} = 0.039147$$

5.3 (i) (a) $i = \left(1 + \dfrac{i^{(2)}}{2}\right)^{2} - 1 = \left(1 + \dfrac{0.11}{2}\right)^{2} - 1 = 0.113025 \quad ie \quad i = 11.3025\%$ [1]

(b) $i = \left(1 + \dfrac{i^{(12)}}{12}\right)^{12} - 1 = \left(1 + \dfrac{0.12}{12}\right)^{12} - 1 = 0.126825 \quad ie \quad i = 12.6825\%$ [1]

[Total 2]

(ii) (a) $i^{(12)} = 12\left((1+i)^{1/12} - 1\right) = 12\left(1.142^{1/12} - 1\right) = 0.133518$ *ie* $i^{(12)} = 13.3518\%$ [1]

(b) If $i^{(3)} = 11\%$, then:

$$i = \left(1 + \frac{i^{(3)}}{3}\right)^3 - 1 = \left(1 + \frac{0.11}{3}\right)^3 - 1 = 0.114083 \quad [1]$$

So:

$$i^{(12)} = 12\left((1+i)^{1/12} - 1\right) = 12\left[1.114083^{1/12} - 1\right] = 0.108519$$

ie $i^{(12)} = 10.8519\%$. [1]

[Total 3]

5.4 We are given $i^{(4)} = 15\%$. This is equivalent to an annual effective rate of:

$$i = \left(1 + \tfrac{0.15}{4}\right)^4 - 1 = 15.8650\%$$

Accumulating £300 for 7 years at this rate gives:

$$\pounds 300 \times 1.158650^7 = \pounds 840.98$$

Alternatively, we could work in quarters. We are given $i^{(4)} = 15\%$ *. This is equivalent to a quarterly effective interest rate of* $\tfrac{15\%}{4} = 3.75\%$ *. Accumulating £300 for 7 years (= 28 quarters) at this rate gives:*

$$\pounds 300 \times 1.0375^{28} = \pounds 840.98$$

5.5 For the first 3 months, we have $d^{(12)} = 18\%$. So the annual effective interest rate is:

$$i = \left(1 - \tfrac{0.18}{12}\right)^{-12} - 1 = 19.88511\%$$

Accumulating £250 for 3 months at this rate gives:

$$\pounds 250 \times 1.1988511^{3/12} = \pounds 261.60$$

For the next 9 months, we have $i^{(4)} = 20\%$. So the annual effective interest rate is:

$$i = \left(1 + \tfrac{0.2}{4}\right)^4 - 1 = 21.550625\%$$

Accumulating £261.60 for 9 months at this rate gives:

$$\pounds 260.61 \times 1.21550625^{9/12} = \pounds 302.83$$

Alternatively, we could work in months for the first 3 months. We are given $d^{(12)}=18\%$. *This is equivalent to a monthly effective discount rate of* $\frac{18\%}{12}=1.5\%$. *Accumulating £250 for 3 months at this rate gives:*

$$£250\times(1-0.015)^{-3}=£261.60$$

Then we could work in quarters for the next 9 months. We have $i^{(4)}=20\%$, *which is equivalent to an effective quarterly interest rate of* $\frac{20\%}{4}=5\%$. *Accumulating £261.60 for 9 months (= 3 quarters) at this rate gives:*

$$£261.60\times1.05^{3}=£302.83$$

5.6 (i) $6.34e^{\frac{3}{12}\times0.09}=£6.48$

(ii) $6.34e^{3\times0.09}=£8.31$

(iii) $6.34e^{\left(7+\frac{5}{365}\right)\times0.09}=£11.92$

5.7 (i) The accumulated values after 6 months are:

(a) $100e^{0.5\delta}=100e^{0.5\times0.05}=£102.53$ [1]

(b) $100\left(1+\dfrac{i^{(12)}}{12}\right)^{6}=100\left(1+\dfrac{0.05}{12}\right)^{6}=£102.53$ [1]

(c) $100(1+i)^{0.5}=100\times1.05^{0.5}=£102.47$ [1]

[Total 3]

(ii) The force of interest in (i)(a) is 5% *pa* and the effective annual rate of interest in (i)(c) is 5% *pa*. The force of interest relates to interest being paid continuously over the period, whereas the effective annual rate of interest relates to interest being paid at the end of the period. [1]

Since the interest received early in the period under the force of interest will earn further interest, the accumulated value is higher than when the interest is all paid at the end of the period (and can therefore not earn further interest). [1]

A force of interest of 5% pa is equivalent to an annual effective interest rate of $i=e^{0.05}-1=5.127\%$, *which is greater than the 5% figure used in (i)(c).*

[Total 2]

5.8 The accumulated value at time 8 is:

$$400A(3,8) = 400A(3,6)A(6,8)$$

$$= 400\exp\left(\int_3^6 0.04\,dt\right)\exp\left(\int_6^8 0.2 - 0.02t\,dt\right)$$

$$= 400\exp\left([0.04t]_3^6\right)\exp\left(\left[0.2t - 0.01t^2\right]_6^8\right)$$

$$= 400e^{0.24-0.12}e^{0.96-0.84}$$

$$= 400e^{0.24}$$

$$= \$508.50$$

5.9 The accumulation factor from time 1 to time 2 is:

$$A(1,2) = \exp\left(\int_1^2 (0.01t + 0.04)dt\right)$$

$$= \exp\left(\left[0.005t^2 + 0.04t\right]_1^2\right)$$

$$= \exp[(0.02 + 0.08) - (0.005 + 0.04)]$$

$$= e^{0.055}$$

Equating this to a one-year accumulation factor in terms of $d^{(2)}$, the nominal rate of discount convertible half-yearly, we find that:

$$\left(1 - \frac{d^{(2)}}{2}\right)^{-2} = e^{0.055} \quad \Rightarrow \quad d^{(2)} = 2\left(1 - \left(e^{0.055}\right)^{-1/2}\right) = 0.054251 \quad ie \quad 5.4251\%$$

5.10 (i) ***Present value***

The present value at time 2 is given by the expression:

$$PV_{t=2} = 1{,}000\exp\left(-\int_2^3 (0.08 - 0.001t)\,dt\right)\exp\left(-\int_3^5 (0.025t - 0.04)\,dt\right)\exp\left(-\int_5^{10} 0.03\,dt\right) \qquad [2]$$

Evaluating the integrals, we find that:

$$PV_{t=2} = 1{,}000\exp\left(-\left[0.08t - 0.0005t^2\right]_2^3\right)\exp\left(-\left[0.0125t^2 - 0.04t\right]_3^5\right)\exp\left(-\left[0.03t\right]_5^{10}\right)$$

$$= 1{,}000e^{-(0.2355-0.158)}e^{-(0.1125+0.0075)}e^{-(0.3-0.15)}$$

$$= 1{,}000e^{-0.3475}$$

$$= £706.45$$ [3]

[Total 5]

(ii) ***Equivalent annual effective interest rate***

If the annual effective rate of interest is i, we have:

$$706.45(1+i)^8 = 1{,}000$$ [1]

Solving for i gives:

$$i = \left(\frac{1{,}000}{706.45}\right)^{1/8} - 1 = 4.44\%\ pa$$ [1]

[Total 2]

5.11 We need separate expressions here for the present values depending on whether t is less than 10, between 10 and 12 or greater than or equal to 12.

When $0 \le t < 10$:

$$v(t) = \exp\left(-\int_0^t 0.04 + 0.002s\,ds\right)$$

$$= \exp\left(-\left[0.04s + 0.001s^2\right]_0^t\right)$$

$$= e^{-0.04t-0.001t^2}$$ [1]

When $10 \le t < 12$, we need the product of two discount factors:

$$v(t) = \exp\left(-\int_0^{10}(0.04+0.002t)\,dt\right)\exp\left(-\int_{10}^t(0.015s-0.08)\,ds\right)$$

$$= \exp\left(-\left[0.04t + 0.001t^2\right]_0^{10}\right)\exp\left(-\left[0.0075s^2 - 0.08s\right]_{10}^t\right)$$

$$= e^{-0.5}e^{-0.0075t^2+0.08t-0.05}$$

$$= e^{-0.0075t^2+0.08t-0.55}$$ [2]

Alternatively, we could have set $t = 10$ in the formula for $0 \le t < 10$ to obtain the factor of $e^{-0.5}$.

When $12 \le t$, we need to the product of three discount factors:

$$v(t) = \exp\left(-\int_0^{10} (0.04 + 0.002t)\, dt\right)\exp\left(-\int_{10}^{12} (0.015s - 0.08)\, ds\right)\exp\left(-\int_{12}^{t} 0.07\, ds\right)$$

$$= \exp\left(-\left[0.04t + 0.001t^2\right]_0^{10}\right)\exp\left(-\left[0.0075s^2 - 0.08s\right]_{10}^{12}\right)\exp\left(-\left[0.07s\right]_{12}^{t}\right)$$

$$= e^{-0.5}e^{-(0.12+0.05)}e^{-0.07t+0.84}$$

$$= e^{-0.67}e^{-0.07t+0.84}$$

$$= e^{-0.07t+0.17}$$ [2]

Alternatively, we could have set $t = 12$ *in the formula for* $10 \le t < 12$ *to obtain the factor* $e^{-0.67}$*.*

In summary:

$$v(t) = \begin{cases} e^{-0.04t-0.001t^2} & 0 \le t < 10 \\ e^{-0.0075t^2+0.08t-0.55} & 10 \le t < 12 \\ e^{-0.07t+0.17} & 12 \le t \end{cases}$$

[Total 5]

6

Real and money interest rates

Syllabus objectives

2.2 Demonstrate a knowledge and understanding of real and money interest rates.

0 Introduction

This chapter looks at two types of interest rate, namely real and money rates. Real rates of interest are needed when inflation needs to be taken into account. Money rates of interest are used when inflation does not need to be taken into account.

Money rates of interest are also sometimes referred to as nominal rates of interest, but these are not related to the nominal rates of interest convertible *p*thly, $i^{(p)}$, that we met in the previous chapter.

An actuary will use either the real or money rate of interest depending on whether inflation has already been allowed for or not.

We will meet real rates of interest again in Chapter 13.

1 Definition of real and money interest rates

Accumulating an investment of 1 for a period of time t from time 0 produces a new total accumulated value $A(0,t)$, say.

Typically the investment of 1 will be a sum of money, say £1 or $1 or €1.

Question

Calculate the accumulated value if $1 is invested for 7 years at an interest rate of 6.5% *pa* effective.

Solution

The accumulated value at time 7 years is:

$$\$1 \times 1.065^7 = \$1.55$$

In this case, if we are given the information on the initial investment of 1 in the specified currency, the period of the investment, and the cash amount of money accumulated, then the underlying interest rate is termed a 'money rate of interest'.

In the above question, we have a money rate of interest of 6.5% *pa* effective.

More generally, given any series of monetary payments accumulated over a period, a money rate of interest is that rate which will have been earned so as to produce the total amount of cash in hand at the end of the period of accumulation.

In practice, most such accumulations will take place in economies subject to inflation, where a given sum of money in the future will have less purchasing power than at the present day. It is often useful, therefore, to reconsider what the accumulated value is worth allowing for the eroding effects of inflation.

Inflation is a measure of the increase in the cost of goods and services, for example the price of a loaf of bread or a litre of petrol.

Purchasing power refers to the goods that a given amount of money can buy. When inflation occurs, we can buy less goods with the same amount of money.

Question

A bunch of flowers costs £13 on 1/1/18 and £14 on 31/12/18.

Calculate the annual rate of inflation on bunches of flowers during 2018.

Solution

The annual rate of inflation, j, is given by:

$$1+j=\frac{14}{13} \quad \Rightarrow \quad j=7.69\%$$

In this question, since there has been inflation during 2018, the same amount of money buys fewer bunches of flowers at the end of the year. For example, £182 would buy $\frac{182}{13}=14$ bunches of flowers on 1/1/18, but only $\frac{182}{14}=13$ bunches of flowers on 31/12/18. So, the purchasing power of £182 is lower at the end of the year than at the start of the year.

Question

Based on the prices of flowers given above, calculate the amount of money at the start of the year that is equivalent to £182 at the end of the year.

Solution

Since £182 can only buy 13 bunches of flowers at the end of the year, it is equivalent in value to $13\times £13=£169$ at the start of the year.

Notice that $\frac{182}{1.0769}=169$*, ie we divide by (1 + the rate of inflation) to calculate how much money at the end of the year is worth.*

Returning to the initial Core Reading example above, suppose the accumulation took place in an economy subject to inflation so that the cash $A(0,t)$ is effectively worth only $A^*(0,t)$ after allowing for inflation, where $A^*(0,t)<A(0,t)$. In this case, the rate of interest at which the original sum of 1 would have to be accumulated to produce the sum A^* is lower than the money rate of interest.

The sum $A^*(0,t)$ is referred to as the real amount accumulated, and the underlying interest rate, reduced for the effects of inflation, is termed a 'real rate of interest'.

Question

A bank offers an effective annual rate of interest on one of its accounts of 4.2%. The rate of inflation is 3% *pa* effective. Calculate the real rate of interest.

Solution

£1 accumulates in the bank account over one year to £1.042.

Since the rate of inflation is 3% *pa*, goods that cost £1 at the start of the year, cost £1.03 at the end of the year. So the quantity of goods that can be purchased at the end of the year is:

$$\frac{1.042}{1.03} = 1.0117$$

times the quantity that can be purchased at the start of the year.

Therefore the real rate of interest is 1.17% *pa* effective.

We will look at these calculations in more detail later in this course.

More generally, given any series of monetary payments accumulated over a period, a real rate of interest is that rate which will have been earned so as to produce the total amount of cash in hand at the end of the period of accumulation reduced for the effects of inflation.

Chapter 13 of this subject describes ways of calculating real rates of interest given money rates of interest (and vice versa).

2 Deflationary conditions

The above descriptions assume that the inflation rate is positive. Where the inflation rate is negative, termed 'deflation', the above theory still applies and $A^*(0,t) > A(0,t)$, giving rise to the conclusion that the real rate of interest in such circumstances would be higher than the money rate of interest.

Negative inflation rates do occur in some countries from time to time, *eg* the UK had a negative inflation rate during parts of 2015.

Question

A bank offers an annual effective rate of interest on one of its accounts of 4.2%. The rate of inflation is -2% *pa* effective. Calculate the real rate of interest.

Solution

£1 accumulates in the bank account over one year to £1.042.

Since the rate of inflation is -2% *pa*, goods that cost £1 at the start of the year, cost £0.98 at the end of the year. So the quantity of goods that can be purchased at the end of the year is:

$$\frac{1.042}{0.98} = 1.0633$$

times the quantity that can be purchased at the start of the year.

Therefore the real rate of interest is 6.33% *pa* effective.

Notice here that (1 + the rate of inflation) < 1 since inflation is negative.

As might be expected, where there is no inflation $A^*(0,t) = A(0,t)$, and the real and money rates of interest are the same.

This is because any amount of money can buy the same amount of goods at both the start and the end of the year.

3 Usefulness of real and money interest rates

We assume here that we have a positive inflation rate.

Which of the two rates of interest, real or money, is the more useful will depend on two main factors:

- **the purpose to which the rate will be put**
- **whether the underlying data have or have not already been adjusted for inflation.**

The purpose to which the rate will be put

Generally, where the actuary is performing calculations to determine how much should be invested to provide for future outgo, the first step will be to determine whether the future outgo is real or monetary in nature. The type of interest rate to be assumed would then be, respectively, a real or a monetary rate.

For example, first suppose that an actuary was asked to calculate the sum to be invested by a person aged 40 to provide for a round-the-world cruise when the person reaches 60, and where the person says the cruise costs £25,000.

Unless the person has, for some reason, already made an allowance for inflation in suggesting a figure of £25,000 then that amount is probably today's cost of the cruise. In this case, the actuary would be wise to assume (checking his understanding with the person) an inflation rate and this could be achieved by assuming a real rate of interest.

Here the future outgo of £25,000 is real in nature in that it is likely to be more than £25,000 in 20 years' time.

As an alternative example, suppose that a person has a mortgage of £50,000 to be paid off in twenty years' time. Here, the party that granted the mortgage would contractually be entitled to only £50,000 in twenty years' time. Accordingly, in working out how much should be invested to repay the outgo in this case, a money rate of interest would be assumed.

Here the future outgo of £50,000 is monetary in nature in that it is going to be exactly £50,000 in 20 years' time.

Whether the underlying data have or have not already been adjusted for inflation

In the first example above, we see that the data may already have been adjusted for inflation and in that case it would not be appropriate to allow for inflation again. A money rate would then be assumed.

More generally in actuarial work, the nature of the data provided must be understood before choosing the type and amount of assumptions to be made.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 6 Summary

Real rates of interest allow for future inflation. Money rates of interest ignore the effects of inflation.

In periods of positive inflation, the real rate of interest is lower than the money rate of interest.

In periods of negative inflation, the real rate of interest is higher than the money rate of interest.

In periods of zero inflation, the real rate of interest is equal to the money rate of interest.

Which of the two rates of interest, real or money, is the more useful will depend on two main factors:

- the purpose to which the rate will be put
- whether the underlying data values have or have not been adjusted for inflation.

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 6 Practice Questions

6.1 (i) Define a real rate of interest.

(ii) Define a money rate of interest.

(iii) State the circumstances in which the real rate of interest would be lower than the money rate of interest.

6.2 An inflation index $Q(t)$ is such that $Q(1/1/13) = 724$ and $Q(1/1/18) = 913$. Calculate the average annual rate of inflation over the period from 1 January 2013 to 1 January 2018.

6.3 In each of the following circumstances, state whether the calculations should use a money or real rate of interest.

(i) An actuarial student wants to invest an amount of money now to buy a new car in one year's time. Today's list price of the car is available.

(ii) A woman wants to invest a lump sum today in order to provide her with a fixed income of £25,000 *pa* for the rest of her life.

(iii) A man buys a zero-coupon bond that will provide him with £100,000 in 10 years' time. He is trying to calculate an appropriate purchase price.

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 6 Solutions

6.1 (i) A real rate of interest is a rate of interest that allows for inflation. It is the rate earned in order to produce the total amount of cash in hand at the end of the period of accumulation reduced for the effect of inflation, and represents the increase in the quantity of goods than can be purchased.

(ii) A money rate of interest is a rate of interest that does not allow for inflation. It is the rate earned in order to produce the total amount of cash in hand at the end of the period of accumulation.

(iii) The real rate of interest is lower than the money rate of interest when (positive) inflation is present.

6.2 The average annual rate of inflation j can be found from the equation:

$$(1+j)^5 = \frac{Q(1/1/18)}{Q(1/1/13)} = \frac{913}{724} \quad \Rightarrow \quad j = \left(\frac{913}{724}\right)^{1/5} - 1 = 4.75\%$$

6.3 (i) A real rate would be used here since the price of the car is known now, but is likely to increase over the year.

(ii) A money rate would be used here since the income of £25,000 *pa* is fixed, and so is known in money terms.

(iii) A money rate would be used here since the payment of £100,000 (the redemption value) is fixed.

7

Discounting and accumulating

Syllabus objectives

2.4 Calculate the present value and accumulated value for a given stream of cashflows under the following individual or combination of scenarios:

2.4.1 Cashflows are equal at each time period.

2.4.2 Cashflows vary with time which may or may not be a continuous function of time.

2.4.3 Some of the cashflows are deferred for a period of time.

2.4.4 Rate of interest or discount is constant.

2.4.5 Rate of interest or discount varies with time which may or may not be a continuous function of time.

0 Introduction

So far we have calculated present values and accumulated values of single payments. This chapter considers present values and accumulated values of a series of payments and of continuous payment streams.

1 Present values of cashflows

In many compound interest problems, one must find the discounted present value of cashflows due in the future. It is important to distinguish between (a) discrete and (b) continuous payments.

In Section 1.1 we will consider discrete payments before looking at continuous payments in Section 1.2.

1.1 Discrete cashflows

We have already seen that the present value of a cashflow, C, due at time t years is Cv^t where $v = 1/(1+i)$ and i is the effective rate of interest per annum. Here we are assuming that the effective rate of interest is constant over the period.

What if we have two payments, C_1 due at time t_1, and C_2 at time t_2? The present value of these payments is the amount we would have to invest in a bank account to be able to pay each of the payments at the times they are required. Rather than investing a single sum into a single bank account to provide for the payments, we could have set up a *separate* bank account to cater for each payment and invested the present value of each payment in the corresponding account.

This alternative arrangement would have exactly the same result. So, we see that the present value of the two payments is just the sum of the individual present values.

More generally, the present value of a series of payments of $c_{t_1}, c_{t_2}, \ldots, c_{t_n}$ due at times $t_1, t_2, \ldots, t_n$ is given by:

$$\sum_{j=1}^{n} c_{t_j} v^{t_j}$$

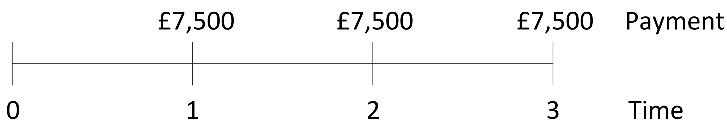
Question

Under its current rental agreement, a company is obliged to make payments of £7,500 at the end of each of the next three years for the building it occupies. The company wishes to provide for these payments by investing a single sum in its bank account that pays interest at a rate of 7.5% *pa* effective.

Calculate the sum that must be invested today.

Solution

The payments can be represented on a timeline as follows:



Here: $v = \dfrac{1}{1+i} = 1.075^{-1}$.

So: PV of payment due at time 1 $= 7{,}500v = 7{,}500 \times 1.075^{-1} = 6{,}976.74$

PV of payment due at time 2 $= 7{,}500v^2 = 7{,}500 \times 1.075^{-2} = 6{,}489.99$

PV of payment due at time 3 $= 7{,}500v^3 = 7{,}500 \times 1.075^{-3} = 6{,}037.20$

So the amount that must be invested in the account today is:

PV all payments $= 6{,}976.74 + 6{,}489.99 + 6{,}037.20 = £19{,}504$

If the effective interest rate is not constant, then we could write the present value in terms of the function $v(t)$, where $v(t)$ is the (discounted) present value of 1 due at time t.

The present value of the sums $c_{t_1}, c_{t_2}, ..., c_{t_n}$ due at times $t_1, t_2, ..., t_n$ (where $0 \le t_1 < t_2 < \ldots < t_n$) is:

$$c_{t_1}v(t_1) + c_{t_2}v(t_2) + \cdots + c_{t_n}v(t_n) = \sum_{j=1}^{n} c_{t_j}v(t_j)$$

If the number of payments is infinite, the present value is defined to be:

$$\sum_{j=1}^{\infty} c_{t_j}v(t_j)$$

provided that this series converges. It usually will in practical problems.

We could also express the present value in terms of a force of interest that varies over time. If the force of interest at time t is $\delta(t)$, then:

$$v(t) = \exp\left(-\int_0^t \delta(s)\,ds\right)$$

and the present value of the sums $c_{t_1}, c_{t_2}, ..., c_{t_n}$ due at times $t_1, t_2, ..., t_n$ is:

$$\sum_{j=1}^{n} c_{t_j} \exp\left(-\int_0^{t_j} \delta(s)\,ds\right)$$

Question

Calculate the value at time $t=0$ of \$250 due at time $t=6$ and \$600 due at time $t=8$ if $\delta(t)=3\%$ *pa* for all *t*.

Solution

If the force of interest, δ, is constant, then the present value of a payment of *C* due at time *t* is:

$$Ce^{-\delta t}$$

Therefore the present value of these two payments is:

$$250e^{-6\times 0.03}+600e^{-8\times 0.03}=\$680.79$$

1.2 Continuously payable cashflows (payment streams)

Suppose that $T>0$ and that between times 0 and *T* an investor will be paid money continuously, the rate of payment at time *t* being £ $\rho(t)$ per unit time. What is the present value of this cashflow?

In order to answer this question, it is essential to understand what is meant by the 'rate of payment' of the cashflow at time *t*. If *M*(*t*) denotes the total payment made between time 0 and time *t*, then by definition:

$$\rho(t)=M'(t) \qquad \text{for all } t$$

Then, if $0\le\alpha<\beta\le T$, the total payment received between time α and time β is:

$$\begin{aligned} M(\beta)-M(\alpha) &= \int_\alpha^\beta M'(t)\,dt \\ &= \int_\alpha^\beta \rho(t)\,dt \end{aligned} \tag{1.1}$$

Thus the rate of payment at any time is simply the derivative of the *total* amount paid up to that time, and the total amount paid between any two times is the integral of the rate of payments over the appropriate time interval.

Intuitively, we can think of the integral in (1.1) as the sum of lots of small payments, each of amount $\rho(t)dt$. It may help to consider the following simple example.

If the rate of payment is a constant £24 *pa*, then in any one year the total amount paid is £24, but this payment is spread evenly over the year. In half a year, the total paid is $24\times\frac{1}{2}$, *ie* £12. In one month, the total paid is $24\times\frac{1}{12}$, *ie* £2. So, in a small time period *dt*, the total paid is $£24\,dt$.

Question

A life insurer starts issuing a new type of 10-year savings policy to young investors who pay weekly premiums of £10. The insurer assumes that it will sell 10,000 policies evenly over each year and that no policyholders will stop paying premiums after taking out the policy.

Calculate the total premium income that will be received during the first 3 years, assuming that there are 52.18 ($=365.25/7$) weeks in each year.

Solution

Let t measure the time in years from when the first policy is sold. By time t, the insurer will have sold $10{,}000t$ policies.

For example, at time 0, the insurer will have sold 0 policies, at time 1 it will have sold 10,000 policies, and at time 0.5 it will have sold 5,000 (as the policies are assumed to be sold evenly over each year).

So the weekly premium income at time t will be $10{,}000t \times £10 = £100{,}000t$, and this corresponds to an annual rate of premium income at time t of:

$$52.18 \times £100{,}000t = £5{,}218{,}000t$$

The total premium income received in the first three years is the integral of the rate of payment of premium income over the time interval:

$$\int_0^3 5{,}218{,}000t\,dt = \left[\frac{5{,}218{,}000t^2}{2}\right]_0^3 = 2{,}609{,}000 \times 9 = £23{,}481{,}000$$

We can check this answer by reasoning in a different way. Since the policies are sold evenly over each year, in the first year there are on average 5,000 policies paying premiums (as no policies have been sold at the start of the year and 10,000 have been sold by the end of the year). These will generate income during the first year of:

$$5{,}000 \times 10 \times 52.18 = £2{,}609{,}000$$

In the second year, there are on average 15,000 policies paying premiums (as 10,000 policies have been sold at the start of the year and 20,000 have been sold by the end of the year). These will generate income during the second year of:

$$15{,}000 \times 10 \times 52.18 = £7{,}827{,}000$$

In the third year, there are on average 25,000 policies paying premiums. These will generate income during the third year of:

$$25{,}000 \times 10 \times 52.18 = £13{,}045{,}000$$

So the total premium income received during the first three year is, as before:

$$2{,}609{,}000 + 7{,}827{,}000 + 13{,}045{,}000 = £23{,}481{,}000$$

(This alternative approach only works because the number of policies sold increases linearly over time.)

Between times t and $t+dt$ the total payment received is $M(t+dt) - M(t)$. If dt is very small this is approximately $M'(t)dt$ or $\rho(t)dt$. Theoretically, therefore, we may consider the present value of the money received between times t and $t+dt$ as $v(t)\rho(t)dt$. The present value of the entire cashflow is obtained by integration as:

$$\int_0^T v(t)\rho(t)dt$$

where T is the time at which the payment stream ends.

Intuitively, we can view this integral as summing, between times $t = 0$ and $t = T$, the present values of an infinite number of payments. At each time point, t, the payment made (in the time interval of length dt) is $\rho(t)dt$. We use $v(t)$ to discount the payment from time t to time 0, and the integral 'sums' the infinite number of payments.

If T is infinite we obtain, by a similar argument, the present value:

$$\int_0^\infty v(t)\rho(t)dt$$

By combining the results for discrete and continuous cashflows, we obtain the formula:

$$\sum c_t v(t) + \int_0^\infty v(t)\rho(t)dt \qquad \textbf{(1.2)}$$

for the present value of a general cashflow (the summation being over those values of t for which c_t, the discrete cashflow at time t, is non-zero).

Assuming a constant interest rate this simplifies slightly to the important result for the present value of a series of discrete cashflows and a continuous cashflow:

$$\sum c_t v^t + \int_0^\infty v^t \rho(t)dt$$

Question

A company expects to receive a continuous cashflow for the next five years, where the rate of payment is 100×0.8^t at time t years.

Calculate the present value of this cashflow assuming a constant force of interest of 8% *pa* throughout the period.

Solution

The present value can be calculated using the formula:

$$\int_0^5 v(t)\rho(t)dt$$

with $\rho(t)=100\times 0.8^t$ and $v(t)=e^{-0.08t}$. Using the properties of exponentials and logs, we can write $0.8^t=e^{\ln 0.8^t}=e^{t\ln 0.8}$, so the present value is:

$$\begin{aligned}\int_0^5 e^{-0.08t}\times 100\times e^{t\ln 0.8}\,dt &= 100\int_0^5 e^{(-0.08+\ln 0.8)t}\,dt\\ &=100\left[\frac{e^{(-0.08+\ln 0.8)t}}{-0.08+\ln 0.8}\right]_0^5\\ &=100\left[\frac{e^{(-0.08+\ln 0.8)\times 5}-1}{-0.08+\ln 0.8}\right]\\ &=£257.42\end{aligned}$$

So far we have assumed that all payments, whether discrete or continuous, are positive. If one has a series of income payments (which may be regarded as positive) and a series of outgoings (which may be regarded as negative) their *net present value* is defined as the difference between the value of the positive cashflow and the value of the negative cashflow.

The net present value will often be abbreviated to NPV. We will study net present values in Chapter 12.

Question

A company expects to receive a continuous cashflow of £350 *pa* for the next five years. It also expects to have to pay out £600 at the end of the first year and £400 at the end of the third year.

Calculate the net present value of these cashflows if $v(t)=1-0.01t$ for $0\le t\le 5$.

Solution

The present value of the income can be calculated using the formula:

$$\int_0^5 v(t)\rho(t)dt$$

with $\rho(t)=350$ and $v(t)=1-0.01t$.

This gives:

$$\int_0^5 350(1-0.01t)dt = \left[350t - 1.75t^2\right]_0^5 = £1,706.25$$

The present value of the outgo is:

$$600v(1) + 400v(3) = 600 \times 0.99 + 400 \times 0.97 = £982$$

So the net present value is:

$$\text{NPV} = 1,706.25 - 982 = £724.25$$

2 Valuing cashflows

Consider times t_1 and t_2 , where t_2 is not necessarily greater than t_1. *The value at time t_1 of the sum C due at time t_2* is defined as:

(a) If $t_1 \geq t_2$, the accumulation of C from time t_2 until time t_1; or

(b) If $t_1 < t_2$, the discounted value at time t_1 of C due at time t_2.

In both cases the value at time t_1 of C due at time t_2 is:

$$C\exp\left[-\int_{t_1}^{t_2}\delta(t)dt\right] \tag{2.1}$$

(Note the convention that, if $t_1 > t_2$, $\int_{t_1}^{t_2}\delta(t)dt = -\int_{t_2}^{t_1}\delta(t)dt$.)

This result was derived in the last chapter. If there is a constant force of interest, *ie* $\delta(t)=\delta$ for all t, then this result becomes:

$$Ce^{-(t_2-t_1)\delta} = Ce^{(t_1-t_2)\delta}$$

Since:

$$\int_{t_1}^{t_2}\delta(t)dt = \int_0^{t_2}\delta(t)dt - \int_0^{t_1}\delta(t)dt$$

it follows immediately from Equation (2.1) that the value at time t_1 of C due at time t_2 is:

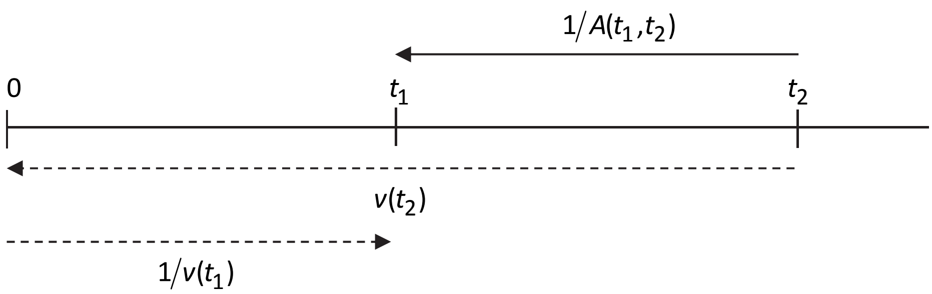
$$C\frac{v(t_2)}{v(t_1)} \tag{2.2}$$

Remembering that $v(t)=1/A(0,t)$, this result can be written in the form $Cv(t_2)A(0,t_1)$.

This tells us that in order to find the value at time t_1 of C due at time t_2, we could first discount back to time 0 by multiplying by $v(t_2)$, and then accumulate to time t_1 by multiplying by $A(0,t_1)$. The last step is equivalent to dividing by $v(t_1)$.

Alternatively, this calculation could be performed directly using the expression $C/A(t_1,t_2)$.

These two approaches are represented diagrammatically below:



Question

Calculate the value at time 4 of a payment of 860 at time 10 if $v(10) = 0.76$ and $v(4) = 0.91$.

Solution

The value at time 4 is:

$$860 \times \frac{v(10)}{v(4)} = 860 \times \frac{0.76}{0.91} = 718.24$$

The value at a general time t_1 of a discrete cashflow of c_t at time *t* (for various values of *t*) and a continuous payment stream at rate $\rho(t)$ per time unit may now be found, by the methods given in Section 1, as:

$$\sum c_t \frac{v(t)}{v(t_1)} + \int_{-\infty}^{\infty} \rho(t) \frac{v(t)}{v(t_1)} dt \qquad \textbf{(2.3)}$$

where the summation is over those values of t for which $c_t \neq 0$.

We note that in the special case when $t_1 = 0$ (the present time), the value of the cashflow is:

$$\sum c_t v(t) + \int_{-\infty}^{\infty} \rho(t) v(t) dt$$

where the summation is over those values of t for which $c_t \neq 0$. This is a generalisation of formula (1.2) to cover the past as well as present or future payments. If there are incoming and outgoing payments, the corresponding *net value* may be defined, as in Section 1, as the difference between the value of the *positive* and the *negative* cashflows. If all the payments are due at or after time t_1, their value at time t_1 may also be called their 'discounted value', and if they are due at or before time t_1, their value may be referred to as their 'accumulation'.

It follows that any value may be expressed as the sum of a discounted value and an accumulation. This fact is helpful in certain problems. Also, if $t_1 = 0$ and all the payments are due at or after the present time, their value may also be described as their '(discounted) present value', as defined by formula (1.2).

It follows from formula (2.2) that the value at any time t_1 of a cashflow may be obtained from its value at another time t_2 by applying the factor $v(t_2)/v(t_1)$, *ie*:

$$\begin{bmatrix} \text{Value at time } t_1 \\ \text{of cashflow} \end{bmatrix} = \begin{bmatrix} \text{Value at time } t_2 \\ \text{of cashflow} \end{bmatrix} \begin{bmatrix} \dfrac{v(t_2)}{v(t_1)} \end{bmatrix}$$

or:

$$\begin{bmatrix} \text{Value at time } t_1 \\ \text{of cashflow} \end{bmatrix} \left[v(t_1) \right] = \begin{bmatrix} \text{Value at time } t_2 \\ \text{of cashflow} \end{bmatrix} \left[v(t_2) \right] \qquad \textbf{(2.4)}$$

Each side of Equation (2.4) is the value of the cashflow at the present time (time 0).

In particular, by choosing time t_2 as the present time and letting $t_1 = t$, we obtain the result:

$$\left[\begin{array}{l}\textbf{Value at time } \boldsymbol{t} \\ \textbf{of cashflow}\end{array}\right] = \left[\begin{array}{l}\textbf{Value at the present} \\ \textbf{time of cashflow}\end{array}\right]\left[\frac{\mathbf{1}}{\boldsymbol{v(t)}}\right]$$

These results are useful in many practical examples. The time 0 and the unit of time may be chosen so as to simplify the calculations.

Question

Consider the following four payments: £100 on 1 January 2017, £130 on 1 January 2018, £150 on 1 January 2020 and £160 on 1 January 2021.

If $t = 0$ on 1 January 2016 and $v(t) = 0.92 - \dfrac{(t-2)^3}{100}$, calculate the value of these payments on 1 January 2019.

Solution

From the formula $v(t) = 0.92 - \dfrac{(t-2)^3}{100}$, we can calculate $v(1) = 0.93$, $v(2) = 0.92$, $v(3) = 0.91$, $v(4) = 0.84$, $v(5) = 0.65$.

The present value of these cashflows at $t = 0$ (*ie* 1 January 2016) is:

$$100v(1) + 130v(2) + 150v(4) + 160v(5) = £442.60$$

The present value of these cashflows at $t = 3$ (*ie* 1 January 2019) is therefore:

$$\frac{1}{v(3)} \times 442.60 = £486.37$$

The first two payments occur before 1 January 2019, so these have been accumulated. The last two payments occur after 1 January 2019, so these have been discounted.

2.1 Constant interest rate

The special case when we assume that interest rates remain constant is of particular importance. Using this assumption $v(t) = v^t$ for all t. Remember also that $v = 1/(1+i)$ and so $1/v^t = (1+i)^t$.

In the diagram below, each of the three payments P_1, P_2 and P_3 has the same present value.

Payments	$P_1 = Xv^n$ ↓	$P_2 = X$ ↓	$P_3 = X(1+i)^n$ ↓
Time	$-n$	0	n

This shows that we can think of the factors $(1+i)^n$ and v^n as a way of adjusting payments to a different point on the timeline.

If the present value of a series of definite payments at a particular date is X, then:

- the accumulated value at a date *n* years later is $X(1+i)^n$
- the present value at a date *n* years earlier is Xv^n.

Note that *n* does not have to be a whole number in these formulae.

Question

Under its current rental agreement, a company is obliged to make annual payments of £7,500 for the building it occupies. Payments are due on 1 January 2020, 1 January 2021 and 1 January 2022. The nominal rate of interest is 8% per annum, convertible quarterly.

Calculate the value of these rental payments on:

(i) 1 January 2019

(ii) 1 January 2018

(iii) 1 July 2033

Solution

A nominal rate of interest of 8% *pa* convertible quarterly is equivalent to a quarterly effective interest rate of $\frac{8\%}{4} = 2\%$.

(i) Working in quarters, the value of the rental payments on 1 January 2019 is:

$$7{,}500(v^4 + v^8 + v^{12}) = 7{,}500(1.02^{-4} + 1.02^{-8} + 1.02^{-12}) = £19{,}243.72$$

(ii) Since 1 January 2018 is 1 year (= 4 quarters) before 1 January 2019, the value of the rental payments on 1 January 2023 is:

$$19{,}243.72 \times 1.02^{-4} = £17{,}778.22$$

(iii) Since 1 July 2033 is 14.5 years (= 58 quarters) after 1 January 2019, the value of the rental payments on 1 July 2033 is:

$$19{,}243.72 \times 1.02^{58} = £60{,}687.46$$

Alternatively, we could first calculate the effective annual rate, i, as $1.02^4 - 1 = 8.243216\%$*, and then work in years. So, for example, the value in part (i) is:*

$$7{,}500(v + v^2 + v^3) = £19{,}243.72 \text{ where } v = 1/1.08243216$$

2.2 Payment streams

In Section 1.2, we saw that the present value of a continuous payment stream received from time 0 to time T, where the rate of payment at time t is $\rho(t)$, is given by:

$$\int_0^T v(t)\rho(t)dt$$

In this section, we will consider a continuous payment stream paid at a rate of $\rho(t)$ from time *a* to time *b*, during which time the force of interest is $\delta(t)$.

The present value at time *a* of this payment stream is:

$$PV_{t=a} = \int_a^b \rho(t)\exp\left(-\int_a^t \delta(s)ds\right)dt$$

Intuitively, we can obtain this formula by first considering the payment at time *t*, which is at a rate of $\rho(t)$. This payment needs to be discounted back to time *a* and the discount factor is $\exp\left(-\int_a^t \delta(s)ds\right)$. Finally, we need to add together all the present values of the payments at the different times. Since we are receiving payments continuously, we integrate these present values between the limits *a* and *b, ie* the times between which the payment comes in.

Question

A continuous payment stream is paid at rate $e^{-0.03t}$ from time $t=0$ to time $t=10$.

Calculate the present value of this payment stream at time $t=0$, given that the force of interest over this time period is 0.04 *pa*.

Solution

The payment stream starts at time 0 and finishes at time 10, so $a=0$ and $b=10$. Using these values in the general formula for the present value of a payment stream with $\rho(t)=e^{-0.03t}$ and $\delta(t)=0.04$ gives:

$$PV_{t=0} = \int_0^{10} e^{-0.03t} e^{-\int_0^t 0.04ds}\, dt = \int_0^{10} e^{-0.03t} e^{-[0.04s]_0^t}\, dt = \int_0^{10} e^{-0.03t} e^{-0.04t}\, dt$$

This integral simplifies and is evaluated as follows:

$$PV_{t=0} = \int_0^{10} e^{-0.07t}\, dt = \left[-\frac{1}{0.07}e^{-0.07t}\right]_0^{10} = \frac{1}{0.07}(1-e^{-0.7}) = 7.192$$

We can also consider the accumulated value of the continuous payment stream paid at a rate of $\rho(t)$ from time *a* to time *b*, during which time the force of interest is $\delta(t)$. The accumulated value at time *b* of this payment stream is:

$$AV_{t=b} = \int_a^b \rho(t)\exp\left(\int_t^b \delta(s)ds\right)dt$$

Intuitively, we can obtain this formula by first considering the payment at time *t*, which is at a rate of $\rho(t)$. This payment needs to be accumulated to time *b* and the accumulation factor will be $\exp\left(\int_t^b \delta(s)ds\right)$. Finally, we need to add together all the accumulated values of the payments at the different times. Since we are receiving payments continuously, we integrate these present values between the limits *a* and *b, ie* the times between which the payment comes in.

Question

The force of interest at time t, where $0 \le t \le 10$, is given by $\delta(t) = 0.07$.

Calculate the accumulated value at time 10 of a payment stream, paid continuously from time 5 to time 10, under which the rate of payment at time t is $10e^{0.05t}$.

Solution

The payment stream starts at time 5 and finishes at time 10, so $a = 5$ and $b = 10$. Using these values in the general formula for the accumulated value of a payment stream with $\rho(t) = 10e^{0.05t}$ and $\delta(t) = 0.07$ gives:

$$AV_{t=10} = \int_5^{10} 10e^{0.05t} e^{\int_t^{10} 0.07ds} dt = \int_5^{10} 10e^{0.05t} e^{\left[0.07s\right]_t^{10}} dt = \int_5^{10} 10e^{0.05t}\, e^{0.7-0.07t} dt$$

This integral simplifies and is evaluated as follows:

$$AV_{t=10} = 10e^{0.7}\int_5^{10} e^{-0.02t}\, dt = 10e^{0.7}\left[\frac{e^{-0.02t}}{-0.02}\right]_5^{10} = 10e^{0.7}\left(\frac{e^{-0.2} - e^{-0.1}}{-0.02}\right) = 86.699$$

One result that can be useful here is the chain rule for differentiation expressed in integral form. The chain rule applied to the exponential of a function tells us that:

$$\frac{d}{dt}e^{f(t)} = f'(t)e^{f(t)}$$

Integrating both sides, we have:

$$\int_a^b f'(t)e^{f(t)}\,dt = \left[e^{f(t)}\right]_a^b$$

Question

The force of interest at time t is given by:

$$\delta(t) = 0.01t + 0.04 \qquad\qquad 0 \le t \le 5$$

Calculate the present value at time 0 of a payment stream, received continuously from time 0 to time 5, under which the rate of payment at time t is $0.5t + 2$.

Solution

We can apply the general formula for the present value of a payment stream with $a = 0$, $b = 5$ and $\rho(t) = 0.5t + 2$ to give:

$$\begin{aligned}
PV_{t=0} &= \int_0^5 (0.5t+2)\exp\left(-\int_0^t (0.01s+0.04)\,ds\right)dt \\
&= \int_0^5 (0.5t+2)\exp\left(-\left[0.005s^2+0.04s\right]_0^t\right)dt \\
&= \int_0^5 (0.5t+2)\exp\left(-\left[0.005t^2+0.04t\right]\right)dt
\end{aligned}$$

Now, using the general result $\int_a^b f'(t)e^{f(t)}\,dt = \left[e^{f(t)}\right]_a^b$, with $f(t) = -\left[0.005t^2+0.04t\right]$, so that $f'(t) = -(0.01t+0.04)$, we know that:

$$\int_0^5 -(0.01t+0.04)\exp\left(-\left[0.005t^2+0.04t\right]\right)dt = \left[\exp\left(-\left[0.005t^2+0.04t\right]\right)\right]_0^5$$

Now, since $0.5t+2=-50\times-(0.01t+0.04)$, we have:

$$PV_{t=0}=\int_0^5 (0.5t+2)\exp\left(-\left[0.005t^2+0.04t\right]\right)dt$$

$$=-50\int_0^5 -(0.01t+0.04)\exp\left(-\left[0.005t^2+0.04t\right]\right)dt$$

$$=-50\left[\exp\left(-\left[0.005t^2+0.04t\right]\right)\right]_0^5$$

$$=-50\left(\exp(-0.125-0.2)-\exp(0)\right)$$

$$=13.87$$

Alternatively, we can evaluate the integral:

$$PV_{t=0}=\int_0^5 (0.5t+2)\exp\left(-\left[0.005t^2+0.04t\right]\right)dt$$

using the substitution $u=-0.005t^2-0.04t$*, so that:*

$$\frac{du}{dt}=-0.01t-0.04 \;\Rightarrow\; du=(-0.01t-0.04)dt \;\Rightarrow\; -50du=(0.5t+2)dt$$

Also, when $t=0$*,* $u=0$ *and when* $t=5$*,* $u=-0.325$*, so the integral becomes:*

$$PV_{t=0}=\int_0^{-0.325} -50e^u\,du=-50\left[e^u\right]_0^{-0.325}=-50\left(e^{-0.325}-1\right)=13.87$$

2.3 Sudden changes in interest rates

In Chapter 5, we saw that when the force of interest is not a continuous function of time, it is necessary to break up the calculations at the points when the force changes. The same idea applies when we are given a rate of interest that changes at certain points in time. We need to take care to ensure that we use the correct interest rate for each period.

Question

Calculate the present value as at 1 January 2020 of the following payments:

(i) a single payment of £2,000 payable on 1 July 2024

(ii) a single payment of £5,000 payable on 31 December 2031.

Assume effective rates of interest of 8% *pa* until 31 December 2026 and 6% *pa* thereafter.

Solution

(i) Here, the interest rate is constant throughout the relevant period, so the present value is just:

$$2{,}000v^{4\frac{1}{2}@8\%} = 2{,}000 \times 1.08^{-4.5} = £1{,}415$$

(ii) Here, we need to break the calculation up at 31 December 2026 when the interest rate changes. So, we discount the payment for 5 years (from 31 December 2031 to 31 December 2026) at 6% *pa*, and then for a further 7 years (from 31 December 2026 to 1 January 2020) at 8% *pa*:

$$5{,}000v^{7@8\%} \times v^{5@6\%} = 5{,}000 \times 1.08^{-7} \times 1.06^{-5} = £2{,}180$$

3 Interest income

Consider now an investor who wishes not to accumulate money but to receive an income while keeping his capital fixed at C. If the rate of interest is fixed at i per time unit, and if the investor wishes to receive income at the end of each time unit, it is clear that the income will be iC per time unit, payable in arrears, until such time as the capital is withdrawn.

This is because the effective rate of interest, i, is defined to be the amount of interest a single initial investment will earn at the end of the time period.

For example, consider an investor who wishes to receive income at the end of each year. If the investor deposits £1,000 in a bank account that pays an effective rate of interest of 8% per annum, the amount of each payment will be £80 (8% of 1,000).

However, if interest is paid continuously with force of interest $\delta(t)$ at time t then the income received between times t and $t+dt$ will be $C\delta(t)\,dt$. So the total interest income from time 0 to time T will be:

$$I(T)=\int_0^T C\delta(t)\,dt \tag{3.1}$$

If interest is paid continuously to the investor, then we are just considering a continuous cashflow with a rate of payment of $C\delta(t)$. The total amount of interest received can therefore be calculated by applying formula (1.1), which gives formula (3.1). The total amount of interest received is the sum, between 0 and T, of lots of small interest payments, each of amount $C\delta(t)dt$.

If the investor withdraws the capital at time T, the present values of the income and capital at time 0 are:

$$C\int_0^T \delta(t)v(t)dt \tag{3.2}$$

and:

$$Cv(T) \tag{3.3}$$

respectively.

Since:

$$\int_0^T \delta(t)v(t)dt=\int_0^T \delta(t)\exp\left[-\int_0^t \delta(s)ds\right]dt=\left[-\exp\left(-\int_0^t \delta(s)ds\right)\right]_0^T=1-v(T)$$

we obtain:

$$C=C\int_0^T \delta(t)v(t)dt+Cv(T) \tag{3.4}$$

as one would expect by general reasoning.

If we invest an amount of capital C, then the present value of the proceeds we receive from this investment should equal our original amount of capital.

Question

An investor deposits £2,000 in a bank account and receives income at the end of each of the next three years. The rate of interest is 4% *pa* effective. The investor withdraws the capital after three years.

Show that the present value of the proceeds from this arrangement is equal to the initial amount deposited.

Solution

At the end of each year the investor receives $0.04 \times 2{,}000 = £80$. The present value of the interest received is:

$$80(v + v^2 + v^3) = 80(1.04^{-1} + 1.04^{-2} + 1.04^{-3}) = £222.01$$

The present value of the capital received after three years is:

$$2{,}000v^3 = 2{,}000 \times 1.04^{-3} = £1{,}777.99$$

The present value of the capital plus the present value of the interest equals the initial investment, *ie*:

$$1{,}777.99 + 222.01 = £2{,}000$$

So far we have described the difference between money returned at the end of the term and the cash originally invested as 'interest'. In practice, however, this quantity may be divided into *interest income* and *capital gains*, the term *capital loss* being used for a negative capital gain.

In return for an investment of capital, an investor will expect to receive interest payments. In addition, the value of the capital may also increase (or decrease). Equities or shares are a good example of this. If an investor buys some shares in a company, then the investor should receive dividends (*ie* interest) from the company. However the capital value that the investor receives back will depend upon the market price of the shares at the time they are sold. We will consider this in more detail later in the course.

Chapter 7 Summary

In many compound interest problems, we may need to determine the discounted present value of cashflows due in the future. It is important to distinguish between discrete and continuous payments.

The present value of a series of discrete payments is the sum of the individual present values. The present value of continuous payments is calculated by integrating the rate of payment multiplied by a discount factor. The formula for the present value is:

$$\sum c_t v(t) + \int_0^\infty v(t)\rho(t)dt$$

The *net present value* is defined as the difference between the value of the positive cashflows and the value of the negative cashflows.

The value of payments that are due after the time of valuation is called a *discounted* value.

The value of payments that are due before the time of valuation is called an *accumulated* value.

The value of a cashflow at one particular time can easily be found from the value of the cashflow at a different time. The formula for moving along the timeline is:

$$\begin{bmatrix}\text{Value at time } t_1 \\ \text{of cashflow}\end{bmatrix} = \begin{bmatrix}\text{Value at time } t_2 \\ \text{of cashflow}\end{bmatrix}\left[\frac{v(t_2)}{v(t_1)}\right]$$

An investor may wish to receive an income while keeping the amount of capital fixed. The present value of the income plus the present value of the returned capital equals the initial capital invested, *ie*:

$$C = C\int_0^T \delta(t)v(t)dt + Cv(T)$$

The practice questions start on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 7 Practice Questions

7.1 Calculate the total present value as at 1 September 2022 of payments of £280 due on 1 September 2024 and £360 due on 1 March 2025, assuming the interest rate is 15% *pa* effective.

7.2 Calculate the total present value as at 1 June 2019 of payments of £100 on 1 January 2020 and £200 on 1 May 2020, assuming a rate of discount of 12% *pa* convertible quarterly.

7.3 An investment of £1,000 made at time 0 is accumulated at the following rates: 8% per annum simple for two years, followed by a rate of discount of 6% per annum convertible monthly for two years. Calculate the accumulated amount of the investment after 4 years.

7.4 A company is contracted to make payments of £1,500 at time 3, £4,000 at time 7 and £5,500 at time 10. The effective annual interest rate is assumed to be:

- 3% from time 0 to time 4
- 5% from time 4 to time 8
- 8% from time 8 onwards.

Calculate the value of the payments as at:

(i) time 0

(ii) time 5.

7.5 A woman deposits £200 in a special bank account. Interest is paid to the woman every year on her birthday for five years. The capital is returned after exactly five years, along with any interest accrued since her last birthday. Interest is calculated at an effective rate of 6% *pa*.

Calculate the present value of the interest received by the woman.

7.6 The force of interest at time t, where t is measured in years, is given by:

$$\delta(t)=\begin{cases}0.002+0.01t+0.0004t^2 & 0\le t<6\\ 0.01+0.003t & 6\le t<10\\ 0.04 & t\ge 10\end{cases}$$

Calculate the present value at time 0 of a continuous payment stream of £120 per annum payable from time 10 to time 15.

7.7 The force of interest at time t is given by:

$$\delta(t)=0.01+0.05t \qquad 0\le t\le 10$$

Calculate the accumulated value at time 10 of a continuous payment stream that is received from time 4 to time 8, under which the rate of payment at time t is $0.3+1.5t$.

7.8 The force of interest at time t is given by:

Exam style

$$\delta(t)=\begin{cases} 0.04 & 0<t\leq 1 \\ 0.05t-0.01 & 1<t\leq 5 \\ 0.24 & t>5 \end{cases}$$

(i) Derive and simplify as far as possible expressions for $A(t)$, where $A(t)$ is the total accumulated value at time t (>0) of an investment of 1 at time 0. [5]

(ii) A continuous payment stream is received at a rate of $25e^{-0.02t}$ units per annum between time 5 and time 10. Calculate the present value (at time 0) of this payment stream. [4]

[Total 9]

Chapter 7 Solutions

7.1 The payment of £280 is made in two years' time, and the payment of £360 is made in 2.5 years' time, so the present value is:

$$280v^2 + 360v^{2.5} = 280 \times 1.15^{-2} + 360 \times 1.15^{-2.5} = £465.56$$

7.2 We are given $d^{(4)} = 12\%$. This is equivalent to:

$$i = \left(1 - \frac{d^{(4)}}{4}\right)^{-4} - 1 = \left(1 - \frac{0.12}{4}\right)^{-4} - 1 = 12.95698\%\ pa$$

The payment of £100 is made in 7 months' time and the payment of £200 is made in 11 months' time, so working in years, the present value is:

$$100v^{7/12} + 200v^{11/12} = 100(1.1295698)^{-7/12} + 200(1.1295698)^{-11/12} = £272.00$$

7.3 The accumulation factor for 2 years in terms of a simple rate of interest i is $(1+2i)$.

Since $1+i = \left(1 - \frac{d^{(12)}}{12}\right)^{-12}$, the accumulation factor for 2 years in terms of $d^{(12)}$ is $\left(1 - \frac{d^{(12)}}{12}\right)^{-24}$.

So the accumulated value of the investment at time 4 years is:

$$1{,}000(1 + 2 \times 0.08)\left(1 - \tfrac{0.06}{12}\right)^{-24} = £1{,}308.29$$

7.4 (i) To calculate the value of the payments at time 0:

- the payment of £1,500 at time 3 needs to be discounted for 3 years at 3% *pa*
- the payment of £4,000 at time 7 needs to be discounted for 7 years in total – 3 years at 5% *pa* (from time 7 back to time 4) and 4 years at 3% *pa* (from time 4 back to time 0)
- the payment of £5,500 at time 10 needs to be discounted for 10 years in total – 2 years at 8% *pa* (from time 10 back to time 8), 4 years at 5% *pa* (from time 8 back to time 4) and 4 years at 3% *pa* (from time 4 back to time 0).

So, the present value at time 0 is:

$$1{,}500v^{3@3\%} + 4{,}000v^{4@3\%}v^{3@5\%} + 5{,}500v^{4@3\%}v^{4@5\%}v^{2@8\%}$$

$$= 1{,}500(1.03)^{-3} + 4{,}000(1.03)^{-4}(1.05)^{-3} + 5{,}500(1.03)^{-4}(1.05)^{-4}(1.08)^{-2}$$

$$= £7{,}889.49$$

(ii) The value at time 5 can be found by accumulating the value at time 0 for 4 years at 3% *pa* and 1 year at 5% *pa*. This gives:

$$7{,}889.49(1.03)^4(1.05) = £9{,}323.68$$

7.5 We know that the present value of the interest received plus the present value of the capital returned will equal the initial deposit. Therefore:

$$200 = PV(\text{interest}) + 200v^5$$

$$\Rightarrow \quad PV(\text{interest}) = 200 - 200v^5 = 200(1-1.06^{-5}) = £50.55$$

This answer does not depend on where the woman's birthday falls during the year.

7.6 The present value of the payment stream at time 10 (when the payment stream begins) is:

$$PV_{t=10} = \int_{10}^{15} 120\exp\left[-\int_{10}^{t} 0.04\,ds\right]dt$$

$$= \int_{10}^{15} 120\exp\left(-[0.04s]_{10}^{t}\right)dt$$

$$= \int_{10}^{15} 120\exp\left(-0.04(t-10)\right)dt$$

$$= 120e^{0.4}\int_{10}^{15} e^{-0.04t}\,dt$$

Carrying out the integration, we have:

$$PV_{t=10} = 120e^{0.4}\left[\frac{e^{-0.04t}}{-0.04}\right]_{10}^{15} = 120e^{0.4}\left(\frac{e^{-0.04\times 15} - e^{-0.04\times 10}}{-0.04}\right) = £543.81$$

We then need to discount the present value at time 10 back to time 0 using the appropriate values of $\delta(t)$:

$$PV_{t=0} = 543.81\exp\left[-\int_{6}^{10}(0.01+0.003t)dt\right]\exp\left[-\int_{0}^{6}(0.002+0.01t+0.0004t^2)dt\right]$$

Now:

$$\exp\left[-\int_{6}^{10}(0.01+0.003t)dt\right] = \exp\left(-\left[0.01t+0.0015t^2\right]_{6}^{10}\right) = e^{-0.136}$$

Also:

$$\exp\left[-\int_0^6 (0.002+0.01t+0.0004t^2)dt\right]=\exp\left(-\left[0.002t+0.005t^2+\frac{0.0004}{3}t^3\right]_0^6\right)=e^{-0.2208}$$

So:

$$PV_{t=0}=543.81\times e^{-0.136}\times e^{-0.2208}=£380.62$$

7.7 We can calculate the accumulated value at time 8 first and then accumulate that value to time 10. Using the general formula for the accumulated value of a payment stream with $a=4$, $b=8$ and $\rho(t)=0.3+1.5t$:

$$AV_{t=8}=\int_4^8 (0.3+1.5t)\exp\left(\int_t^8 (0.01+0.05s)ds\right)dt$$

$$=\int_4^8 (0.3+1.5t)\exp\left(\left[0.01s+0.025s^2\right]_t^8\right)dt$$

$$=\int_4^8 (0.3+1.5t)\exp\left(\left[1.68-0.01t-0.025t^2\right]\right)dt$$

Now, using the general result that $\int_a^b f'(t)e^{f(t)}\,dt=\left[e^{f(t)}\right]_a^b$, we know:

$$\int_4^8 -(0.01+0.05t)\exp\left(1.68-0.01t-0.025t^2\right)dt=\left[\exp\left(1.68-0.01t-0.025t^2\right)\right]_4^8$$

So, since $0.3+1.5t=-30\times-(0.01+0.05t)$, we have:

$$AV_{t=8}=-30\int_4^8 -(0.01+0.05t)\exp\left(1.68-0.01t-0.025t^2\right)dt$$

$$=-30\left[\exp\left(1.68-0.01t-0.025t^2\right)\right]_4^8$$

Evaluating this gives:

$$AV_{t=8}=-30\left(\exp(1.68-0.08-1.6)-\exp(1.68-0.04-0.4)\right)$$

$$=-30\left(e^0-e^{1.24}\right)$$

$$=73.6684$$

We now need to accumulate this value at time 8 forward to time 10:

$$\begin{aligned}
AV_{t=10} &= AV_{t=8} \times A(8,10) \\
&= 73.6684 \times \exp\left(\int_8^{10} (0.01 + 0.05t)\, dt \right) \\
&= 73.6684 \times \exp\left(\left[0.01t + 0.025t^2 \right]_8^{10} \right) \\
&= 73.6684 \times \exp\big((0.1 + 2.5) - (0.08 + 1.6) \big) \\
&= 73.6684 e^{0.92} \\
&= 184.855
\end{aligned}$$

Alternatively, we can carry out the integral:

$$AV_{t=8} = \int_4^8 (0.3 + 1.5t) \exp\left(\left[1.68 - 0.01t - 0.025t^2 \right] \right) dt$$

using the substitution $u = 1.68 - 0.01t - 0.025t^2$*, so that:*

$$\frac{du}{dt} = -0.01 - 0.05t \;\Rightarrow\; du = (-0.01 - 0.05t)\, dt \;\Rightarrow\; -30du = (0.3 + 1.5t)\, dt$$

Also, when $t = 4$*,* $u = 1.24$ *and when* $t = 8$*,* $u = 0$*, so the integral becomes:*

$$AV_{t=8} = \int_{1.24}^{0} -30e^u\, du = -30\left[e^u \right]_{1.24}^{0} = -30\left(1 - e^{1.24} \right) = 73.6684$$

7.8 (i) We have to break down the expression for $A(t)$ at times when the force of interest changes. Since $A(t)$ represents the accumulated value at time t of an investment of 1 at time 0, we have:

$0 < t \le 1$:

$$A(t) = e^{0.04t} \qquad [1]$$

$1 < t \le 5$:

$$\begin{aligned} A(t) &= A(1) \times \exp\left[\int_1^t (0.05s - 0.01)ds\right] \\ &= e^{0.04} \times \exp\left[\int_1^t (0.05s - 0.01)ds\right] \\ &= e^{0.04} \times \exp\left[0.025s^2 - 0.01s\right]_1^t \\ &= e^{0.04} \times e^{0.025t^2 - 0.01t - 0.025 + 0.01} \\ &= e^{0.025t^2 - 0.01t + 0.025} \end{aligned} \qquad [2]$$

$5 < t$:

$$\begin{aligned} A(t) &= A(5) \times \exp\left[\int_5^t 0.24\,ds\right] \\ &= e^{0.025 \times 5^2 - 0.01 \times 5 + 0.025} \times \exp\left[\int_5^t 0.24\,ds\right] \\ &= e^{0.6} e^{0.24t - 1.2} \\ &= e^{0.24t - 0.6} \end{aligned} \qquad [2]$$

[Total 5]

(ii) We will calculate the present value at time 5 first and then discount it back to time 0. The present value at time 5 is:

$$PV_{t=5} = \int_5^{10} 25e^{-0.02t} \exp\left[-\int_5^t 0.24\,ds\right] dt \qquad [1]$$

Carrying out the integration:

$$PV_{t=5} = \int_5^{10} 25e^{-0.02t} \exp\left(-\left[0.24s\right]_5^t\right) dt$$

$$= \int_5^{10} 25e^{-0.02t} e^{-0.24t+1.2} dt$$

$$= 25e^{1.2} \int_5^{10} e^{-0.26t} dt$$

$$= 25e^{1.2} \left[\frac{e^{-0.26t}}{-0.26}\right]_5^{10}$$

$$= \frac{25e^{1.2}}{-0.26}\left[e^{-2.6} - e^{-1.3}\right]$$

$$= 63.2924$$ [2]

We then need to discount this back to time 0. We know from part (i) that the accumulation factor from time 0 to time 5 is:

$$A(5) = e^{0.025\times 5^2 - 0.01\times 5 + 0.025} = e^{0.6}$$

So the discount factor from time 5 to time 0 is $e^{-0.6}$. The present value at time 0 of the payment stream is therefore:

$$PV_{t=0} = 63.2924e^{-0.6} = 34.7356$$ [1]

[Total 4]

Alternatively, we could calculate the present value of the payment stream at time 0 all in one go. From part (i) we know that for $t > 5$:

$$A(t) = e^{0.24t - 0.6}$$

So the discount factor from time t to time 0 is:

$$v(t) = \frac{1}{A(t)} = e^{-0.24t+0.6}$$

and the present value of the payment stream at time 0 is obtained from the integral:

$$PV_{t=0} = \int_5^{10} 25e^{-0.02t} v(t) dt = \int_5^{10} 25e^{-0.02t} e^{-0.24t+0.6} dt = 25e^{0.6} \int_5^{10} e^{-0.26t} dt$$

8

Level annuities

Syllabus objectives

2.5 Define and derive the following compound interest functions (where payments can be in advance or in arrears) in terms of i , v , n , d , δ , $i^{(p)}$ and $d^{(p)}$:

2.5.1 $a_{\overline{n}|}$, $s_{\overline{n}|}$, $a^{(p)}_{\overline{n}|}$, $s^{(p)}_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}$, $\ddot{s}_{\overline{n}|}$, $\ddot{a}^{(p)}_{\overline{n}|}$, $\ddot{s}^{(p)}_{\overline{n}|}$, $\bar{a}_{\overline{n}|}$ and $\bar{s}_{\overline{n}|}$

2.5.2 ${}_{m|}a_{\overline{n}|}$, ${}_{m|}a^{(p)}_{\overline{n}|}$, ${}_{m|}\ddot{a}_{\overline{n}|}$, ${}_{m|}\ddot{a}^{(p)}_{\overline{n}|}$ and ${}_{m|}\bar{a}_{\overline{n}|}$

0 Introduction

In the next two chapters we will learn how to calculate the present value of a series of payments. We will also meet some actuarial symbols representing compound interest functions, which we will use very frequently in this course.

Terminology

Here is a brief summary of the key terms that we will use:

An *annuity* is a regular series of payments. An *annuity-certain* is an annuity payable for a definite period of time: the payments do not depend on some factor, such as whether a person is alive or not.

If payments are made at the *end* of each time period, they are paid *in arrears*. If they are made at the *beginning* of each time period, they are paid *in advance*. An annuity paid in advance is also known as an *annuity-due*.

Where the first payment is made during the first time period, this is an *immediate* annuity. Where no payments are made during the first time period, this is a *deferred* annuity.

If each payment is for the same amount, this is a *level* annuity. If payments increase (decrease) each time by the same amount, this is a *simple increasing (decreasing)* annuity.

So, for example, payments of £2 made at the start and halfway through each of the next five years can be described as a level immediate annuity payable half-yearly in advance for five years

This chapter covers level annuities – both immediate and deferred. Chapter 9 then covers increasing and decreasing annuities, and ultimately deals with variable payments that can be evaluated using similar techniques to the ones detailed here.

1 Present values

1.1 Payments made in arrears

Consider a series of n payments, each of amount 1, to be made at time intervals of one unit, the first payment being made at time $t+1$.

	1	**1**	**1**	**…**	**1**	**1**	**Payment**
t	$t+1$	$t+2$	$t+3$	**…**	$t+n-1$	$t+n$	**Time**

Such a sequence of payments is illustrated in the diagram above, in which the rth payment is made at time $t+r$.

The value of this series of payments *one unit of time before the first payment is made* is denoted by $a_{\overline{n}|}$.

So, in the above example, $a_{\overline{n}|}$ is the value at time t of the payments shown.

The symbol $a_{\overline{n}|}$ (pronounced 'A.N.') represents the present value of an annuity consisting of n payments of 1 unit made at the end of each of the next n time periods. This is called an annuity paid in arrears.

Clearly, if $i=0$, then $a_{\overline{n}|}=n$; otherwise:

$$\begin{aligned} a_{\overline{n}|} &= v+v^2+v^3+\cdots+v^n \\ &= \frac{v(1-v^n)}{1-v} \\ &= \frac{1-v^n}{v^{-1}-1} \\ &= \frac{1-v^n}{i} \end{aligned} \qquad \textbf{(1.1)}$$

The derivation above uses the fact that the terms on the right-hand side of the first equation form a geometric progression, so we can use the formula for the sum of the first n terms of a geometric progression:

$$a+ar+ar^2+\cdots+ar^{n-1}=\frac{a\left(1-r^n\right)}{1-r}$$

where: $a=\text{first term}=v$

and: $r=\text{common ratio}=v$.

An alternative way is prove result (1.1) is to multiply the first equation through by $(1+i)$, so that v^3 becomes v^2 *etc*. This gives a series very similar to the original series. We can then obtain a formula for $a_{\overline{n}|}$ by subtracting the two equations:

$$\begin{aligned} a_{\overline{n}|} &= \quad v+v^2+v^3+\cdots+v^{n-1}+v^n \\ (1+i)a_{\overline{n}|} &= 1+v+v^2+v^3+\cdots+v^{n-1} \\ \therefore \quad ia_{\overline{n}|} &= 1 \qquad\qquad\qquad\qquad -v^n \end{aligned}$$

So:

$$a_{\overline{n}|}=\frac{1-v^n}{i}$$

If $n=0$, $a_{\overline{n}|}$ is defined to be zero.

Thus $a_{\overline{n}|}$ is the value at the start of any period of length *n* of a series of *n* payments, each of amount 1, to be made *in arrears* at unit time intervals over the period. It is common to refer to such a series of payments, made in arrears, as an *immediate annuity-certain* and to call $a_{\overline{n}|}$ the present value of the immediate annuity-certain. When there is no possibility of confusion with a life annuity (*ie* a series of payments dependent on the survival of one or more human lives), the term *annuity* may be used as an alternative to annuity-certain.

$a_{\overline{n}|}$ is called an immediate annuity even though the first payment is made at the end of the first time period, and hence not 'immediately'. Immediate annuities are annuities where the first payment is made during the first time period, including at the end of the period. If no payments are made during the first time period then the annuity is *deferred*. Deferred annuities are covered in Section 7.

In the *Tables*, values of $a_{\overline{n}|}$ are tabulated at various interest rates from ½% to 25%. For example, the value of $a_{\overline{10}|}$ calculated at 4% interest is given on page 56 of the *Tables* as 8.1109. We can verify this using the formula obtained above:

$$a_{\overline{10}|}=\frac{1-v^{10}}{i}=\frac{1-1.04^{-10}}{0.04}=8.1109$$

We can denote $a_{\overline{n}|}$ calculated at a rate of interest of *i* % effective as $a_{\overline{n}|}^{@i\%}$, $a_{\overline{n}|@i\%}$ or $a_{\overline{n}|,i\%}$.

Question

Calculate the present value as at 1 March 2020 of a series of payments of £1,000 payable on the first day of each month from April 2020 to December 2020 inclusive, assuming a rate of interest of 6% *pa* convertible monthly.

Solution

An interest rate of 6% *pa* convertible monthly is equivalent to an effective monthly interest rate of ½%. There are 9 payments of £1,000 each, starting in one month's time. So, working in months, the PV of the payments is:

$$1{,}000a_{\overline{9}|}^{@½\%}$$

We can look up $a_{\overline{9}|}^{@½\%}$ in the *Tables* or alternatively apply the formula:

$$1{,}000a_{\overline{9}|}^{@½\%}=1000\times\frac{1-1.005^{-9}}{0.005}=1000\times 8.7791=£8{,}779$$

1.2 Payments made in advance

Now consider a series of n payments made at the start of each time period as represented on the timeline below. The first payment is made at time zero and the last at time $n-1$.

1	1	1	1	…	1	Payment
0	1	2	3	…	*n*–1	Time

The value of this series of payments *at the time the first payment is made* is denoted by $\ddot{a}_{\overline{n}|}$.

The symbol $\ddot{a}_{\overline{n}|}$ is pronounced 'A due n'.

If $i=0$, then $\ddot{a}_{\overline{n}|}=n$; otherwise:

$$\begin{aligned}\ddot{a}_{\overline{n}|}&=1+v+v^2+\cdots+v^{n-1}\\&=\frac{1-v^n}{1-v}\\&=\frac{1-v^n}{d}\end{aligned}\qquad\textbf{(1.2)}$$

Again, the derivation of the formula above uses the fact that the n terms on the right-hand side form a geometric progression with first term 1, and common ratio v.

Thus $\ddot{a}_{\overline{n|}}$ is the value at the start of any given period of length *n* of a series of *n* payments, each of amount 1, to be made *in advance* at unit time intervals over the period. It is common to refer to such a series of payments, made in advance, as an *annuity-due* and to call $\ddot{a}_{\overline{n|}}$ the present value of the annuity-due.

It follows directly from the above definitions that:

$$\left.\begin{aligned} &\ddot{a}_{\overline{n|}} = (1+i)a_{\overline{n|}} \\ &\text{and that, for } n \geq 2: \\ &\ddot{a}_{\overline{n|}} = 1 + a_{\overline{n-1|}} \end{aligned}\right\} \qquad (1.3)$$

The first of the equations (1.3) can be obtained by reasoning that the payments for $\ddot{a}_{\overline{n|}}$ correspond exactly with those for $a_{\overline{n|}}$, except that each payment is made one year earlier, *ie* each payment has a present value that is greater by a factor of $(1+i)$. So $\ddot{a}_{\overline{n|}} = (1+i)a_{\overline{n|}}$.

Alternatively, since $(1+i)v = 1$, we can obtain this algebraically:

$$\begin{aligned} \ddot{a}_{\overline{n|}} &= 1 + v + v^2 + \cdots + v^{n-1} \\ &= (1+i)(v + v^2 + \cdots + v^{n-1} + v^n) \\ &= (1+i)a_{\overline{n|}} \end{aligned}$$

The second of the equations (1.3) can be obtained by noting that $\ddot{a}_{\overline{n|}}$, which is the PV at time $t = 0$ of a series of payments of 1 payable at times $t = 0, 1, 2, \ldots, n-1$, is the same as:

- an initial payment of 1 (which has a PV of 1) plus
- a series of payments of 1, payable at times $t = 1, 2, \ldots, n-1$ (which has a PV of $a_{\overline{n-1|}}$).

So $\ddot{a}_{\overline{n|}} = a_{\overline{n-1|}} + 1$.

Alternatively, algebraically, we have:

$$\ddot{a}_{\overline{n|}} = 1 + v + v^2 + \cdots + v^{n-1} = 1 + (v + v^2 + \cdots + v^{n-1}) = 1 + a_{\overline{n-1|}}$$

Question

Calculate $a_{\overline{25|}}$ and $\ddot{a}_{\overline{15|}}$ using an interest rate of 13½% *pa* effective.

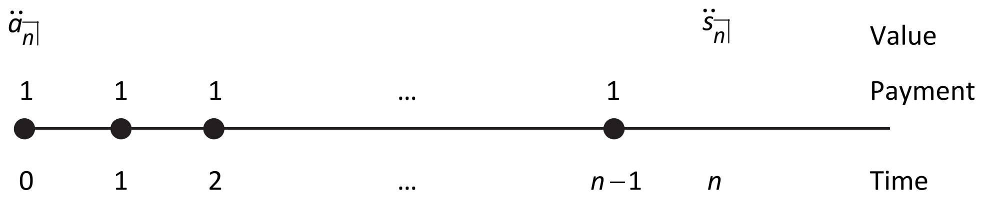
Solution

$$a_{\overline{25|}} = \frac{1 - v^{25}}{i} = \frac{1 - 1.135^{-25}}{0.135} = 7.095 \qquad \text{and} \qquad \ddot{a}_{\overline{15|}} = \frac{1 - v^{15}}{d} = \frac{1 - 1.135^{-15}}{0.135/1.135} = 7.149$$

2 Accumulations

The value of the series of payments *at the time the last payment is made* is denoted by $s_{\overline{n}|}$. The value *one unit of time after the last payment is made* is denoted by $\ddot{s}_{\overline{n}|}$.

In other words, $s_{\overline{n}|}$ considers the same series of payments as $a_{\overline{n}|}$ but it is the accumulated value at time *n*, as opposed to the present value at time 0. Similarly, $\ddot{s}_{\overline{n}|}$ is the accumulated value at time *n* of the annuity we looked at above when defining $\ddot{a}_{\overline{n}|}$, as shown on the timeline:



If $i=0$ then $s_{\overline{n}|}=\ddot{s}_{\overline{n}|}=n$; otherwise

$$\begin{aligned} s_{\overline{n}|} &= (1+i)^{n-1}+(1+i)^{n-2}+(1+i)^{n-3}+\cdots+1 \\ &= (1+i)^n a_{\overline{n}|} \\ &= \frac{(1+i)^n-1}{i} \end{aligned} \tag{2.1}$$

and:

$$\begin{aligned} \ddot{s}_{\overline{n}|} &= (1+i)^n+(1+i)^{n-1}+(1+i)^{n-2}+\cdots+(1+i) \\ &= (1+i)^n \ddot{a}_{\overline{n}|} \\ &= \frac{(1+i)^n-1}{d} \end{aligned} \tag{2.2}$$

Thus $s_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$ are the values at the end of any period of length *n* of a series of *n* payments, each of amount 1, made at unit time intervals over the period, where the payments are made in arrears and in advance respectively. Sometimes $s_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$ are called the 'accumulation' (or the 'accumulated amount') of an immediate annuity and an annuity-due respectively. When $n=0$, $s_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$ are defined to be zero. It is an immediate consequence of the above definition that:

$$\ddot{s}_{\overline{n}|}=(1+i)s_{\overline{n}|}$$

and that:

$$s_{\overline{n+1}|}=1+\ddot{s}_{\overline{n}|}$$

or:

$$\ddot{s}_{\overline{n}|}=s_{\overline{n+1}|}-1$$

Question

Calculate $s_{\overline{10}|}$ and $\ddot{s}_{\overline{13}|}$ using an interest rate of 3½% *pa* effective.

Solution

$$s_{\overline{10}|} = \frac{(1+i)^{10} - 1}{i} = \frac{1.035^{10} - 1}{0.035} = 11.731 \qquad \text{and} \qquad \ddot{s}_{\overline{13}|} = \frac{(1+i)^{13} - 1}{d} = \frac{1.035^{13} - 1}{0.035/1.035} = 16.677$$

3 Continuously payable annuities

Let n be a non-negative number. The value at time 0 of an annuity payable continuously between time 0 and time n, where the rate of payment per unit time is constant and equal to 1, is denoted by $\bar{a}_{\overline{n|}}$.

This is a mathematical idealisation, which makes calculations easier. With a continuously payable annuity, the payments are considered as a continuous cashflow that is payable at a given *rate* over a given period of time. The symbol $\bar{a}_{\overline{n|}}$ is pronounced 'A bar n'.

We can obtain a formula for $\bar{a}_{\overline{n|}}$ using the formula for the present value of a continuous payment stream we met in Chapter 7.

Clearly:

$$\bar{a}_{\overline{n|}} = \int_0^n 1.v^t dt = \int_0^n e^{-\delta t} dt = \left[-\frac{1}{\delta} e^{-\delta t} \right]_0^n$$

$$= \frac{1 - e^{-\delta n}}{\delta}$$

$$= \frac{1 - v^n}{\delta} \qquad \text{(if } \delta \neq 0\text{)} \qquad \textbf{(3.1)}$$

Note that $\bar{a}_{\overline{n|}}$ is defined even for non-integral values of n. If $\delta = 0$ (or, equivalently, $i = 0$), $\bar{a}_{\overline{n|}}$ is of course equal to n.

Since equation (3.1) may be written as:

$$\bar{a}_{\overline{n|}} = \frac{i}{\delta} \left(\frac{1 - v^n}{i} \right)$$

it follows immediately that, if n is an integer:

$$\bar{a}_{\overline{n|}} = \frac{i}{\delta} a_{\overline{n|}} \qquad \text{(if } \delta \neq 0\text{)}$$

Note that this relationship can be very useful, especially when using the *Tables*. Although values of $\bar{a}_{\overline{n|}}$ are not tabulated, values of $a_{\overline{n|}}$ and i/δ are.

The *accumulated amount* of such an annuity at the time the payments cease is denoted by $\bar{s}_{\overline{n|}}$. By definition, therefore:

$$\bar{s}_{\overline{n|}} = \int_0^n e^{\delta(n-t)} dt$$

Hence:

$$\bar{s}_{\overline{n|}} = (1+i)^n \bar{a}_{\overline{n|}}$$

If the rate of interest is non-zero:

$$\boldsymbol{\bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta}}$$

$$\boldsymbol{= \frac{i}{\delta} s_{\overline{n}|}}$$

Question

A payment of £1 is made at the beginning of each week for 1 year.

By approximating these weekly payments as a continuously payable annuity, and assuming that there are 52.18 weeks in a year, calculate the present value of these payments, using an effective rate of interest of 8% *pa*.

Solution

Using the assumption given, £52.18 will be paid in total over the year. The PV of the weekly payments is then:

$$52.18\bar{a}_{\overline{1}|}^{@8\%} = 52.18\left(\frac{1-v}{\delta}\right) = 52.18 \times \frac{1-1.08^{-1}}{\ln(1.08)} = 52.18 \times 0.96244 = £50.22$$

Using this approximation, we would have arrived at exactly the same answer if the payments were made at the ***end*** *of each week.*

Notice the similarity between the formulae for the present and accumulated values of annuities:

$$a_{\overline{n}|} = \frac{1-v^n}{i} \qquad \ddot{a}_{\overline{n}|} = \frac{1-v^n}{d} \qquad \bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta}$$

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i} \qquad \ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d} \qquad \bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta}$$

The numerators are always consistent – the only difference between the formulae is in the denominators.

4 Annuities payable *p*thly

4.1 Present values

Where annuity payments are made p times a year (*eg* $p=12$ for a monthly annuity), a superscript (p) is added in the top right-hand corner of the symbol. The annuity is still payable for n years and still refers to a total *annual* amount of 1 unit, *ie* the annuity consists of np payments, each of amount $1/p$ units.

If *p* and *n* are positive integers, the notation $a_{\overline{n}|}^{(p)}$ is used to denote the value at time 0 of a level annuity payable *p*thly in arrears at the rate of 1 per unit time over the time interval $[0,n]$. For this annuity the payments are made at times $1/p, 2/p, 3/p, \ldots, n$ and the amount of each payment is $1/p$.

By definition, a series of *p* payments, each of amount $i^{(p)}/p$ in arrears at *p*thly subintervals over any unit time interval, has the same value as a single payment of amount *i* at the end of the interval. By proportion, *p* payments, each of amount $1/p$ in arrears at *p*thly subintervals over any unit time interval, have the same value as a single payment of amount $i/i^{(p)}$ at the end of the interval.

Consider now that annuity for which the present value is $a_{\overline{n}|}^{(p)}$. The remarks in the preceding paragraph show that the *p* payments after time $r-1$ and not later than time *r* have the same value as a single payment of amount $i/i^{(p)}$ at time *r*. This is true for $r=1, 2, \ldots, n$, so the annuity has the same value as a series of *n* payments, each of amount $i/i^{(p)}$, at times $1, 2, \ldots, n$. This means that:

$$a_{\overline{n}|}^{(p)} = \frac{i}{i^{(p)}} a_{\overline{n}|} \qquad \textbf{(4.1)}$$

Note that $i/i^{(p)}$ is tabulated in the *Tables* and so $a_{\overline{n}|}^{(p)}$ can quickly be calculated for some interest rates by looking up the values of $i/i^{(p)}$ and $a_{\overline{n}|}$.

An alternative approach, from first principles, is to write:

$$a_{\overline{n}|}^{(p)} = \sum_{t=1}^{np} \frac{1}{p} v^{t/p} = \frac{1}{p} \frac{v^{1/p}(1-v^n)}{1-v^{1/p}}$$

$$= \frac{1-v^n}{p\left[(1+i)^{1/p}-1\right]}$$

$$= \frac{1-v^n}{i^{(p)}} \qquad \textbf{(4.2)}$$

which confirms equation (4.1).

The first line in the above derivation uses the formula for the sum of a geometric progression of np terms, with first term $\frac{1}{p}v^{1/p}$ and common ratio $v^{1/p}$.

Question

Calculate $a_{\overline{6}|}^{(4)}$ at 1½% *pa*, both with and without using the *Tables*.

Solution

Using the *Tables*:

$$a_{\overline{6}|}^{(4)} = \frac{i}{i^{(4)}} a_{\overline{6}|} = 1.005608 \times 5.6972 = 5.729$$

Without using the *Tables*:

$$a_{\overline{6}|}^{(4)} = \frac{1-v^6}{i^{(4)}} = \frac{1-1.015^{-6}}{4(1.015^{0.25}-1)} = 5.729$$

Likewise, we define $\ddot{a}_{\overline{n}|}^{(p)}$ to be the present value of a level annuity-due payable *p*thly at the rate of 1 per unit time over the time interval $[0,n]$. (The annuity payments, each of amount $1/p$, are made at times $0,\ 1/p,\ 2/p, \ldots, n-(1/p)$.)

A series of *p* payments, each of amount $d^{(p)}/p$, in advance at *p*thly subintervals over any unit time interval has the same value as a single payment of amount *i* at end of the interval. Hence, by proportion, *p* payments, each of amount $1/p$ in advance at *p*thly subintervals, have the same value as a single payment of amount $i/d^{(p)}$ at the end of the interval. This means (by an identical argument to that above) that:

$$\ddot{a}_{\overline{n}|}^{(p)} = \frac{i}{d^{(p)}} a_{\overline{n}|} \tag{4.3}$$

Alternatively, from first principles, we may write:

$$\ddot{a}_{\overline{n}|}^{(p)} = \sum_{t=1}^{np} \frac{1}{p} v^{(t-1)/p}$$

$$= \frac{1-v^n}{d^{(p)}} \tag{4.4}$$

(on simplification), which confirms equation (4.3).

The simplification referred to here involves using the formula for the sum of a geometric progression of np terms, with first term $\frac{1}{p}$ and common ratio $v^{1/p}$.

Note that:

$$a_{\overline{n}|}^{(p)} = v^{1/p} \ddot{a}_{\overline{n}|}^{(p)} \qquad \textbf{(4.5)}$$

each expression being equal to $\dfrac{(1-v^n)}{i^{(p)}}$.

Equation (4.5) can be derived by general reasoning. The payments for $\ddot{a}_{\overline{n}|}^{(p)}$ correspond exactly with those for $a_{\overline{n}|}^{(p)}$, except that each payment is made a period of length $1/p$ earlier, *ie* each payment has a present value that is greater by a factor of $(1+i)^{1/p}$. So $\ddot{a}_{\overline{n}|}^{(p)} = (1+i)^{1/p} a_{\overline{n}|}^{(p)}$, which is equivalent to equation (4.5).

Note that, since:

$$\lim_{p\to\infty} i^{(p)} = \lim_{p\to\infty} d^{(p)} = \delta$$

it follows immediately from equation (4.2) and (4.4) that:

$$\lim_{p\to\infty} a_{\overline{n}|}^{(p)} = \lim_{p\to\infty} \ddot{a}_{\overline{n}|}^{(p)} = \bar{a}_{\overline{n}|}$$

Question

Calculate the present value as at 1 January 2019 of a series of payments of £100 payable on the first day of each month during 2020, 2021 and 2022, assuming an effective rate of interest of 8% per annum.

Solution

The present value of the payments as at 1 January 2020 is $1{,}200\ddot{a}_{\overline{3}|}^{(12)}$.

So the present value as at 1 January 2019 is:

$$1{,}200v\ddot{a}_{\overline{3}|}^{(12)} = 1{,}200v\frac{1-v^3}{d^{(12)}} = 1{,}200\times 0.92593\times\frac{1-0.79383}{0.076714} = £2{,}986$$

using the values given in the *Tables*.

Since no payments are received during 2019, this is an example of a deferred annuity. These are dealt with in more detail in Section 7 of this chapter.

4.2 Accumulations

Similarly, we define $s_{\overline{n}|}^{(p)}$ and $\ddot{s}_{\overline{n}|}^{(p)}$ to be the accumulated amounts of the corresponding *p*thly immediate annuity and annuity-due respectively. Thus:

$$s_{\overline{n}|}^{(p)} = (1+i)^n a_{\overline{n}|}^{(p)} = (1+i)^n \frac{i}{i^{(p)}} a_{\overline{n}|} \qquad \textbf{(by (4.1))}$$

$$= \frac{i}{i^{(p)}} s_{\overline{n}|}$$

Using the formula for $s_{\overline{n}|}$ gives:

$$s_{\overline{n}|}^{(p)} = \frac{(1+i)^n - 1}{i^{(p)}}$$

Also:

$$\ddot{s}_{\overline{n}|}^{(p)} = (1+i)^n \ddot{a}_{\overline{n}|}^{(p)} = (1+i)^n \frac{i}{d^{(p)}} a_{\overline{n}|} \qquad \textbf{(by (4.3))}$$

$$= \frac{i}{d^{(p)}} s_{\overline{n}|}$$

Again, using the formula for $s_{\overline{n}|}$ gives:

$$s_{\overline{n}|}^{(p)} = \frac{(1+i)^n - 1}{d^{(p)}}$$

As previously, notice the similarity between the formulae for the present and accumulated values of annuities:

$$a_{\overline{n}|}^{(p)} = \frac{1-v^n}{i^{(p)}} \qquad \ddot{a}_{\overline{n}|}^{(p)} = \frac{1-v^n}{d^{(p)}}$$

$$s_{\overline{n}|}^{(p)} = \frac{(1+i)^n - 1}{i^{(p)}} \qquad \ddot{s}_{\overline{n}|}^{(p)} = \frac{(1+i)^n - 1}{d^{(p)}}$$

The numerators are always consistent – the only difference between the formulae is in the denominators.

Question

An investment manager has invested £1,000 in a fixed-interest security that pays interest of £40 at the end of each half-year. In addition, the full £1,000 is returned on redemption in 12 years' time. The investment manager deposits all of the proceeds from the security in a bank account that pays an effective rate of interest of 8% *pa*.

Calculate the amount of money in the bank account after 10 years.

Solution

The interest payments are £80 *pa* payable half-yearly in arrears. After 10 years, the accumulated value of these will be:

$$80 s_{\overline{10|}}^{(2)@8\%} = 80 \frac{i}{i^{(2)}} s_{\overline{10|}} = 80 \times 1.019615 \times 14.4866 = £1{,}182$$

using values from the *Tables*.

No payments received after time 10 years are included in this calculation.

The above proportional arguments may be applied to other varying series of payments. Consider, for example, an annuity payable annually in arrears for *n* years, the payment in the *t* th year being x_t. The present value of this annuity is obviously:

$$a = \sum_{t=1}^{n} x_t v^t \tag{4.6}$$

Consider now a second annuity, also payable for *n* years with the payment in the *t* th year, again of amount x_t, being made in *p* equal instalments in arrears over that year. If $a^{(p)}$ denotes the present value of this second annuity, by replacing the *p* payments for year *t* (each of amount x_t / p) by a single equivalent payment at the end of the year of amount $x_t[i / i^{(p)}]$, we immediately obtain:

$$a^{(p)} = \frac{i}{i^{(p)}} a$$

where *a* is given by equation (4.6) above.

Question

A series of payments of \$1 at time 1, \$4 at time 2, \$9 at time 3, and so on up to \$100 at time 10 has a present value of \$245.32 when evaluated at an effective interest rate of 6% *pa*.

Calculate the present value of the series of payments if, instead of being paid at the end of each year, each annual amount is split into three equal instalments paid at the end of each third of a year.

Solution

The present value will be a factor of $i / i^{(3)}$ greater. Therefore:

$$PV = 245.32 \times \frac{i}{i^{(3)}} = 245.32 \times \frac{0.06}{0.058838} = \$250.16$$

4.3 Annuities payable *p*thly where *p* < 1

If $p<1$, then we are considering an annuity payable less frequently than annually. For example, if $p=0.5$, the annuity is payable every two years. The formulae for $a_{\overline{n}|}^{(p)}$ and $\ddot{a}_{\overline{n}|}^{(p)}$ are still valid for $p<1$.

In Section 4.1 the symbol $a_{\overline{n}|}^{(p)}$ was introduced. Intuitively, with this notation one considers *p* to be an integer greater than 1 and assumes that the product $n.p$ is also an integer. (This, of course, will be true when *n* itself is an integer, but one might, for example, have $p=4$ and $n=5.75$ so that $np=23$.) Then $a_{\overline{n}|}^{(p)}$ denotes the value at time 0 of $n.p$ payments, each of amount $1/p$, at times $1/p, 2/p, \ldots, (np)/p$.

Non-integer values of *n* will be looked at more closely in the next section.

From a theoretical viewpoint it is perhaps worth noting that when *p* is the reciprocal of an integer and $n.p$ is also an integer (*eg* when $p=0.25$ and $n=28$), $a_{\overline{n}|}^{(p)}$ still gives the value at time 0 of $n.p$ payments, each of amount $1/p$, at times $1/p, 2/p, \ldots, (np)/p$.

For example, the value at time 0 of a series of seven payments, each of amount 4, at times 4, 8, 12 ,.., 28 may be denoted by $a_{\overline{28}|}^{(0.25)}$.

It follows that this value equals:

$$\frac{1-v^{28}}{(0.25)\left[(1+i)^4-1\right]}$$

This last expression may be written in the form:

$$\left[\frac{4}{\frac{(1+i)^4-1}{i}}\right].\frac{1-v^{28}}{i}=\frac{4}{s_{\overline{4}|}}.a_{\overline{28}|}$$

Although $a_{\overline{28}|}^{(0.25)}$ *can* be written in the form immediately above, it is unlikely that we would ever choose to do so.

5 Non-integer values of *n*

Let p be a positive integer. Until now the symbol $a_{\overline{n}|}^{(p)}$ has been defined only when n is a positive integer. For certain non-integral values of n the symbol $a_{\overline{n}|}^{(p)}$ has an intuitively obvious interpretation. For example, it is not clear what meaning, if any, may be given to $a_{\overline{23.5}|}$, but the symbol $a_{\overline{23.5}|}^{(4)}$ ought to represent the present value of an immediate annuity of 1 per annum payable quarterly in arrears for 23.5 years (*ie* a total of 94 quarterly payments, each of amount 0.25). On the other hand, $a_{\overline{23.25}|}^{(2)}$ has no obvious meaning.

Suppose that n is an integer multiple of $1/p$, say $n = r/p$, where r is an integer. In this case we define $a_{\overline{n}|}^{(p)}$ to be the value at time 0 of a series of r payments, each of amount $1/p$, at times $1/p, 2/p, 3/p, ..., r/p = n$. If $i = 0$, then clearly $a_{\overline{n}|}^{(p)} = n$. If $i > 0$, then:

$$
\begin{aligned}
a_{\overline{n}|}^{(p)} &= \frac{1}{p}(v^{1/p} + v^{2/p} + v^{3/p} + \cdots + v^{r/p}) \\
&= \frac{1}{p} v^{1/p} \left(\frac{1 - v^{r/p}}{1 - v^{1/p}} \right) \\
&= \frac{1}{p} \left[\frac{1 - v^{r/p}}{(1+i)^{1/p} - 1} \right]
\end{aligned}
$$

The derivation above uses the formula for the sum of a geometric progression of r terms, with first term $v^{1/p}$ and common ratio $v^{1/p}$.

Thus:

$$
a_{\overline{n}|}^{(p)} = \begin{cases} \dfrac{1 - v^n}{i^{(p)}} & \text{if } i > 0 \\ n & \text{if } i = 0 \end{cases} \qquad \textbf{(5.1)}
$$

The standard formula for $a_{\overline{n}|}^{(p)}$ therefore applies for non-integer values of *n* when *n* is an integer multiple of $\dfrac{1}{p}$.

Note that, by working in terms of a new time unit equal to $\frac{1}{p}$ times the original time unit and with the equivalent effective interest rate of $\frac{i^{(p)}}{p}$ per new time unit, we see that:

$$a_{\overline{n}|}^{(p)} \text{ (at rate } i \text{) = } \frac{1}{p} a_{\overline{np}|} \text{ (at rate } \frac{i^{(p)}}{p} \text{)}$$

This formula is useful when $\frac{i^{(p)}}{p}$ is a tabulated rate of interest.

Question

Calculate $a_{\overline{3.5}|}^{(12)}$ given that $i = 19.5618\%$.

Solution

We could evaluate this directly using (5.1):

$$a_{\overline{3.5}|}^{(12)} = \frac{1-v^{3.5}}{i^{(12)}} = \frac{1-1.195618^{-3.5}}{12(1.195618^{1/12}-1)} = 2.5828$$

Alternatively, we could spot that when $i = 19.5618\%$, $i^{(12)} = 12(1.195618^{1/12}-1) = 18.00\%$ and $\frac{i^{(12)}}{12} = 1.5\%$. So, using values from the *Tables*:

$$a_{\overline{3.5}|}^{(12)} = \tfrac{1}{12} a_{\overline{42}|}^{@1.5\%} = \tfrac{1}{12} \times 30.9941 = 2.5828$$

Note that the definition of $a_{\overline{n}|}^{(p)}$ given by equation (5.1) is mathematically meaningful for *all* non-negative values of *n*. For our present purpose, therefore, it is convenient to adopt equation (5.1) as a definition of $a_{\overline{n}|}^{(p)}$ for all *n*.

This is only a mathematical definition. It is not easily translated into the present value of a series of payments.

If *n* is not an integer multiple of $\frac{1}{p}$, there is no universally recognised definition of $a_{\overline{n}|}^{(p)}$. For example, if $n = n_1 + f$, where n_1 is an integer multiple of $1/p$ and $0 < f < 1/p$, some writers define $a_{\overline{n}|}^{(p)}$ as:

$$a_{\overline{n_1}|}^{(p)} + fv^n$$

With this alternative definition:

$$a^{(2)}_{\overline{23.75}|} = a^{(2)}_{\overline{23.5}|} + 1/4\, v^{23.75}$$

which is the present value of an annuity of 1 per annum, payable half-yearly in arrears for 23.5 years, together with a final payment of 0.25 after 23.75 years. Note that this is *not* equal to the value obtained from definition (5.1).

For example, using $i = 0.03$, definition (5.1) gives:

$$a^{(2)}_{\overline{23.75}|} = \frac{1-v^{23.75}}{i^{(2)}} = \frac{1-1.03^{-23.75}}{2(1.03^{1/2}-1)} = 16.9391$$

but using the alternative definition instead gives:

$$a^{(2)}_{\overline{23.75}|} = a^{(2)}_{\overline{23.5}|} + \tfrac{1}{4}v^{23.75} = \frac{1-1.03^{-23.5}}{2(1.03^{1/2}-1)} + \tfrac{1}{4}\times 1.03^{-23.75} = 16.9395$$

We can extend the above results to develop formulae for $\ddot{a}^{(p)}_{\overline{n}|}$, $s^{(p)}_{\overline{n}|}$ and $\ddot{s}^{(p)}_{\overline{n}|}$ for all non-negative n.

If $i > 0$, we define for all non-negative *n*:

$$\left.\begin{aligned}
\ddot{a}^{(p)}_{\overline{n}|} &= (1+i)^{1/p}\, a^{(p)}_{\overline{n}|} = \frac{(1-v^n)}{d^{(p)}} \\
s^{(p)}_{\overline{n}|} &= (1+i)^n\, a^{(p)}_{\overline{n}|} = \frac{(1+i)^n - 1}{i^{(p)}} \\
\ddot{s}^{(p)}_{\overline{n}|} &= (1+i)^n\, \ddot{a}^{(p)}_{\overline{n}|} = \frac{(1+i)^n - 1}{d^{(p)}}
\end{aligned}\right\} \qquad \textbf{(5.2)}$$

If $i = 0$, each of these last three functions is defined to equal *n*.

Whenever *n* is an integer multiple of $1/p$, say $n = r/p$, then $\ddot{a}^{(p)}_{\overline{n}|}$, $s^{(p)}_{\overline{n}|}$, $\ddot{s}^{(p)}_{\overline{n}|}$, are values at different times of an annuity-certain of *r* payments, each of amount $1/p$, at intervals of $1/p$ time unit.

As before, we use the simpler notations $a_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}$, $s_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$ to denote $a^{(1)}_{\overline{n}|}$, $\ddot{a}^{(1)}_{\overline{n}|}$, $s^{(1)}_{\overline{n}|}$ and $\ddot{s}^{(1)}_{\overline{n}|}$ respectively, thus extending the definition of $a_{\overline{n}|}$ *etc,* to all non-negative values of *n*.

It is a trivial consequence of our definitions that the formulae:

$$\left.\begin{aligned}
a^{(p)}_{\overline{n}|} &= \frac{i}{i^{(p)}}\, a_{\overline{n}|} \\
\ddot{a}^{(p)}_{\overline{n}|} &= \frac{i}{d^{(p)}}\, a_{\overline{n}|} \\
s^{(p)}_{\overline{n}|} &= \frac{i}{i^{(p)}}\, s_{\overline{n}|} \\
\ddot{s}^{(p)}_{\overline{n}|} &= \frac{i}{d^{(p)}}\, s_{\overline{n}|}
\end{aligned}\right\} \qquad \textbf{(5.3)}$$

(valid when $i > 0$) now hold for *all* values of *n*.

6 Perpetuities

We can also consider an annuity that is payable forever. This is called a *perpetuity*. For example, consider an equity that pays a dividend of £10 at the end of each year. An investor who purchases the equity pays an amount equal to the present value of the dividends. The present value of the dividends is:

$$10v + 10v^2 + 10v^3 + \cdots$$

This can be summed using the formula for an infinite geometric progression:

$$10v + 10v^2 + 10v^3 + \cdots = \frac{10v}{1-v} = \frac{10}{i}$$

The formula for the sum to infinity of a geometric progression with first term a and common ratio r is $\dfrac{a}{1-r}$ provided $|r|<1$.

Recall the formula for the present value of an annuity of £10 *pa* that continues for *n* years:

$$10a_{\overline{n}|} = \frac{10(1-v^n)}{i}$$

We have let $n \to \infty$ in this expression in order to arrive at the formula $\dfrac{10}{i}$.

Note that this formula only holds when i is positive.

Question

Calculate the present value of an annuity that pays £150 *pa* annually in arrears forever using an annual effective rate of interest of 8%.

Solution

The present value is:

$$150a_{\overline{\infty}|} = \frac{150}{i} = \frac{150}{0.08} = £1,875$$

We could alternatively calculate this using the sum to infinity of a geometric progression with first term $150v$ and common ratio v :

$$150v + 150v^2 + 150v^3 + \cdots = \frac{150v}{1-v} = £1,875$$

In general, the present value of payments of 1 *pa* payable at the start and end of each year forever is $\dfrac{1}{d}$ and $\dfrac{1}{i}$, respectively. That is, $\ddot{a}_{\overline{\infty}|} = \dfrac{1}{d}$ and $a_{\overline{\infty}|} = \dfrac{1}{i}$.

Question

Calculate the present value of payments of £2,000 at times $0,1,2,\ldots$ using $i = 7.6\%$ *pa* effective.

Solution

The present value is:

$$2{,}000\ddot{a}_{\overline{\infty}|} = \frac{2{,}000}{d} = \frac{2{,}000}{0.076/1.076} = £28{,}315.79$$

We could alternatively calculate this using the sum to infinity of a geometric progression with first term $2{,}000$ and common ratio v:

$$2{,}000 + 2{,}000v + 2{,}000v^2 + 2{,}000v^3 + \cdots = \frac{2{,}000}{1-v} = £28{,}315.79$$

Perpetuities payable pthly

The present value of payments of 1 *pa* payable in instalments of $\frac{1}{p}$ at the end of each *p*thly time period forever is:

$$a^{(p)}_{\overline{\infty}|} = \frac{1}{i^{(p)}}$$

The present value of payments of 1 *pa* payable in instalments of $\frac{1}{p}$ at the start of each *p*thly time period forever is:

$$\ddot{a}^{(p)}_{\overline{\infty}|} = \frac{1}{d^{(p)}}$$

Question

Calculate the present value of an annuity that pays £300 *pa* monthly in arrears forever using an annual effective rate of interest of 6%.

Solution

The present value is:

$$300a^{(12)}_{\overline{\infty}|} = \frac{300}{i^{(12)}} = \frac{300}{0.0584106} = £5{,}136.05$$

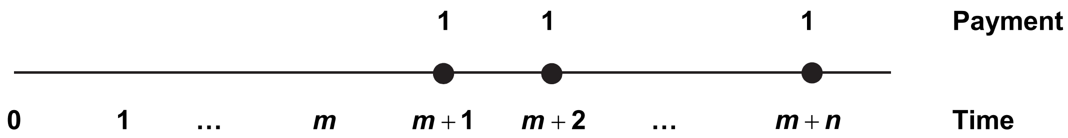
We could alternatively calculate this using the sum to infinity of a geometric progression with first term $\frac{300}{12}v^{\frac{1}{12}}$ and common ratio $v^{\frac{1}{12}}$:

$$\frac{300}{12}v^{\frac{1}{12}}+\frac{300}{12}v^{\frac{2}{12}}+\frac{300}{12}v^{\frac{3}{12}}+\cdots=\frac{300}{12}\times\frac{v^{\frac{1}{12}}}{1-v^{\frac{1}{12}}}=£5,136.05$$

7 Deferred annuities

7.1 Annual payments

Suppose that m and n are non-negative integers. The value at time 0 of a series of n payments, each of amount 1, due at times $(m+1), (m+2), \ldots, (m+n)$ is denoted by ${}_{m|}a_{\overline{n}|}$ (see the figure below).



Such a series of payments may be considered as an immediate annuity, deferred for m time units. When $n > 0$:

$$\begin{aligned}{}_{m|}a_{\overline{n}|} &= v^{m+1} + v^{m+2} + v^{m+3} + \cdots + v^{m+n} \\ &= (v + v^2 + v^3 + \cdots + v^{m+n}) - (v + v^2 + v^3 + \cdots + v^m) \\ &= v^m (v + v^2 + v^3 + \cdots + v^n)\end{aligned}$$

The last two equations show that:

$${}_{m|}a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|} \qquad \textbf{(7.1)}$$

and:

$${}_{m|}a_{\overline{n}|} = v^m a_{\overline{n}|} \qquad \textbf{(7.2)}$$

Either of these two equations may be used to determine the value of a deferred immediate annuity. Together they imply that:

$$a_{\overline{m+n}|} = a_{\overline{m}|} + v^m a_{\overline{n}|}$$

This formula could easily be deduced using general reasoning. The present value of a series of $(n+m)$ payments of one unit payable at the end of each time period is equal to the sum of:

(a) the present value of m payments of one unit payable at the end of each time period, and

(b) the present value of n payments of one unit payable at the end of each time period, deferred for m years.

Question

Using both equations (7.1) and (7.2), calculate the value of ${}_{8|}a_{\overline{12}|}$ using an interest rate of 6.2% *pa* convertible half-yearly.

Solution

An interest rate of 6.2% *pa* convertible half-yearly is equivalent to an effective annual interest rate of:

$$i = \left(1 + \frac{i^{(2)}}{2}\right)^2 - 1 = 1.031^2 - 1 = 6.2961\%$$

Using Equation (7.1), we have:

$$_{8|}a_{\overline{12|}} = a_{\overline{20|}} - a_{\overline{8|}} = \frac{1 - v^{20}}{i} - \frac{1 - v^8}{i} = 11.19923 - 6.13767 = 5.06156$$

Using Equation (7.2), we have:

$$_{8|}a_{\overline{12|}} = v^8 a_{\overline{12|}} = 0.613566 \times 8.249406 = 5.06156$$

We may define the corresponding deferred annuity-due as:

$$_{m|}\ddot{a}_{\overline{n|}} = v^m \ddot{a}_{\overline{n|}}$$

Since $\ddot{a}_{\overline{n|}} = (1+i)a_{\overline{n|}}$ (from equation 1.3), we can also write:

$$_{m|}\ddot{a}_{\overline{n|}} = v^m (1+i) a_{\overline{n|}} = v^{m-1} a_{\overline{n|}}$$

7.2 Continuously payable annuities

If *m* is a non-negative number, we use the symbol $_{m|}\bar{a}_{\overline{n|}}$ to denote the present value of a continuously payable annuity of 1 per unit for *n* time units, deferred for *m* time units. Thus:

$$\begin{aligned} _{m|}\bar{a}_{\overline{n|}} &= \int_m^{m+n} e^{-\delta t} dt \\ &= e^{-\delta m} \int_0^n e^{-\delta s} ds \\ &= \int_0^{m+n} e^{-\delta t} dt - \int_0^m e^{-\delta t} dt \end{aligned}$$

Hence:

$$\begin{aligned} _{m|}\bar{a}_{\overline{n|}} &= \bar{a}_{\overline{m+n|}} - \bar{a}_{\overline{m|}} \\ &= v^m \bar{a}_{\overline{n|}} \end{aligned}$$

Question

An alien on the planet Xi is currently aged exactly 48 years old. When the alien retires at exact age 60, it will receive an income of 500 per year, payable continuously.

Given that all aliens on the planet Xi die at exact age 76, calculate the present value of the retirement benefits using an interest rate of 10% *pa* effective.

Solution

The present value of the retirement benefits is:

$$500 \times {}_{12|}\bar{a}_{\overline{16}|} = 500v^{12}\left(\frac{1-v^{16}}{\delta}\right) = 500 \times 1.1^{-12} \times \frac{1-1.1^{-16}}{\ln 1.1} = 1{,}308$$

7.3 Annuities payable *p*thly

The present values of an immediate annuity and an annuity-due, payable *p*thly at the rate of 1 per unit time for *n* time units and deferred for *m* time units, are denoted by:

$$\left.\begin{aligned} & {}_{m|}a_{\overline{n}|}^{(p)} = v^m a_{\overline{n}|}^{(p)} \\ \text{and} \quad & {}_{m|}\ddot{a}_{\overline{n}|}^{(p)} = v^m \ddot{a}_{\overline{n}|}^{(p)} \end{aligned}\right\} \quad \textbf{(7.3)}$$

respectively.

Question

Calculate ${}_{5|}a_{\overline{10}|}^{(2)}$ and ${}_{6|}\ddot{a}_{\overline{10}|}^{(2)}$ using an interest rate of 5% *pa* effective, and explain why the value of ${}_{5|}a_{\overline{10}|}^{(2)}$ is higher.

Solution

Evaluating the functions using $i = 0.05$, we have:

$${}_{5|}a_{\overline{10}|}^{(2)} = v^5 a_{\overline{10}|}^{(2)} = 1.05^{-5} \times \frac{1-1.05^{-10}}{2(1.05^{1/2}-1)} = 6.125$$

and

$${}_{6|}\ddot{a}_{\overline{10}|}^{(2)} = v^6 \ddot{a}_{\overline{10}|}^{(2)} = 1.05^{-6} \times \frac{1-1.05^{-10}}{2(1-1.05^{-1/2})} = 5.977$$

Both annuity functions value a ten-year annuity payable half-yearly. The first payment of ${}_{5|}a_{\overline{10}|}^{(2)}$ is at time 5½, whereas the first payment of ${}_{6|}\ddot{a}_{\overline{10}|}^{(2)}$ is at time 6. Therefore since the payments are made sooner, ${}_{5|}a_{\overline{10}|}^{(2)}$ has the greater value.

In fact, since the first payment of ${}_{5|}a_{\overline{10}|}^{(2)}$ is made half a year sooner than the first payment of ${}_{6|}\ddot{a}_{\overline{10}|}^{(2)}$, we have:

$$ {}_{5|}a_{\overline{10}|}^{(2)} = (1+i)^{0.5}\, {}_{6|}\ddot{a}_{\overline{10}|}^{(2)} $$

7.4 Non-integer values of *n*

We may also extend the definitions of ${}_{m|}a_{\overline{n}|}^{(p)}$ and ${}_{m|}\ddot{a}_{\overline{n}|}^{(p)}$ to all values of *n* by the formulae:

$$ \left. \begin{aligned} {}_{m|}\boldsymbol{a}_{\overline{n}|}^{(p)} &= \boldsymbol{v}^{m}\boldsymbol{a}_{\overline{n}|}^{(p)} \\ {}_{m|}\ddot{\boldsymbol{a}}_{\overline{n}|}^{(p)} &= \boldsymbol{v}^{m}\ddot{\boldsymbol{a}}_{\overline{n}|}^{(p)} \end{aligned} \right\} \qquad \textbf{(7.4)} $$

and so:

$$ \left. \begin{aligned} {}_{m|}\boldsymbol{a}_{\overline{n}|}^{(p)} &= \boldsymbol{a}_{\overline{n+m}|}^{(p)} - \boldsymbol{a}_{\overline{m}|}^{(p)} \\ {}_{m|}\ddot{\boldsymbol{a}}_{\overline{n}|}^{(p)} &= \ddot{\boldsymbol{a}}_{\overline{n+m}|}^{(p)} - \ddot{\boldsymbol{a}}_{\overline{m}|}^{(p)} \end{aligned} \right\} \qquad \textbf{(7.5)} $$

This is easily proved:

$$ v^m a_{\overline{n}|}^{(p)} = v^m\left(\frac{1-v^n}{i^{(p)}}\right) = \frac{v^m - v^{n+m}}{i^{(p)}} = \frac{(1-v^{n+m})-(1-v^m)}{i^{(p)}} = a_{\overline{n+m}|}^{(p)} - a_{\overline{m}|}^{(p)} $$

and similarly for $v^m\ddot{a}_{\overline{n}|}^{(p)}$.

7.5 Sudden changes in interest rates

We considered sudden changes in interest rates for single payments earlier in the course. We now consider the impact of changes in interest rates on a series of payments. This involves the use of deferred annuities.

Question

Calculate the present value as at 1 January 2019 of payments of £100 on the first day of each quarter during calendar years 2021 to 2030 inclusive.

Assume the effective rate of interest is 8% per annum until 31 December 2025 and 6% per annum thereafter.

Solution

Here we must value a ten-year annuity payable quarterly in advance, deferred for two years. We have to split the ten-year annuity into two five-year annuities to allow for the change in interest rates.

The present value here is:

$$400v^{2@8\%}(\ddot{a}_{\overline{5|}}^{(4)@8\%} + v^{5@8\%}\ \ddot{a}_{\overline{5|}}^{(4)@6\%})$$

$$= 400 \times 0.85734\left(\frac{1-0.68058}{0.076225} + 0.68058 \times \frac{1-0.74726}{0.057847}\right)$$

$$= 400 \times 6.1420 = £2,457$$

Chapter 8 Summary

An annuity consists of a regular series of payments. Payments continue for a specified period. The amounts of the payments may be level, increasing or decreasing. Continuous annuities involve continuous payments paid at specified rates.

The symbol $a_{\overline{n}|}$ ($\ddot{a}_{\overline{n}|}$) represents the PV of an annuity consisting of n payments of 1 unit made at the end (start) of each of the next n time periods. This is called an annuity paid in arrears (advance). The formulae for the present values are:

$$a_{\overline{n}|} = \frac{1-v^n}{i} \qquad \ddot{a}_{\overline{n}|} = \frac{1-v^n}{d} = \frac{i}{d}a_{\overline{n}|} = (1+i)a_{\overline{n}|} = a_{\overline{n-1}|} + 1$$

$s_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$ are the accumulated values at the end of any period of length n of a series of n payments, each of amount 1, made at unit time intervals over the period, where the payments are made in arrears and in advance respectively. The formulae for the accumulated values are:

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i} \qquad \ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d} = \frac{i}{d}s_{\overline{n}|} = (1+i)s_{\overline{n}|} = s_{\overline{n+1}|} - 1$$

The value at time 0 of an annuity payable continuously between time 0 and time n, where the rate of payment per unit time is constant and equal to 1, is denoted by $\bar{a}_{\overline{n}|}$. The formulae for continuous payments are:

$$\bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta} = \frac{i}{\delta}a_{\overline{n}|} \qquad \bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta} = \frac{i}{\delta}s_{\overline{n}|} \qquad PV = \int_0^n \rho(t)v^t\,dt$$

If p and n are positive integers, the notation $a^{(p)}_{\overline{n}|}$ is used to denote the value at time 0 of a level annuity payable $p$thly in arrears at the rate of 1 per unit time over the time interval $[0,n]$. For this annuity the payments are made at times $1/p, 2/p, 3/p, \ldots, n$ and the amount of each payment is $1/p$. The required formulae are:

$$a^{(p)}_{\overline{n}|} = \frac{1-v^n}{i^{(p)}} = \frac{i}{i^{(p)}}a_{\overline{n}|} \qquad s^{(p)}_{\overline{n}|} = \frac{(1+i)^n - 1}{i^{(p)}} = \frac{i}{i^{(p)}}s_{\overline{n}|}$$

$$\ddot{a}^{(p)}_{\overline{n}|} = \frac{1-v^n}{d^{(p)}} = \frac{i}{d^{(p)}}a_{\overline{n}|} \qquad \ddot{s}^{(p)}_{\overline{n}|} = \frac{(1+i)^n - 1}{d^{(p)}} = \frac{i}{d^{(p)}}s_{\overline{n}|}$$

There is more than one definition of $a^{(p)}_{\overline{n}|}$ when n is not an integer multiple of $1/p$.

An annuity that is payable forever is called a perpetuity. The required formulae are:

$$a_{\overline{\infty}|} = \frac{1}{i} \qquad \ddot{a}_{\overline{\infty}|} = \frac{1}{d} \qquad a^{(p)}_{\overline{\infty}|} = \frac{1}{i^{(p)}} \qquad \ddot{a}^{(p)}_{\overline{\infty}|} = \frac{1}{d^{(p)}}$$

Deferred annuities are annuities where no payment is made during the first time period.

The value at time 0 of a series of n payments, each of amount 1, due at times $(m+1), (m+2), \ldots, (m+n)$ is denoted by ${}_{m|}a_{\overline{n}|}$:

$${}_{m|}a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|} = v^m a_{\overline{n}|}$$

Other functions exist for annuities payable in advance, continuously and *p*thly:

$${}_{m|}\ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{m+n}|} - \ddot{a}_{\overline{m}|} = v^m \ddot{a}_{\overline{n}|} \qquad {}_{m|}\bar{a}_{\overline{n}|} = \bar{a}_{\overline{m+n}|} - \bar{a}_{\overline{m}|} = v^m \bar{a}_{\overline{n}|}$$

$${}_{m|}a^{(p)}_{\overline{n}|} = v^m a^{(p)}_{\overline{n}|} \qquad {}_{m|}\ddot{a}^{(p)}_{\overline{n}|} = v^m \ddot{a}^{(p)}_{\overline{n}|}$$

Chapter 8 Practice Questions

8.1 (i) Calculate the following functions at $i = 9\%$:

(a) $a^{(4)}_{\overline{3}|}$

(b) $\ddot{a}^{(4)}_{\overline{4}|}$

(c) $\bar{s}_{\overline{10}|}$

(ii) Calculate the following functions at $i = 25\%$:

(a) $\ddot{a}^{(12)}_{\overline{10}|}$

(b) $a^{(12)}_{\overline{6.5}|}$

8.2 Calculate the accumulated value as at 1 January 2020 of payments of £100 paid every two years from 1 January 1980 to 1 January 2018 inclusive, using an interest rate of 12% *pa* effective.

8.3 Calculate the present value on 1 June 2019 of payments of £1,000 payable on the first day of each month from July 2019 to December 2019 inclusive, assuming a rate of interest of 8% per annum convertible quarterly.

8.4 An annuity provides payments of $40 at the end of each month forever. If the interest rate is 10% *pa* convertible quarterly, calculate the present value of the annuity.

8.5 An annuity-certain is payable monthly in advance for 40 years. The annuity is to be paid at the rate of £100 *pa* for the first 20 years, £120 *pa* for the next 5 years and £200 *pa* for the last 15 years.

Determine whether each of the following expressions gives the correct present value of the payments as at the commencement date of the annuity, assuming an annual effective interest rate of i .

(i) $(100a_{\overline{20}|} + 120v^{20}a_{\overline{5}|} + 200v^{25}a_{\overline{15}|})\dfrac{i}{i^{(12)}}$

(ii) $200\ddot{a}^{(12)}_{\overline{40}|} - 80\ddot{a}^{(12)}_{\overline{25}|} - 20\ddot{a}^{(12)}_{\overline{20}|}$

(iii) $100(1 + a^{(12)}_{\overline{39}|}) + 20v^{20}(1 + a^{(12)}_{\overline{19}|}) + 80v^{25}(1 + a^{(12)}_{\overline{14}|})$

8.6 Calculate the present value as at 1 June 2018 of 41 monthly payments each of £100 commencing on 1 January 2019, assuming a rate of interest of 10% *pa* convertible half-yearly.

8.7 Payments of £1,000 *pa* are payable quarterly in arrears from 1/1/20 to 31/12/25.

The annual effective rate of interest is 3.4% for calendar years 2020-2023 (inclusive) and 4.2% thereafter. Calculate:

(i) the present value of the payments at 1/1/20

(ii) the accumulated value of the payments at 1/1/27.

8.8 X denotes the present value of an annuity consisting of payments of £2,000 payable at the end of each of the next 8 years, valued using an interest rate of 8% *pa* convertible quarterly.

Y denotes the present value of an annuity consisting of payments of £4,000 payable at the end of every fourth year for the next 16 years, valued using an interest rate of 8% *pa* convertible half-yearly.

Calculate the ratio X/Y.

8.9 Using an interest rate of 12% *pa* convertible monthly, calculate:

Exam style

(i) the combined present value of an immediate annuity payable monthly in arrears such that payments are £1,000 *pa* for the first 6 years and £400 *pa* for the next 4 years, together with a lump sum of £2,000 at the end of the 10 years. [3]

(ii) the amount of the level annuity payable continuously for 10 years having the same present value as the payments in (i). [3]

(iii) the accumulated value of the first 7 years' payments at the end of the 7th year for the payments in (i) and (ii). [3]

[Total 9]

Chapter 8 Solutions

8.1 (i)(a) $a_{\overline{3}|}^{(4)} = \dfrac{1-v^3}{i^{(4)}} = \dfrac{1-1.09^{-3}}{0.087113} = 2.6152$

(i)(b) $\ddot{a}_{\overline{4}|}^{(4)} = \dfrac{1-v^4}{d^{(4)}} = \dfrac{1-1.09^{-4}}{0.085256} = 3.4200$

(i)(c) $\bar{s}_{\overline{10}|} = \dfrac{(1+i)^{10}-1}{\delta} = \dfrac{1.09^{10}-1}{\ln(1.09)} = 15.8668$

(ii)(a) $\ddot{a}_{\overline{10}|}^{(12)} = \dfrac{1-v^{10}}{d^{(12)}} = \dfrac{1-1.25^{-10}}{0.221082} = 4.0375$

(ii)(b) $a_{\overline{6.5}|}^{(12)} = \dfrac{1-v^{6.5}}{i^{(12)}} = \dfrac{1-1.25^{-6.5}}{0.225231} = 3.3989$

8.2 Working in two-yearly time periods with an effective two-yearly interest rate of:

$$i = 1.12^2 - 1 = 25.44\%$$

we can calculate the present value as:

$$100\ddot{s}_{\overline{20}|} = 100 \times \frac{1.2544^{20}-1}{0.2544/1.2544} = 100 \times 453.89 = £45,389$$

Alternatively, we could work in years and calculate $50\ddot{s}_{\overline{40}|}^{(½)}$ @12%.

8.3 As at 1 June 2019, the payments of £1,000 are made at the end of each of the next 6 months. We are given $i^{(4)} = 8\%$, so the effective annual interest rate is:

$$i = \left(1 + \frac{i^{(4)}}{4}\right)^4 - 1 = 1.02^4 - 1 = 8.2432\%$$

and the effective monthly interest rate is:

$$1.082432^{1/12} - 1 = 0.66227\%$$

So, working in months, the present value of the payments on 1 June 2019 is:

$$1{,}000a_{\overline{6}|}^{@0.66227\%} = 1{,}000 \times \frac{1-1.0066227^{-6}}{0.0066227} = £5,863$$

Alternatively, the present value can be expressed as $1{,}000v\ddot{a}_{\overline{6}|}^{@0.66227\%}$ *or* $12{,}000a_{\overline{0.5}|}^{(12)\ @8.2432\%}$.

8.4 We are given that $i^{(4)}=0.1$, so the effective annual interest rate is:

$$i=\left(1+\frac{i^{(4)}}{4}\right)^4-1=10.381\%$$

Using an interest rate of 10.381%, the present value is:

$$40\times 12a_{\overline{\infty}|}^{(12)}=\frac{40\times 12}{i^{(12)}}=\frac{40}{(1+i)^{\frac{1}{12}}-1}=\$4{,}839.78$$

We could alternatively calculate this using the sum to infinity of a geometric progression with first term $40v^{\frac{1}{12}}$ *and common ratio* $v^{\frac{1}{12}}$*:*

$$40v^{\frac{1}{12}}+40v^{\frac{2}{12}}+40v^{\frac{3}{12}}+\cdots=\frac{40v^{\frac{1}{12}}}{1-v^{\frac{1}{12}}}=\$4{,}839.78$$

8.5 (i) This expression is incorrect. It is the present value of the same series of payments made monthly in arrears, not monthly in advance. As the payments are made monthly in advance, the ratio outside the bracket should be $\dfrac{i}{d^{(12)}}$ not $\dfrac{i}{i^{(12)}}$.

(ii) This expression is correct.

(iii) This expression is incorrect. A correct expression is:

$$100\ddot{a}_{\overline{40}|}^{(12)}+20v^{20}\ddot{a}_{\overline{20}|}^{(12)}+80v^{25}\ddot{a}_{\overline{15}|}^{(12)}$$

but note that $1+a_{\overline{39}|}^{(12)}\neq\ddot{a}_{\overline{40}|}^{(12)}$, for example.

8.6 10% *pa* convertible half-yearly is equivalent to an effective annual interest rate of:

$$i=\left(1+\frac{0.1}{2}\right)^2-1=10.25\%$$

So, working in years, the present value of the annuity is:

$$1{,}200v^{7/12}\ddot{a}_{\overline{41/12}|}^{(12)}@10.25\%=1{,}200\times 1.1025^{-7/12}\times 2.91729=£3{,}307$$

Alternatively, working in months, with an effective monthly interest rate of:

$$1.1025^{1/12}-1=0.81648\%$$

the present value of the payments is:

$$100v^7\ddot{a}_{\overline{41}|}@0.81648\%=100\times 1.0081648^{-7}\times 35.00751=£3{,}307$$

We could alternatively use either of the following expressions to evaluate this present value:

- $1{,}200v^{6/12}a^{(12)}_{\overline{41/12}|}$ @10.25% *(working in years)*
- $100v^6 a_{\overline{41}|}$ @0.81648% *(working in months)*

8.7 (i) The present value is:

$$1{,}000a^{(4)}_{\overline{4}|\,@3.4\%} + 1{,}000v^4_{@3.4\%}a^{(4)}_{\overline{2}|\,@4.2\%}$$

Evaluating these:

$$i^{(4)}_{@3.4\%} = 4\left(1.034^{\frac{1}{4}} - 1\right) = 0.03357$$

$$i^{(4)}_{@4.2\%} = 4\left(1.042^{\frac{1}{4}} - 1\right) = 0.04135$$

The present value is then:

$$1{,}000 \times \frac{1-1.034^{-4}}{0.03357} + 1{,}000 \times 1.034^{-4} \times \frac{1-1.042^{-2}}{0.04135} = £5{,}399.40$$

(ii) We need to accumulate the answer to part (i) by 7 years. Four of these years have an interest rate of 3.4% *pa* and the remainder have an interest rate of 4.2% *pa*:

$$5{,}399.40 \times 1.034^4 \times 1.042^3 = £6{,}982.81$$

8.8 Consider X first. An interest rate of 8% *pa* convertible quarterly corresponds to an effective interest rate of 2% per quarter.

Working in years, using an effective annual interest rate of $1.02^4 - 1 = 8.2432\%$, gives:

$$X = 2{,}000a^{@8.2432\%}_{\overline{8}|} = 2{,}000 \times \frac{1-(1.02^4)^{-8}}{1.02^4 - 1} = £11{,}388$$

Now consider Y. An interest rate of 8% *pa* convertible half-yearly corresponds to an effective rate of 4% per half-year. Working in 4-year time periods, using an effective 4-yearly interest rate of $1.04^8 - 1 = 36.8569\%$, gives:

$$Y = 4{,}000a^{@36.8569\%}_{\overline{4}|} = 4{,}000 \times \frac{1-(1.04^8)^{-4}}{1.04^8 - 1} = £7{,}759$$

So $\dfrac{X}{Y} = \dfrac{11{,}388}{7{,}759} = 1.468$.

8.9 (i) An interest rate of 12% *pa* convertible monthly is equivalent to a monthly effective interest rate of 1%.

So, working in months, the present value is:

$$PV = \frac{1{,}000}{12} a_{\overline{72}|} + \frac{400}{12} v^{72} a_{\overline{48}|} + 2{,}000 v^{120} \quad @1\% \qquad [1½]$$

$$= \frac{1{,}000}{12} \times 51.1504 + \frac{400}{12} \times 1.01^{-72} \times 37.9740 + 2{,}000 \times 1.01^{-120}$$

$$= 4{,}262.53 + 618.34 + 605.99 = £5{,}486.86 \qquad [1½]$$

[Total 3]

(ii) If the annual rate of payment is X, then, still working in months:

$$5{,}486.86 = \frac{X}{12} \bar{a}_{\overline{120}|} = \frac{X}{12}\left(\frac{1-v^{120}}{\delta}\right) = \frac{X}{12} \times 70.0484 \qquad [2]$$

So:

$$X = 5{,}486.86 \times \frac{12}{70.0484} = £939.95 \qquad [1]$$

[Total 3]

(iii) Working in months, the accumulated value at time 7 years of the first 7 years' (*ie* 84 months') payments in (i) is:

$$AV = (1+i)^{84}\left(\frac{1{,}000}{12} a_{\overline{72}|} + \frac{400}{12} v^{72} a_{\overline{12}|}\right)$$

$$= (1.01)^{84}\left(\frac{1{,}000}{12} \times 51.1504 + \frac{400}{12} \times 1.01^{-72} \times 11.2551\right) = £10{,}255.23 \qquad [2]$$

Using the value of X calculated in (ii) rounded to the nearest pence, the accumulated value for (ii) is:

$$\frac{939.95}{12} \bar{s}_{\overline{84}|} = \frac{939.95}{12} \times 131.325 = £10{,}286.54 \qquad [1]$$

[Total 3]

9

Increasing annuities

Syllabus objectives

2.5 Define and derive the following compound interest functions (where payments can be in advance or in arrears) in terms of i, v, n, d, δ, $i^{(p)}$ and $d^{(p)}$:

2.5.3 $(Ia)_{\overline{n|}}$, $(I\ddot{a})_{\overline{n|}}$, $(I\bar{a})_{\overline{n|}}$, $(\overline{Ia})_{\overline{n|}}$ and the respective deferred annuities.

0 Introduction

In this chapter we will derive formulae for calculating the present values of simple increasing annuities. A simple increasing annuity is an annuity where the payments increase each time by a fixed amount. For example, a payment of £3 at time 1, £6 at time 2, £9 at time 3 *etc*.

At the end of the chapter we give some special cases that use the techniques that have been developed so far in the course.

1 Varying annuities

1.1 Annual payments

For an annuity in which the payments are not all of an equal amount, it is a simple matter to find the present (or accumulated) value from first principles. Thus, for example, the present value of such an annuity may always be evaluated as:

$$\sum_{i=1}^{n} X_i v^{t_i}$$

where the *i*th payment, of amount X_i, is made at time t_i.

If there are also continuous payments, then the present value may be calculated as:

$$\sum_{i=1}^{n} X_i v^{t_i} + \int_{-\infty}^{\infty} \rho(t) v^t dt$$

where $\rho(t)$ is the rate of payment per time unit at time *t*.

In the particular case when $X_i = t_i = i$, the annuity is known as an 'increasing annuity' and its present value is denoted by $(Ia)_{\overline{n|}}$.

$(Ia)_{\overline{n|}}$, therefore, represents the present value of payments of 1 at the end of the first time period, 2 at the end of the second time period, ..., *n* at the end of the *n*th time period.

Thus:

$$\left.\begin{aligned} (Ia)_{\overline{n|}} &= v + 2v^2 + 3v^3 + \cdots + nv^n \\ \text{Hence: } (1+i)(Ia)_{\overline{n|}} &= 1 + 2v + 3v^2 + \cdots + nv^{n-1} \end{aligned}\right\} \qquad (1.1)$$

By subtraction, we obtain:

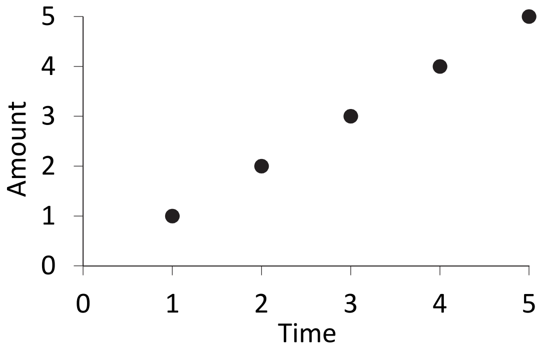
$$\begin{aligned} i(Ia)_{\overline{n|}} &= 1 + v + v^2 + \cdots + v^{n-1} - nv^n \\ &= \ddot{a}_{\overline{n|}} - nv^n \end{aligned}$$

So:

$$(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i}$$

This formula can be found in the *Tables* on page 31.

The graph below shows the pattern of the payments of $(Ia)_{\overline{5}|}$.



The present value of any annuity payable in arrears for *n* time units for which the amounts of successive payments form an arithmetic progression can be expressed in terms of $a_{\overline{n}|}$ and $(Ia)_{\overline{n}|}$. If the first payment of such an annuity is *P* and the second payment is $(P+Q)$, the *t*th payment is $(P-Q)+Qt$, then the present value of the annuity is

$$(P-Q)a_{\overline{n}|}+Q(Ia)_{\overline{n}|} \tag{1.2}$$

Alternatively, the present value of the annuity can be derived from first principles.

On a timeline we can show the payments as:

	P	$P+Q$	$P+2Q$	$P+3Q$	...	$P+(n-1)Q$	Payments
0	1	2	3	4	...	n	Time

Alternatively, these payments can be thought of as the sum of the following two sets of payments:

	$P-Q$	$P-Q$	$P-Q$	$P-Q$	...	$P-Q$	Payments
0	1	2	3	4	...	n	Time

	Q	$2Q$	$3Q$	$4Q$	...	nQ	Payments
0	1	2	3	4	...	n	Time

ie we have a level annuity with payments of $P-Q$ and an increasing annuity that increases by Q each time.

Question

Calculate the present value of a series of 10 annual payments, where the first payment is £500 made in one year's time, and each payment is £100 higher than the previous one. Assume an effective rate of interest of 8% *pa*.

Solution

We can think of this series of payments as a combination of:

- a level annuity of £400 payable annually in arrears and
- an increasing annuity of £100 payable annually in arrears.

So the present value of the payments is:

$$400a_{\overline{10}|}+100(Ia)_{\overline{10}|}=400a_{\overline{10}|}+100\left(\frac{\ddot{a}_{\overline{10}|}-10v^{10}}{i}\right)$$

$$=400\times 6.7101+100\times\frac{7.2469-10\times 1.08^{-10}}{0.08}$$

$$=£5,953$$

or this can be calculated using values from the *Tables*.

The notation $(I\ddot{a})_{\overline{n}|}$ is used to denote the present value of an increasing annuity-due payable for *n* time units, the *t*th payment (of amount *t*) being made at time $t-1$. Thus:

$$\begin{aligned}(I\ddot{a})_{\overline{n}|}&=1+2v+3v^2+\cdots+nv^{n-1}\\&=(1+i)(Ia)_{\overline{n}|}\\&=1+a_{\overline{n-1}|}+(Ia)_{\overline{n-1}|}\end{aligned}$$

While the third of the expressions above is true, it is not especially useful for calculation purposes.

The second line above follows by general reasoning by noting that the payments for $(I\ddot{a})_{\overline{n}|}$ are the same as for $(Ia)_{\overline{n}|}$, but advanced by 1 year. This gives us the formula:

$$(I\ddot{a})_{\overline{n}|}=(1+i)(Ia)_{\overline{n}|}=(1+i)\frac{\ddot{a}_{\overline{n}|}-nv^n}{i}=\frac{\ddot{a}_{\overline{n}|}-nv^n}{i/(1+i)}=\frac{\ddot{a}_{\overline{n}|}-nv^n}{d}$$

We can alternatively derive this formula for $(I\ddot{a})_{\overline{n}|}$ using the same approach that we used to obtain a formula for $(Ia)_{\overline{n}|}$.

Starting from:

$$(I\ddot{a})_{\overline{n}|} = 1 + 2v + 3v^2 + \cdots + (n-1)v^{n-2} + nv^{n-1}$$

we can multiply through by v :

$$v(I\ddot{a})_{\overline{n}|} = v + 2v^2 + \cdots + (n-2)v^{n-2} + (n-1)v^{n-1} + nv^n$$

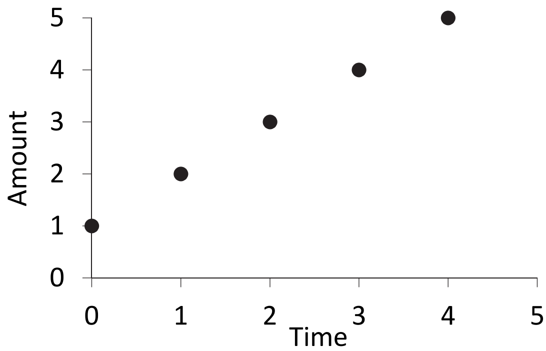
and subtract the second equation from the first to give:

$$(1-v)(I\ddot{a})_{\overline{n}|} = 1 + v + v^2 + \cdots + v^{n-2} + v^{n-1} - nv^n$$

Since $1 - v = d$ and $1 + v + v^2 + \cdots + v^{n-2} + v^{n-1} = \ddot{a}_{\overline{n}|}$, this gives:

$$d(I\ddot{a})_{\overline{n}|} = \ddot{a}_{\overline{n}|} - nv^n \quad \Rightarrow \quad (I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}$$

The graph below shows the pattern of the payments of $(I\ddot{a})_{\overline{5}|}$.



Question

Calculate the present value of payments of £50 at time 0, £60 at time 1 year, £70 at time 2 years and so on. The last payment is at time 10 years. Assume that the annual effective rate of interest is 4.2%.

Solution

First note that 11 payments are made in total. These can be shown on a timeline as follows:

We can think of this series of payments as a level annuity-due with payments of 40, and an increasing annuity-due with increases of 10 each time.

The present value of these payments is therefore:

$$40\ddot{a}_{\overline{11}|} + 10(I\ddot{a})_{\overline{11}|}$$

We have:

$$\ddot{a}_{\overline{11}|} = \frac{1-1.042^{-11}}{0.042/1.042} = 9.03074$$

$$(I\ddot{a})_{\overline{11}|} = \frac{\ddot{a}_{\overline{11}|} - 11v^{11}}{d} = \frac{9.03074 - 11 \times 1.042^{-11}}{0.042/1.042} = 50.48174$$

So the present value is:

$$40 \times 9.03074 + 10 \times 50.48174 = £866.05$$

1.2 Continuously payable annuities

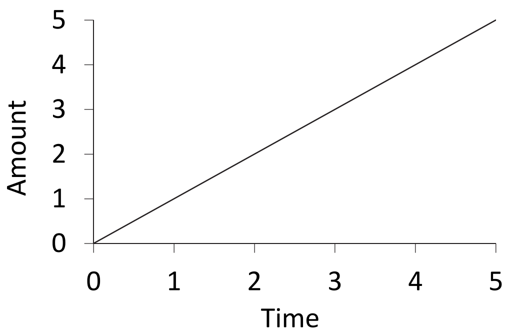
For increasing annuities which are payable continuously, it is important to distinguish between an annuity which has a constant rate of payment *r* (per unit time) throughout the *r*th period and an annuity which has a rate of payment t at time t. For the former the rate of payment is a step function taking the discrete values $1, 2, \ldots$. For the latter the rate of payment itself increases continuously. If the annuities are payable for *n* time units, their present values are denoted by $(I\bar{a})_{\overline{n}|}$ and $(\overline{Ia})_{\overline{n}|}$ respectively.

A bar over the *a* indicates that the payments are made continuously and a bar over the *I* indicates that increases occur continuously, rather than at the end of the year.

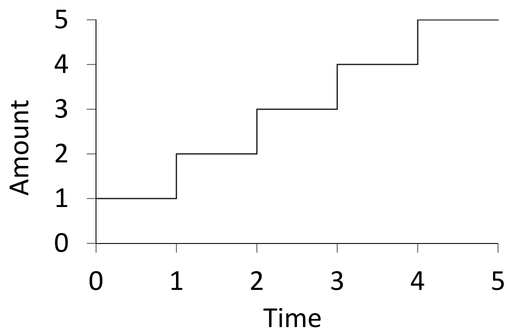
$(\overline{Ia})_{\overline{n}|}$ may also be written as $(\bar{I}\bar{a})_{\overline{n}|}$ – the two are equivalent.

The graphs below show the profiles of the payments.

$(\overline{Ia})_{\overline{5}|}$



$(I\bar{a})_{\overline{5}|}$



Clearly:

$$(I\bar{a})_{\overline{n}|} = \sum_{r=1}^{n}\left(\int_{r-1}^{r} rv^t dt\right)$$

and:

$$(\bar{I}\bar{a})_{\overline{n}|} = \int_0^n tv^t dt$$

and it can be shown that:

$$(I\bar{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{\delta}$$

and:

$$(\bar{I}\bar{a})_{\overline{n}|} = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta}$$

The formula for $(\overline{Ia})_{\overline{n}|}$ is derived by integrating by parts. Using the formula on page 3 of the *Tables*, with $u = t$ and $\dfrac{dv}{dt} = e^{-\delta t}$:

$$\begin{aligned}(\overline{Ia})_{\overline{n}|} &= \int_0^n tv^t\ dt = \int_0^n te^{-\delta t}\ dt \\ &= \left[t\left(-\frac{1}{\delta}e^{-\delta t}\right)\right]_0^n - \int_0^n -\frac{1}{\delta}e^{-\delta t}\ dt \\ &= -\frac{nv^n}{\delta} + \frac{1}{\delta}\int_0^n v^t\ dt = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta}\end{aligned}$$

Question

A man agrees to make investments continuously for the next 10 years. He decides that he can afford to invest £$20t$ at time t , $0 \le t \le 10$. Calculate the:

(i) present value of the investment at time 0

(ii) accumulated value of the investment at time 10.

Assume that the annual effective rate of interest is 3.7% throughout the 10 years.

Solution

(i) The present value of these payments is:

$$20(I\bar{a})_{\overline{10}|} = 20 \times \frac{\bar{a}_{\overline{10}|} - 10v^{10}}{\delta} = 20 \times \frac{\left(\dfrac{1-1.037^{-10}}{\ln 1.037}\right) - 10 \times 1.037^{-10}}{\ln 1.037} = 787.82$$

(ii) The accumulated value is the present value accumulated for 10 years:

$$787.82 \times 1.037^{10} = 1{,}132.96$$

The formula given above for for $(I\bar{a})_{\overline{n}|}$ can be derived by breaking it up into individual years:

$$\begin{aligned}(I\bar{a})_{\overline{n}|} &= \int_0^1 v^t\,dt + \int_1^2 2v^t\,dt + \cdots + \int_{n-1}^{n} nv^t\,dt \\ &= \bar{a}_{\overline{1}|} + 2v\bar{a}_{\overline{1}|} + \cdots + nv^{n-1}\bar{a}_{\overline{1}|} \\ &= \bar{a}_{\overline{1}|}\left(1 + 2v + \cdots + nv^{n-1}\right) \\ &= \bar{a}_{\overline{1}|} \times (I\ddot{a})_{\overline{n}|} = \frac{1-v}{\delta} \times \frac{\ddot{a}_{\overline{n}|} - nv^n}{d} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{\delta}\end{aligned}$$

since $1 - v = d$.

Alternatively, by general reasoning, the payments made in each year under $(I\bar{a})_{\overline{n}|}$ are the same as for $(Ia)_{\overline{n}|}$, but they are paid continuously throughout the year, rather than at the end of the year. So, by proportioning:

$$(I\bar{a})_{\overline{n}|} = (Ia)_{\overline{n}|} \times \frac{\bar{a}_{\overline{1}|}}{a_{\overline{1}|}} = (Ia)_{\overline{n}|} \times \frac{\frac{1-v}{\delta}}{\frac{1-v}{i}} = (Ia)_{\overline{n}|} \times \frac{i}{\delta}$$

which simplifies to the formula given.

The formula $(I\bar{a})_{\overline{n}|} = (Ia)_{\overline{n}|} \times \dfrac{i}{\delta}$ may be useful for calculations since values of both $(Ia)_{\overline{n}|}$ and $\dfrac{i}{\delta}$ are given in the *Tables* at various rates of interest.

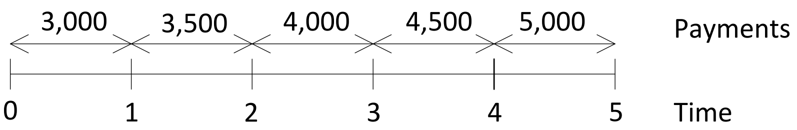
Question

Rent on a property is payable continuously for 5 years. The rent in the first year is £3,000, thereafter the annual rent increases by £500 *pa*.

Calculate the present value of the rent at the start of the 5 years, using an annual effective rate of interest of 6%.

Solution

Shown on a timeline, the rental payments are as follows:



The present value of these payments is:

$$2{,}500\bar{a}_{\overline{5}|} + 500(I\bar{a})_{\overline{5}|}$$

Evaluating these annuities:

$$\bar{a}_{\overline{5}|} = \frac{1-1.06^{-5}}{\ln 1.06} = 4.3375$$

$$\ddot{a}_{\overline{5}|} = \frac{1-1.06^{-5}}{0.06/1.06} = 4.4651$$

$$(I\bar{a})_{\overline{5}|} = \frac{4.4651 - 5\times 1.06^{-5}}{\ln 1.06} = 12.5078$$

So the present value is:

$$2{,}500 \times 4.3375 + 500 \times 12.5078 = £17{,}097.66$$

Alternatively, we can use values from the Tables as follows:

$$\bar{a}_{\overline{5}|} = \frac{i}{\delta} \times a_{\overline{5}|} = 1.029709 \times 4.2124 = 4.3375$$

$$(I\bar{a})_{\overline{5}|} = \frac{i}{\delta} \times (Ia)_{\overline{5}|} = 1.029709 \times 12.1469 = 12.5078$$

The present values of deferred increasing annuities are defined in the obvious manner, for example:

$$_{m|}(Ia)_{\overline{n}|} = v^m (Ia)_{\overline{n}|}$$

1.3 Decreasing payments

We can also use increasing annuities to calculate the present values of annuities where the payments decrease by a fixed amount each time.

Question

A man makes payments into an investment account of \$200 at time 5, \$190 at time 6, \$180 at time 7, and so on until a payment of \$100 at time 15. Assuming an annual effective rate of interest of 3.5%, calculate:

(i) the present value of the payments at time 4,

(ii) the present value of the payments at time 0,

(iii) the accumulated value of the payments at time 15.

Solution

There are 11 payments in total. The payments can be thought of as:

210 at time 5, 210 at time 6, 210 at time 7, …, 210 at time 15

LESS the following payments:

10 at time 5, 20 at time 6, 30 at time 7, …, 110 at time 15

(i) The present value of these payments at time 4 (which is one time period before the first payment is made) is:

$$210a_{\overline{11}|} - 10(Ia)_{\overline{11}|}$$

Evaluating these, we have:

$$a_{\overline{11}|} = \frac{1-1.035^{-11}}{0.035} = 9.0016$$

$$\ddot{a}_{\overline{11}|} = \frac{1-1.035^{-11}}{0.035/1.035} = 9.3166 \quad (\text{or } \ddot{a}_{\overline{11}|} = 1.035 \times a_{\overline{11}|} = 9.3166)$$

$$(Ia)_{\overline{11}|} = \frac{9.3166 - 11 \times 1.035^{-11}}{0.035} = 50.9201$$

So the present value is:

$$210 \times 9.0016 - 10 \times 50.9201 = 1{,}381.13$$

(ii) To find the present value at time 0, we need to discount the answer to part (i) by 4 years:

$$1{,}381.13 \times 1.035^{-4} = 1{,}203.57$$

(iii) To find the accumulated value at time 15, we need to accumulate the answer to (i) by 11 years:

$$1{,}381.13 \times 1.035^{11} = 2{,}016.40$$

2 Special cases

2.1 Irregular payments

When the interest rate is constant, we can use the approach illustrated in the following example. This involves converting the payments into a simpler series of payments with the same present value.

Question

Write down an expression in terms of annuity functions for the present value as at 1 January 2019 of the following payments under the operation of a constant rate of interest:

£100 on 1 January, 1 April, 1 July and 1 October 2019

£200 on 1 January, 1 April, 1 July and 1 October 2020

£300 on 1 January, 1 April, 1 July and 1 October 2021

£400 on 1 January, 1 April, 1 July and 1 October 2022

£500 on 1 January, 1 April, 1 July and 1 October 2023

Solution

We can convert the payments for each calendar year to an equivalent single payment with the same present value payable on 1 January that year. For example, the payments in 2021 are equivalent to a single payment of $1{,}200\ddot{a}_{\overline{1}|}^{(4)}$ payable on 1 January 2021.

So, the payments are equivalent (in terms of present value) to the following five payments:

- $1 \times 400\ddot{a}_{\overline{1}|}^{(4)}$ on 1 January 2019,
- $2 \times 400\ddot{a}_{\overline{1}|}^{(4)}$ on 1 January 2020,
- *etc*
- $5 \times 400\ddot{a}_{\overline{1}|}^{(4)}$ on 1 January 2023.

This is a simple increasing annuity (payable annually in advance) where the payment amounts increase by $400\ddot{a}_{\overline{1}|}^{(4)}$ each year. So the present value is $(I\ddot{a})_{\overline{5}|} \times 400\ddot{a}_{\overline{1}|}^{(4)}$.

2.2 Compound increasing annuities

We have already looked at *simple* increasing annuities, such as $(Ia)_{\overline{n}|}$, where the payments increase by a constant *amount* each time. We also need to be able to value *compound* increasing annuities where the payments increase by a constant *factor* each time.

Question

Calculate the present value of an annuity payable annually in arrears for 15 years, where the first payment is 500 and subsequent payments increase by 3% per annum compound.

The effective annual rate of interest is 10%.

Solution – Method 1

The payment at the end of the first year is 500; the payment at the end of the second year is 500×1.03; the payment at the end of the third year is 500×1.03^2, and so on until the final payment of 500×1.03^{14} is made at the end of the fifteenth year.

From first principles, we can write down an expression for the present value of this annuity as follows:

$$PV = 500v + 500\times 1.03v^2 + 500\times 1.03^2 v^3 + \cdots + 500\times 1.03^{14} v^{15} \qquad (*)$$

The terms being summed in this present value expression form a geometric progression of 15 terms, with first term $500v$ and common ratio $1.03v = \frac{1.03}{1.1}$. So the present value is equal to:

$$PV = 500v \times \frac{1-\left(\frac{1.03}{1.1}\right)^{15}}{1-\frac{1.03}{1.1}} = \frac{500}{1.1}\times 9.853407 = 4{,}479$$

Method 2

We'll now consider a slightly different way of solving this problem. The equation (*) could be rearranged and written as:

$$\begin{aligned} PV &= \frac{500}{1.03}\left(1.03v + 1.03^2 v^2 + \cdots 1.03^{15} v^{15}\right) \\ &= \frac{500}{1.03}\left(v' + v'^2 + \cdots v'^{15}\right) \end{aligned}$$

where $v' = 1.03v$.

Remember that:

$$a_{\overline{n}|} = v + v^2 + \cdots + v^n \qquad @i\% \qquad \text{where } v = \frac{1}{1+i}$$

If we introduce a new interest rate i', we can define:

$$a_{\overline{n|}}{}' = v' + v'^2 + \cdots + v'^n \quad @i'\% \quad \text{where} \quad v' = \frac{1}{1+i'} = 1.03v \qquad (**)$$

This is the value, $a_{\overline{n|}}{}'$, of an annuity-certain, calculated at interest rate i'. From (**) we get:

$$\frac{1}{1+i'} = \frac{1.03}{1.1} \quad \Rightarrow \quad i' = \frac{1.1}{1.03} - 1 = 6.7961\%$$

So we can now write the present value as:

$$PV = \frac{500}{1.03} a_{\overline{15|}}{}' \quad @6.7961\%$$

$$= \frac{500}{1.03}\left(\frac{1-\left(\frac{1}{1.067961}\right)^{15}}{0.067961}\right) = \frac{500}{1.03} \times 9.2264 = 4{,}479$$

This example shows how a compound increasing annuity can be valued as a level annuity at a different interest rate.

Method 3

Finally, we'll consider a slight adaptation of Method 2, using an annuity-due instead of an annuity in arrears. Here, we rewrite the equation (*) as:

$$PV = 500v\left(1 + 1.03v + \cdots 1.03^{14}v^{14}\right) = 500v\left(1 + v' + v'^2 + \cdots v'^{14}\right)$$

where $v' = 1.03v$.

This time the expression in brackets is an annuity-due and so we can write:

$$PV = 500v\ddot{a}_{\overline{15|}}{}'$$

We must be slightly careful when evaluating this expression. The v in the expression must be calculated at 10%, whereas the annuity must be calculated at our new interest rate $i' = 6.7961\%$ as before. So:

$$PV = 500v\left(\frac{1-v'^{15}}{d'}\right) = \frac{500}{1.1} \times \left(\frac{1-\left(\frac{1}{1.067961}\right)^{15}}{0.067961/1.067961}\right) = \frac{500}{1.1} \times 9.8534 = 4{,}479$$

We have considered three slightly different methods here. There is no 'best method' to use. While Methods 2 and 3 may appear to be more complicated, it's worth trying them out as they're not quite as complex as they initially appear, and we will use a similar approach when considering life assurances and life annuities with compound increasing benefits later in the course.

Let's take another look at Method 2 in a more general context.

Consider an annuity under which the payment at time t is $(1+e)^t$, where e is any constant and $t=1,2,\ldots,n$. Then the present value of the single payment paid at time t is $\dfrac{(1+e)^t}{(1+i)^t}=\left(\dfrac{1+e}{1+i}\right)^t$. If we introduce a new interest rate i' defined by the equation $\dfrac{1}{1+i'}=\dfrac{1+e}{1+i}$, we find that the present value of the payment at time t can be expressed as $\left(\dfrac{1}{1+i'}\right)^t$, *ie* it is $(v')^t$, calculated using the interest rate i'.

Similarly, the present value of this compound increasing annuity is:

$$v'+(v')^2+\cdots+(v')^n$$

which is the value of an annuity-certain, $a_{\overline{n|}}'$, calculated at the interest rate i'.

We can rearrange the equation for i' above to find an expression for i' in terms of i and e:

$$\frac{1}{1+i'}=\frac{1+e}{1+i} \qquad \Rightarrow \qquad i'=\frac{1+i}{1+e}-1=\frac{(1+i)-(1+e)}{1+e}=\frac{i-e}{1+e}$$

Question

Calculate the present value (at time $t=0$) of payments of $£20{,}000\times 1.0381^{t-1}$ payable at times $t=1,2,3,\ldots,10$, where time is measured in years, assuming a constant annual effective rate of interest of 9%.

Solution

The payment at time 1 is £20,000; the payment at time 2 is $£20{,}000\times 1.0381$; the payment at time 3 is $£20{,}000\times 1.0381^2$, and so on.

The present value of these payments is:

$$PV=20{,}000\left(v+1.0381v^2+\cdots+1.0381^9v^{10}\right)$$

The terms in brackets form a geometric progression of 10 terms, with first term v and common ratio $1.0381v=\dfrac{1.0381}{1.09}$. So the present value is equal to:

$$PV=20{,}000v\times\frac{1-\left(\frac{1.0381}{1.09}\right)^{10}}{1-\frac{1.0381}{1.09}}=\frac{20{,}000}{1.09}\times 8.107974=£148{,}770$$

Alternatively, we can write the present value as:

$$PV = 20{,}000\left(v + 1.0381v^2 + \cdots + 1.0381^9 v^{10}\right)$$

$$= \frac{20{,}000}{1.0381}\left(1.0381v + 1.0381^2 v^2 + \cdots + 1.0381^{10} v^{10}\right)$$

$$= \frac{20{,}000}{1.0381} a_{\overline{10}|}'$$

In this case we have to divide by 1.0381 to ensure that the power of 1.0381 matches the power of v in each term of the expression.

The interest rate to use for the annuity is:

$$i' = \frac{i-e}{1+e} = \frac{0.09 - 0.0381}{1+0.0381} = 0.05000$$

ie almost exactly 5%. So the present value of the payments is:

$$\frac{20{,}000}{1.0381} a_{\overline{10}|@5\%} = \frac{20{,}000}{1.0381} \times 7.7217 = £148{,}770$$

As another alternative, we can write the present value as:

$$PV = 20{,}000\left(v + 1.0381v^2 + \cdots + 1.0381^9 v^{10}\right)$$

$$= 20{,}000v\left(1 + 1.0381v + \cdots + 1.0381^9 v^9\right)$$

$$= 20{,}000 \times \frac{1}{1.09} \times \ddot{a}_{\overline{10}|@5\%}$$

Evaluating this expression gives the same answer as before.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 9 Summary

$(Ia)_{\overline{n|}}$ represents the present value of payments of 1 at time 1, 2 at time 2, …, n at time n:

$$(Ia)_{\overline{n|}} = \sum_{t=1}^{n} tv^t = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i}$$

An increasing annuity but with payments in advance is given by:

$$(I\ddot{a})_{\overline{n|}} = \sum_{t=0}^{n-1} (t+1)v^t = \frac{\ddot{a}_{\overline{n|}} - nv^n}{d} = (1+i)(Ia)_{\overline{n|}}$$

For the continuous annuity $(I\bar{a})_{\overline{n|}}$, the rate of payment is a step function taking the discrete values 1, 2, …, n. For $(\overline{Ia})_{\overline{n|}}$, the rate of payment itself increases continuously. The rate of payment at time t is t. The formulae are:

$$(I\bar{a})_{\overline{n|}} = \sum_{r=1}^{n} \left(\int_{r-1}^{r} rv^t dt \right) = \frac{\ddot{a}_{\overline{n|}} - nv^n}{\delta} = \frac{i}{\delta}(Ia)_{\overline{n|}} \qquad (\overline{Ia})_{\overline{n|}} = \int_0^n tv^t \, dt = \frac{\bar{a}_{\overline{n|}} - nv^n}{\delta}$$

The present value of a compound increasing annuity can be found either by using the formula for the sum of terms in a geometric progression, or by writing it as a level annuity at an adjusted rate of interest.

The sum of the first n terms of a geometric progression with first term a and common ratio r is:

$$\frac{a(1-r^n)}{1-r}$$

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 9 Practice Questions

9.1 (i) Calculate the following at $i = 9\%$:

(a) $(Ia)_{\overline{60}|}$

(b) $(I\ddot{a})_{\overline{60}|}$

(c) $(I\bar{a})_{\overline{100}|}$

(d) $(\overline{I}\bar{a})_{\overline{100}|}$

(ii) Calculate the following at $i = 7\%$:

(a) $(Is)_{\overline{1}|}$

(b) $(I\ddot{s})_{\overline{10}|}$

9.2 A series of payments is to be received annually in advance. The first payment is £10. Thereafter, payments increase by £2 per annum. The last payment is made at the beginning of the tenth year.

Determine whether each of the following is a correct expression for the present value of the annuity.

(i) $\sum_{t=0}^{9} 8v^t + 2\sum_{t=0}^{9} tv^t$

(ii) $10\ddot{a}_{\overline{10}|} + 2(Ia)_{\overline{9}|}$

(iii) $8\ddot{a}_{\overline{10}|} + 2(I\ddot{a})_{\overline{10}|}$

9.3 Given that $\delta(t) = 0.01t$ for $0 \le t \le 10$, calculate the value of $(\overline{I}\bar{a})_{\overline{10}|}$.

9.4 An annuity is payable annually in advance for a term of 20 years. The payment is £500 in year 1, £550 in year 2, and so on, increasing by £50 each year.

Exam style

Calculate the present value of this annuity, assuming that the effective rate of interest is 5% *pa* for the first twelve years and 7% *pa* thereafter. [5]

9.5 A continuous payment stream is such that the level rate of payment in year t is $100 \times 1.05^{t-1}$, for $t = 1,2,\ldots,10$. Calculate the present value of this payment stream as at its commencement date, assuming a rate of interest of 10% *pa.*

9.6 A series of 10 payments is received at times 5, 6, 7,…, 14. The first payment is \$200. Each of the next five payments is 5.7692% greater than the previous one, and thereafter each payment is 6.7961% greater than the previous one.

Calculate the present value of these payments at time 0 using an interest rate of 10% *pa* effective.

9.7 An investor in property expects to receive rental payments for the next 50 years in line with the following assumptions:

Exam style

- The current level of rental payments is £20,000 per annum, paid quarterly in advance.
- Payments will remain fixed for 5-year periods. At the end of each 5-year period the payments will rise in line with total inflationary growth over the previous five years.
- Inflation is assumed to be constant at 3% per annum.
- The interest rate is 12% per annum effective.

Calculate the present value of the rental income the investor expects to receive. [6]

9.8 (i) Assuming a rate of interest of 6% *pa* effective, calculate the present value as at 1 January 2020 of the following annuities, each with a term of 25 years:

Exam style

(a) an annuity payable annually, where the first payment is £3,000 made on 1 January 2021, and payments increase by £500 on each subsequent 1 January.

(b) an annuity as in (i), but only 10 increases are to be made, the annuity then remaining level for the remainder of the term. [5]

(ii) An investor is to receive a special annual annuity for a term of 10 years in which payments are increased by 5% compound each year to allow for inflation. The first payment is to be £1,000 on 1 November 2021. Calculate the accumulated value of the annuity payments as at 31 October 2038 if the investor achieves an effective rate of return of 4% per half-year. [4]

[Total 9]

Chapter 9 Solutions

9.1 (i)(a) Using the formula for the present value of an increasing annuity:

$$(Ia)_{\overline{60}|}=\frac{\ddot{a}_{\overline{60}|}-60v^{60}}{i}=\frac{12.0423-60\times 1.09^{-60}}{0.09}=130.02$$

The value of $(Ia)_{\overline{60}|}$ *can also be looked up directly in the Tables.*

(i)(b) This can be calculated from the previous answer:

$$(I\ddot{a})_{\overline{60}|}=1.09(Ia)_{\overline{60}|}=1.09\times 130.02=141.72$$

(i)(c) Using the formula:

$$(I\bar{a})_{\overline{100}|}=\frac{\ddot{a}_{\overline{100}|}-100v^{100}}{\delta}=\frac{12.1089-100\times 1.09^{-100}}{\ln(1.09)}=140.30$$

Alternatively, using values from the Tables:

$$(I\bar{a})_{\overline{100}|}=\frac{i}{\delta}(Ia)_{\overline{100}|}=1.044354\times 134.3426=140.30$$

(i)(d) Using the formula:

$$(\bar{I}\bar{a})_{\overline{100}|}=\frac{\bar{a}_{\overline{100}|}-100v^{100}}{\delta}=\frac{11.6018-100\times 1.09^{-100}}{\ln(1.09)}=134.42$$

(ii)(a) This is the accumulated value at time 1 of a payment of 1 unit made at time 1:

$$(Is)_{\overline{1}|}=1$$

(ii)(b) This is the accumulated value of a 10-year increasing annuity due:

$$(I\ddot{s})_{\overline{10}|}=1.07^{10}(I\ddot{a})_{\overline{10}|}$$

$$=1.07^{10}\left(\frac{\ddot{a}_{\overline{10}|}-10v^{10}}{0.07/1.07}\right)=1.07^{10}\left(\frac{7.515232-10\times 1.07^{-10}}{0.07/1.07}\right)=73.12$$

Alternatively, using values from the Tables:

$$(I\ddot{s})_{\overline{10}|}=1.07^{10}(I\ddot{a})_{\overline{10}|}=1.07^{10}\times 1.07(Ia)_{\overline{10}|}=1.07^{11}\times 34.7391=73.12$$

9.2 The present value of the annuity is:

$$PV = 10 + 12v + 14v^2 + \cdots + 26v^8 + 28v^9$$

Expression (i) is not correct since the first term (obtained when $t = 0$) is 8, not 10.

Expression (ii) is correct since it can be rearranged to give the expression above:

$$10\ddot{a}_{\overline{10}|} + 2(Ia)_{\overline{9}|} = 10(1 + v + \cdots + v^9) + 2(v + 2v^2 + \cdots + 9v^9) = 10 + 12v + \cdots + 28v^9$$

Expression (iii) is also correct since it can be rearranged to give the expression above:

$$8\ddot{a}_{\overline{10}|} + 2(I\ddot{a})_{\overline{10}|} = 8(1 + v + \cdots + v^9) + 2(1 + 2v + \cdots + 10v^9) = 10 + 12v + \cdots + 28v^9$$

9.3 The discount factor for a payment at time t is:

$$v(t) = \exp\left(-\int_0^t 0.01s\, ds \right) = e^{-0.005t^2}$$

This continuously-increasing, continuously-payable annuity has a rate of payment of t at time t , so its present value can be expressed as an integral as:

$$(\bar{I}\bar{a})_{\overline{10}|} = \int_0^{10} t\,v(t)\,dt = \int_0^{10} te^{-0.005t^2} dt$$

Now, using the general result:

$$\int_a^b f'(t)e^{f(t)} dt = \left[e^{f(t)} \right]_a^b$$

we know that:

$$\int_0^{10} (-0.01t)e^{-0.005t^2}\ dt = \left[e^{-0.005t^2} \right]_0^{10}$$

So, we have:

$$(\bar{I}\bar{a})_{\overline{10}|} = \int_0^{10} te^{-0.005t^2} dt = -100 \int_0^{10} (-0.01t)e^{-0.005t^2} dt$$

$$= -100 \left[e^{-0.005t^2} \right]_0^{10} = -100\left(e^{-0.5} - 1 \right) = 39.35$$

Alternatively, we can use the substitution $u=-0.005t^2$*, so that:*

$$\frac{du}{dt}=-0.01t \;\Rightarrow\; du=-0.01t\,dt \;\Rightarrow\; -100\,du=t\,dt$$

When $t=0$*,* $u=0$*, and when* $t=10$*,* $u=-0.5$*, so we have:*

$$(I\bar{a})_{\overline{10}|}=\int_0^{10} te^{-0.005t^2}dt=\int_0^{-0.5} -100e^u\,du=-100\Big[e^u\Big]_0^{-0.5}=-100\left(e^{-0.5}-1\right)=39.35$$

9.4 The present value of the payments can be expressed as:

$$450\ddot{a}_{\overline{12}|}^{@5\%}+50(I\ddot{a})_{\overline{12}|}^{@5\%}+v^{12@5\%}(1{,}050\ddot{a}_{\overline{8}|}^{@7\%}+50(I\ddot{a})_{\overline{8}|}^{@7\%}) \qquad [2]$$

The numerical values of the components are:

$$\ddot{a}_{\overline{12}|}^{@5\%}=\frac{1-v^{12}}{d}=\frac{1-0.55684}{0.05/1.05}=9.3064 \qquad [½]$$

$$(I\ddot{a})_{\overline{12}|}^{@5\%}=\frac{\ddot{a}_{\overline{12}|}-12v^{12}}{d}=\frac{9.3065-6.6821}{0.05/1.05}=55.1117 \qquad [½]$$

$$\ddot{a}_{\overline{8}|}^{@7\%}=\frac{1-v^{8}}{d}=\frac{1-0.58201}{0.07/1.07}=6.3893 \qquad [½]$$

$$(I\ddot{a})_{\overline{8}|}^{@7\%}=\frac{\ddot{a}_{\overline{8}|}-8v^{8}}{d}=\frac{6.3893-4.6561}{0.07/1.07}=26.4935 \qquad [½]$$

Substituting in these values gives a present value of:

$$450\times 9.3064+50\times 55.1117+0.55684(1{,}050\times 6.3893+50\times 26.4935)=£11{,}417 \qquad [1]$$

[Total 5]

9.5 The rate of payment is 100 in the first year, 100×1.05 in the second year, 100×1.05^2 in the third year, and so on. The present value of the payment stream is therefore:

$$100\bar{a}_{\overline{1}|}+100\times 1.05v\bar{a}_{\overline{1}|}+100\times 1.05^2v^2\bar{a}_{\overline{1}|}+\cdots+100\times 1.05^9v^9\bar{a}_{\overline{1}|}$$

$$=100\bar{a}_{\overline{1}|}\left(1+1.05v+1.05^2v^2+\cdots+1.05^9v^9\right)$$

The terms in brackets form a geometric progression of 10 terms with first term 1 and common ratio $1.05v$, so the present value is:

$$100\times\frac{1-v}{\delta}\times\frac{1-(1.05v)^{10}}{1-1.05v}=100\times 0.95382\times 8.1838=780.59$$

Alternatively, this could be evaluated as:

$$100\bar{a}_{\overline{1}|}^{@10\%}\left(1+1.05v+1.05^2v^2+\cdots+1.05^9v^9\right)=100\bar{a}_{\overline{1}|}^{@10\%}\left(1+V+V^2+\cdots+V^9\right)$$

$$=100\bar{a}_{\overline{1}|}^{@10\%}\ddot{a}_{\overline{10}|}^{@4.7619\%}$$

where $V=\dfrac{1}{1+I}=\dfrac{1.05}{1.1} \Rightarrow I=\dfrac{1.1}{1.05}-1=4.7619\%$.

9.6 Let $e=5.7692\%$ and $f=6.7961\%$. The present value of the payments is:

$$200v^5+200(1+e)v^6+200(1+e)^2v^7+\cdots+200(1+e)^5v^{10}$$

$$+200(1+e)^5(1+f)v^{11}+200(1+e)^5(1+f)^2v^{12}+\cdots+200(1+e)^5(1+f)^4v^{14}$$

This splits naturally into two geometric progressions. The first relates to the first line above – it contains 6 terms, has first term $200v^5$ and common ratio $(1+e)v$. The second relates to the second line above – it contains 4 terms, has first term $200(1+e)^5(1+f)v^{11}$ and common ratio $(1+f)v$. So the present value can be written as:

$$\frac{200v^5\left(1-\left((1+e)v\right)^6\right)}{1-(1+e)v}+\frac{200(1+e)^5(1+f)v^{11}\left(1-\left((1+f)v\right)^4\right)}{1-(1+f)v}$$

Evaluating this using $v=1.1^{-1}$, the present value is:

$$677.030+379.405=\$1,056.44$$

Alternatively, we can split up the series and pull out common factors:

$$\frac{200v^4}{(1+e)}\left((1+e)v+(1+e)^2v^2+\cdots+(1+e)^6v^6\right)$$

$$+200(1+e)^5v^{10}\left((1+f)v+(1+f)^2v^2+\cdots+(1+f)^4v^4\right)$$

$$=\frac{200v^4}{(1+e)}a'_{\overline{6}|}+200(1+e)^5v^{10}a''_{\overline{4}|}$$

where one dash represents an interest rate of $\dfrac{0.1-0.057692}{1.057692}=4\%$ *pa, and two dashes represents an interest rate of* $\dfrac{0.1-0.067961}{1.067961}=3\%$ *pa.*

The present value can then be evaluated as:

$$\frac{200\times 1.1^{-4}}{1.057692}\times\frac{1-1.04^{-6}}{0.04}+200(1.057692)^5\times 1.1^{-10}\times\frac{1-1.03^{-4}}{0.03}$$

$$=129.15167\times 5.24214+102.07026\times 3.71710$$

$$=\$1,056.44$$

9.7 The present value of the first five years' worth of payments (working in thousands of pounds) is $20\ddot{a}^{(4)}_{\overline{5}|}$ (calculated at $i=12\%$).

The present value of the next 5 years' worth of payments is:

$$20\times 1.03^5\times\ddot{a}^{(4)}_{\overline{5}|}\times v^5$$

We must increase the annual payment by 1.03^5 *because we are given an annual rate of inflation and we are told that 'the payments will rise in line with* ***total*** *inflationary growth over the previous five years'.*

The next 5 years' worth of payments will be worth:

$$20\times 1.03^{10}\times\ddot{a}^{(4)}_{\overline{5}|}\times v^{10}$$

and so on. So the total present value will be:

$$20\,\ddot{a}^{(4)}_{\overline{5}|@12\%}\left[1+1.03^5v^5+1.03^{10}v^{10}+\dots+1.03^{45}v^{45}\right]$$ [2]

The terms in brackets form a geometric progression of 10 terms, with first term 1 and common ratio $1.03^5v^5=\dfrac{1.03^5}{1.12^5}$. So:

$$1+1.03^5v^5+1.03^{10}v^{10}+\dots+1.03^{45}v^{45}=\frac{1-\left(\frac{1.03^5}{1.12^5}\right)^{10}}{1-\frac{1.03^5}{1.12^5}}=2.877967$$ [2]

Also:

$$\ddot{a}^{(4)}_{\overline{5}|@12\%}=\frac{1-v^5}{d^{(4)}}=\frac{1-1.12^{-5}}{4\left(1-1.12^{-1/4}\right)}=3.871305$$ [1]

So the present value of the rental income is:

$$20\times 3.871305\times 2.877967=222.830$$

ie £222,830. [1]

[Total 6]

Alternatively, the present value expression can be written as:

$$20\,\ddot{a}^{(4)}_{\overline{5}|@12\%}\left[1+1.03^5v^5+1.03^{10}v^{10}+\cdots+1.03^{45}v^{45}\right]=20\,\ddot{a}^{(4)}_{\overline{5}|@12\%}\,\ddot{a}_{\overline{10}|@\,j\%}$$

where the last annuity term above is calculated at the interest rate j for which

$v_j=\dfrac{1}{1+j}=\left(\dfrac{1.03}{1.12}\right)^5$. *This gives $j=52.021\%$. So the total present value is:*

$$20\,\ddot{a}^{(4)}_{\overline{5}|@12\%}\,\ddot{a}_{\overline{10}|@\,j\%}=20\times\frac{1-v^5}{d^{(4)}}\times\frac{1-v_j^{10}}{d_j}=20\times3.871305\times2.877967=222.830$$

ie £222,830, as before.

9.8 (i)(a) The present value of the payments can be expressed as:

$$PV=2{,}500a_{\overline{25}|}+500(Ia)_{\overline{25}|}$$ [1]

Using annuity values from the *Tables*:

$$PV=2{,}500\times12.7834+500\times128.7565=£96{,}337$$ [1]

(i)(b) The present value of the payments is:

$$PV=2{,}500a_{\overline{25}|}+500(Ia)_{\overline{11}|}+5{,}500v^{11}a_{\overline{14}|}$$ [2]

So, using annuity values from the *Tables*:

$$PV=2{,}500\times12.7834+500\times42.7571+5{,}500\times1.06^{-11}\times9.2950=£80{,}268$$ [1]

[Total 5]

Alternatively, we could use the expression:

$$PV=2{,}500a_{\overline{11}|}+500(Ia)_{\overline{11}|}+8{,}000v^{11}a_{\overline{14}|}$$

(ii) The first payment (of £1,000) needs to be accumulated for 17 years (or 34 half-years). The second payment (of $£1{,}000\times1.05$) needs to be accumulated for 16 years (or 32 half-years), and so on, until the tenth payment (of $£1{,}000\times1.05^9$), which needs to be accumulated for 8 years (or 16 half-years).

The accumulated value of the payments is therefore:

$$AV=1{,}000\times1.04^{34}+1{,}000\times1.05\times1.04^{32}+\cdots+1{,}000\times1.05^9\times1.04^{16}$$ [2]

This is a geometric progression of 10 terms, with first term $1{,}000 \times 1.04^{34}$ and common ratio 1.05×1.04^{-2}, so the accumulated value is:

$$AV = 1{,}000 \times 1.04^{34} \times \frac{1-(1.05 \times 1.04^{-2})^{10}}{1-1.05 \times 1.04^{-2}} = £33{,}324 \qquad [2]$$

[Total 4]

Alternatively, the accumulated value can be calculated using:

$$AV = 1{,}000 \times 1.04^{34} \left(1 + \frac{1.05}{1.04^2} + \frac{1.05^2}{1.04^4} + \cdots + \frac{1.05^9}{1.04^{18}}\right)$$

$$= 1{,}000 \times 1.04^{34} \left(1 + V + V^2 + \cdots + V^9\right)$$

$$= 1{,}000 \times 1.04^{34}\, \ddot{a}_{\overline{10|}}^{@3.0095\%}$$

where $V = \dfrac{1}{1+I} = \dfrac{1.05}{1.04^2} \;\Rightarrow\; I = \dfrac{1.04^2}{1.05} - 1 = 3.0095\%$.

End of Part 1

What next?

1. Briefly **review** the key areas of Part 1 and/or re-read the **summaries** at the end of Chapters 1 to 9.
2. Ensure you have attempted some of the **Practice Questions** at the end of each chapter in Part 1. If you don't have time to do them all, you could save the remainder for use as part of your revision.
3. Attempt **Assignment X1**.

10

Equations of value

Syllabus objectives

3.1 Define an equation of value.

- 3.1.1 Define an equation of value, where payment or receipt is certain.
- 3.1.2 Describe how an equation of value can be adjusted to allow for uncertain receipts or payments.
- 3.1.3 Understand the two conditions required for there to be an exact solution to an equation of value.

0 Introduction

An *equation of value* equates the present value of money received to the present value of money paid out:

'PV income = PV outgo'

or equivalently:

'*PV income – PV outgo = 0'*

Equations of value are used throughout actuarial work. For example:

- the 'fair price' to pay for an investment such as a fixed-interest security or an equity (*ie PV outgo*) equals the present value of the proceeds from the investment, discounted at the rate of interest required by the investor.
- the premium for an insurance policy is calculated by equating the present value of the expected amounts received in premiums to the present value of the expected benefits and other outgo.

1 The equation of value and the yield on a transaction

1.1 The theory

Consider a transaction that provides that, in return for outlays of amount $a_{t_1}, a_{t_2}, \dots, a_{t_n}$ at time $t_1, t_2, \dots, t_n$, an investor will receive payments of $b_{t_1}, b_{t_2}, \dots, b_{t_n}$ at these times respectively. (In most situations only one of a_{t_r} and b_{t_r} will be non-zero.) At what force or rate of interest does the series of outlays have the same value as the series of receipts? At force of interest δ the two series are of equal value if and only if:

$$\sum_{r=1}^{n} a_{t_r} e^{-\delta t_r} = \sum_{r=1}^{n} b_{t_r} e^{-\delta t_r}$$

This equation may be written as:

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta t_r} = 0 \text{ (1.1)}$$

where $c_{t_r} = b_{t_r} - a_{t_r}$ is the amount of the *net cashflow* at time t_r. (We adopt the convention that a negative cashflow corresponds to a payment *by* the investor and a positive cashflow represents a payment *to* the investor.)

Equation (1.1), which expresses algebraically the condition that, at force of interest δ, the total value of the net cashflows is 0, is called the equation of value for the force of interest implied by the transaction. If we let $e^{\delta} = 1+i$, the equation may be written as:

$$\sum_{r=1}^{n} c_{t_r} (1+i)^{-t_r} = 0 \qquad \textbf{(1.2)}$$

The latter form is known as the equation of value for the rate of interest or the 'yield equation'. Alternatively, the equation may be written as:

$$\sum_{r=1}^{n} c_{t_r} v^{t_r} = 0$$

In relation to continuous payment streams, if we let $\rho_1(t)$ and $\rho_2(t)$ be the rates of paying and receiving money at time *t* respectively, we call $\rho(t) = \rho_2(t) - \rho_1(t)$ the net rate of cashflow at time *t*. The equation of value (corresponding to Equation (1.1)) for the force of interest is:

$$\int_0^{\infty} \rho(t) e^{-\delta t} dt = 0$$

When both discrete and continuous cashflows are present, the equation of value is:

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta t_r} + \int_0^\infty \rho(t) e^{-\delta t} dt = 0 \tag{1.3}$$

and the equivalent yield equation is:

$$\sum_{r=1}^{n} c_{t_r} (1+i)^{-t_r} + \int_0^\infty \rho(t)(1+i)^{-t} dt = 0 \tag{1.4}$$

For any given transaction, Equation (1.3) may have no roots, a unique root, or several roots. If there is a unique root, δ_0 say, it is known as the force of interest implied by the transaction, and the corresponding rate of interest $i_0 = e^{\delta_0} - 1$ is called the 'yield' per unit time. (Alternative terms for the yield are the 'internal rate of return' and the 'money-weighted rate of return' for the transaction.) Thus the yield is defined if and only if Equation (1.4) has precisely one root greater than –1 and, when such a root exists, it is the yield.

The yield must be greater than –1, since $e^{\delta_0} > 0$, so $i_0 = e^{\delta_0} - 1 > -1$. A yield of –1 corresponds to a return of –100%, *ie* losing all the money originally invested.

Question

An investor pays £100 now in order to receive £60 in 5 years' time and £60 in 10 years' time.

Calculate the annual effective rate of interest earned on this investment.

Solution

The equation of value is:

$$100 = 60v^5 + 60v^{10}$$

This is a quadratic in v^5, which can be solved to give:

$$v^5 = (1+i)^{-5} = \frac{-60 \pm \sqrt{60^2 + 4 \times 60 \times 100}}{120} = 0.8844 \text{ or } -1.884$$

Rearranging, this gives i to be 2.49% or –188%. Since i must be greater than –1, the annual effective rate of interest is 2.49%.

The analysis of the equation of value for a given transaction may be somewhat complex depending on the shape of the function $f(i)$ denoting the left-hand side of Equation (1.4). However, when the equation $f(i) = 0$ is such that f is a monotonic function, its analysis is particularly simple. The equation has a root if and only if we can find i_1 and i_2 with $f(i_1)$ and $f(i_2)$ of opposite sign. In this case, the root is unique and lies between i_1 and i_2. By choosing i_1 and i_2 to be 'tabulated' rates sufficiently close to each other, we may determine the yield to any desired degree of accuracy.

A monotonic function is one without any turning points, so it either always increases, or always decreases.

Having identified suitable values of i_1 and i_2, linear interpolation could be used to obtain the yield i_0 from these. Examples of this method are given a little later in this chapter.

It should be noted that, after multiplication by $(1+i)^{t_0}$, Equation (1.2) takes the equivalent form:

$$\sum_{r=1}^{n} c_{t_r}(1+i)^{t_0-t_r}=0$$

This slightly more general form may be called the *equation of value at time t_0*. It is of course directly equivalent to the original equation (which is now seen to be the equation of value at time 0). In certain problems a particular choice of t_0 may simplify the solution.

1.2 Solving for an unknown quantity

Many problems in actuarial work can be reduced to solving an equation of value for an unknown quantity. We will look at how to do this using examples based on a hypothetical fixed-interest security which operates as follows:

Security *S*

A price P is paid (by the investor) in return for a series of interest payments of D payable at the end of each of the next n years and a final redemption payment of R payable at the end of the n years. The investor earns an annual effective rate of return of i.

					R
Cashflows	$-P$	D	...	D	D
Time	0	1	...	$n-1$	n

The equation of value for this investment is:

$$P=Da_{\overline{n}|}+Rv^n \qquad \text{calculated at interest rate } i$$

In the remainder of this section we will consider how to solve this equation when each of the quantities *P*, *D* or *R*, *n* and i is unknown.

Solving for the present value (*P*)

The present value (which, in this case, represents the price) can be found using formulae we derived earlier in the course.

Question

Calculate P, given that $D=5$, $R=125$, $i=10\%$ and $n=10$.

Solution

The price P can be calculated directly (using 10% interest):

$$P = 5a_{\overline{10|}} + 125v^{10} = 5 \times 6.1446 + 125 \times 0.38554 = £78.92$$

Note from this example that the equation of value holds for the values $P = 78.92$, $D = 5$, $R = 125$, $i = 10\%$ and $n = 10$. We will treat these values as our reference values.

The result shown in the following question is sometimes useful.

Question

For Security *S*, show algebraically that if $D = iR$, then $P = R$.

Solution

The price P is given by the equation of value:

$$P = iRa_{\overline{n|}} + Rv^n \quad \text{(calculated at interest rate } i\text{)}$$

Using the formula for the annuity and simplifying gives:

$$P = iR\left(\frac{1-v^n}{i}\right) + Rv^n = R(1-v^n) + Rv^n = R$$

We can also obtain this result by general reasoning.

Suppose an investor deposits a sum of money R into a bank account that pays an effective annual interest rate i. The investor leaves the money in the account for n years. If interest is paid at the end of each year, and is withdrawn as soon as it is paid, the investor will receive interest payments of iR at the end of each year and the initial deposit of R will be repaid at the end of n years.

Under this arrangement the cashflows the investor receives exactly match the cashflows received by investing an amount P in Security S. Also, the rate of return obtained from the bank account will be i (by definition), which is the same as the interest rate i required from Security S.

So, investing R in the bank account or P in Security S leads to the same cashflows and gives the same rate of return. So P and R must be equal.

Question

Calculate P, given that $D=10$, $R=125$, $i=8\%$ and $n=10$.

Solution

The value of P is:

$$P=10a_{\overline{10}|}+125v^{10}=10\times 6.7101+125\times 0.46319=£125.00$$

This calculation verifies the result just proved, since here $D=10=0.08\times 125=iR$ *and we find that* $P=125=R$.

Solving for the amount of a payment (*D* or *R*)

Solving the equation of value for D or R is straightforward.

Question

Calculate D, given that $P=127.12$, $R=125$, $i=7.75\%$ and $n=10$.

Solution

The equation of value is:

$$127.12=Da_{\overline{10}|}+125v^{10}$$

So: $$127.12=D\times\frac{1-1.0775^{-10}}{0.0775}+125\times 1.0775^{-10}$$

ie $$127.12=D\times 6.7864+125\times 0.47405$$

This can be rearranged to find D:

$$D=\frac{127.12-125\times 0.47405}{6.7864}=10.00$$

Solving for the timing of a payment (n)

We can solve the equation of value for n by expressing the annuity function in terms of v.

Question

Calculate n, given that $P = 78.92$, $D = 5$, $R = 125$ and $i = 0.10$.

Solution

The equation of value is:

$$78.92 = 5a_{\overline{n}|} + 125v^n$$

Substituting the formula for $a_{\overline{n}|}$ gives:

$$78.92 = 5 \times \frac{1 - v^n}{0.10} + 125v^n$$

ie $$78.92 = 50(1 - v^n) + 125v^n = 50 + 75v^n$$

This can be rearranged to find v^n:

$$v^n = \frac{78.92 - 50}{75} = 0.38560$$

ie $$1.10^{-n} = 0.38560$$

Taking logs, and using the result $\log a^b = b \log a$, we find:

$$-n \log 1.10 = \log 0.38560 \quad ie \quad n = -\frac{\log 0.38560}{\log 1.10} = 10.00$$

Solving for the interest rate (i)

Finding the interest rate is the hardest type of calculation, since the equation of value cannot usually be solved explicitly. If the equation of value cannot be solved explicitly, we could use trial and error, based on a rough initial guess.

To obtain an initial guess, we can approximate the interest rate by combining the cashflows into a single payment, payable on an average payment date. This is illustrated in the following question.

Question

Given that $P = 78.92$, $D = 5$, $R = 125$ and $n = 10$, determine a rough estimate for the value of i.

Solution

The equation of value is:

$$78.92 = 5a_{\overline{10|}} + 125v^{10} \quad \text{(calculated at interest rate } i\text{)}$$

Here, a payment of 5 is received at the end of each of years 1 to 10 (roughly equivalent to a total of 50 paid on average at time 5½), and in addition a payment of 125 is received at the end of year 10. Combining these (and weighting the timings by amounts) gives a single payment of 175 (*ie* $50 + 125$) at time 8.7 (*ie* $(50 \times 5\tfrac{1}{2} + 125 \times 10)/175$). This gives an equation we can solve more easily:

$$78.92 \approx 175v^{8.7} \text{ (calculated at rate } i\text{)}$$

So:

$$1 + i \approx \left(\frac{78.92}{175}\right)^{-\frac{1}{8.7}} = 1.096 \quad \textit{ie} \quad i \approx 9.6\%$$

This rough estimate is quite close to the exact value of 10% (which we know from earlier questions).

An alternative method for finding a first guess is to use a first-order binomial expansion, replacing $(1+i)^n$ by $(1+ni)$. However, this is better suited to equations of value that contain no annuities. For example, using this method here we would have attained a first guess of 7.7%:

$$78.92 = 5\left(\frac{1-(1+i)^{-10}}{i}\right) + 125(1+i)^{-10}$$

$$\Rightarrow \quad 78.92 \approx 5\left(\frac{1-(1-10i)}{i}\right) + 125(1-10i)$$

$$\Rightarrow \quad 78.92 \approx 50 + 125(1-10i)$$

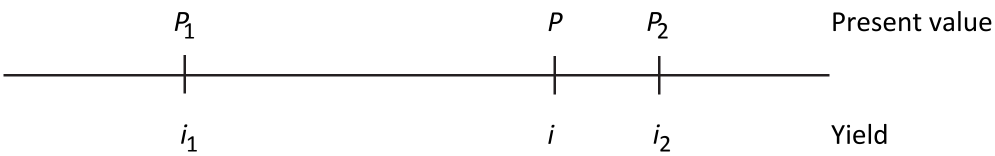
$$\Rightarrow \quad i \approx \frac{96.08}{1{,}250} = 7.7\%$$

Once an initial estimate has been obtained, a more accurate solution can then be found from the exact equation by linear interpolation, using the initial estimate as a starting point.

Estimating an unknown interest rate using linear interpolation

Suppose that the present values, calculated at interest rates i_1 and i_2, are P_1 and P_2 respectively, and we wish to work out the approximate interest rate corresponding to a present value of P.

This situation is illustrated on the following diagram



If the present value is a linear function of the interest rate, then the proportionate change in the interest rates will equal the proportionate change in the present values:

$$\frac{i-i_1}{i_2-i_1}=\frac{P-P_1}{P_2-P_1}$$

Rearranging this relationship gives the approximate value of i:

$$i \approx i_1+\frac{P-P_1}{P_2-P_1}\times(i_2-i_1)$$

This approximation works best if the trial values are close to the true value, *eg* values that are 1% apart. This formula also works even if the true value does not lie *between* the two trial values, but we would not recommend this approach (*ie* extrapolation) in the exam. We recommend that you interpolate between values that are either side of the true value and are a maximum of 1% apart. It is even better if the two values are 0.5% apart.

Question

Given that $P=75$, $D=5$, $R=125$ and $n=10$, calculate the value of i.

Solution

The equation of value is:

$$75=5a_{\overline{10|}}+125v^{10}$$

We know that when $i=10\%$, the right-hand side of this equation is equal to 78.92.

The price paid (75) is lower than this. So the value of i must be greater than 10%. Using $i=11\%$, the right-hand side is:

$$5a_{\overline{10|}}+125v^{10}=5\left(\frac{1-1.11^{-10}}{0.11}\right)+125(1.11)^{-10}=73.47$$

Interpolating linearly using these two values gives:

$$i \approx 10\% + \frac{75 - 78.92}{73.47 - 78.92} \times (11\% - 10\%) = 10.7\%$$

1.3 Example applications

Later in the course we will use equations of value in the context of the most common types of investment: fixed-interest bonds, index-linked bonds, equities (*ie* shares) and property.

Here, as an example, we will look at a question involving a property investment.

Question

A company has just bought an office block for £5m, which it will rent out to a number of small businesses. The total rent for the first year will be £100,000, and this is expected to increase by 4% *pa* compound in each future year. The office block is expected to be sold after 20 years for £7.5m.

Assuming that rent is paid in the middle of each year, calculate the yield the company will obtain on this investment.

Solution

Working in £000s, the equation of value here is:

$$5{,}000 = 100(v^{½} + 1.04v^{1½} + 1.04^2 v^{2½} + \cdots + 1.04^{19} v^{19.5}) + 7{,}500v^{20}$$

The terms in brackets form a geometric progression of 20 terms, with first term $v^{½}$ and common ratio $1.04v$. So the equation of value can be written:

$$5{,}000 = \frac{100v^{½}\left(1 - (1.04v)^{20}\right)}{1 - 1.04v} + 7{,}500v^{20} \qquad \text{provided } v \neq \frac{1}{1.04}$$

We can solve this by trial and error. At 4%, the right-hand side (using the first expression) is:

$$100\left(1.04^{-½} + 1.04^{-½} + \cdots + 1.04^{-½}\right) + \frac{7{,}500}{1.04^{20}} = 100 \times 20 \times 1.04^{-½} + \frac{7{,}500}{1.04^{20}} = 5{,}384.06$$

At 5%, the right-hand side (using the second expression) is 4,611.57. Interpolating between these two values, we obtain:

$$i \approx 4\% + \frac{5{,}000 - 5{,}384.06}{4{,}611.57 - 5{,}384.06} \times (5\% - 4\%) = 4.5\%$$

2 Uncertain payment or receipt

If there is uncertainty about the payment or receipt of a cashflow at a particular time, allowance can be made in one of two ways:

- **apply a probability of payment/receipt to the cashflow at each time, or**
- **use a higher rate of discount.**

2.1 Probability of cashflow

The probability of payment/receipt can be allowed for by adapting the earlier equations. For example, Equation (1.4) can be revised to produce:

$$\sum_{r=1}^{n} p_{t_r} c_{t_r} (1+i)^{-t_r} + \int_0^\infty p(t)\rho(t)(1+i)^{-t}\,dt = 0 \qquad (2.1)$$

where p_t and $p(t)$ represent the probability of a cashflow at time *t*.

Where the force of interest is constant, and we can say that the probability is itself in the form of a discounting function, then Equation (1.3) can be generalised as:

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta t_r} e^{-\mu t_r} + \int_0^\infty \rho(t) e^{-\delta t} e^{-\mu t}\,dt = 0 \qquad (2.2)$$

where μ is a constant force, rather than rate, of the probability of a cashflow at time *t*.

These probabilities of cashflows may often be estimated by consideration of the experience of similar cashflows. For example, this approach is used to assess the probabilities of cashflows that are dependent on the survival of a life – this is the theme of later chapters.

In other cases, there may be lack of data from which to determine an accurate probability for a cashflow. Instead a more approximate probability, or likelihood, may be determined after careful consideration of the risks.

In some cases, it may be spurious to attempt to determine the probability of each cashflow and so more approximate methods may be justified.

Wherever the uncertainty about the probability of the amount or timing of a cashflow could have significant financial effect, a sensitivity analysis may be performed. This involves calculations performed using different possible values for the likelihood and the amounts of the cashflows. Alternatively, a stochastic approach could be used to indicate possible outcomes.

Stochastic models were introduced in Chapter 2, and involve setting up some of the key assumptions as random variables. The model is then run many times to produce a distribution of outputs.

Question

A lottery ticket costs £1. The following table shows the different prizes available together with the chance of winning and the delay before receiving the prize money.

Prize	Probability of winning	Time before payment
£20	1 in 50	1 day
£200	1 in 1,000	1 day
£2,000	1 in 50,000	1 week
£200,000	1 in 2 million	2 weeks
£2 million	1 in 14 million	4 weeks

Calculate the expectation of the present value of the prize money assuming an effective rate of interest of 0.016% per day.

Solution

To calculate the expectation of the present value of the prize money, we take each possible present value, multiply it by the probability that it occurs, and then sum over all possibilities.

So, working in days, the expectation of the present value of the prize money is:

$$\frac{1}{50}\times 20v+\frac{1}{1{,}000}\times 200v+\frac{1}{50{,}000}\times 2{,}000v^7$$

$$+\frac{1}{2{,}000{,}000}\times 200{,}000v^{14}+\frac{1}{14{,}000{,}000}\times 2{,}000{,}000v^{28}$$

$$=£0.88$$

2.2 Higher discount rate

As the discounting functions and the probability functions in Equations (2.1) and (2.2) are both dependent on time, they can be combined into a single time-dependent function. In cases where there is insufficient information to objectively produce the probability functions, this combined function can be viewed as an adjusted discounting function that makes an implicit allowance for the probability of the cashflow.

Where the probability of the cashflow is a function that is of similar form to the discounting function, the combination can be treated as if a different discount rate were being used. For example, Equation (2.2) becomes:

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta' t_r} + \int_0^\infty \rho(t) e^{-\delta' t} dt = 0$$

where $\delta' = \delta + \mu$. The revised force of discount is therefore greater than the actual force of discount, as μ must be positive in order to give a probability between 0 and 1. It can therefore be shown that the rate of discount that is effectively used is greater than the actual rate of discount before the implicit allowance for the probability of the cashflow.

Question

A woman has invested some money in a company run by some ex-criminals. In return for the investment she expects to receive £100 at the end of each of the next ten years. The annual effective interest rate is 5%.

Calculate the present value of her investment by:

(i) ignoring the possibility that the payments might not be made.

(ii) assuming the probability of receiving the first payment is 0.95, the second payment is 0.9, the third payment is 0.85 *etc*.

(iii) increasing the force of interest by 0.04652.

Solution

(i) $PV = 100a_{\overline{10}|}^{@5\%} = 100 \times 7.7217 = £772.17$

(ii) The present value allowing for the probabilities of payment is:

$$PV = 100v \times 0.95 + 100v^2 \times 0.9 + \cdots + 100v^{10} \times 0.5$$
$$= 95v + 90v^2 + \cdots + 50v^{10}$$

This is equivalent to an annuity in arrears with decreases of 5 each year. So the present value is:

$$PV = 100a_{\overline{10}|} - 5(Ia)_{\overline{10}|} = 100 \times 7.7217 - 5 \times 39.3738 = £575.30$$

(iii) The new force of interest is $\ln(1.05) + 0.04652 = \ln(1.1)$. Therefore we can use an effective rate of interest of 10% *pa*:

$$PV = 100a_{\overline{10}|}^{@10\%} = 100 \times 6.1446 = £614.46$$

Chapter 10 Summary

An equation of value equates the present value of money received to the present value of money paid out:

$$PV\ income - PV\ outgo = 0$$

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta t_r} + \int_0^\infty \rho(t) e^{-\delta t} dt = 0$$

$$\sum_{r=1}^{n} c_{t_r} (1+i)^{-t_r} + \int_0^\infty \rho(t)(1+i)^{-t} dt = 0$$

To calculate the yield on a transaction from an equation of value of the form $f(i)=0$, we need:

- to find i_1 and i_2 such that $f(i_1)$ and $f(i_2)$ are of opposite sign, and
- the final value obtained for i to be greater than -1.

Some equations of value cannot be solved algebraically. In such cases, we might calculate the yield using a trial and error approach, in conjunction with linear interpolation. The formula for linear interpolation is:

$$i \approx i_1 + \frac{P - P_1}{P_2 - P_1} \times (i_2 - i_1)$$

If there is uncertainty about the payment or receipt of a cashflow at a particular time, allowance can be made in one of two ways:

- apply a probability of payment/receipt to the cashflow at each time
- use a higher rate of discount such as a new force of interest δ', where $\delta' = \delta + \mu$.

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 10 Practice Questions

Questions 10.1 to 10.3 relate to financial security *S*, which operates as follows:

A price P is paid (by the investor) in return for a series of interest payments of *D* payable at the end of each of the next n years, and a final redemption payment of R payable at the end of the n years. The investor earns an annual effective rate of return of i .

10.1 Calculate P , given that $D=5$, $R=125$, $i=10\%$ and $n=20$.

10.2 Calculate n , given that $P=83.73$, $D=4$, $R=101$ and $i=6\%$.

10.3 Calculate i , given that $P=75$, $D=5$, $R=120$ and $n=10$.

10.4 An investor is to pay £80,000 for a property. The investor will then be entitled to receive rental payments at the end of each year for 99 years. The rental payment will be fixed for the first 33 years, increasing to double the original amount for the next 33 years, and three times the original amount for the remaining 33 years. The value of the property at the end of the 99 years is expected to be £1,500,000.

Calculate the amount of the rent payable in the first year, if the investor expects to obtain a rate of return of 8% *pa* effective on the purchase.

10.5 An investor deposits £2,000 into an account, then withdraws level annual payments starting one year after the deposit is made. Immediately after the 11th annual withdrawal, the investor has £400 left in the account. Calculate the amount of each withdrawal, given that the effective annual rate of interest is 8%.

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 10 Solutions

10.1 The price P can be calculated directly as follows:

$$P = 5a_{\overline{20}|} + 125v^{20} = 5 \times 8.5136 + 125 \times 0.14864 = 61.15$$

10.2 The term n satisfies the equation:

$$83.73 = 4a_{\overline{n}|} + 101v^n = 4\left(\frac{1-1.06^{-n}}{0.06}\right) + 101 \times 1.06^{-n}$$

Rearranging gives:

$$\left(101 - \frac{4}{0.06}\right)1.06^{-n} = 83.73 - \frac{4}{0.06} \quad \Rightarrow \quad 1.06^{-n} = 0.49699$$

Taking logs, we find:

$$-n\ln 1.06 = \ln 0.49699 \quad \Rightarrow \quad n = -\frac{\ln 0.49699}{\ln 1.06} = 12.00 \text{ years}$$

10.3 To find the yield, we must solve the equation of value:

$$75 = 5a_{\overline{10}|} + 120v^{10}$$

At 10%, RHS = 76.99.

At 11%, RHS = 71.71.

Interpolating, we find that $i \approx 0.10 + \dfrac{76.99-75}{76.99-71.71}(0.11-0.10) = 0.1038$.

So *i* is approximately 10.4% *pa*.

10.4 If the amount of the rent payable in the first year is X, the equation of value is:

$$80{,}000 = Xa_{\overline{33}|} + 2Xv^{33}a_{\overline{33}|} + 3Xv^{66}a_{\overline{33}|} + 1{,}500{,}000v^{99}$$

ie $$80{,}000 = X(1 + 2v^{33} + 3v^{66})a_{\overline{33}|} + 1{,}500{,}000v^{99}$$

So:

$$80{,}000 = X(1 + 2 \times 1.08^{-33} + 3 \times 1.08^{-66}) \times 11.5139 + 1{,}500{,}000 \times 1.08^{-99}$$

This can be rearranged to find X:

$$X = \frac{80{,}000 - 736.444}{13.545} = £5{,}852$$

10.5 If the amount of the annual withdrawal is X, then we need to solve the equation:

$$2{,}000 = Xa_{\overline{11|}} + 400v^{11}$$

So:

$$2{,}000 = 7.1390X + 400 \times 1.08^{-11}$$

Rearranging to find X gives:

$$X = \frac{1{,}828.45}{7.1390} = £256.12$$

11

Loan schedules

Syllabus objectives

3.2 Use the concept of equation of value to solve various practical problems.

3.2.1 Apply the equation of value to loans repaid by regular instalments of interest and capital. Obtain repayments, interest and capital components, the effective interest rate (APR) and construct a schedule of repayments.

0 Introduction

A very common transaction involving compound interest is a loan that is repaid by regular instalments, at a fixed rate of interest, for a predetermined term.

Loans are mostly used by companies or individuals to raise funds, usually to buy buildings or equipment.

Most loans operate like a repayment mortgage, where the initial capital is repaid during the term of the loan. This is done by making repayments that are greater than the amount of interest due. The remainder of each repayment is used to repay part of the capital.

Some loans operate like interest-only mortgages, where the repayments represent interest only. This means that at the end of the term of the loan, the borrower will need to repay the capital using money from elsewhere.

Repayment loans and interest-only loans were first introduced in Chapter 3.

1 An example

This topic is best introduced through an example. We will consider a more general case afterwards.

Consider a very simple example. Assume a bank lends an individual £1,000 for three years, in return for three payments of £*X*, say, one at the end of each year. The bank will charge an effective rate of interest of 7% per annum.

The equation of value for the transaction gives:

$$\mathbf{1{,}000 = Xa_{\overline{3}|} \quad \Rightarrow \quad X = 381.05}$$

Question

Verify the value of X obtained above.

Solution

Using the *Tables*, we see that $a_{\overline{3}|} = 2.6243$ at 7%. Therefore:

$$X = \frac{1{,}000}{2.6243} = 381.05$$

So, the borrower pays £381.05 at times $t = 1, 2$ and 3 in return for the loan of £1,000 at time 0. These three payments cover both the interest due and the £1,000 capital.

It is helpful to see how this works in detail.

Each payment is first used to pay any interest due since the last payment, and then to reduce the amount of capital outstanding.

The initial amount of capital outstanding is £1,000 and the first payment is at time 1. Interest accrues before the first payment at 7% *pa*.

At time 1 the interest due on the loan of £1,000 is £70. The total payment made is £381.05. This leaves £311.05 that is available to repay some of the capital. The capital outstanding after this payment is then $£(1{,}000 - 311.05) = £688.95$.

At time 2 the interest due is now only 7% of $£688.95 = £48.22$, as the borrower does not pay interest on the capital that is already repaid, only on the amount outstanding. This leaves $£(381.05 - 48.22) = £332.83$ available to repay capital. The capital outstanding after this payment is then $£(688.95 - 332.83) = £356.12$.

Finally, at time 3 the interest due is 7% of $£356.12 = £24.93$, leaving $£(381.05 - 24.93) = £356.12$ available to pay the outstanding sum of £356.12, and the capital is precisely repaid.

One important point is that each repayment must pay first for interest due on the outstanding capital. The balance is then used to repay some of the capital outstanding. Each payment, therefore, comprises both interest and capital repayment. It may be necessary to identify the separate elements of the payments – for example if the tax treatment of interest and capital differs. Notice also that, where repayments are level, the interest component of the repayment instalments will decrease as capital is repaid, with the consequence that the capital payment will increase.

In this example, the interest payments reduce from £70 to £24.93 and the capital payments increase from £311.05 to £356.12. All payments, including the first payment and the last payment, include an interest element and a capital element.

Question

A bank lends a company £5,000 at a fixed rate of interest of 10% *pa* effective. The loan is to be repaid by five level annual payments.

Calculate the interest and capital payments of each repayment.

Solution

First, we calculate the amount of each repayment, Y.

$$Ya_{\overline{5}|} = 5,000 \quad \Rightarrow \quad Y = \frac{5,000}{3.7908} = £1,318.98$$

The following table shows how each repayment of £1,318.98 is split between interest and capital payments.

Year	Loan outstanding at start of the year (L)	Interest due at the end of the year (I = 10% of L)	Capital repaid at the end of the year ($C = 1,318.98 - I$)	Loan outstanding at end of the year (L–C)
1	5,000	500	818.98	4,181.02
2	4,181.02	418.10	900.88	3,280.14
3	3,280.14	328.01	990.97	2,289.17
4	2,289.17	228.92	1,090.06	1,199.11
5	1,199.11	119.91	1,199.07	0.04*

* This is non-zero due to rounding.

2 Calculating the capital outstanding

2.1 Introduction

In the example in the previous section, we calculated the capital outstanding just after each payment by rolling forward the loan contract year by year. This was fine for a short-term loan but if the term of the loan had been, say, 20 years, then it would have been very time-consuming. There are two much quicker ways to calculate the capital outstanding immediately after a repayment has been made:

(a) by calculating the accumulated value of the original loan less the accumulated value of the repayments made to date – called the *retrospective* method

(b) by calculating the present value of future repayments – called the *prospective* method.

Applying these two methods to the example in Section 1, the capital outstanding just after the first payment is:

(a) $1{,}000(1.07) - 381.05 = 688.95$

(b) $381.05 a_{\overline{2|}} = 381.05 \times 1.80802 = 688.95$

These values agree with those found earlier.

Question

Use the retrospective and prospective methods to verify that the capital outstanding just after the second payment is £356.12.

Solution

(a) $1{,}000(1.07)^2 - 381.05 s_{\overline{2|}} = 1{,}000(1.07)^2 - 381.05 \times 2.0700 = 356.13$

(b) $381.05v = \dfrac{381.05}{1.07} = 356.12$

The small difference is due to using a rounded repayment amount.

These results will now be proved for the more general case when the loan payments are not necessarily level.

2.2 The theory

Let L_t be the amount of the loan outstanding at time $t = 0, 1, \ldots, n$, immediately after the repayment at *t*. The repayments are assumed to be in regular instalments, of amount X_t at time *t*, $t = 1, 2, 3, \ldots, n$. (Note that we are not assuming all instalments are the same amount.) Let *i* be the effective rate of interest, per time unit, charged on the loan. Let f_t be the capital repaid at *t*, and let b_t be the interest paid at *t*, so that $X_t = f_t + b_t$.

The equation of value for the loan at time 0 is:

$$L_0 = X_1 v + X_2 v^2 + \cdots + X_n v^n \tag{2.1}$$

If the loan is repaid by level regular instalments so that $X_t = X$ for all *t*, the above equation simplifies to:

$$L_0 = X a_{\overline{n|}}$$

We can find the loan outstanding at *t prospectively* or *retrospectively.*

Prospective loan calculation

Calculating the loan prospectively involves looking forward and calculating the present value at the current point in time of future cashflows.

Consider the loan transactions at time *n*, which is the end of the contract term. After the final instalment of capital and interest the loan is exactly repaid. So the final instalment X_n must exactly cover the capital that remains outstanding after the instalment paid at $n-1$, together with the interest due on that capital. That is:

$$b_n = iL_{n-1} \quad ; \quad f_n = L_{n-1}$$

so that:

$$X_n = iL_{n-1} + L_{n-1} = (1+i)L_{n-1} \quad \Rightarrow \quad L_{n-1} = X_n v$$

So the capital outstanding at time $n-1$, L_{n-1}, is equal to the present value at time $n-1$ of the future payment at time *n*.

Similarly, at any time $t+1$, $t \le n-2$, we know that the capital repaid is $L_t - L_{t+1}$, so that the instalment X_{t+1} is:

$$X_{t+1} = iL_t + (L_t - L_{t+1}) \quad \Rightarrow \quad L_t = (L_{t+1} + X_{t+1})v$$

Similarly, $L_{t+1} = (L_{t+2} + X_{t+2})v$, and working forward, successively substituting for L_{t+r} until we get to $L_n = 0$, we get:

$$\begin{aligned} L_t &= (L_{t+1} + X_{t+1})v \\ &= ((L_{t+2} + X_{t+2})v + X_{t+1})v = X_{t+1}v + X_{t+2}v^2 + L_{t+2}v^2 \\ &= X_{t+1}v + X_{t+2}v^2 + X_{t+3}v^3 + L_{t+3}v^3 \\ &= \vdots \\ &= X_{t+1}v + X_{t+2}v^2 + X_{t+3}v^3 + \cdots + X_n v^{n-t} \end{aligned}$$

If the loan is repaid by level regular instalments so that $X_t = X$ for all *t*, the loan outstanding at time *t* is:

$$L_t = X(v + v^2 + v^3 + \cdots + v^{n-t}) = Xa_{\overline{n-t|}}$$

This gives the 'prospective method' for calculating the loan outstanding. What this equation tells us is that, for calculating the loan outstanding immediately after the repayment at *t*, say, we have:

Prospective Method: The loan outstanding at time *t* is the present (or discounted) value at time *t* of the future repayment instalments.

Note the condition for this method – the present value must be calculated at a repayment date.

Retrospective loan calculation

Calculating the loan retrospectively involves looking backwards and calculating the accumulated value of past cashflows.

At $t = 1$ the interest due is $b_1 = iL_0$, so the capital repaid is $f_1 = X_1 - iL_0$, leaving a loan outstanding of:

$$L_1 = L_0 - (X_1 - iL_0) = L_0(1+i) - X_1$$

In general, at time $t \geq 1$ the interest due is $b_t = iL_{t-1}$, leaving capital repaid at t of $X_t - iL_{t-1}$, giving:

$$L_t = L_{t-1}(1+i) - X_t$$

Similarly, $L_{t-1} = L_{t-2}(1+i) - X_{t-1}$ and, working back from t to 0 we have:

$$\begin{aligned} L_t &= L_{t-1}(1+i) - X_t \\ &= (L_{t-2}(1+i) - X_{t-1})(1+i) - X_t = L_{t-2}(1+i)^2 - X_{t-1}(1+i) - X_t \\ &= L_o(1+i)^t - (X_1(1+i)^{t-1} + X_2(1+i)^{t-2} + \cdots + X_{t-1}(1+i) + X_t) \end{aligned}$$

If the loan is repaid by level regular instalments so that $X_t = X$ for all *t*, the loan outstanding at time *t* is:

$$L_t = L_0(1+i)^t - X((1+i)^{t-1} + (1+i)^{t-2} + \cdots + (1+i) + 1) = L_0(1+i)^t - Xs_{\overline{t|}}$$

This gives the 'retrospective method' of calculating the outstanding loan. This may be described in words as:

Retrospective Method: The loan outstanding at time *t* is the accumulated value at time *t* of the original loan less the accumulated value at time *t* of the repayments to date.

Both approaches are very useful in calculating the capital outstanding at any time. Neither result depends on the interest rate being constant. It may be useful to work through the equations assuming the interest charged on the loan in year $r-1$ to r is i_r, say.

Since both methods calculate the loan outstanding at time *t*, they must both give the same result. This is fairly easy to show algebraically.

Consider Equation (2.1) and multiply it by $(1+i)^t$ giving:

$$L_0(1+i)^t = X_1(1+i)^{t-1} + X_2(1+i)^{t-2} + \cdots + X_{t-1}(1+i) + X_t + X_{t+1}v + \cdots + X_n v^{n-t}$$

Rearranging gives:

$$L_0(1+i)^t - (X_1(1+i)^{t-1} + X_2(1+i)^{t-2} + \cdots + X_{t-1}(1+i) + X_t)$$

$$= X_{t+1}v + X_{t+2}v^2 + X_{t+3}v^3 + \cdots + X_n v^{n-t}$$

which shows that the retrospective result equals the prospective result.

It is not really necessary to memorise the formulae given above. When looking at questions about loans, it is usually easier to apply the principles given here, rather than trying to use the general formulae.

Question

A loan of $50,000 is repayable by level annual payments at the end of each of the next 5 years. Interest is 8% *pa* effective for the first three years and 12% *pa* effective thereafter.

Calculate the loan outstanding immediately after the second repayment.

Solution

Let the amount of each repayment be X so that:

$$50{,}000 = X(a_{\overline{3}|8\%} + v^3_{8\%}\, a_{\overline{2}|12\%})$$

$$\Rightarrow \quad X = \frac{50{,}000}{2.5771 + 1.08^{-3} \times 1.6901}$$

$$\Rightarrow \quad X = \$12{,}759.15$$

Using the prospective approach, the loan outstanding immediately after the second payment is:

$$12{,}759.15 v_{8\%}(1 + a_{\overline{2}|12\%}) = 12{,}759.15 \times 1.08^{-1} \times (1 + 1.6901) = \$31{,}781$$

Alternatively, using the retrospective approach:

$$50{,}000(1.08)^2 - 12{,}759.15((1.08) + 1) = \$31{,}781$$

Note that, unless specifically mentioned, exam questions do not require the calculation to be performed both ways, but it is a good way to check an answer.

3 Calculating the interest and capital elements

Given the outstanding capital at any time, we can calculate the interest and capital element of any instalment.

For example, consider the instalment X_t at time *t*. We can calculate the interest element contained in this payment by calculating the loan outstanding immediately after the previous instalment, at $t-1$, L_{t-1}. The interest due on capital of L_{t-1} for one unit of time at effective rate *i* per time unit is iL_{t-1}, and this is the interest paid at *t*. The capital repaid may be found using $X_t - iL_{t-1}$, or by $L_{t-1} - L_t$.

In the example in Section 1 of this chapter, the capital repaid was calculated by deducting the interest due from each instalment. Alternatively, the capital repaid can be calculated by taking the difference between the capital outstanding before and after the instalment.

Similarly, it is a simple matter to calculate the interest paid and capital repaid over several instalments. For example, consider the five instalments from $t+1$ to $t+5$, inclusive. The loan outstanding immediately before the first instalment is L_t. The loan outstanding after the fifth instalment is L_{t+5}. The total capital repaid is therefore $L_t - L_{t+5}$. The total capital and interest paid is $X_{t+1} + X_{t+2} + \cdots + X_{t+5}$. Hence, the total interest paid is:

$$\sum_{k=t+1}^{t+5} b_k = (X_{t+1} + X_{t+2} + \cdots + X_{t+5}) - (L_t - L_{t+5})$$

Note the key differences between working out the interest paid and capital repaid in a single payment or in a series of payments.

For a single payment:

- calculate the loan outstanding after the previous payment
- calculate the interest paid by multiplying by the effective interest rate
- calculate the capital repaid by subtracting the interest paid from the repayment.

For a series of payments:

- calculate the capital repaid by subtracting the loan outstanding after the payments from the loan outstanding before the payments
- calculate the interest paid by subtracting the capital repaid from the total of the payments made.

Question

A loan of 16,000 is repayable by ten level payments, made annually in arrears. The annual effective rate of interest is 4%. Calculate:

(i) the interest element of the 4th payment

(ii) the capital element of the 7th payment

(iii) the capital repaid in the last five years of the loan

(iv) the total interest paid over the whole loan.

Solution

If the annual repayment is X, then:

$$Xa_{\overline{10}|} = 16{,}000 \quad \Rightarrow \quad X = \frac{16{,}000}{a_{\overline{10}|}} = \frac{16{,}000}{8.1109} = 1{,}972.66$$

(i) The capital outstanding after the 3rd payment (working prospectively) is:

$$1{,}972.66a_{\overline{7}|} = 1{,}972.66 \times 6.0021 = 11{,}840.01$$

The interest element of the 4th payment is:

$$0.04 \times 11{,}840.01 = 473.60$$

(ii) The capital outstanding after the 6th payment (working prospectively) is:

$$1{,}972.66a_{\overline{4}|} = 1{,}972.66 \times 3.6299 = 7{,}160.55$$

The interest element of the 7th payment is:

$$0.04 \times 7{,}160.55 = 286.42$$

Therefore the capital element of the 7th payment is:

$$1{,}972.66 - 286.42 = 1{,}686.24$$

(iii) The capital repaid over the last five years of the loan must be the capital outstanding after the 5th payment, *ie*:

$$1{,}972.66a_{\overline{5}|} = 1{,}972.66 \times 4.4518 = 8{,}781.93$$

(iv) The total interest payable over the whole loan is the total payment made less the capital borrowed, *ie*:

$$10 \times 1{,}972.66 - 16{,}000 = 3{,}726.60$$

4 The loan schedule

In the solution to one of the questions in Section 1, we set out the loan outstanding and the capital and interest part of each payment in a table. This type of table is called a *loan schedule*. It will now be defined in a more general context.

The loan payments can be expressed in the form of a table, or 'schedule', as follows.

Year $r \to r+1$	Loan outstanding at r	Instalment at $r+1$	Interest due at $r+1$	Capital repaid at $r+1$	Loan outstanding at $r+1$
$0 \to 1$	L_0	X_1	iL_0	$X_1 - iL_0$	L_1 $= L_0 - (X_1 - iL_0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t \to t+1$	L_t	X_{t+1}	iL_t	$X_{t+1} - iL_t$	L_{t+1} $= L_t - (X_{t+1} - iL_t)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1 \to n$	L_{n-1}	X_n	iL_{n-1}	$X_n - iL_{n-1}$	0

With spreadsheet software, it is a simple matter to construct the entire schedule for any loan.

Without a spreadsheet, it is quite time-consuming to have to construct a complete schedule for a loan with more than four or five repayments.

Question

A loan of amount L is to be repaid by level annual payments at the end of each of the next n years.

The annual effective interest rate is i, and the annual repayment $P = \dfrac{L}{a_{\overline{n|}}}$.

Set out the loan schedule for this loan, simplifying expressions where possible.

Solution

Using the loan schedule structure outlined above:

Year $r \to r+1$	Loan outstanding at r	Instalment at $r+1$	Interest due at $r+1$	Capital repaid at $r+1$	Loan outstanding at $r+1$
$0 \to 1$	$L = Pa_{\overline{n}\rceil}$	$P = \dfrac{L}{a_{\overline{n}\rceil}}$	$iL = iPa_{\overline{n}\rceil}$ $= P(1-v^n)$	Pv^n	$P(a_{\overline{n}\rceil} - v^n) = Pa_{\overline{n-1}\rceil}$
$1 \to 2$	$Pa_{\overline{n-1}\rceil}$	$P = \dfrac{L}{a_{\overline{n}\rceil}}$	$iPa_{\overline{n-1}\rceil}$ $= P(1-v^{n-1})$	Pv^{n-1}	$Pa_{\overline{n-2}\rceil}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t \to t+1$	$Pa_{\overline{n-t}\rceil}$	$P = \dfrac{L}{a_{\overline{n}\rceil}}$	$P(1-v^{n-t})$	Pv^{n-t}	$Pa_{\overline{n-t-1}\rceil}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1 \to n$	$Pa_{\overline{1}\rceil} = Pv$	$P = \dfrac{L}{a_{\overline{n}\rceil}}$	$P(1-v)$	Pv	0

Here:

- the interest due is calculated as the annual effective interest rate i multiplied by the loan outstanding at the start of the year,
- the capital repaid is the repayment amount P minus the interest due, and
- the loan outstanding at the end of the year is calculated as the loan outstanding at the start of the year minus the capital repaid.

We see that the capital repaid increases by a factor of $(1+i)$ each year.

5 Instalments payable more frequently than annually

Most loans will be repaid in quarterly, monthly or weekly instalments. No new principles are involved where payments are made more frequently than annually, but care needs to be taken in calculating the interest due at any instalment date.

If the rate of interest used is effective over the same time unit as the frequency of the repayment instalments, then the calculations proceed exactly as above, with the time unit redefined appropriately.

For the case where the interest is expressed as an effective annual rate, with repayment instalments payable *p*thly, we have the equation of value for the loan, given repayments of X_t at time $t = \frac{1}{p}, \frac{2}{p}, \frac{3}{p}, \ldots, n$:

$$L_0 = X_{1/p}v^{1/p} + X_{2/p}v^{2/p} + X_{3/p}v^{3/p} + \cdots + X_n v^n$$

In the case where the loan is repaid by level instalments of amount X payable $p$thly (so that the total repayment each year is pX), the loan equation simplifies to:

$$L_0 = X(v^{1/p} + v^{2/p} + v^{3/p} + \cdots + v^n) = pXa_{\overline{n}|}^{(p)}$$

Question

A loan of £900 is repayable by level monthly payments for 3 years, with interest payable at 18.5% *pa* effective. Calculate the amount of each monthly payment.

Solution

Let M equal the monthly payment. Then:

$$900 = 12Ma_{\overline{3}|}^{(12)} = 12M\left(\frac{1-v^3}{i^{(12)}}\right)$$

$$\Rightarrow \quad M = \frac{900i^{(12)}}{12(1-v^3)} = \frac{900 \times 12(1.185^{1/12}-1)}{12(1-1.185^{-3})} = £32.13$$

It is easy to show that the two basic principles for calculating the loan outstanding hold when repayments are more frequent than annual. That is, the loan outstanding at any repayment date, immediately after an instalment has been paid, may still be calculated as the present value of the remaining repayment instalments, or as the accumulated value of the original loan less the repayments made to date.

Prospectively:

$$L_t = X_{t+1/p}v^{1/p} + X_{t+2/p}v^{2/p} + \cdots + X_n v^{n-t}$$

Retrospectively:

$$L_t = L_0(1+i)^t - (X_{1/p}(1+i)^{t-1/p} + X_{2/p}(1+i)^{t-2/p} + \cdots + X_{t-1/p}(1+i)^{1/p} + X_t)$$

If the loan is repaid by level instalments of amount X payable $p$thly, these two expressions for the loan outstanding at time t simplify to:

Prospectively: $L_t = pXa_{\overline{n-t|}}^{(p)}$

Retrospectively: $L_t = L_0(1+i)^t - pXs_{\overline{t|}}^{(p)}$

Capital and interest elements

Given an annual effective rate of interest of i, the effective rate of interest over a period $\frac{1}{p}$ is $(1+i)^{1/p} - 1$, which is equal to $i^{(p)}/p$. The interest due at $t+\frac{1}{p}$, given capital outstanding of L_t at some repayment date t, is therefore $b_{t+1/p} = ((1+i)^{1/p} - 1)L_t$. The capital repaid at $t+\frac{1}{p}$ is then:

$$f_{t+1/p} = X_{t+1/p} - ((1+i)^{1/p} - 1)L_t = X_{t+1/p} - \frac{i^{(p)}}{p}L_t$$

This is just the total payment at time $t+\frac{1}{p}$ less the interest due at time $t+\frac{1}{p}$.

The capital repaid at $t+\frac{1}{p}$ is also equal to the capital outstanding at time t less the capital outstanding at time $t+\frac{1}{p}$, *ie* $L_t - L_{t+1/p}$.

Question

Calculate the interest and capital portions of the thirteenth repayment of the loan introduced in the previous question.

Solution

We calculated the monthly repayment to be £32.13. Working prospectively, the loan outstanding immediately after the twelfth payment is:

$$12 \times 32.13a_{\overline{2|}}^{(12)} = 385.56 \times \frac{1-v^2}{i^{(12)}} = 385.56 \times 1.6839 = £649.25$$

The interest portion of the thirteenth payment is therefore:

$$649.25 \times \frac{i^{(12)}}{12} = 649.25 \times (1.185^{1/12} - 1) = £9.25$$

The capital portion is:

$$32.13 - 9.25 = £22.88$$

We can also use the same approach to calculate the capital and interest elements of a series of repayments that we used when the repayments were made annually.

Question

A loan of £4,000 is repayable by equal monthly payments for 5 years. Interest is payable at a rate of 7% *pa* effective.

Calculate the interest paid and the capital repaid in the 4th year.

Solution

Calculating the monthly payment, X:

$$12Xa_{\overline{5}|}^{(12)} = 4,000 \quad \Rightarrow \quad X = \frac{4,000}{12 \times 4.2301} = 78.80$$

Working prospectively, the capital outstanding at the start of the 4th year (*ie* at time 3) is:

$$12 \times 78.80 \times a_{\overline{2}|}^{(12)} = 12 \times 78.80 \times 1.8653 = 1,763.84$$

The capital outstanding at the end of the 4th year is:

$$12 \times 78.80 \times a_{\overline{1}|}^{(12)} = 12 \times 78.80 \times 0.9642 = 911.75$$

So the capital repaid in the 4th year is:

$$1,763.84 - 911.75 = 852.09$$

The interest paid is the difference between the total payment and the capital repaid, *ie*:

$$12 \times 78.80 - 852.09 = 945.60 - 852.09 = 93.51$$

---

6 Consumer credit: APR

As we have seen in earlier chapters, there are various ways to display an interest rate. For example, it could be nominal but compounded monthly, or effective. We also noticed that 10% compounded monthly has an effective rate greater than 10% compounded quarterly.

Question

Calculate the annual effective interest rate that is equivalent to:

(i) a nominal rate of interest of 10% per annum convertible monthly

(ii) a nominal rate of interest of 10% per annum convertible quarterly.

Solution

(i) If $i^{(12)} = 10\%$:

$$i = \left(1 + \frac{i^{(12)}}{12}\right)^{12} - 1 = \left(1 + \frac{0.1}{12}\right)^{12} - 1 = 10.4713\%$$

(ii) If $i^{(4)} = 10\%$:

$$i = \left(1 + \frac{i^{(4)}}{4}\right)^{4} - 1 = \left(1 + \frac{0.1}{4}\right)^{4} - 1 = 10.3813\%$$

So, two interest rates that appear, at a glance, to be the same, might actually relate to different annual effective rates.

This affects the interest rate being advertised for a loan. Moreover, loans may have additional costs, such as opening costs or display simpler rates (for example, 1% daily for a pay-day loan). Since such an advertised rate ignores the effects of compounding or other costs, it will be considerably lower than the true effective rate of interest charged on the loan.

A 'pay-day' loan is a loan that is designed to be short-term in nature, to give the borrower sufficient funds to pay essential bills. The intention is that the loan will be repaid in full from the borrower's next pay packet. Such loans often charge very high rates of interest, to reflect the risk that the borrower may be unable to repay the loan.

Question

A loan provider quotes an interest rate of 1% per day effective.

(i) Calculate the annual effective interest rate on this loan.

(ii) Comment on why the loan provider has chosen to quote the interest rate as a daily rate.

Solution

(i) Assuming there are 365 days in a year, the annual effective interest rate equivalent to 1% per day effective is:

$$i = 1.01^{365} - 1 = 36.7834 \quad ie \ \ 3{,}678.34\%$$

(ii) An interest rate of 1% per day effective does not appear to be very high, and so is unlikely to put potential borrowers off taking out the loan. With the effect of compounding, the interest charged on an annual basis is very high indeed, and, if disclosed, this would cause most borrowers to look elsewhere.

To ensure that consumers can make informed judgements about the interest rates charged, lenders are required (in most circumstances) to give information about the effective rate of interest charged. In the UK, this is in the form of the Annual Percentage Rate of charge, or APR, which is defined as the effective annual rate of interest, rounded to the nearer 1/10th of 1%.

If all loan providers are required to quote the interest rate charged in the same format, it is easy for consumers to compare the rates and avoid being misled.

The APR is the rate of interest at which the present value of the amount borrowed equals the present value of the repayments (including all other charges), rounded to the nearer 0.1%.

Question

A motorist borrows £5,000 to buy a car. The loan is repaid by level payments of £458.33 at the end of each of the next 12 months. Calculate the APR paid by the motorist.

Solution

To determine the annual effective rate of interest on the loan, we need to solve the equation of value:

$$12 \times 458.33 a_{\overline{1|}}^{(12)} = 5{,}000 \quad ie \quad a_{\overline{1|}}^{(12)} = 0.90910$$

At 20%, $a_{\overline{1|}}^{(12)} = 0.90721$. This is too low. In order to increase the present value, we need to decrease the interest rate.

At 19.5%, $a_{\overline{1|}}^{(12)} = 0.90921$, and at 19.6%, $a_{\overline{1|}}^{(12)} = 0.90881$.

Since the 19.5% value is closer to 0.90910 than the 19.6% value, the APR is 19.5%.

To double check the rounding, we see that at 19.55%, $a_{\overline{1|}}^{(12)} = 0.90901$*, so the true value lies between 19.5% and 19.55%, ie 19.5% to 1 decimal place.*

In the question above, we see that the APR is effectively calculated by trial and error. To ease the calculation of the APR, we can consider a different measure of the interest charged on a loan, to give us an indication of where to start the trial and error process.

The *flat rate of interest* is calculated as:

$$\text{flat rate} = \frac{\text{total interest}}{\text{original loan} \times \text{term in years}} = \frac{\text{total repayment} - \text{original loan}}{\text{original loan} \times \text{term in years}}$$

This is quite a straightforward calculation to perform.

Question

Calculate the flat rate on the loan taken out by the motorist in the previous question.

Solution

The original loan amount is £5,000 and the term of the loan is 1 year. The total repayment made is 12×458.33. So, the flat rate is:

$$\frac{12 \times 458.33 - 5{,}000}{5{,}000 \times 1} = 10.0\%$$

We see that the APR (of 19.5%) is roughly equal to twice the flat rate (of 10%). Since this is often the case, when calculating an APR, we could start by calculating the flat rate, and then double it to provide a first guess for the APR.

To see why the APR is roughly twice the flat rate, we can rewrite the formula for the flat rate as:

$$\text{flat rate} = \frac{\text{total interest}}{\text{original loan} \times \text{term in years}} = \frac{\text{average annual interest}}{\text{original loan}}$$

This is because the total interest paid over the whole term of the loan divided by the term in years is equal to the average annual interest paid. However, to approximate the annual effective interest rate on the loan (*ie* the APR), we should divide the average annual interest by the average loan amount, rather than the original loan amount.

The loan outstanding varies from the original loan amount at outset to 0 at the end of the term, so the average loan amount is about one half of the original loan amount. Therefore, the APR is approximately:

$$\text{APR} \approx \frac{\text{average annual interest}}{\text{average loan amount}} = \frac{\text{average annual interest}}{\frac{1}{2} \times \text{original loan}} = 2 \times \frac{\text{average annual interest}}{\text{original loan}}$$

which is twice the flat rate.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 11 Summary

A very common transaction involving compound interest is a loan that is repaid by regular instalments, at a fixed rate of interest, for a predetermined term.

Each repayment must first pay the interest due on the outstanding capital; the balance is then used to repay some of the capital outstanding.

We can find the loan outstanding prospectively or retrospectively.

A prospective method involves finding the present value of future repayments.

A retrospective method involves calculating the accumulated value of the initial loan less the accumulated value of the repayments to date.

We can calculate the interest element contained in a single payment by calculating the loan outstanding immediately after the previous instalment and multiplying it by the rate of interest. The capital element is the total payment less the interest payment.

We can calculate the capital repaid in a period where there is more than one payment by subtracting the capital outstanding at the end of the period from the capital outstanding at the start of the period. The interest paid in this period is then the total payment less the capital repaid.

The interest and capital components in the repayments for a loan can be set out in the form of a loan schedule.

No new principles are involved where payments are made more frequently than annually, but care needs to be taken when calculating the interest due at any instalment date, to ensure the correct interest rate is used.

Lenders are usually required to give information about the effective rate of interest charged.

In the UK, this is in the form of the Annual Percentage Rate of charge, or APR, which is defined as the effective annual rate of interest, rounded to the nearest 1/10th of 1%.

The practice questions start on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 11 Practice Questions

11.1 A loan of £120,000 is repayable by equal quarterly payments for 25 years. The effective rate of interest is 6% *pa*.

Calculate the interest portion of the first payment.

11.2 A loan of £1,000 is to be repaid by level monthly instalments over 10 years using an interest rate of 10% *pa* effective.

Calculate the capital repaid in the sixth year.

11.3 A bank issues a 10-year loan for £100,000 to a businessman. The loan is to be repaid by annual repayments, payable in arrears, calculated using an interest rate of 8% *pa* effective. The repayment schedule has been designed so that half the capital will have been repaid by the end of the term. The remaining £50,000 will be repaid at the end of the term using funds from other sources.

Calculate the annual repayment.

11.4 A loan of £80,000 is repayable by eight annual payments, with the first payment being made in one year's time. The first three payments are half as much as the remaining five payments, and the annual effective interest rate is 4.5%.

Calculate the loan outstanding one year before the loan is completely repaid.

11.5 A customer borrows £4,000 under a consumer credit loan. Repayments are calculated based on an APR of 15.4%, and are paid monthly in arrears for 5 years.

Calculate the amount of each monthly repayment.

11.6 A man borrows £7,500 to buy a car. He repays the loan by 24 monthly instalments of £368.75, payable in arrears.

Exam style

Calculate the APR on this transaction. [4]

11.7 A loan of £50,000 is repaid over a period of 10 years by a series of level monthly instalments. Interest is charged on the loan at the rate of interest of 8% *pa* effective.

Exam style

(i) Calculate the monthly repayment. [2]

(ii) Calculate the amount of interest paid in the first year. [3]

After the payment at the end of 7 years, the borrower takes a 2-month payment break, *ie* the borrower does not pay the next 2 monthly instalments.

(iii) Calculate the extra amount that needs to be paid each month in order to fully repay the debt by the end of the 10th year. [4]

[Total 9]

11.8 Exam style

A loan is to be repaid by payments at the end of each of the next 15 years. The first payment is £100 and the payments increase by £20 *pa* thereafter. Repayments are calculated using a rate of interest of 5% *pa* effective.

(i) Calculate the amount of the loan. [3]

(ii) Set out a loan schedule showing the capital and interest elements in, and the amount of loan outstanding after, the 6th and 7th payments. [5]

(iii) Calculate the capital and interest element of the last instalment. [2]
[Total 10]

11.9 Exam style

An actuarial student takes out a mortgage for £250,000 with a term of 25 years. The mortgage is repayable by level instalments made monthly in arrears. Interest is charged at a rate of 6% *pa* effective.

(i) Calculate the monthly repayment. [2]

(ii) (a) Calculate the capital repaid in the fourth year.

(b) Calculate the interest element of the 49th repayment. [4]

After completing her exams, six years after taking out the mortgage, the newly-qualified actuary reviews her finances and realises that she can afford to make repayments at twice the rate calculated in (i).

(iii) Calculate the length of time by which this course of action reduces the remaining term of the loan. [4]

(iv) Calculate the amount of the final repayment, and hence the interest saved by the actuary, if she follows this course of action. [5]
[Total 15]

Chapter 11 Solutions

11.1 The interest portion of the first payment is:

$$120{,}000(1.06^{1/4}-1)=£1{,}760.86$$

The amount of each instalment is not needed here because we know the loan outstanding at the start.

11.2 Let X be the monthly payment, then:

$$12Xa_{\overline{10|}}^{(12)}=1{,}000 \quad \Rightarrow \quad X=\frac{1{,}000}{12\times 6.4213}=£12.98$$

The capital outstanding at the start of the sixth year (*ie* with 5 years still to run) is:

$$12\times 12.98a_{\overline{5|}}^{(12)}=12\times 12.98\times 3.9615=617.05$$

Similarly, the capital outstanding at the end of the sixth year (*ie* with 4 years still to run) is:

$$12\times 12.98a_{\overline{4|}}^{(12)}=12\times 12.98\times 3.3127=515.98$$

So the capital repaid during the sixth year is $617.05-515.98=£101.07$.

11.3 The annual repayment R can be found by setting the initial loan amount equal to the present value of all the repayments (*ie* the annual repayments plus the lump sum of £50,000):

$$100{,}000=Ra_{\overline{10|}}+50{,}000v^{10} \quad @8\%$$

ie:

$$100{,}000=6.7101R+50{,}000\times 1.08^{-10}$$

So:

$$R=\frac{76{,}840}{6.7101}=£11{,}451$$

11.4 Let X equal the amount of the first instalment, then:

$$80{,}000=Xa_{\overline{8|}}+Xv^3a_{\overline{5|}} \quad \Rightarrow \quad 80{,}000=X\left(\frac{1-1.045^{-8}}{0.045}+1.045^{-3}\times\frac{1-1.045^{-5}}{0.045}\right)$$

Therefore:

$$X=\frac{80{,}000}{6.5959+0.8763\times 4.3900}=7{,}660.77$$

Alternatively, the equation for X can be written:

$$80,000 = Xa_{\overline{3}|} + 2Xv^3 a_{\overline{5}|}$$

Working prospectively, the loan outstanding one year before the end of the term is equal to the present value at that time (*ie* time 7) of the final repayment:

$$2 \times 7,660.77v = £14,662$$

11.5 We can calculate the amount of each monthly repayment, M, from the equation of value:

$$4,000 = 12M\, a^{(12)}_{\overline{5}|@15.4\%} = 12M \frac{1-1.154^{-5}}{12(1.154^{1/12}-1)} = 42.5877M$$

$$\Rightarrow \quad M = £93.92$$

11.6 The APR is the annual effective rate of interest that solves the equation of value:

$$12 \times 368.75 a^{(12)}_{\overline{2}|} = 7,500 \qquad ie \quad a^{(12)}_{\overline{2}|} = 1.6949$$ [1]

To obtain a first guess for the APR, we can calculate the flat rate, as follows:

$$\frac{24 \times 368.75 - 7,500}{7,500 \times 2} = 9\%$$

The APR is roughly twice the flat rate, so as a first guess we can try 18%:

$$a^{(12)}_{\overline{2}|@18\%} = 1.6909$$

Then trying 17%:

$$a^{(12)}_{\overline{2}|@17\%} = 1.7052$$

Interpolating between these two values gives an interest rate of 17.7%. Now trying 17.7% and 17.8%:

$$a^{(12)}_{\overline{2}|@17.7\%} = 1.6952$$ [1]

$$a^{(12)}_{\overline{2}|@17.8\%} = 1.6938$$ [1]

Since the 17.7% value is closer to 1.6949 than the 17.8% value, the APR is 17.7%. [1]

[Total 4]

To double check the rounding, we see that at 17.75%, $a^{(12)}_{\overline{2}|} = 1.6945$*, so the true value lies between 17.7% and 17.75%, ie 17.7% to 1 decimal place.*

11.7 (i) Let P denote the monthly repayment. Then, working in years:

$$12Pa_{\overline{10}|}^{(12)} @8\% = 50,000 \qquad [1]$$

Solving for P:

$$P = \frac{50,000}{12 \times 6.9527} = £599.29 \qquad [1]$$

[Total 2]

Alternatively, working in months, the equation of value is:

$$Pa_{\overline{120}|} = 50,000$$

where the annuity is calculated using the effective monthly interest rate:

$$i = 1.08^{1/12} - 1 = 0.00643403$$

(ii) Working prospectively, the amount outstanding at the end of the first year is given by:

$$(12 \times 599.29)a_{\overline{9}|}^{(12)} @8\% = 12 \times 599.29 \times 6.4728 = £46,548.71 \qquad [1]$$

The capital repaid in the first year is then:

$$50,000 - 46,548.71 = £3,451.29 \qquad [1]$$

and the interest paid in the first year is:

$$(12 \times 599.29) - 3,451.29 = £3,740.19 \qquad [1]$$

[Total 3]

Alternatively, working in months, the amount outstanding at the end of the first year is given by:

$$599.29a_{\overline{108}|} @0.643403\%$$

(iii) Working prospectively, the amount outstanding after 7 years (just before the payment break) is:

$$(12 \times 599.29)a_{\overline{3}|}^{(12)} @8\% = 12 \times 599.29 \times 2.6703 = £19,203.25 \qquad [1]$$

As no payments are made for the next 2 months, the capital outstanding after 7 years and 2 months is:

$$19,203.25 \times 1.08^{2/12} = 19,451.15 \qquad [1]$$

This is to be repaid in equal monthly instalments of Q over the next 2 years and 10 months. So:

$$12Qa^{(12)}_{\overline{2\frac{10}{12}|}} @8\% = 19{,}451.15 \quad \Rightarrow \quad Q = \frac{19{,}451.15}{12 \times 2.5375} = £638.78 \qquad [1]$$

The extra monthly payment is therefore $638.78 - 599.29 = £39.49$ [1]

[Total 4]

Alternatively, working in months, the amount outstanding after 7 years is given by:

$$599.29a_{\overline{36|}} @0.643403\%$$

and the equation to solve for Q is:

$$Qa_{\overline{34|}} @0.643403\% = 19{,}451.15$$

11.8 (i) The amount of the loan, L, is the present value of all the repayments:

$$L = 80a_{\overline{15|}} + 20(Ia)_{\overline{15|}} \qquad [2]$$

Using values from the *Tables*:

$$L = 80 \times 10.3797 + 20 \times 73.6677 = £2{,}303.73 \qquad [1]$$

[Total 3]

(ii) We first calculate the capital outstanding immediately after the 5th payment has been made. The 6th payment is £200; the 7th payment is £220 and so on. Again, using values from the *Tables*:

$$180a_{\overline{10|}} + 20(Ia)_{\overline{10|}} = 180 \times 7.7217 + 20 \times 39.3738 = £2{,}177.38$$

The interest element of the 6th payment is:

$$0.05 \times 2{,}177.38 = £108.87$$

Hence, the capital element is $200 - 108.87 = £91.13$, and the capital outstanding after the 6th payment is $2{,}177.38 - 91.13 = £2{,}086.25$.

Similarly, the interest element of the 7th payment is:

$$0.05 \times 2{,}086.25 = £104.31$$

and the capital element is $220 - 104.31 = 115.69$. The capital outstanding after the 7th payment is therefore $2{,}086.25 - 115.69 = £1{,}970.56$.

Expressing these amounts in a loan schedule gives:

Payment	***Interest element (£)***	***Capital element (£)***	***Capital outstanding after payment (£)***
5			2,177.38
6	108.87	91.13	2,086.25
7	104.31	115.69	1,970.56

[1 mark for row 5, 2 marks each for rows 6 and 7]
[Total 5]

(iii) As there are 14 increases of £20 after the first repayment of £100, the last payment is £380. Working prospectively, the capital outstanding immediately after the penultimate payment is therefore:

$$380v = £361.90 \qquad [1]$$

This must also be the capital element of the last payment if that payment is to pay off the loan. Thus the interest element is:

$$380 - 361.90 = £18.10 \qquad [1]$$

[Total 2]

11.9 (i) ***Monthly repayment***

Let M be the monthly repayment. The equation of value is:

$$250{,}000 = 12Ma_{\overline{25}|}^{(12)} \qquad [1]$$

Using $a_{\overline{25}|}^{(12)} = 13.1312$, this gives:

$$M = £1{,}586.55 \qquad [1]$$

[Total 2]

(ii)(a) ***Capital repaid in the fourth year***

The capital outstanding at the start of the fourth year is calculated (prospectively) as:

$$12Ma_{\overline{22}|}^{(12)} = 12 \times 1{,}586.55 \times 12.36924 = £235{,}493.04 \qquad [1]$$

The capital outstanding at the end of the fourth year is calculated (prospectively) as:

$$12Ma_{\overline{21}|}^{(12)} = 12 \times 1{,}586.55 \times 12.08419 = £230{,}065.97 \qquad [1]$$

The capital repaid in the fourth year is therefore:

$$235{,}493.04 - 230{,}065.97 = £5{,}427 \qquad [1]$$

(ii)(b) ***Interest element in the 49th repayment***

To calculate the interest element in the 49th repayment, the capital outstanding immediately after the previous (*ie* 48th) repayment is needed. The 48th repayment is made at the end of four years, so the capital outstanding at that time is £230,065.97 from (ii)(a).

So the interest element in the 49th repayment is:

$$230{,}065.97\times(1.06^{1/12}-1)=£1{,}120 \qquad [1]$$

[Total 4]

(iii) ***Reduction in payment term***

After six years, when the student has qualified, the remaining term is 19 years. The capital outstanding at this point is:

$$12Ma_{\overline{19}|}^{(12)}=12\times1{,}586.55\times11.46174=£218{,}215.42 \qquad [1]$$

If the actuary makes monthly repayments at twice the original rate, the equation of value is:

$$218{,}215.42=12\times2\times1{,}586.55\times a_{\overline{n}|}^{(12)}$$

where n is the reduced payment term. So:

$$a_{\overline{n}|}^{(12)}=\frac{1-v^n}{i^{(12)}}=5.73087 \qquad [1]$$

$$\Rightarrow 1-v^n=0.33474$$

$$\Rightarrow v^n=0.66526$$

Taking logs of both sides:

$$-n\ln(1.06)=\ln(0.66526)$$

$$\Rightarrow n=6.9949 \qquad [1]$$

Therefore, the final repayment will be made 7 years after the increased payments commence.

This means the payment term is shortened by 12 years. [1]

[Total 4]

(iv) ***Final repayment amount and total interest saved***

Now assume that the actuary makes twice the original monthly repayments, and let P be the amount of the final repayment made.

P can be found by solving the equation of value:

$$218{,}215.42=12\times2\times1{,}586.55\times a_{\overline{6\frac{11}{12}}|}^{(12)}+Pv^7 \qquad [1]$$

Using $a^{(12)}_{\overline{6\frac{11}{12}}|} = 5.67886$, gives:

$$Pv^7 = 1,980.313 \quad \Rightarrow \quad P = £2,977.66 \qquad [1]$$

The total interest paid is equal to the difference between the total repayments made and the total capital repaid.

Hence, if the actuary is making twice the original monthly repayments, the total interest paid after the end of the sixth year is:

$$12 \times 2 \times 1,586.55 \times 6\tfrac{11}{12} + 2,977.66 - 218,215.42 = £48,129.54 \qquad [1]$$

If the actuary continues making only the original repayments, the total interest paid after the end of the sixth year is:

$$12 \times 1,586.55 \times 19 - 218,215.42 = £143,517.98 \qquad [1]$$

Hence, the total interest saved by following the new course of action is:

$$143,517.98 - 48,129.54 = £95,388 \qquad [1]$$

[Total 5]

Alternatively, we could calculate the total interest paid over the whole term of the loan, under each of the repayment schedules.

Where only the original repayments are made, the total interest is:

$$12 \times 1,586.55 \times 25 - 250,000 = £225,965.00$$

Where twice the original repayments are made after six years, the total interest is:

$$12 \times 1,586.55 \times 6 + 12 \times 2 \times 1,586.55 \times 6\tfrac{11}{12} + 2,977.66 - 250,000 = £130,576.56$$

Subtracting these gives the total interest saved as £95,388 as above.

12

Project appraisal

Syllabus objectives

3.3 Show how discounted cashflow and equation of value techniques can be used in project appraisals.

3.3.1 Calculate the net present value and accumulated profit of the receipts and payments from an investment project at given rates of interest.

3.3.2 Calculate the internal rate of return, payback period and discounted payback period and discuss their suitability for assessing the suitability of an investment project.

0 Introduction

This chapter looks at methods that can be used to decide between alternative investment projects. We consider the following criteria:

- net present value
- accumulated profit
- internal rate of return
- payback period
- discounted payback period.

Estimating cashflows

Suppose an investor is considering the merits of an investment or business project. The investment or project will normally require an initial outlay and possibly other outlays in future, which will be followed by receipts, although in some cases the pattern of income and outgo is more complicated. The cashflows associated with the investment or business venture may be completely fixed (as in the case of a secure fixed-interest security maturing at a given date) or they may have to be estimated.

For an organisation issuing a fixed-interest security, there will be an initial positive cashflow, a single known negative cashflow on a specified future date, and a series of smaller known negative cashflows on a regular set of specified future dates.

The estimation of the cash inflows and outflows associated with a business project usually requires considerable experience and judgement. All the relevant factors (such as taxation and investment grants) and risks (such as construction delays) should be considered by the actuary, with assistance from experts in the relevant field (*eg* civil engineering for building projects). The identification and assessment of the risks may be conducted using the Risk Analysis and Management for Projects (RAMP) approach for risk analysis and management that has been developed by, and published on behalf of, the actuarial and civil engineering professions.

There's no need to know any more about RAMP than this for the Subject CM1 exam. However, more information about RAMP and initiatives between the actuarial and civil engineering professions can be found on the Institute and Faculty of Actuaries' website.

Considerable uncertainty will exist in the assessment of many of the risks, so it is prudent to perform calculations on more than one set of assumptions, *eg* on the basis of 'optimistic', 'average', and 'pessimistic' forecasts respectively.

A set of optimistic assumptions is often called a 'weak' basis and a pessimistic set of assumptions is often called a 'strong', 'prudent' or 'cautious' basis. Average assumptions are also called 'best-estimate' or 'realistic' assumptions.

More complicated techniques (using statistical theory) are available to deal with this kind of uncertainty. Precision is not attainable in the estimation of cashflows for many business projects and hence extreme accuracy is out of place in many calculations.

For example, there is no point in quoting a final answer to eight decimal places if the figures used in the calculation have only been estimated to the nearest million.

Net cashflow c_t at time t (measured in suitable time units) is:

$$c_t = \text{cash inflow at time } t - \text{cash outflow at time } t$$

If any payments may be regarded as continuous then $\rho(t)$, the net rate of cashflow per unit time at time t, is defined as:

$$\rho(t) = \rho_1(t) - \rho_2(t)$$

where $\rho_1(t)$ and $\rho_2(t)$ denote the rates of inflow and outflow at time t respectively.

We will now start looking at methods that can be used to decide between alternative investment projects. We will use the following two hypothetical projects as examples. Both relate to a small software company that has been asked to set up a new computer system for a major client.

Project R

Project R delegates all the development work to outside companies. The estimated cashflows for Project R are (where brackets indicate expenditure):

Beginning of Year 1	(£150,000)	(contractors' fees)
Beginning of Year 2	(£250,000)	(contractors' fees)
Beginning of Year 3	(£250,000)	(contractors' fees)
End of Year 3	£1,000,000	(sales)

Project S

Project S carries out all the development work in-house by purchasing the necessary equipment and using the company's own staff. The estimated cashflows for Project S are:

Beginning of Year 1	(£325,000)	(new equipment)
Throughout Year 1	(£75,000)	(staff costs)
Throughout Year 2	(£90,000)	(staff costs)
Throughout Year 3	(£120,000)	(staff costs)
End of Year 3	£1,000,000	(sales)

The staff costs can be assumed to be paid continuously throughout each year.

The next few sections will refer back to these two examples.

1 Fixed interest rates

Having ascertained or estimated the net cashflows of the investment or project under scrutiny, the investor will wish to measure its profitability in relation to other possible investments or projects. In particular, the investor may wish to determine whether or not it is prudent to borrow money to finance the venture.

Assume for the moment that the investor may borrow or lend money at a fixed rate of interest *i* per unit time. The investor could accumulate the net cashflows connected with the project in a separate account in which interest is payable or credited at this fixed rate. By the time the project ends (at time *T*, say), the balance in this account will be:

$$\sum c_t (1+i)^{T-t} + \int_0^T \rho(t)(1+i)^{T-t}\,dt \qquad \textbf{(1.1)}$$

where the summation extends over all t such that $c_t \neq 0$.

1.1 Accumulated value

One criterion that can be used to assess an investment project involves calculating the 'accumulated profit' at the end of the project. This is the accumulated value of the net cashflows (as at the time of the last payment).

The accumulated value, at time *T*, of a cashflow can be expressed as:

$$A(T) = \sum c_t (1+i)^{T-t} + \int_0^T \rho(t)(1+i)^{T-t}\,dt$$

The accumulated profit for a project has the intuitive appeal that it represents the final amount 'left over' if all the payments for the project were transacted through a bank account that earned interest at a rate *i*.

However, accumulated profit calculations suffer from the disadvantage that they can only be used in situations where there is a definite fixed time horizon for the project. This will not be the case if the time horizon (*ie* the time until the last cashflow payment) is:

- unlimited, or
- the timing of the payments is uncertain.

An example relating to the first bullet point is where an investor is considering purchasing a fixed-interest security with coupon payments that continue forever (known as an undated or irredeemable security). In order to determine a value for the accumulated profit, the investor would have to pick an arbitrary date and assume that the holding would be sold on that date (otherwise the accumulated profit would be infinite). The value calculated for the accumulated profit will then depend crucially on which date is selected.

An example relating to the second bullet point is where a retired man is considering using a lump sum to buy a pension payable for the rest of his life. Since he does not know when he will die, he cannot know the date to which he should accumulate the payments.

Question

Calculate the accumulated profit after 20 years of a project in which \$20,000 is paid out at time 0 and \$5,000 is received at times 5 to 15, inclusive. Assume an annual effective rate of interest of 3%.

Solution

There are 11 payments of \$5,000 and the accumulated value of these at the time the last payment is received (*ie* at time 15) is $5{,}000 s_{\overline{11}|}$. The accumulated profit is therefore:

$$-20{,}000 \times 1.03^{20} + 5{,}000 s_{\overline{11}|} \times 1.03^{5} = \$38{,}117$$

where $s_{\overline{11}|} = 12.8078$ when $i = 0.03$.

A further problem associated with accumulated profit calculations is that the accumulated profits for two different projects cannot be compared directly if they have different time horizons, since the calculated values will relate to different dates. However, this problem can be avoided by accumulating all the profits to the date of the last payment for the longer project.

These problems can be avoided by calculating the 'net present value' instead.

1.2 Net present values

The present value at rate of interest *i* of the net cashflows is called the 'net present value' at rate of interest *i* of the investment or business project, and is usually denoted by $NPV(i)$. Hence:

$$NPV(i) = \sum c_t (1+i)^{-t} + \int_0^T \rho(t)(1+i)^{-t}\,dt \qquad (1.2)$$

The rate of interest *i* used to calculate the net present value is often referred to as the *risk discount rate*. Note that the risk discount rate is a rate of interest (*i*) not a rate of discount (*d*).

The net present value is similar to the accumulated profit, the only difference being that we are now looking at the value at the outset (which, by definition, is a fixed date), rather than the value at the end of the project. A higher net present value indicates a more profitable project.

(If the project continues indefinitely, the accumulation (1.1) is not defined, but the net present value may be defined by Equation (1.2) with $T = \infty$.) If $\rho(t) = 0$, we obtain the simpler formula:

$$NPV(i) = \sum c_t v^t$$

where $v = (1+i)^{-1}$.

Since the equation:

$$NPV(i) = 0$$

is the equation of value for the project at the present time, the yield i_0 on the transaction is the solution of this equation, provided that a unique solution exists.

It may readily be shown that $NPV(i)$ is a smooth function of the rate of interest *i* and that $NPV(i) \to c_0$ as $i \to \infty$.

Question

Calculate the net present value for both Project R and Project S (outlined in the introduction to this chapter) using a risk discount rate of 20% *pa*.

Based on the net present values obtained, comment on which project is preferable.

Solution

For Project R, the net present value (in £000s) is:

$$NPV_R = -150 - 250v - 250v^2 + 1{,}000v^3 \quad @20\%$$

$$= -150 - 250 \times 1.2^{-1} - 250 \times 1.2^{-2} + 1{,}000 \times 1.2^{-3} = 46.759$$

For Project S, the net present value (in £000s) is:

$$NPV_S = -325 - 75\bar{a}_{\overline{1}|} - 90v\bar{a}_{\overline{1}|} - 120v^2\bar{a}_{\overline{1}|} + 1{,}000v^3 \quad @20\%$$

$$= -325 - (75 + 90v + 120v^2)\bar{a}_{\overline{1}|} + 1{,}000v^3$$

$$= -325 - (75 + 90 \times 1.2^{-1} + 120 \times 1.2^{-2}) \times 0.9141 + 1{,}000 \times 1.2^{-3}$$

$$= 40.405$$

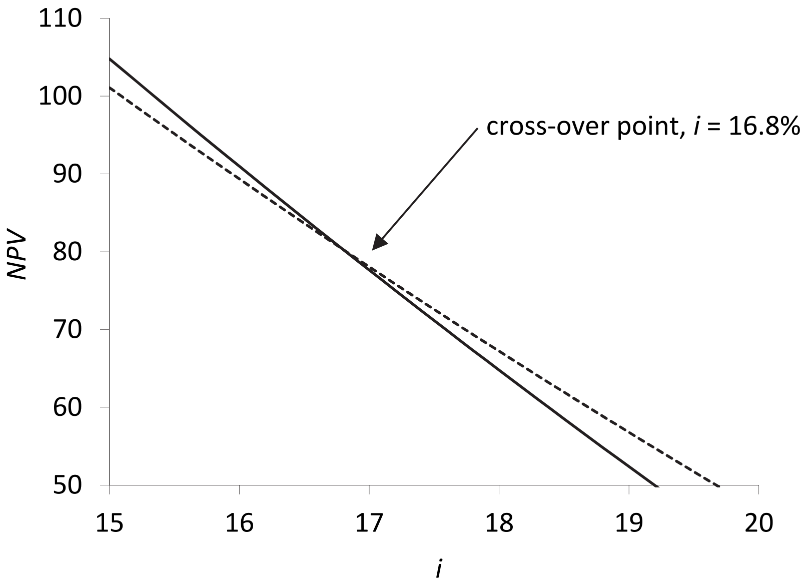
The net present values are £46,759 for Project R and £40,405 for Project S.

Using a risk discount rate of 20%, Project R has the higher net present value and so appears more favourable.

The net present value will depend on the risk discount rate used.

The following graph shows the net present values for Project R and Project S for different risk discount rates. There is a crossover point corresponding to about 16.8%. If the risk discount rate used exceeds 16.8%, the net present value is greater for Project R than for Project S. If the risk discount rate is less than 16.8%, the net present value is greater for Project S.

In the graph, the dotted line represents Project R and the solid line represents Project S.



In the graph above, as is the case for most projects in the real world, the net present value decreases as the risk discount rate (i) increases. The reason for this is that in Project R and Project S the income occurs later in time than the outgo, meaning it is discounted for a longer period, so its present value is more affected by the change in the risk discount rate. Therefore, as the risk discount rate increases, the present value of the project's income reduces by more than the present value of the project's outgo, and the net present value falls.

Question

Let the cashflow at time t be denoted C_t (where time t is measured in years and the amounts are in £000). The cashflows for two business ventures are as follows:

Venture 1: $C_0 = -100$, $C_1 = -40$, $C_2 = +50$, $C_3 = +120$

Venture 2: $C_1 = -45$, $C_3 = +25$, $C_4 = +25$, $C_5 = +25$

Calculate the accumulated profit at time 5 and the net present value for each of these ventures using a risk discount rate of 15% *pa*.

Solution

The accumulated profit for Venture 1 at time 5 is:

$$AP_1 = -100(1+i)^5 - 40(1+i)^4 + 50(1+i)^3 + 120(1+i)^2 \quad @15\% = -36.352$$

The accumulated profit for Venture 2 at time 5 is:

$$AP_2 = -45(1+i)^4 + 25(1+i)^2 + 25(1+i) + 25 \quad @15\% = 8.107$$

So the accumulated profits are –£36,352 (*ie* a loss) for Venture 1 and £8,107 for Venture 2.

The net present values are:

$$NPV_1 = -100 - 40v + 50v^2 + 120v^3 \quad @15\% = -18.073$$

$$NPV_2 = -45v + 25v^3 + 25v^4 + 25v^5 \quad @15\% = 4.031$$

So the NPVs are –£18,073 for Venture 1 and £4,031 for Venture 2.

Alternatively, the net present values could be calculated using the relationship:

$$NPV_i = AP_i \times 1.15^{-5} \quad \text{for} \quad i = 1,2$$

1.3 Internal rate of return

In economics and accountancy the yield per annum is often referred to as the 'internal rate of return' (IRR) or the 'yield to redemption'. The latter term is frequently used when dealing with fixed-interest securities, for which the 'running' (or 'flat') yield is also considered.

We will leave the definition of the running yield until later in the course.

The internal rate of return for an investment project is the effective rate of interest that equates the present value of income and outgo, *ie* it makes the net present value of the cashflows equal to zero.

If all the payments for the project were transacted through a bank account that earned interest at the same rate as the internal rate of return, the net proceeds at the end of the project (*ie* the accumulated profit) would be zero.

For most projects, there is a unique solution to the equation defining the internal rate of return, since the quantity 'PV payments in $-$ PV payments out' generally decreases as i increases, as we saw in the graph on the previous page.

The internal rate of return need not be positive. A zero return implies that the investor receives no return on the investment and if the yield is negative then the investor loses money on the investment. It is difficult, however, to find a practical interpretation for a yield less than –1, and so if there is not a solution to the equation greater than –1, the yield is undefined.

In some cases, it is possible for there to be more than one solution. In such cases the smallest positive solution is usually used. Also, if there are only inflows of cash (*ie* no outflow), the internal rate of return will be infinite.

Usually, the equation of value cannot be solved directly to find the interest rate. In these cases, an approximate solution can be found using the methods we looked at in Chapter 10. A more accurate value can then be found using linear interpolation by calculating the net present value for interest rates close to the initial estimate.

Question

Calculate the internal rate of return for both Project R and Project S (outlined in the introduction to this chapter) and, based on the values obtained, comment on which project is preferable.

Solution

Project R

For Project R, we need to calculate the interest rate i that satisfies the equation of value:

$$NPV_R(i) = -150 - 250v - 250v^2 + 1{,}000v^3 = 0$$

From an earlier question, we already know that $NPV_R(0.2) = 46.759$.

As this is too high, we need to try a higher rate. We find that $NPV_R(0.25) = 2.000$ and $NPV_R(0.26) = -5.977$.

We can approximate i by linearly interpolating using these two values:

$$i \approx 25\% + \frac{0 - 2.000}{-5.977 - 2.000} \times (26\% - 25\%) = 25.25\%$$

Now, $NPV_R(0.2525) = -0.022$, which is very close to 0, so the internal rate of return for Project R is approximately 25.25% *pa*.

Project S

For Project S, we need to calculate the interest rate that satisfies the equation of value:

$$NPV_S(i) = -325 - (75 + 90v + 120v^2)\bar{a}_{\overline{1}|} + 1{,}000v^3 = 0$$

From an earlier question, we already know that $NPV_S(0.2) = 40.405$.

As this is too high, we need to try a higher rate. We find that $NPV_S(0.23) = 6.898$ and $NPV_S(0.24) = -3.520$.

We can approximate i by linearly interpolating using these two values:

$$i \approx 23\% + \frac{0 - 6.898}{-3.520 - 6.898} \times (24\% - 23\%) = 23.66\%$$

Now $NPV_S(0.2366) = -0.018$, which is very close to 0, so the internal rate of return for Project S is approximately 23.66% *pa*.

Since the internal rate of return (or yield) for Project R exceeds that for Project S, Project R appears more favourable based on this criterion.

The practical interpretation of the net present value function $NPV(i)$ and the yield is as follows. Suppose that the investor may lend or borrow money at a fixed rate of interest i_1. Since, from Equation (1.2), $NPV(i_1)$ is the present value at rate of interest i_1 of the net cashflows associated with the project, we conclude that the project will be profitable if and only if:

$$NPV(i_1) > 0$$

Also, if the project ends at time *T*, then the profit (or, if negative, loss) at that time is (by Expression (1.1)):

$$NPV(i_1)(1+i_1)^T$$

Let us now assume that, as is usually the case in practice, the yield i_0 exists and $NPV(i)$ changes from positive to negative when $i = i_0$. Under these conditions it is clear that the project is profitable if and only if:

$$i_1 < i_0$$

***ie* the yield exceeds that rate of interest at which the investor may lend or borrow money.**

Many projects will need to provide a return to shareholders and so there will not be a specific fixed rate of interest that has to be exceeded. Instead a target, or hurdle, rate of return may be set for assessing whether a project is likely to be sufficiently profitable.

1.4 The comparison of two investment projects

Suppose now that an investor is comparing the merits of two possible investments or business ventures, which we call projects *A* and *B* respectively. We assume that the borrowing powers of the investor are not limited.

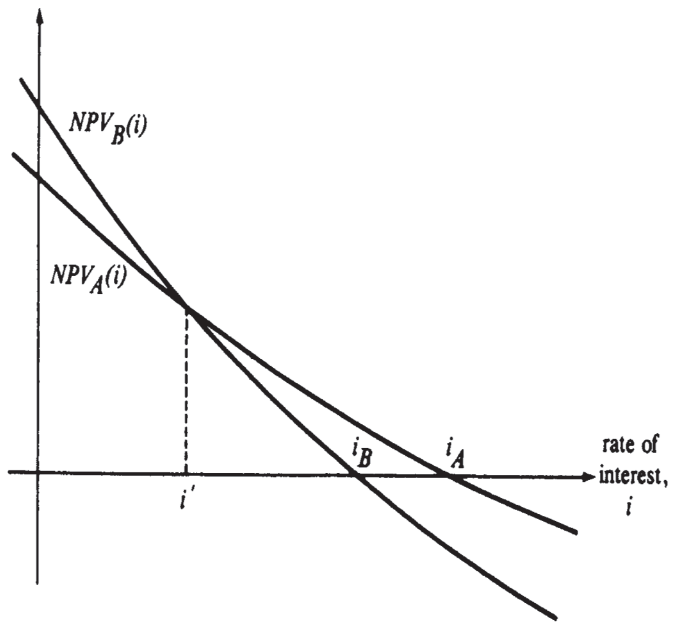
There are therefore no restrictions on how much money the investor can borrow.

Let $NPV_A(i)$ and $NPV_B(i)$ denote the respective net present value functions and let i_A and i_B denote the yields (which we shall assume to exist). It might be thought that the investor should always select the project with the higher yield, but this is not invariably the best policy. A better criterion to use is the *profit at time T* (the date when the later of the two projects ends) or, equivalently, the *net present value*, calculated at the rate of interest i_1 at which the investor may lend or borrow money. This is because *A* is the more profitable venture if:

$$NPV_A(i_1) > NPV_B(i_1)$$

The fact that $i_A > i_B$ may not imply that $NPV_A(i_1) > NPV_B(i_1)$ is illustrated in the following diagram. Although i_A is larger than i_B, the $NPV(i)$ functions 'cross-over' at i'. It follows that $NPV_B(i_1) > NPV_A(i_1)$ for any $i_1 < i'$, where i' is the *cross-over rate*. There may even be more than one cross-over point, in which case the range of interest rates for which project *A* is more profitable than project *B* is more complicated.

The following graph is similar to the graph in Section 1.2, although it has been extended to show the yields for the two projects.



We now give a final example for this section.

Example

An investor is considering whether to invest in either or both of the following loans:

Loan X: **For a purchase price of £10,000, the investor will receive £1,000 per annum payable quarterly in arrears for 15 years.**

Loan Y: **For a purchase price of £11,000, the investor will receive an income of £605 per annum, payable annually in arrears for 18 years, and a return of his outlay at the end of this period.**

The investor may lend or borrow money at 4% per annum. Would you advise the investor to invest in either loan, and, if so, which would be the more profitable?

Solution

We first consider loan *X*:

$$NPV_X(i) = -10,000 + 1,000 a_{\overline{15}|}^{(4)}$$

and the yield is found by solving the equation $NPV_X(i) = 0$, or $a_{\overline{15}|}^{(4)} = 10$, which gives $i_X \approx 5.88\%$.

This value for i_X can be found using trial and error, and is easily checked by calculating $a_{\overline{15}|}^{(4)}$ using an interest rate of 5.88% *pa* effective:

$$a_{\overline{15}|}^{(4)} = \frac{1-v^{15}}{i^{(4)}} = \frac{1-1.0588^{-15}}{4(1.0588^{1/4}-1)} = 10.00$$

For loan *Y* we have:

$$NPV_Y(i) = -11{,}000 + 605a_{\overline{18|}} + 11{,}000v^{18}$$

and the yield (*ie* the solution of $NPV_Y(i) = 0$) is $i_Y = 5.5\%$.

The equation $NPV_Y(i) = 0$ can be rearranged and solved for i as follows:

$$-11{,}000 + 605a_{\overline{18|}} + 11{,}000v^{18} = 0$$

$$\Rightarrow \quad 605a_{\overline{18|}} = 11{,}000(1 - v^{18})$$

$$\Rightarrow \quad 605\left(\frac{1 - v^{18}}{i}\right) = 11{,}000(1 - v^{18})$$

$$\Rightarrow \quad i = \frac{605}{11{,}000} = 5.5\%$$

The rate of interest at which the investor may lend or borrow money is 4% per annum, which is less than both i_X and i_Y, so we compare $NPV_X(0.04)$ and $NPV_Y(0.04)$.

Now $NPV_X(0.04) = £1{,}284$ and $NPV_Y(0.04) = £2{,}089$, so it follows that, although the yield on loan *Y* is less than on loan *X*, the investor will make a larger profit from loan *Y*. We should, therefore, advise him that an investment in either loan would be profitable, but that, if only one of them is to be chosen, then loan *Y* will give the higher profit.

The above example illustrates the fact that the choice of investment depends very much on the rate of interest i_1 at which the investor may lend or borrow money. If this rate of interest were 5¾%, say, then loan *Y* would produce a loss to the investor, while loan *X* would give a profit.

2 Different interest rates for lending and borrowing

We have assumed so far that the investor may borrow or lend money at the same rate of interest i_1. In practice, however, the investor will probably have to pay a higher rate of interest (j_1, say) on borrowings than the rate (j_2, say) he receives on investments.

This is because banks make profits by borrowing money from savers at one rate of interest and lending it out for mortgages, business loans, *etc* at a higher rate.

The difference $j_1 - j_2$ between these rates of interest depends on various factors, including the credit-worthiness of the investor and the expense of raising a loan.

The concepts of net present value and yield are in general no longer meaningful in these circumstances. We must calculate the accumulation of net cashflows from first principles, the rate of interest depending on whether the investor's account is in credit. In many practical problems the balance in the investor's account (*ie* the accumulation of net cashflows) will be negative until a certain time t_1 and positive afterwards, except perhaps when the project ends.

In some cases the investor must finance his investment or business project by means of a fixed-term loan without an early repayment option. In these circumstances the investor cannot use a positive cashflow to repay the loan gradually, but must accumulate this money at the rate of interest applicable on lending, *ie* j_2.

Question

A company must choose between Project C and Project D, both of which would be financed by a loan, repayable only at the end of the project. The company must pay interest at a rate of 6.25% *pa* effective on money borrowed, but can only earn interest at a rate of 4% *pa* effective on money invested in its deposit account.

The cashflows for Project C, which has a term of 5 years, are:

Outgo		Income	
£100,000	(start of year 1)	£140,000	(end of year 5)

The cashflows for Project D, which has a term of 3 years, are:

Outgo		Income	
£80,000	(start of year 1)	£10,000	(end of year 1)
£20,000	(start of year 2)	£30,000	(end of year 2)
£5,000	(start of year 3)	£87,000	(end of year 3)

Calculate the accumulated profit at the end of 5 years for each project.

Solution

Project C

Since Project C does not generate any income, the company will be relying on the loan throughout the 5-year term. So only the borrowing rate of 6.25% will be relevant here.

The accumulated profit at time 5 years will be:

$$140,000 - 100,000 \times 1.0625^5 = £4,592$$

Project D

Since Project D does generate income during the project's term, we need to consider the company's net assets at the end of each year to see whether there are any excess funds available to invest.

At the end of year 1, there is an income payment of £10,000. However, at that time the company needs to pay interest of $80,000 \times 0.0625 = £5,000$ on the initial loan and it also has further outgo of £20,000 at the start of year 2. The net *outgo* at this time is therefore:

$$5,000 + 20,000 - 10,000 = £15,000$$

So further borrowing of £15,000 is required. This takes the total borrowing to:

$$80,000 + 15,000 = £95,000$$

At the end of year 2, there is an income payment of £30,000. However, at that time the company needs to pay interest of $95,000 \times 0.0625 = £5,937.5$ on its borrowing and it also has further outgo of £5,000 at the start of year 3. The net *income* at this time is therefore:

$$30,000 - 5,937.5 - 5,000 = £19,062.5$$

Since the loan can only be repaid at the end of the project, this money cannot be used to reduce the loan outstanding. Instead, the company has £19,062.5 available for investment.

So, just after the outgo of £5,000 at the start of year 3, the company has:

- loans totalling £95,000
- investments of £19,062.5.

At the end of the project (*ie* time 3 years), the company receives income of £87,000. The money invested at the start of year 3 has grown to:

$$19,062.5 \times 1.04 = £19,825$$

So the total amount at the company's disposal is $87,000 + 19,825 = £106,825$. From this, the company must repay the loan of £95,000 plus the interest accrued on this loan over the year. This leaves an amount at time 3 years of:

$$106,825 - 95,000 \times 1.0625 = £5,887.5$$

To obtain the accumulated profit at time 5, we take this money at time 3 and accumulate it for 2 years to time 5 (using the investment rate of interest of 4%):

$$5{,}887.5 \times 1.04^2 = £6{,}368$$

We now also consider the accumulated profit from Project C and Project D assuming that it is possible to use spare funds to repay part of the loan at any time.

Project C

The accumulated profit for Project C is unchanged, since there is no opportunity for repaying the loan early, as no income is received before the end of the project.

Project D

The calculation for the first year is unchanged, since there are no excess funds in the first year.

However, at the end of year 2, the net income of £19,062.5 could be used to repay part of the loan. This reduces the loan outstanding to $95{,}000 - 19{,}062.5 = £75{,}937.5$.

At the end of year 3, income of £87,000 is received, which can be used to repay the loan of £75,937.5 plus the interest accrued on this loan over the year. This leaves an amount at time 3 years of:

$$87{,}000 - 75{,}937.5 \times 1.0625 = £6{,}316.41$$

The accumulated profit at the end of year 5 is then:

$$6{,}316.41 \times 1.04^2 = £6{,}832$$

We see that the accumulated profit for Project D is greater in the case where the loan can be repaid early. This is because the cost of interest payments will be reduced, and the excess funds can be invested for a longer period.

2.1 Payback periods

Another quantity that is useful to calculate when an investment project is financed by outside borrowing is the discounted payback period (DPP). The DPP tells us how long it takes for the project to move into a position of profit.

In many practical problems the net cashflow changes sign only once, this change being from negative to positive. In these circumstances the balance in the investor's account will change from negative to positive at a unique time t_1, or it will always be negative, in which case the project is not viable. If this time t_1 exists, it is referred to as the 'discounted payback period' (DPP). It is the smallest value of *t* such that $A(t) \geq 0$, where:

$$A(t) = \sum_{s \leq t} c_s (1 + j_1)^{t-s} + \int_0^t \rho(s)(1 + j_1)^{t-s}\, ds \qquad \textbf{(2.1)}$$

Note that t_1 does not depend on j_2 but only on j_1, the rate of interest applicable to the investor's borrowings. Suppose that the project ends at time *T*. If $A(T) < 0$ (or, equivalently, if $NPV(j_1) < 0$) the project has no discounted payback period and is not profitable. If the project is viable (*ie* there is a discounted payback period t_1) the *accumulated profit* when the project ends at time *T* is:

$$P = A(t_1)(1+j_2)^{T-t_1} + \sum_{t>t_1} c_t(1+j_2)^{T-t} + \int_{t_1}^{T} \rho(t)(1+j_2)^{T-t}\,dt$$

This follows since the net cashflow is accumulated at rate j_2 after the discounted payback period has elapsed.

Other things being equal, a project with a shorter discounted payback period is preferable to a project with a longer discounted payback period because it will start producing profits earlier.

Question

Derive a formula for the accumulated value of an investment project that has cashflows at times $t_1, t_2, \ldots, t_{10}$ (where $t_1 < t_2 < \cdots < t_{10}$), given that the first 3 cashflows are negative while the remainder are positive, and that the discounted payback period is t_7. Assume that the project is financed by borrowing at rate j that can be repaid at any time, and that excess funds can be invested at rate i ($i < j$).

Solution

Up to time t_7 (*ie* during the DPP), the project has to be funded by borrowing (at rate j).

So the accumulated value at this time is:

$$AV(t_7) = \sum_{k=1}^{7} C_{t_k}(1+j)^{t_7-t_k}$$

Thereafter, there are excess funds that can be invested at rate i up to time t_{10}.

So the accumulated value at the end of the project is:

$$AV(t_{10}) = AV(t_7)(1+i)^{t_{10}-t_7} + \sum_{k=8}^{10} C_{t_k}(1+i)^{t_{10}-t_k}$$

If interest is ignored in formula (2.1) (*ie* if we put $j_1 = 0$), the resulting period is called the 'payback period'. However, its use instead of the discounted payback period often leads to erroneous results and is therefore not to be recommended.

So, the payback period is the earliest time at which the total value of the income received to date is greater than or equal to the total value of the outgo to date. Since interest is ignored in the calculation of the payback period, it takes no account of the time value of money.

Question

The business plan for a new company that has obtained a 5-year lease for operating a local bus service is shown in the table below. Items marked with an asterisk represent continuous cashflows.

Cashflow item	*Timing*	*Amount (£000)*
Initial set up costs	Immediate	–250
Fees from advertising contracts	1 month	+200
Purchase of vehicles	3 months	–2,000
Fares from passengers*	From 3 months onwards	+1,000 *pa*
Staff costs & other operating costs*	From 3 months onwards	– 400 *pa*
Resale value of assets	5 years	+500

Determine the discounted payback period for this project assuming that it will be financed by a loan based on an effective annual interest rate of 10%, and that the loan can be repaid at any time.

Solution

If the DPP exists for this project, it will be at some point after 3 months, as the income from fares is needed to recoup the initial expenditure.

Remembering that the starred cashflows are continuous, the accumulated value of cashflows up to time t (where 3 months $< t < 5$ years) is:

$$AV(t) = -250\times 1.10^{t} + 200\times 1.10^{t-1/12} - 2{,}000\times 1.10^{t-3/12} + (1{,}000-400)\bar{s}_{\overline{t-3/12}|}$$

$$= (-250\times 1.10^{3/12} + 200\times 1.10^{2/12} - 2{,}000)\times 1.10^{t-3/12} + 600\bar{s}_{\overline{t-3/12}|}$$

$$= -2{,}052.83\times 1.10^{t-3/12} + 600\bar{s}_{\overline{t-3/12}|}$$

To determine the DPP, we need to find the value of t for which the accumulated value of the cashflows up to that time is 0. This occurs when:

$$2{,}052.83\times 1.10^{t-3/12} = 600\bar{s}_{\overline{t-3/12}|}$$

Since $\bar{s}_{\overline{n}|} = (1+i)^n \bar{a}_{\overline{n}|}$, we can divide through by $1.10^{t-3/12}$ and rearrange to obtain:

$$\bar{a}_{\overline{t-3/12}|} = \frac{2{,}052.83}{600} = 3.42138$$

Now:

$$\frac{1-1.10^{-(t-3/12)}}{\ln 1.10}=3.42138 \quad \Rightarrow \quad 1.10^{-(t-3/12)}=0.67391$$

Taking logs and solving, we find:

$$-(t-3/12)\ln 1.10=\ln 0.67391 \quad \Rightarrow \quad t=-\frac{\ln 0.67391}{\ln 1.10}+\frac{3}{12}=4.39$$

So the discounted payback period is 4.39 years.

Instead of working with accumulated values, we could work with present values. The DPP is the time t when $NPV_{\text{up to } t}=0$. The equation to solve is:

$$NPV=-250+200v^{1/12}-2{,}000v^{3/12}+600v^{3/12}\bar{a}_{\overline{t-3/12|}}=0$$

This gives the same answer as above.

The discounted payback period is often employed when considering a single investment of *C*, say, in return for a series of payments each of *R*, say, payable annually in arrears for *n* years. The discounted payback period t_1 years is clearly the smallest integer *t* such that $A^*(t)\geq 0$, where:

$$A^*(t)=-C(1+j_1)^t+Rs_{\overline{t|}} \qquad \text{at rate } j_1$$

***ie* the smallest integer *t* such that:**

$$Ra_{\overline{t|}}\geq C \qquad \text{at rate } j_1$$

The project is therefore viable if $t_1 \leq n$, in which case the accumulated profit after *n* years is clearly:

$$P=A^*(t_1)(1+j_2)^{n-t_1}+Rs_{\overline{n-t_1|}} \quad \text{at rate } j_2$$

Question

A speculator borrows £50,000 at an effective interest rate of 8% per annum to finance a project that is expected to generate £7,500 at the end of each year for the next 15 years.

Calculate the discounted payback period for this investment.

Solution

The accumulated profit at the end of year t will be:

$$AP(t)=-50{,}000(1+i)^t+7{,}500s_{\overline{t|}}$$

We need to find the first time t such that this accumulated profit is greater than or equal to 0:

$$-50{,}000(1+i)^t + 7{,}500 s_{\overline{t}|} \geq 0$$

Since $s_{\overline{n}|} = (1+i)^n a_{\overline{n}|}$, we can simplify this by dividing through by $7{,}500(1+i)^t$:

$$a_{\overline{t}|} \geq \frac{50{,}000}{7{,}500} = 6.6666$$

Looking at the *Tables* (at 8%), we see that $a_{\overline{9}|} = 6.2469$ and $a_{\overline{10}|} = 6.7101$. So the discounted payback period is 10 years.

Instead of working with accumulated values, we could work with present values, in which case we want the first time t when $NPV_{\text{up to } t} \geq 0$. The equation to solve is:

$$-50{,}000 + 7{,}500 a_{\overline{t}|} \geq 0$$

This gives the same answer as above.

3 Other considerations

At the simplest level, for projects involving similar amounts of money and with similar time horizons, the project that results in the highest accumulated profit will be the most favourable.

Where the project can be funded without the need for external borrowing, this is equivalent to selecting the project with the highest net present value. The internal rate of return will provide a useful secondary criterion.

Where external borrowing is involved, the accumulated profit must be calculated directly by looking at the cashflows and taking into account the precise conditions of the loan. The discounted payback period will provide a useful secondary criterion.

However, it may not be a straightforward decision for the owners of a business to decide between alternative investment projects purely on the basis of net present values, internal rates of return and discounted payback periods. In many cases a comparison of net present values or internal rates of return for alternative projects will not lead to a decisive conclusion, since the values may be very close or they may conflict. So other considerations will have to be brought into the decision.

Other factors for a company to consider when deciding between different projects include:

Cashflows

- Are the cashflow requirements for the project consistent with the business's other needs?
- Over what period will the profits be produced and how will the profits be used?
- Is it worth carrying out the project if the potential profit is very small in money terms?

Borrowing requirements

- Can the business raise the necessary cash at the times required?
- What rate of interest will the business have to pay on borrowed funds?
- Are time limits or other restrictions imposed on borrowing?

Resources

- Are the other resources required for the project available?
- Does the business have the necessary staff, technical expertise and equipment?

Risk

- What are the financial risks involved in going ahead with the project (and in doing nothing)?
- How certain is the business about the appropriate risk discount rate to use?
- Is it possible that the project might make an unacceptably large loss?
- Can suppliers be relied on to fulfil their contracts according to the agreed timetable and budget?

Investment conditions

- What is the economic climate?
- Are interest rates likely to rise or fall?

Indirect benefits

- Will the project bring any additional benefits?
- Will the equipment purchased and the skills developed be of value to the business in the future?

The following question illustrates these points.

Question

A company is considering investing in two projects: Project C and Project E.

The cashflows for Project C, which has a term of 5 years, are:

- Initial outgo: £100,000
- Income (at the end of year 5): £140,000

The cashflows for Project E, which has a term of 3 years, are:

- Initial outgo: £100,000
- Income (at the end of each of the next 3 years): £38,850

(i) For each of the projects, calculate:

(a) the internal rate of return

(b) the range of interest rates at which money can be borrowed in order for the projects to be viable

(c) the accumulated profit at the end of 5 years, assuming that the projects are financed by a loan subject to interest at 6.25% *pa* effective.

(ii) Outline other considerations that might be taken into account when deciding between Project C and Project E.

Solution

(i)(a) The internal rate of return for Project C, i_C, is the interest rate that solves the equation:

$$100{,}000 = 140{,}000(1+i_C)^{-5} \quad \Rightarrow \quad i_C = \left(\frac{140{,}000}{100{,}000}\right)^{1/5} = 7.0\% \text{ (to 1 dp)}$$

The internal rate of return for Project E, i_E , is the interest rate that solves the equation:

$$100{,}000 = 38{,}850 a_{\overline{3}|} \quad \Rightarrow \quad a_{\overline{3}|} = 2.574$$

Using trial and error, we find that $i_E = 8.1\%$ (to 1 dp).

(i)(b) If the borrowing rate is less than 7%, both projects will be profitable.

If the borrowing rate is between 7% and 8.1%, Project E will be profitable, but Project C will not.

If the borrowing rate exceeds 8.1%, neither project will be profitable.

(i)(c) The accumulated value of the profits at the end of 5 years, using a rate of interest of 6.25%, are:

Project C: $140{,}000 - 100{,}000 \times 1.0625^5 = £4{,}592$

Project E: $38{,}850 \times 1.0625^2 s_{\overline{3}|} - 100{,}000 \times 1.0625^5 = £4{,}561$

(ii) Other considerations include:

1. Although $i_E > i_C$, the internal rates of return are quite close. So the decision based on this criterion is not clear cut.

2. The higher yield available under Project E applies for only 3 years, whereas Project C will produce profits over a 5-year period. If the company invests in Project C, it will receive a yield of 7.0% for the full 5 years. So, if interest rates turn out to be lower in years 4 and 5, Project C may prove to be a better investment.

3. Project C may not be desirable if it will leave the company short of cash during the next 5 years, until the income is received at the end of the project.

4. If the company is not able to borrow more than £100,000 and is required to make regular repayments on money borrowed, then Project C will not be viable, since it does not produce any income with which to pay the interest.

Chapter 12 Summary

The profitability of an investment project can be assessed by calculating the accumulated profit:

$$\textit{Accumulated profit} = \textit{AV income} - \textit{AV outgo}$$

The net present value (NPV) of an investment project is the present value of the net cashflows, calculated at the risk discount rate.

$$\textit{Net present value} = \textit{PV income} - \textit{PV outgo} \qquad (\textit{@ risk discount rate})$$

The internal rate of return for an investment project is the effective rate of interest that equates the present value of income and outgo, *ie* it makes the net present value of the cashflows equal to zero.

The discounted payback period for an investment project is the earliest time after the start of the project when the accumulated value of the past cashflows (positive and negative), calculated using the borrowing rate, becomes greater or equal to zero.

$$\textit{Acc. profit at discounted payback period} \geq 0 \quad (\textit{@ borrowing rate})$$

The payback period is similar to the discounted payback period, but it ignores any interest paid. It is the earliest time that the total value of the past cashflows (positive and negative) becomes greater than or equal to zero.

There are several other factors that must be taken into account when comparing alternative investment projects. These include the cashflow requirements of the business, the borrowing requirements, other resources required, risks involved, investment conditions and indirect benefits.

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 12 Practice Questions

12.1 An investor borrows money at an effective rate of interest of 10% *pa* to invest in a 6-year project. The cashflows for the project are:

- an initial outlay of £25,000
- regular income of £10,000 *pa* during the first 5 years (assumed to be received continuously)
- regular expenditure of £2,000 *pa* during the first 5 years (assumed to be payable continuously)
- a decommissioning expense of £5,000 at the end of the 6th year.

Calculate:

(i) the net present value of the project's cashflows

(ii) the discounted payback period

(iii) the internal rate of return.

12.2 An investor borrows £1,000 by taking out an interest-only loan at an effective rate of interest of 6% *pa*, and invests the money in a project. The loan is to be repaid in full after 2 years (with no early repayment option) and interest on the money borrowed is paid at the end of each month. The project will provide income of £50 at the end of each month for 24 months and the investor can invest spare funds at an interest rate of 5% *pa* effective. Calculate the accumulated profit at the end of 2 years.

12.3 (i) Define:

Exam style

(a) the discounted payback period

(b) the payback period

of an investment project. [2]

(ii) Describe the disadvantages of using these two measures for determining whether to proceed with an investment project. [3]

[Total 5]

12.4 Exam style

A property development company has just purchased a retail outlet for \$4,000,000. A further \$900,000 will be spent refurbishing the outlet in six months' time.

An agreement has been made with a prospective tenant who will occupy the outlet beginning one year after the purchase date. The tenant will pay rent to the owner for five years and will then immediately purchase the outlet from the property development company for \$6,800,000. The initial rent will be \$360,000 per annum and this will be increased by the same percentage compound rate at the beginning of each successive year. The rental income is received quarterly in advance.

Calculate the compound percentage increase in the annual rent required to earn the company an internal rate of return of 12% per annum effective. [9]

12.5 Exam style

An insurance company borrows £50 million at an effective interest rate of 9% per annum. The insurance company uses the money to invest in a capital project that pays £6 million per annum payable half-yearly in arrears for 20 years. The income from the project is used to repay the loan. Once the loan has been repaid, the insurance company can earn interest at an effective interest rate of 7% per annum.

(i) Calculate the discounted payback period for this investment. [4]

(ii) Calculate the accumulated profit the insurance company will have made at the end of the term of the capital project. [5]

[Total 9]

12.6 Exam style

A company is considering investing in the following project. The company has to make an initial investment of three payments, each of £85,000. The first is due at the start of the project, the second one year later, and the third payment is due two years after the start of the project.

After 10 years it is assumed that a major refurbishment of the infrastructure will be required, costing £125,000.

The project is expected to provide no income in the first two years, an income received continuously of £30,000 in the third year, £32,000 in the fourth year, £34,000 in the fifth year and £36,000 in the sixth year. Thereafter the income is expected to increase by 2% per annum (compound) at the start of each year.

The income is expected to cease at the end of the 20th year from the start of the project.

The cashflow within each year is assumed to be received at a constant rate.

(i) Calculate the net present value of the project at a rate of interest of 7% *pa* effective. [6]

(ii) Show that the discounted payback period does not fall within the first 10 years, assuming an effective rate of interest of 7% *pa*. [5]

(iii) Calculate the discounted payback period for the project, assuming an effective rate of interest of 7% *pa*. [5]

[Total 16]

Chapter 12 Solutions

12.1 (i) ***Net present value***

The net present value of the project's cashflows, evaluated using an interest rate of 10% *pa* effective, is:

$$NPV = -25{,}000 + 10{,}000\bar{a}_{\overline{5}|} - 2{,}000\bar{a}_{\overline{5}|} - 5{,}000v^6$$

$$= -25{,}000 + 8{,}000\bar{a}_{\overline{5}|} - 5{,}000v^6$$

$$= -25{,}000 + 8{,}000 \times 3.97732 - 5{,}000 \times 1.1^{-6}$$

$$= £3{,}996.16$$

(ii) ***Discounted payback period***

To calculate the DPP, we need to find the point in time at which the net present value of the project's cashflows (calculated at the borrowing rate of 10%) equals zero, *ie* we need to find the value of t for which:

$$-25{,}000 + 8{,}000\bar{a}_{\overline{t}|} = 0 \quad @10\%$$

*Note that since the NPV of **all** the project's cashflows (as calculated in (i)) exceeds 0, the DPP must fall at some point **before** the end of the project, so we can exclude the decommissioning expense from the above equation.*

This can be simplified to:

$$\bar{a}_{\overline{t}|}^{@10\%} = \frac{25{,}000}{8{,}000} = 3.125$$

Solving, we find:

$$\frac{1-1.1^{-t}}{\ln 1.1} = 3.125 \quad \Rightarrow \quad 1.1^{-t} = 0.70216 \quad \Rightarrow \quad -t\ln 1.1 = \ln 0.70216 \quad \Rightarrow \quad t = 3.71 \text{ years}$$

So the DPP is 3.71 years.

(iii) ***Internal rate of return***

The internal rate of return is the interest rate at which the net present value equals 0, *ie*:

$$NPV = -25{,}000 + 8{,}000\bar{a}_{\overline{5}|} - 5{,}000v^6 = 0 \quad @IRR$$

Using an interest rate of 10% (in part (i)), gives a positive NPV. To reduce the NPV, we need to increase the interest rate:

At 15%: $NPV = -25{,}000 + 8{,}000 \times 3.59771 - 5{,}000 \times 1.15^{-6} = 1{,}620.06$

At 18%: $NPV = -25{,}000 + 8{,}000 \times 3.40086 - 5{,}000 \times 1.18^{-6} = 354.69$

At 19%: $NPV = -25{,}000 + 8{,}000 \times 3.33969 - 5{,}000 \times 1.19^{-6} = -43.17$

Linearly interpolating between the values at 18% and 19% gives:

$$IRR \approx 18\% + \left(\frac{0 - 354.69}{-43.17 - 354.69} \right) (19\% - 18\%) = 18.9\%$$

12.2 The amount of each monthly interest payment required under the loan is:

$$1{,}000(1.06^{1/12} - 1) = £4.87$$

So the investor's net monthly income is:

$$50 - 4.87 = £45.13$$

The accumulated profit at the end of 2 years (when the £1,000 borrowed must be repaid) will be:

$$AV = 12 \times 45.13 s_{\overline{2}|}^{(12)} @5\% - 1{,}000 = 12 \times 45.13 \times 2.0966 - 1{,}000 = £135$$

12.3 (i) ***Definitions***

(a) The discounted payback period for a project is the smallest time t for which the present (or accumulated) value of the income up to time t exceeds the present (or accumulated) value of the outgo up to time t. [1]

(b) The payback period is the same as the discounted payback period, except that the present value calculation (or accumulation) is carried out using an interest rate of 0%. In other words, it is the earliest time for which the monetary value of the income exceeds the monetary value of the outgo. [1]

[Total 2]

(ii) ***Disadvantages***

Neither the DPP nor the PP give any indication of how profitable a project is, as they ignore cashflows after the accumulated value of zero is reached. [1]

The PP can give misleading results as it does not take into account the time value of money. [1]

There may not be one unique time when the balance in the investor's account changes from negative to positive, so there may not be a unique DPP or PP. [1]

[Total 3]

12.4 *This question is Subject CT1, April 2015, Question 9.*

Working in $millions, the total present value of the rental payments is:

$$PV = 0.36v\, \ddot{a}_{\overline{1}|}^{(4)} \left[1 + (1+b)v + \cdots + (1+b)^4 v^4\right] \qquad [1]$$

where b is the compound increase in the rental rate.

Using the formula for the sum of five terms of a geometric progression with first term 1 and common ratio $(1+b)v$, we have a total present value of the rental payments of:

$$PV = 0.36v\, \ddot{a}_{\overline{1}|}^{(4)} \left[\frac{1 - \left[(1+b)v\right]^5}{1 - (1+b)v}\right] \qquad [1]$$

The net present value of all the project's cashflows must equal 0 at the internal rate of return of 12%. So:

$$0 = -4 - 0.9v^{½} + 0.36v\, \ddot{a}_{\overline{1}|}^{(4)} \left[\frac{1 - \left[(1+b)v\right]^5}{1 - (1+b)v}\right] + 6.8v^6 \qquad [1]$$

Evaluating the compound interest functions at 12%:

$$v^{½} = 0.944911$$

$$v^6 = 0.506631$$

$$d^{(4)} = 4\left[1 - 1.12^{-¼}\right] = 0.111738$$

$$\ddot{a}_{\overline{1}|}^{(4)} = \frac{1-v}{d^{(4)}} = \frac{1 - 1.12^{-1}}{0.111738} = 0.958873$$

So the equation of value becomes:

$$0 = -4 - 0.9 \times 0.944911 + 0.36 \times \frac{1}{1.12} \times 0.958873 \times \frac{1 - X^5}{1 - X} + 6.8 \times 0.506631$$

where $X = \dfrac{1+b}{1.12}$.

Simplifying this, we obtain:

$$0 = -4 - 0.850420 + 0.308209 \times \frac{1 - X^5}{1 - X} + 3.445092$$

Rearranging this equation, we find that:

$$\frac{1-X^5}{1-X}=4.5597$$ [2]

We now solve this by trial and error. Choosing an initial starting value of $X=0.9$, we obtain the following values:

X	$\frac{1-X^5}{1-X}$
0.9	4.0951
0.92	4.2615
0.94	4.4349
0.96	4.6157
0.953	4.5516
0.954	4.5607

[2]

Interpolating, we obtain:

$$X \simeq 0.953+\frac{4.5597-4.5516}{4.5607-4.5516}\times 0.001=0.95389$$ [1]

Finally this gives us a compound growth rate of:

$$b=1.12\times 0.95389-1=0.06835$$

The required compound growth rate is 6.84%. [1]

[Total 9]

12.5 *This question is Subject CT1, April 2014, Question 8.*

(i) ***Discounted payback period***

The discounted payback period is the point in time when the balance on our account first turns positive. So, we require the smallest value of n for which:

$$NPV=-50+6a_{\overline{n}|}^{(2)}>0 \quad\Rightarrow\quad a_{\overline{n}|}^{(2)}>8.333$$ [1]

Solving for n we find:

$$\frac{1-1.09^{-n}}{2(1.09^{½}-1)}>8.333 \quad\Rightarrow\quad 0.2662>1.09^{-n} \quad\Rightarrow\quad \ln 0.2662>-n\ln 1.09 \quad\Rightarrow\quad n>15.36$$ [2]

Since the payments are half-yearly, the DPP is 15.5 years. [1]

[Total 4]

(ii) ***Accumulated profit***

Working in £millions, the NPV of the cashflows occurring up to and including time 15.5 years is:

$$-50+6a^{(2)}_{\overline{15.5|}\,@9\%}=-50+6\times\frac{1-v^{15.5}}{i^{(2)}}=-50+6\times 8.369627=0.217762 \quad [2]$$

So the accumulated value at time 15.5 years of these cashflows is:

$$0.217762\times 1.09^{15.5}=0.828119 \quad [1]$$

So the total accumulated profit at time 20 years will be:

$$0.828119\times 1.07^{4.5}+6s^{(2)}_{\overline{4.5|}\,@7\%}=1.122845+6\times 5.171726=32.153200 \quad [2]$$

The accumulated profit at the end of the project is £32.153 million. [Total 5]

12.6 (i) ***Net present value***

The PV of the outgo (in £000s) is:

$$PV\ outgo=85(1+v+v^2)+125v^{10}\ @7\%=302.225 \quad [1]$$

The PV of the income (in £000s) is:

$$PV\ income=30v^2\,\bar{a}_{\overline{1|}}+32v^3\,\bar{a}_{\overline{1|}}+34v^4\,\bar{a}_{\overline{1|}}+36v^5\,\bar{a}_{\overline{1|}}$$
$$+36\times 1.02v^6\bar{a}_{\overline{1|}}+36\times 1.02^2v^7\bar{a}_{\overline{1|}}+\cdots+36\times 1.02^{14}v^{19}\bar{a}_{\overline{1|}}$$

This can be simplified to:

$$PV\ income=(30v^2+32v^3+34v^4)\bar{a}_{\overline{1|}}+36v^5(1+1.02v+1.02^2v^2+\cdots+1.02^{14}v^{14})\bar{a}_{\overline{1|}} \quad [2]$$

The first part can be calculated directly and the second part can be summed as a geometric series of 15 terms with first term 1 and common ratio $1.02v$, to give:

$$PV\ income=78.263\times 0.96692+36\times 1.07^{-5}\times\frac{1-1.02^{15}v^{15}}{1-1.02v}\times 0.96692=347.709 \quad [2]$$

So the NPV is $347{,}709-302{,}225=£45{,}484$. [1]

[Total 6]

(ii) ***Show the discounted payback period does not fall within the first 10 years***

The PV of the income (in £000s) received up to time 10 years can be calculated similarly as:

$$PV\ income\ (10\,yrs)=78.263\times 0.96692+36\times 1.07^{-5}\times\frac{1-1.02^5v^5}{1-1.02v}\times 0.96692$$
$$=188.698 \quad [3]$$

The PV of the outgo (in £000s) excluding the refurbishment payment at time 10 years is:

$$PV\ outgo\ (10\ yrs) = 85(1+v+v^2) = 238.682 \qquad [1]$$

As the PV of the income is less than the PV of the outgo up to the end of the 10th year, and the initial outgo (*ie* the 3 payments of £85,000) precedes the start of the income, the accumulated value of the project's cashflows is negative throughout the first 10 years. So the DPP does not fall within the first 10 years. [1]

[Total 5]

(iii) ***Discounted payback period***

We first establish in which year the DPP falls, by considering complete years. Equating the PV of the income for the first n complete years of the project ($n > 10$) to the PV of the outgo gives the following equation:

$$78.263 \times 0.96692 + 36 \times 1.07^{-5} \times \frac{1-1.02^{n-5}v^{n-5}}{1-1.02v} \times 0.96692 = 302.225$$

$$\Rightarrow \quad 75.674 + 24.818 \times \frac{1-1.02^{n-5}v^{n-5}}{1-1.02v} = 302.225 \qquad [1]$$

Note that this equation will only be exact if n is an integer.

Solving this, we find that:

$$(1.02v)^{n-5} = 0.57344 \quad \Rightarrow \quad (n-5)\ln(1.02v) = \ln 0.57344 \quad \Rightarrow \quad n = 16.6 \qquad [1]$$

So, the DPP ends somewhere between time 16 and time 17. To find the exact value of the DPP, we need to value the payments in the final part year (of length t , say) exactly, which gives the equation:

$$75.674 + 24.818 \times \frac{1-1.02^{11}v^{11}}{1-1.02v} + 36 \times 1.02^{11} v^{16} \bar{a}_{\overline{t|}} = 302.225 \qquad [1]$$

Solving this, we find:

$$\bar{a}_{\overline{t|}} = \frac{1-1.07^{-t}}{\ln 1.07} = 0.60514 \quad \Rightarrow \quad 1.07^{-t} = 0.95906$$

$$\Rightarrow \quad -t\ln 1.07 = \ln 0.95906 \quad \Rightarrow \quad t = 0.618 \qquad [1]$$

So the discounted payback period is 16.618 years. [1]

[Total 5]

13

Bonds, equity and property

Syllabus objectives

3.2 Use the concept of equation of value to solve various practical problems.

3.2.2 Calculate the price of, or yield (nominal or real allowing for inflation) from, a bond (fixed-interest or index-linked) where the investor is subject to deduction of income tax on coupon payments and redemption payments are subject to deduction of capital gains tax.

3.2.3 Calculate the running yield and the redemption yield for the financial instrument as described in 3.2.2.

3.2.4 Calculate the upper and lower bounds for the present value of the financial instrument as described in 3.2.2 when the redemption date can be a single date within a given range at the option of the borrower.

3.2.5 Calculate the present value or yield (nominal or real allowing for inflation) from an ordinary share or property, given constant or variable rate of growth of dividends or rents.

This chapter also deals with real rates of interest as required in syllabus objective 2.2.

0 Introduction

In this chapter, we show how to calculate the price to be paid for, or the yield obtained from, the following investments:

- fixed-interest bonds
- index-linked bonds
- equities (*ie* shares)
- property.

The cashflows relating to the first three of these investments were discussed in Chapter 3.

Investment in property involves outgo in the form of the purchase price of the building, any legal or other fees, and maintenance costs. Income is received in the form of rent. Investors in property also hope to make a gain from selling the property for more than its purchase price.

1 Fixed-interest securities

First, we will apply the theory we have developed to fixed-interest securities, *ie* investments that make regular interest payments at a fixed rate, followed by a capital repayment.

The interest payments on a bond are called coupons and the capital repayment is called the redemption payment. Prices for bonds are often calculated 'per £100 nominal'. The coupon and redemption payments are expressed as a percentage of the nominal amount and not as a percentage of the purchase price.

For example, if an investor owns £100 nominal of a fixed-interest bond, which pays coupons annually in arrears at a rate of 6% per annum and has a redemption rate of 102%, the investor will receive:

- £6 at the end of each year of the bond's term, and
- £102 at maturity.

If the redemption payment is equal to the nominal value, the bond is said to be 'redeemed at par', and an investor owning £100 nominal of the bond would receive £100 at maturity.

As in other compound interest problems, one of two questions may be asked:

(1) What price P per unit nominal, should be paid by an investor to secure a net yield of i per annum?

(2) Given that the investor pays a price P per unit nominal, what net yield per annum will be obtained?

1.1 Calculating the price and yield

The price, P, to be paid to achieve a yield of i per annum is equal to:

$$P = \begin{pmatrix} \text{Present value, at rate} \\ \text{of interest } i \text{ per annum,} \\ \text{of net interest payments} \end{pmatrix} + \begin{pmatrix} \text{Present value, at rate} \\ \text{of interest } i \text{ per annum,} \\ \text{of net capital payments} \end{pmatrix} \qquad (1.1)$$

In other words, the price paid is the present value of all the payments the investor receives net of any tax.

Question

A tax-exempt investor purchases £10,000 nominal of a newly issued 5-year fixed-interest bond that is redeemable at par and pays coupons of 8% *pa* half-yearly in arrears.

Calculate the price the investor should pay to obtain an effective annual yield of 10%.

Solution

Based on owning £10,000 nominal of the bond, the coupons are $8\% \times 10{,}000 = £800$ per year. The coupons are payable half-yearly in arrears, so £400 is paid at the end of each half-year. Since the bond is newly issued, the first coupon is due in 6 months' time.

As the bond is redeemable at par, the investor receives a redemption payment of £10,000 at the end of the 5-year term.

The investor is tax-exempt and will therefore receive the full coupon and redemption payments. Hence the price the investor should pay for £10,000 nominal of this bond is:

$$P = 800a_{\overline{5}|}^{(2)} + 10{,}000v^5 \qquad @10\%$$

$$= 800 \times 3.8833 + 10{,}000 \times 1.1^{-5}$$

$$= £9{,}315.85$$

The yield available on a stock that can be bought at a given price, P, can be found by solving equation (1.1) for the net yield i.

If the investor is not subject to taxation, the yield i is referred to as a *gross* yield. The yield on a security is sometimes referred to as the *yield to redemption* or the *redemption yield* to distinguish it from the *flat* (or *running*) *yield*, which is defined as D/P, the ratio of the coupon rate to the price per unit nominal of the stock.

The letter D is used here to denote the coupon rate on the bond.

A redemption yield is the yield obtained by an investor who holds the bond until redemption whereas the running yield is the yield generated by the payment of the coupons alone (ignoring the redemption payment).

Question

Calculate the running yield for the investor in the previous question.

Solution

The investor pays £9,315.85 for £10,000 nominal of the bond and receives coupons of £800 each year. So the running yield is:

$$\frac{800}{9{,}315.85} = 8.59\%$$

Since the coupons are paid half-yearly, this running yield is convertible half-yearly.

A redemption yield that is calculated without making any allowance for tax is called a *gross redemption yield* (GRY). If tax is incorporated in the calculation, this gives the *net redemption yield* (NRY).

We can determine the gross redemption yield for a fixed-interest stock by trial and error and then interpolation.

Question

A tax-exempt investor pays £10,500 for £10,000 nominal of a newly issued 5-year fixed-interest bond that is redeemable at par and pays coupons of 8% *pa* half-yearly in arrears.

Calculate the gross redemption yield obtained by the investor.

Solution

The gross redemption yield is the interest rate i that satisfies the equation of value:

$$10,500 = 800a_{\overline{5}|}^{(2)} + 10,000v^5$$

This is the bond referred to in earlier questions, and we've already seen that if the yield is 10%, the price is £9,315.85. Here the investor is paying more than £9,315.85, so the yield is lower than 10%. However, we can get a better first guess by considering the coupon rate and the redemption payment.

If the purchase price were £10,000, the redemption yield, convertible half-yearly, would be 8% because an investment of £10,000 would be receiving interest of £800 each year payable half-yearly, followed by a return of the initial investment. The effective yield in this case would be just greater than 8%. Since the investor pays more than £10,000, the redemption yield must be less than 8%.

At 6%: $RHS = 10,892.28$

At 7%: $RHS = 10,466.45$

We can approximate i by linearly interpolating using these two values:

$$i \approx 6\% + \frac{10,500 - 10,892.28}{10,466.45 - 10,892.28} \times (7\% - 6\%) = 6.9\%$$

So the GRY is approximately 6.9%.

1.2 No tax

Consider an n year fixed-interest security which pays coupons of D per annum, payable $p$thly in arrears and has redemption amount R.

The price of this bond, at an effective rate of interest i per annum, with no allowance for tax (*ie* i represents the gross yield) is:

$$P = Da_{\overline{n|}}^{(p)} + Rv^n \quad \text{at rate } i \text{ per annum} \tag{1.2}$$

We looked at an example of this earlier, as our tax-exempt investor received the full amount of the coupon and redemption payments.

Note: One could also work with a period of half a year. The corresponding equation of value would then be:

$$P = \frac{D}{2}a_{\overline{2n|}} + Rv^{2n} \quad \text{at rate } i' \text{ where } (1+i')^2 = 1+i$$

Question

A tax-exempt investor purchases £10,000 nominal of a newly issued 5-year fixed-interest bond that is redeemable at par and pays coupons of 8% *pa* half-yearly in arrears.

Calculate the price the investor should pay to obtain an effective annual yield of 10%.

Solution

We have already calculated the answer to this question by working in years. The price worked out to be £9,315.85. We will confirm this by working in half-years, using a half-yearly effective interest rate.

First we need to calculate the half-yearly effective rate:

$$1.1^{1/2} - 1 = 4.880885\%$$

So the price to be paid is:

$$P = 400a_{\overline{10|}} + 10{,}000v^{10} \quad @\ 4.880885\%$$

$$= 400 \times 7.7666 + 10{,}000 \times 1.04880885^{-10}$$

$$= £9{,}315.85$$

1.3 Income tax

Suppose an investor is liable to income tax at rate t_1 on the coupons, which is due at the time that the coupons are paid. The price, P', of this bond, at an effective rate of interest i per annum, where i now represents the net yield, is now:

$$P' = (1-t_1)Da_{\overline{n}|}^{(p)} + Rv^n \qquad \text{at rate } i \text{ per annum} \tag{1.3}$$

The coupon payments are income and hence are liable to income tax. Since investors only want to pay for the coupons they actually receive, they calculate the price based on the *net* coupons received after tax has been deducted.

The redemption payment is a return of the capital lent by the investor (*ie* purchaser of the bond) to the borrower (*ie* issuer of the bond) and so is not liable to income tax (even if the redemption proceeds are greater than the price originally paid).

Question

An investor liable to income tax at 25% purchases £10,000 nominal of a newly issued 5-year fixed-interest bond that is redeemable at par and pays coupons of 8% *pa* half-yearly in arrears.

Calculate the price the investor should pay to obtain a net yield of 10% *pa* effective.

Solution

Again, this is the same bond referred to in earlier questions. However, this time the investor is liable to tax on income.

Using equation (1.3) we have:

$$P = (1-0.25)800a_{\overline{5}|}^{(2)} + 10,000v^5 \qquad @10\%$$

$$= 600 \times 3.8833 + 10,000 \times 1.1^{-5}$$

$$= £8,539.19$$

We can see that this investor receives only £600 *pa* (=75% of the gross coupons) and so to obtain the same yield as the tax-exempt investor considered before, this investor must buy the stock at a lower price.

As well as calculating the price paid for a stock by an investor subject to income tax, we could be asked to calculate the net yield obtained by such an investor given the price paid.

Question

Suppose the investor in the previous question actually purchases the stock for £9,000.

Calculate the net redemption yield obtained on the investment.

Solution

We saw that the price was £8,539.19 when the net yield was 10% *pa*. If the price is £9,000, the net yield is lower than 10% *pa*.

The net redemption yield is the interest rate i that solves the equation of value:

$$9{,}000 = (1-0.25)800a_{\overline{5}|}^{(2)} + 10{,}000v^5$$

At 8%: $RHS = 9{,}248.45$

At 9%: $RHS = 8{,}884.48$

We can approximate i by linearly interpolating using these two values:

$$i \approx 8\% + \frac{9{,}000 - 9{,}248.45}{8{,}884.48 - 9{,}248.45} \times (9\% - 8\%) = 8.7\%$$

So the net redemption yield is approximately 8.7%.

It is possible in some countries that the tax is paid at some later date, for example at the calendar year end.

This does not cause any particular problems as we follow the usual procedure – identify the cashflow amounts and dates and set out the equation of value.

For example, suppose that income tax on the bond each year **is paid in a single** annual **instalment, due, say, k years after the second half-yearly coupon payment each year. Then the equation of value for a given net yield i and price (or value) P' is, immediately after a coupon payment,**

$$P' = Da_{\overline{n}|}^{(p)} + Rv^n - t_1 Dv^k a_{\overline{n}|}$$

Other arrangements may be dealt with similarly from first principles.

Question

An investor purchases £100 nominal of a 10-year bond. The coupon rate is 6% *pa*, and coupons are payable half-yearly in arrears. The bond is redeemable at par. The investor pays 15% tax on income, and tax payments are due four months after each coupon is received.

Calculate the price paid by the investor to achieve a net redemption yield of 9% *pa* effective.

Solution

The price for £100 nominal, P, can be calculated from the equation:

$$P = 6a_{\overline{10}|}^{(2)} + 100v^{10} - 6 \times 0.15 \times v^{4/12} a_{\overline{10}|}^{(2)} \quad @9\%$$

$$= 6 \times 6.5589 + 100 \times 1.09^{-10} - 0.9 \times 1.09^{-4/12} \times 6.5589$$

$$= £75.86$$

1.4 Capital gains tax

If the price paid for a bond is less than the redemption (or sale price if sold earlier), then the investor has made a capital gain.

Capital gains tax is a tax levied on the capital gain. In contrast to income tax, this tax is normally payable once only in respect of each disposal, at the date of sale or redemption.

So capital gains tax is paid at redemption or at the time the bond is sold. As with income tax, the investor only wants to pay for the redemption payment actually received, and so will therefore calculate the price based on the *net* redemption payment received after tax has been deducted.

However, the price depends on whether there is a capital gain, and knowing whether there's a capital gain depends on the price. So we use a test to determine whether there is a capital gain *before* we calculate the price.

Capital gains test

Consider an n year fixed-interest security which pays coupons of D per annum, payable p thly in arrears and has redemption amount R. An investor, liable to income tax at rate t_1 (due at the same time the coupons are paid), **purchases the bond at price P'. If $R > P'$ then there is a capital gain and from (1.3), we have:**

$$R > (1-t_1)Da_{\overline{n}|}^{(p)} + Rv^n$$

$$\Rightarrow \quad R(1-v^n) > (1-t_1)D\frac{1-v^n}{i^{(p)}}$$

$$\Rightarrow \quad R > (1-t_1)\frac{D}{i^{(p)}}$$

$$\Rightarrow \quad i^{(p)} > (1-t_1)\frac{D}{R} \qquad \textbf{(1.4)}$$

An intuitive way of thinking about this is that the overall return on a bond comes from both the coupons and any capital gain. If the return we require is greater than the net coupons we receive, then we must be getting more back than we paid, *ie* by having a capital gain.

If $i^{(p)} < (1-t_1)\frac{D}{R}$, then there is a capital loss, *ie* the investor pays more for the bond than is received back at redemption. If $i^{(p)} = (1-t_1)\frac{D}{R}$, there is neither a capital gain nor a capital loss, *ie* the investor pays the same for the bond as is received back at redemption.

Sometimes g is used in place of $\frac{D}{R}$.

Question

An investor, who is liable to income tax at 25%, purchases £10,000 nominal of a newly issued 5-year fixed-interest bond that is redeemable at par and pays coupons of 8% *pa* half-yearly in arrears.

Determine whether there is a capital gain if the investor requires a net yield of 10% *pa* effective.

Solution

The investor requires a net yield of 10% *pa* and coupons are paid half-yearly, so:

$$i^{(2)} \text{ @}10\% = 2\left(1.1^{½} - 1\right) = 0.0976177$$

Income tax is 25%, and for £10,000 nominal we receive coupons of £800 each year and a redemption payment of £10,000. So:

$$(1-t_1)\frac{D}{R} = (1-0.25)\frac{800}{10,000} = 0.06$$

Since $i^{(2)} > (1-t_1)\frac{D}{R}$ we have a capital gain.

We looked at this bond in an earlier question, and found that an investor who is liable to income tax at 25% and requires a net yield of 10% *pa* effective would pay £8,539.19 for £10,000 nominal. Since the investor receives a redemption payment of £10,000, this confirms the result of the test – there is a capital gain.

If the investor were also liable to capital gains tax (as well as income tax), then the price would have to be less than £8,539.19 to receive the same 10% *pa* net return.

If the investor is also subject to tax at rate t_2 $(0 < t_2 < 1)$ on the capital gains, then let the price payable, for a given net yield i, be P''.

If $i^{(p)} > (1-t_1)\dfrac{D}{R}$ then there is a capital gain. At the redemption date of the loan there is therefore an additional liability of $t_2(R - P'')$.

Capital gains tax is paid *only* on the capital gain and not on the whole of the redemption amount.

In this case:

$$P'' = (1-t_1)Da_{\overline{n}|}^{(p)} + Rv^n - t_2(R-P'')v^n \qquad \text{at rate } i \text{ per annum} \qquad (1.5)$$

Question

An investor liable to income tax at 25% and capital gains tax at 20% purchases £10,000 nominal of a newly issued 5-year fixed-interest bond that is redeemable at par and pays coupons of 8% *pa* half-yearly in arrears.

Calculate the price the investor should pay to obtain a net yield of 10% *pa* effective.

Solution

This follows on from the previous question, in which we showed that there is a capital gain:

$$i^{(2)} \text{ @}10\% = 0.0976177 > (1-t_1)\frac{D}{R} = (1-0.25)\frac{800}{10,000} = 0.06$$

Now we can use equation (1.5) to calculate the price paid:

$$P = (1-0.25)800a_{\overline{5}|}^{(2)} + 10,000v^5 - 0.2(10,000-P)v^5 \qquad \text{@}10\%$$

$$= 600a_{\overline{5}|}^{(2)} + 8,000v^5 + 0.2Pv^5$$

The price paid, P, appears on both sides of the equation. Rearranging:

$$(1-0.2v^5)P = 600a_{\overline{5}|}^{(2)} + 8,000v^5$$

$$\Rightarrow \quad P = \frac{600a_{\overline{5}|}^{(2)} + 8,000v^5}{1-0.2v^5} = \frac{600 \times 3.8833 + 8,000 \times 1.1^{-5}}{1-0.2 \times 1.1^{-5}} = £8,332.06$$

As expected, this is less than the price paid by an investor who is liable for income tax at 25% only (which we calculated to be £8,539.19).

The price calculated is critically dependent on the net yield required by the investor – this affects not only the annuity and discount factors in the equation, but also the capital gains test itself.

Question

An investor liable to income tax at 25% and capital gains tax at 20% purchases £10,000 nominal of a newly issued 5-year fixed-interest bond that is redeemable at par and pays coupons of 8% *pa* half-yearly in arrears.

Calculate the price the investor should pay to obtain a net yield of 5% *pa* effective.

Solution

This is the same scenario as the previous question, but the net yield required is now 5% *pa*, rather than 10% *pa*.

Carrying out the capital gains test:

$$i^{(2)} \text{ @}5\% = 0.0493902$$

$$(1-t_1)\frac{D}{R} = (1-0.25)\frac{800}{10,000} = 0.06$$

Since $i^{(2)} < (1-t_1)\frac{D}{R}$, there is no capital gain.

Since there is no capital gain, the investor will not be liable for capital gains tax, and so we need to use equation (1.3):

$$P = (1-0.25)800a_{\overline{5}|}^{(2)} + 10,000v^5 \qquad \text{@}5\%$$

$$= 600 \times 4.382935 + 10,000 \times 1.05^{-5}$$

$$= \pounds 10,465.02$$

The value of P obtained is greater than £10,000, which confirms that no capital gains tax is payable.

Note that if a stock is sold before the final maturity date, the capital gains tax liability will in general be different, since it will be calculated with reference to the sale proceeds rather than the corresponding redemption amount.

If an investor who paid £9,000 to purchase £10,000 nominal of a bond, sold the bond before maturity for £9,700, the capital gain would be $\pounds 9,700 - \pounds 9,000 = \pounds 700$. If the investor were liable to pay capital gains tax at a rate of 30%, the capital gains tax liability would be $0.3 \times 700 = \pounds 210$.

If $i^{(p)} \le (1-t_1)\frac{D}{R}$ then there is no capital gain and no capital gains tax liability due at redemption. Hence $P'' = P'$ in (1.3), *ie* we only pay income tax. We saw this in the above question.

(We are assuming that it is not permissible to offset the capital loss against any other capital gain.)

Finding the yield when there is capital gains tax

An investor who is liable to capital gains tax may wish to determine the net yield on a particular transaction in which he has purchased a loan at a given price.

One possible approach is to determine the price on two different net yield bases and then estimate the actual yield by interpolation. This approach is not always the quickest method. Since the purchase price is known, so too is the amount of the capital gains tax, and the net receipts for the investment are thus known. In this situation one may more easily write down an equation of value which will provide a simpler basis for interpolation, as illustrated by the next question.

Question

A loan of £1,000 bears interest of 6% per annum payable yearly and will be redeemed at par after ten years. An investor, liable to income tax and capital gains tax at the rates of 40% and 30% respectively, buys the loan for £800. What is his net effective annual yield?

Solution

Note that the net income each year of £36 is 4.5% of the purchase price. Since there is a gain on redemption, the net yield is clearly greater than 4.5%.

The gain on redemption is £200, so that the capital gains tax payable will be £60 and the net redemption proceeds will be £940. The net effective yield *pa* is thus that value of i for which:

$$800 = 36a_{\overline{10}|} + 940v^{10}$$

If the net gain on redemption (*ie* £140) were to be paid in equal instalments over the ten-year duration of the loan rather than as a lump sum, the net receipts each year would be £50 (*ie* £36 + £14). Since £50 is 6.25% of £800, the net yield actually achieved is less than 6.25%.

The net yield is less than 6.25% because, in reality, the capital gain is received right at the end of the term, rather than in equal instalments over the term, and being further in the future, this is less valuable to the investor.

When $i = 0.055$, the right-hand side of the above equation takes the value 821.66, and when $i = 0.06$ the value is 789.85.

By interpolation, we estimate the net yield as:

$$i = 0.055 + \frac{821.66 - 800}{821.66 - 789.85} 0.005 = 0.0584$$

The net yield is thus 5.84% per annum.

Alternatively, we may find the prices to give net yields of 5.5% and 6% per annum. These prices are £826.27 and £787.81, respectively.

For example, using a net yield of 5.5% *pa* effective, the price of the loan is:

$$P = 36a_{\overline{10}|} + 1{,}000v^{10} - 0.3(1{,}000 - P)v^{10}$$

$$\Rightarrow \quad P = \frac{36 \times 7.5376 + 700 \times 1.055^{-10}}{1 - 0.3 \times 1.055^{-10}} = £826.27$$

The yield may then be obtained by interpolation. However, this alternative approach is somewhat longer than the first method.

1.5 Optional redemption dates

Sometimes a security is issued without a fixed redemption date. In such cases the terms of issue may provide that the borrower can redeem the security *at the borrower's option* at any interest date on or after some specified date. Alternatively, the issue terms may allow the borrower to redeem the security at the borrower's option at any interest date on or between two specified dates (or possibly on any one of a series of dates between two specified dates).

In such cases, the loan will be redeemed at the time considered to be most favourable by the borrower (*ie* the issuer). If the interest rate payable on the loan is high relative to market rates, it will be cheaper for the borrower to repay the loan and borrow from elsewhere. Conversely, if the interest rate payable is relatively low, it will be cheaper to allow the loan to continue.

The latest possible redemption date is called the *final redemption date* of the stock, and if there is no such date, then the stock is said to be *undated*. It is also possible for a loan to be redeemable between two specified interest dates, or on or after a specified interest date, at the option of the lender, but this arrangement is less common than when the borrower chooses the redemption date.

The loan will be redeemed at the time when the party with the choice of date will obtain the greatest yield.

An investor who wishes to purchase a loan with redemption dates at the option of the borrower cannot, at the time of purchase, know how the market will move in the future and hence when the borrower will repay the loan. The investor thus cannot know the yield which will be obtained. However, by using (1.4) the investor can determine either:

(1) The maximum price to be paid, if the net yield is to be at least some specified value;

or

(2) The minimum net yield the investor will obtain, if the price is some specified value.

Consider a fixed-interest security which pays coupons of D per annum, payable p thly in arrears and has redemption amount R. The security has an outstanding term of n years, which may be chosen by the borrower subject to the restriction that $n_1 \le n \le n_2$. (We assume that n_1 and n_2 are integer multiples of $1/p$.) Suppose that an investor, liable to income tax at rate t_1, wishes to achieve a net annual yield of at least i.

It follows from equations (1.3) and (1.4) that if $i^{(p)} > (1-t_1)\dfrac{D}{R}$, then the purchaser will receive a capital gain when the security is redeemed. From the investor's viewpoint, the sooner a capital gain is received the better. The investor will therefore obtain a greater yield on a security which is redeemed first.

Question

An investor purchases £100 nominal of a zero-coupon bond for £80. Calculate the yield obtained if the bond is redeemed after (a) five years, and (b) ten years.

Solution

If the purchase price is £80, we must find i that solves the equation:

$$80 = 100v^n$$

(a) $n=5 \;\Rightarrow\; (1+i)^{-5} = 0.8 \;\Rightarrow\; i = 0.8^{-1/5} - 1 = 4.56\%\ pa$

(b) $n=10 \;\Rightarrow\; (1+i)^{-10} = 0.8 \;\Rightarrow\; i = 0.8^{-1/10} - 1 = 2.26\%\ pa$

This illustrates that, if there is a capital gain, then the sooner the bond is redeemed, the higher the yield.

So to *ensure* the investor receives a net annual yield of *at least* i, they should assume the worst case result: that the redemption money is paid as late as possible, *ie* $n = n_2$.

Similarly if $i^{(p)} < (1-t_1)\dfrac{D}{R}$ then there will be a capital loss when the security is redeemed. The investor will wish to defer this loss as long as possible, and will therefore obtain a greater yield on a security which is redeemed later.

Question

An investor purchases £100 nominal of a zero-coupon bond for £120. Calculate the yield obtained if the bond is redeemed after (a) five years, and (b) ten years.

Solution

If the purchase price is £120, we must find i that solves the equation:

$$120 = 100v^n$$

(a) $n=5 \;\Rightarrow\; (1+i)^{-5} = 1.2 \;\Rightarrow\; i = 1.2^{-1/5} - 1 = -3.58\%\ pa$

(b) $n=10 \;\Rightarrow\; (1+i)^{-10} = 1.2 \;\Rightarrow\; i = 1.2^{-1/10} - 1 = -1.81\%\ pa$

This illustrates that if there is a capital loss, then the later the bond is redeemed, the higher the yield.

So to *ensure* the investor receives a net annual yield of *at least* i, they should assume the worst case result: that the redemption money is paid as soon as possible, *ie* $n = n_1$.

Finally, if $i^{(p)} = (1-t_1)\frac{D}{R}$, then there is neither a capital gain nor a capital loss. So it will make no difference to the investor when the security is redeemed. The net annual yield will be i irrespective of the actual redemption date chosen.

Question

An investor purchases £100 nominal of a zero-coupon bond for £100. Calculate the yield obtained if the bond is redeemed after (a) five years, and (b) ten years.

Solution

If the purchase price is £100, we must find i that solves the equation:

$$100 = 100v^n$$

(a) $n = 5 \;\Rightarrow\; (1+i)^{-5} = 1 \;\Rightarrow\; i = 1^{-1/5} - 1 = 0\%\ pa$

(b) $n = 10 \;\Rightarrow\; (1+i)^{-10} = 1 \;\Rightarrow\; i = 1^{-1/10} - 1 = 0\%\ pa$

This illustrates that if there is neither a capital gain nor a capital loss, then the yield is the same irrespective of when the bond is redeemed. In the question immediately above, the yield is zero regardless of the term to redemption as it is a zero-coupon bond. This would not be the case for a coupon-paying bond.

Question

A fixed-interest stock with a coupon of 8% per annum payable half-yearly in arrears can be redeemed at the option of the borrower at any time between 10 and 15 years from the date of issue. The stock is redeemable at par.

An investor, who is subject to tax at 25% on income only, wishes to purchase £100 nominal of this stock. Calculate the maximum price the investor should pay in order to obtain a net yield of at least 7% per annum.

Solution

Carrying out the capital gains test:

$$i^{(2)} = 2(1.07^{0.5} - 1) = 0.0688 \qquad (1-t_1)\frac{D}{R} = 0.75 \times \frac{8}{100} = 0.06$$

Since $i^{(2)} > (1-t_1)\frac{D}{R}$, there is a capital gain on redemption. The worst case scenario is the latest possible redemption date (*ie* time 15 years). This will give the minimum yield. So the maximum price is:

$$P = 0.75 \times 8a_{\overline{15|}}^{(2)} + 100v^{15} \ @7\% = 6 \times 9.2646 + 100 \times 1.07^{-15} = £91.83$$

Suppose, alternatively, that the price of the loan is given. The minimum net annual yield is obtained by again assuming the worst case result for the investor. So if:

(a) $\boldsymbol{P < R}$**, then the investor receives a capital gain when the security is redeemed. The worst case is that the redemption money is repaid at the *latest* possible date. If this does in fact occur, the net annual yield will be that calculated. If redemption takes place at an earlier date, the net annual yield will be greater than that calculated.**

(b) $\boldsymbol{P > R}$**, then the investor receives a capital loss when the security is redeemed. The worst case is that the redemption money is repaid at the *earliest* possible date. The actual yield obtained will be at least the value calculated on this basis.**

(c) $\boldsymbol{P = R}$**, then the investor receives neither a capital gain nor a capital loss. The net annual yield is** $\boldsymbol{i}$**, where** $\boldsymbol{i^{(p)} = (1-t_1)\frac{D}{R}}$**, irrespective of the actual redemption date chosen.**

Question

A fixed-interest stock with a coupon of 8% per annum payable half-yearly in arrears can be redeemed at the option of the borrower at any time between 10 and 15 years from the date of issue. The stock is redeemable at par.

An investor, who is subject to tax at 25% on income only, purchases £100 nominal of this stock for £110. Calculate the minimum net yield this investor could expect to obtain.

Solution

Here there will be a loss on redemption (as the redemption payment of £100 is less than the price paid). So the earliest redemption date is the worst case scenario and will give the minimum yield.

This means we need to determine the yield i that satisfies the equation:

$$110 = 0.75 \times 8a_{\overline{10|}}^{(2)} + 100v^{10}$$

In the previous question, we saw that the right-hand side of this equation is equal to £91.83 when $i = 7\%$. So the value of i to give a price of £110 will be lower than 7%.

Using $i = 5\%$ gives 108.29 and $i = 4.5\%$ gives 112.40. Interpolating gives the minimum net yield:

$$i \approx 4.5\% + \frac{110 - 112.40}{108.29 - 112.40} \times (5\% - 4.5\%) = 4.8\%$$

Note that a capital gains tax liability does not change any of this. For example, an investment which has a capital gain before allowing for capital gains tax must still have a net capital gain after allowing for the capital gains tax liability, so that the 'worst case' for the investor is still the latest redemption.

Question

A fixed-interest stock with a coupon of 8% per annum payable half-yearly in arrears can be redeemed at the option of the borrower at any time between 10 and 15 years from the date of issue. The stock is redeemable at par.

An investor, who is subject to tax at 25% on income and capital gains, purchases £100 nominal of this stock for £110. Calculate the price the investor should pay in order to obtain a net yield of at least 7% per annum.

Solution

Carrying out the capital gains test:

$$i^{(2)} = 2(1.07^{0.5} - 1) = 0.0688 \qquad (1 - t_1)\frac{D}{R} = 0.75 \times \frac{8}{100} = 0.06$$

Since $i^{(2)} > (1 - t_1)\frac{D}{R}$, there is a capital gain on redemption. The worst case scenario is the latest possible redemption date (*ie* time 15 years). This will give the minimum yield. So:

$$P = 0.75 \times 8a_{\overline{15}|}^{(2)} + 100v^{15} - 0.25(100 - P)v^{15} \quad @7\%$$

$$= 6a_{\overline{15}|}^{(2)} + 75v^{15} + 0.25Pv^{15}$$

Rearranging gives:

$$P(1 - 0.25v^{15}) = 6a_{\overline{15}|}^{(2)} + 75v^{15} \quad \Rightarrow \quad P = \frac{6 \times 9.2646 + 75 \times 1.07^{-15}}{1 - 0.25 \times 1.07^{-15}} = £91.02$$

However, in some cases, such as if the redemption price varies, the simple strategy described above will not be adequate, and several values may need to be calculated to determine which is lowest.

For example, we may have a bond with a redemption rate of:

- 120% if it is redeemed between 5 and 10 years from now
- 110% if it is redeemed between 10 and 15 years from now.

2 Uncertain income securities

Securities with uncertain income include:

1. ***Equities*, which have regular declarations of *dividends*. The dividends vary according to the performance of the company issuing the stocks and may be zero.**

2. ***Property* which carries regular payments of rent, which may be subject to regular review.**

3. ***Index-linked bonds* which carry regular coupon payments and a final redemption payment, all of which are increased in proportion to the increase in a relevant index of inflation.**

For all of these investments, investors may be interested in calculating the yield for a given price, or the price or value of the security for a given yield. In order to calculate the value or the yield it is necessary to make assumptions about the future income.

Given the uncertain nature of the future income, one method of modelling the cashflows is to assume statistical distributions for, say, the inflation or dividend growth rate. In this Subject, however, we will make simpler assumptions – for example that dividends increase at a constant rate. It is important to recognise that modelling random variables deterministically, ignoring the variability of the payments and the uncertainty about the expected growth rate, is not adequate for many purposes and stochastic methods will be required.

In all three cases, using this deterministic approach means that we estimate the future cashflows and then solve the equation of value using the estimated cashflows.

Index-linked bonds differ slightly from the other two in that the income is certain in *real* terms. These are therefore covered separately, in Section 4.

2.1 Equities

This section relates to the material covered in Chapter 8 on *perpetuities*.

Given deterministic assumptions about the growth of dividends, we can estimate the future dividends for any given equity, and then solve the equation of value using estimated cashflows for the yield or the price or value.

So, let the value of an equity just after a dividend payment be P, and let D be the amount of this dividend payment. Assume that dividends grow in such a way that the dividend due at time t is estimated to be D_t . We generally value the equity assuming dividends continue in perpetuity, and without explicit allowance for the possibility that the company will default and the dividend payments will cease. In this case, assuming annual dividends:

$$P = \sum_{t=1}^{\infty} D_t v_i^t$$

where i is the return on the share, given price P.

If we assume a constant dividend growth rate of *g*, say, then $D_t = (1+g)^t D$ and:

$$P = D\left(\frac{(1+g)}{(1+i)} + \frac{(1+g)^2}{(1+i)^2} + \frac{(1+g)^3}{(1+i)^3} + \cdots\right)$$

This is a compound increasing annuity. So:

$$\boldsymbol{P = Da_{\overline{\infty}|\,i'} \qquad \text{where} \qquad i' = \frac{1+i}{1+g} - 1}$$

$$\boldsymbol{\Rightarrow \quad P = \frac{D(1+g)}{i-g}}$$

since $a_{\overline{\infty}|\,i'} = \dfrac{1}{i'}$.

At certain times close to the dividend payment date the equity may be offered for sale *excluding* the next dividend. This allows for the fact that there may not be time between the sale date and the dividend payment date for the company to adjust its records to ensure the buyer receives the dividend. An equity which is offered for sale without the next dividend is called ex-dividend or 'xd'. The valuation of ex-dividend stocks requires no new principles.

Question

A French investor, who is taxed at 35% on income, has just purchased 500 shares in a small company ex-dividend. Dividends are paid annually and the next dividend is due in one month's time. The last dividend was €8 per share and dividends are expected to rise by 4% *pa*.

Calculate:

(i) the price paid by the investor if the expected yield is 12% *pa* effective

(ii) the yield the investor would expect to obtain if the price per share is €120.

Solution

(i) ***Price paid***

The last dividend was €8, so the next dividend (due in one month) is €8×1.04, and the one after that (due in one year and one month) is €8×1.04^2.

The investor does not receive the dividend due in one month because the shares are purchased ex-dividend, so the first dividend received by the investor is after one year and one month.

The price of the shares (after taking off the income tax paid) is therefore:

$$P = 500v^{\frac{1}{12}}\left(8 \times 1.04^2 \times 0.65v + 8 \times 1.04^3 \times 0.65v^2 + \cdots\right)$$

The terms in brackets form an infinite geometric progression with first term $8\times1.04^2\times0.65v$ and common ratio $1.04v$.

So the price of the shares is:

$$P=500v^{\frac{1}{12}}\times\frac{8\times1.04^2\times0.65v}{1-1.04v}=€34,822$$

This price can also be calculated using annuities as follows:

$$P=500v^{\frac{1}{12}}\left(8\times1.04^2\times0.65v+8\times1.04^3\times0.65v^2+\cdots\right)$$

$$=500\times8\times1.04\times0.65v^{\frac{1}{12}}\left(1.04v+1.04^2v^2+\cdots\right)$$

$$=500\times5.408v^{\frac{1}{12}}a_{\overline{\infty}|i'}$$

where $i'=\dfrac{1.12}{1.04}-1=\dfrac{0.08}{1.04}$. Therefore:

$$P=500\times5.408\times1.12^{-1/12}\times\frac{1.04}{0.08}=€34,822 \qquad \text{since} \qquad a_{\overline{\infty}|i'}=\frac{1}{i'}=\frac{1.04}{0.08}$$

(ii) ***Yield***

Using the geometric progression approach from above, if the price per share is €120, the yield i can be found from the equation:

$$120=v^{\frac{1}{12}}\times\frac{8\times1.04^2\times0.65v}{1-1.04v}$$

Since a yield of 12% *pa* gives a price of $34,822/500=€69.64$ per share, the yield to give a price of €120 per share will be lower than 12%.

Using $i=8\%$, the right-hand side of the above equation gives €139.71, and using $i=9\%$, the right-hand side gives €111.68. Linearly interpolating, we find the yield to be:

$$i\approx8\%+\frac{120-139.71}{111.68-139.71}\times(9\%-8\%)=8.7\%$$

2.2 Property

The valuation of property by discounting future income follows very similar principles to the valuation of equities. Both require some assumption about the increase in future income; both have income which is related to the rate of inflation (both property rents and company profits will be broadly linked to inflation, over the long term); in both cases we use a deterministic approach.

The major differences between the approach to the property equation of value, compared with the equity equation of value, are:

(1) property rents are generally fixed for a number of years at a time and

(2) some property contracts may be fixed term, so that after a certain period the property income ceases and ownership passes back to the original owner (or another investor) with no further payments.

Let P be the price immediately after receipt of the periodic rental payment. Let m be the frequency of the rental payments each year. We estimate the future cashflows, such that D_t/m is the rental income at time t, $t=\frac{1}{m},\frac{2}{m},\ldots$. If the rents cease after some time n then clearly $D_t=0$ for $t>n$.

Then the equation of value is:

$$P=\sum_{k=1}^{\infty}\frac{1}{m}D_{k/m}v^{k/m}$$

It will usually be much easier to work from first principles than try and apply this formula to every question about property values.

Question

The rent for the next five years of an eighty-year property contract is set at £4,000 per month. Thereafter the rent will increase by 20% compound every five years.

Calculate the price that should be paid for the contract by an investor who wishes to achieve a yield of 7% per annum effective.

Solution

The price to pay is given by:

$$P=48{,}000a_{\overline{5}|}^{(12)}+48{,}000\times1.2\times v^5a_{\overline{5}|}^{(12)}+\cdots+48{,}000\times1.2^{15}\times v^{75}a_{\overline{5}|}^{(12)}$$

Note that the monthly rent is £4,000 and so the annual rent is £48,000.

This formula can be simplified quite easily, as follows:

$$P=48{,}000a_{\overline{5}|}^{(12)}\left(1+1.2v^5+1.2^2v^{10}+\cdots+1.2^{15}v^{75}\right)$$

The terms in brackets form a geometric progression of 16 terms, with first term 1 and common ratio $1.2v^5$. So, using $i=7\%$, the price to pay is:

$$P=48{,}000a_{\overline{5}|}^{(12)}\times\frac{1-(1.2v^5)^{16}}{1-1.2v^5}=48{,}000\times4.2301\times\frac{1-(1.2\times1.07^{-5})^{16}}{1-1.2\times1.07^{-5}}=£1{,}290{,}000$$

Alternatively, we could calculate the price by converting the summation in brackets into an annuity:

$$P = 48{,}000 a_{\overline{5}|}^{(12)} \left(1 + 1.2v^5 + 1.2^2 v^{10} + \cdots + 1.2^{15} v^{75}\right) = 48{,}000 a_{\overline{5}|\,7\%}^{(12)} \ddot{a}_{\overline{16}|\,j\%}$$

where $\dfrac{1}{1+j} = \dfrac{1.2}{1.07^5} \quad \Rightarrow \quad j = 16.879\%$.

Therefore:

$$P = 48{,}000 \times 4.2301 \times \frac{1 - 1.16879^{-16}}{0.16879/1.16879} = £1{,}290{,}000$$

3 Real rates of interest

An investor's basic objective is to maximise the rate of return. So, if an investor has to choose between a number of possible investment opportunities then, other things being equal, the higher the interest rate the better.

However, this is not the whole story. Where price inflation is present, the 'purchasing power' of a specified sum of money tends to be eroded as time passes.

Question

A pensioner has just invested £3,000 in a government savings account that guarantees to provide a rate of return of 7.25% per annum over the next 5 years.

(i) Calculate the accumulated amount in the account at the end of the 5 years.

Toasters currently cost £30 each and are expected to increase in price by 2.5% per annum over the next 5 years.

(ii) Calculate the number of toasters that could be bought now with the initial investment and with the proceeds at the end of the 5 years.

(iii) Comment on your answers to (i) and (ii).

Solution

(i) By the end of the 5 years the account will have accumulated to:

$$3{,}000 \times 1.0725^5 = 3{,}000 \times 1.4190 = £4{,}257$$

(ii) The £3,000 invested now could buy $3{,}000/30 = 100$ toasters.

At the end of the 5 years toasters will cost $£30 \times 1.025^5 = £33.94$ each. So the proceeds will be able to buy $4{,}257/33.94 = 125$ toasters.

(iii) In money terms, the fund has grown by 41.9%, but in terms of the number of toasters it will buy it has only grown by 25%, *ie* there has been an erosion in purchasing power caused by price inflation.

When considering investments, it is often more useful to look at the rate of return earned *after* taking into account the erosion caused by inflation. This is done by looking at the *real rate of interest* (or *real rate of return*).

The idea of a real rate of interest, as distinct from a money rate of interest, was introduced in Chapter 6. Ways of calculating real rates of interest will now be examined.

The real rate of interest of a transaction is the rate of interest after allowing for the effect of inflation on a payment series.

3.1 Inflation-adjusted cashflows

The effect of inflation means that a unit of money at, say, time 0 has different purchasing power than a unit of money at any other time. We find the real rate of interest by first adjusting all payment amounts for inflation, so that they are all expressed in units of purchasing power at the same date.

As a simple example, consider a transaction represented by the following payment line:

Time:	**0**	**1**
Payment:	**–100**	**120**

That is, for an investment of 100 at time 0 an investor receives 120 at time 1.

The effective rate of interest on this transaction is clearly 20% per annum. The real rate of interest is found by first expressing both payments in units of the same purchasing power. Suppose that inflation over this one year period is 5% per annum. This means that 120 at time 1 has a value of $120/1.05 = 114.286$ in terms of time 0 money units.

So, in 'real' terms, that is, after adjusting for the rate of inflation, the transaction is represented as:

Time:	**0**	**1**
Payment:	**–100**	**114.286**

Hence, the real rate of interest is 14.286%.

Question

Calculate the real rate of return on this transaction if inflation had been 5% *pa* for the first nine months of the year but only 3% *pa* for the remaining three months.

Solution

In this case, 120 at time 1 has a value of $\dfrac{120}{1.05^{9/12} \times 1.03^{3/12}} = 114.84$ in terms of time 0 money units.

Therefore the real rate of return is 14.84%.

3.2 Calculating real yields using an inflation index

Where the rates of inflation are known (that is, we are looking back in time at a transaction that is complete) we may adjust payments for the rate of inflation by reference to a relevant inflation index.

For example, assume we have an inflation index, $Q(t_k)$ at time t_k, and a payment series as follows:

Time, *t*:	**0**	**1**	**2**	**3**
Payment:	**-100**	**8**	**8**	**108**
$Q(t)$	**150**	**156**	**166**	**175**

Clearly the rate of interest on this transaction is 8%.

This is because (assuming the time period is 1 year) for an initial investment of 100, we receive interest payments of 8 at the end of each year plus a return of capital at the end of three years. We could check this by showing that 8% solves the equation of value:

$$100 = 8a_{\overline{3}|} + 100v^3$$

Now we can change all these amounts into time 0 money values by dividing the payment at time *t* by the proportional increase in the inflation index from 0 to *t*. For example the inflation-adjusted value of the payment of 8 at time 1 is $8 \div (Q(1)/Q(0))$. The series of payments in time 0 money values is then as follows:

Time, *t*:	**0**	**1**	**2**	**3**
Payment:	**-100**	**7.6923**	**7.2289**	**92.5714**

This gives a yield equation for the real yield:

$$-100 + 7.6923v_{i'} + 7.2289v_{i'}^2 + 92.5714v_{i'}^3 = 0$$

where i' is the real rate of interest, which can be solved using numerical methods to give $i' = 2.63\%$.

In general, the real yield equation for a series of cashflows $\{C_{t_1}, C_{t_2}, \ldots, C_{t_n}\}$, given associated inflation index values $\{Q(0), Q(t_1), Q(t_2), \ldots, Q(t_n)\}$ is, using time 0 money units:

$$\sum_{k=1}^{n} C_{t_k} \frac{Q(0)}{Q(t_k)} v_{i'}^{t_k} = 0 \qquad \Rightarrow \qquad \sum_{k=1}^{n} \frac{C_{t_k}}{Q(t_k)} v_{i'}^{t_k} = 0$$

Until now we have been expressing all payments in terms of time 0 money units. However, this choice was arbitrary, as we could have chosen any date on which to value all the payments.

The second equation here, in which all terms are divided by $Q(0)$, demonstrates that the solution of the yield equation is independent of the date the payment units are adjusted to.

3.3 Calculating real yields given constant inflation assumptions

If we are considering future cashflows, the actual inflation experience will not be known, and some assumption about future inflation will be required. For example, if it is assumed that a constant rate of inflation of j per annum will be experienced, then a cashflow of, say, 100 due at t has value $100(1+j)^{-t}$ in time 0 money values.

So, for a fixed net cashflow series $\{C_{t_k}\}$, $k=1,2,\ldots,n$, assuming a rate of inflation of j per annum, the real, effective rate of interest, i', is the solution of the real yield equation:

$$\sum_{k=1}^{n} C_{t_k} v_j^{t_k} v_{i'}^{t_k} = 0$$

We also know that the effective rate of interest with no inflation adjustment, which may be called the 'money yield' to distinguish from the real yield, is i where:

$$\sum_{k=1}^{n} C_{t_k} v_i^{t_k} = 0$$

So the relationship between the real yield i', the rate of inflation j and the money yield i is $v_i = v_j v_{i'}$.

Therefore:

$$\frac{1}{1+i} = \frac{1}{1+j} \times \frac{1}{1+i'} \quad \Rightarrow \quad 1+i' = \frac{1+i}{1+j} \quad \Rightarrow \quad i' = \frac{i-j}{1+j}$$

This is sometimes approximated to $i' \approx i - j$ if only estimates are required, as this is a reasonable approximation if j is small.

Question

Ten years ago, a saver invested £5,000 in an investment fund operated by an insurance company. Over this period the rate of return earned by the fund has averaged 12% per annum. If prices have increased by 80% in total over this period, calculate the average annual real rate of interest earned by the fund over this period.

Solution

The average annual rate of inflation j over the 10-year period can be found from:

$$(1+j)^{10} = 1.8 \quad \Rightarrow \quad j = 1.8^{1/10} - 1 = 0.0605 \quad ie\ \ 6.05\%$$

The average annual (money) rate of interest is $i = 0.12$.

So the average annual real rate is:

$$i' = \frac{i-j}{1+j} = \frac{0.12-0.0605}{1.0605} = 0.0561 \quad ie\ \ 5.61\%$$

Conversely, if we know the real yield i' which we have obtained from an equation of value using inflation-adjusted cashflows then we can calculate the money yield as follows.

The real growth factor $(1+i')$ can be viewed as the monetary growth factor $(1+i)$, adjusted for the effects of inflation:

$$1+i' = \frac{1+i}{1+j}$$

Rearranging the above equation gives:

$$i = (1+i')(1+j) - 1 \quad \Rightarrow \quad \boldsymbol{i = i' + j\left(1+i'\right)}$$

Question

For the last 10 years an investor has paid £50 at the start of each month into a savings account that has achieved a real rate of interest of 3% per annum over this period.

If inflation has been at a constant rate of 5% per annum for the 10 years, calculate the balance of the investor's account today.

Solution

The real rate of interest over the period, i', has been 3% *pa* and inflation, j, has been 5% *pa*. So, the actual (money) rate of interest, i, has been:

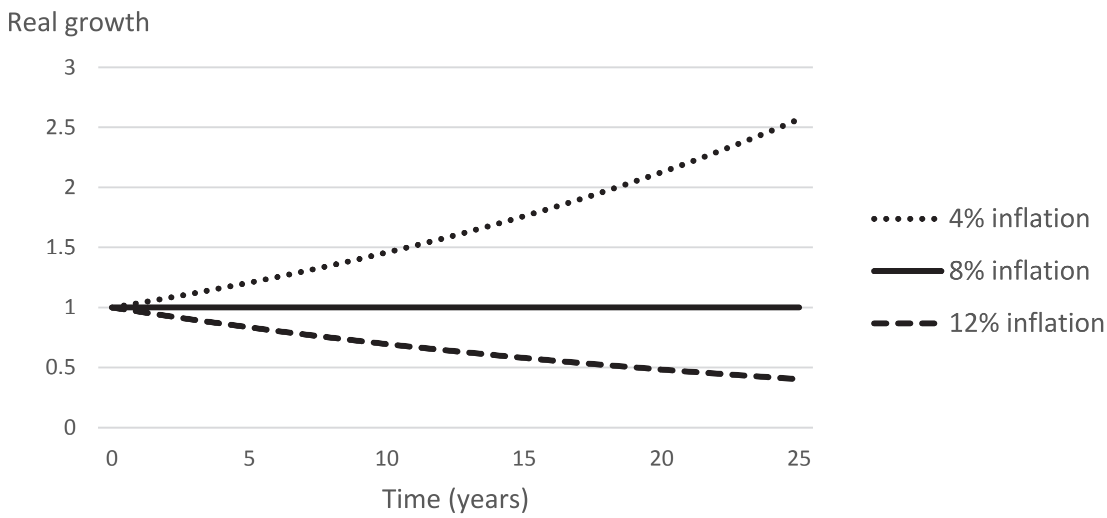
$$i = (1+i')(1+j) - 1 = (1.03)(1.05) - 1 = 0.0815 \quad ie\ \ 8.15\%\ pa$$

and the accumulated value of the investor's account today is:

$$(50 \times 12)\ddot{s}^{(12)}_{\overline{10|}\ @8.15\%} = 600 \times \frac{1.0815^{10} - 1}{12(1 - 1.0815^{-1/12})} = 600 \times 15.2265 = £9{,}136$$

In some cases a combination of known inflation index values and an assumed future inflation rate may be used to find the real rate of interest.

The following graph shows the real growth over time of a single investment when the rate of inflation is (a) higher than, (b) equal to and (c) less than the money rate of interest (taken to be 8% *pa*).



If an investment is achieving a positive real rate of return ($i > j$), then it is outstripping inflation. If it is achieving a negative real rate of return ($i < j$), then it is falling behind inflation (in which case it would be better to spend the money now, rather than 'investing' it).

3.4 Payments related to the rate of inflation

Some contracts specify that the cashflows will be adjusted to allow for future inflation, usually in terms of a given inflation index. The index-linked government security is an example. The actual cashflows will be unknown until the inflation index at the relevant dates are known. The contract cashflows will be specified in terms of some nominal amount to be paid at time *t*, say c_t. If the inflation index at the base date is $Q(0)$ and the relevant value for the time *t* payment is $Q(t)$ then the actual cashflow is:

$$C_t = c_t \frac{Q(t)}{Q(0)}$$

It is easy to show that if the real yield i' is calculated by reference to the same inflation index as is used to inflate the cashflows, then i' is the solution of the real yield equation:

$$\sum_{k=1}^{n} C_{t_k} \frac{Q(0)}{Q(t_k)} v_{i'}^{t_k} = 0 \quad \Rightarrow \quad \sum_{k=1}^{n} c_{t_k} v_{i'}^{t_k} = 0$$

In other words we can solve the yield equation using the nominal amounts.

However, it is *not* always the case that the index used to inflate the cashflows is the same as that used to calculate the real yield. For example the index-linked UK government security has coupons inflated by reference to the inflation index value three months before the payment is made. The real yield, however, is calculated using the inflation index at the actual payment dates.

The lag of 3 months in the UK means that the amount of the next coupon can be determined before it is received. Consider coupons arriving in January and July. The July coupon will use the index from April and the January coupon will use the index from the preceding October.

This makes things a little more complicated and is illustrated in the following question.

Question

A three-year index-linked security is issued at time 0. The security pays nominal coupons of 4% annually in arrears and is redeemed at par. The coupons and capital payment are increased by reference to the inflation index value 3 months before the payment is made. The inflation index value 3 months before issue was 110. The table below shows the index values at other times.

Time	0	$\frac{9}{12}$	1	$1\frac{9}{12}$	2	$2\frac{9}{12}$	3
Index	112	115	116	118	119	122	125

Calculate the real yield if the price of the security is £100.

Solution

First, we calculate the monetary amount of each payment using the inflation index values 3 months before the payment date. We then express these amounts in terms of time 0 money units. The results are shown in the following table.

Time	1	2	3
Nominal payment (*ie* cash actually received)	$4\times\frac{115}{110}=4.1818$	$4\times\frac{118}{110}=4.2909$	$104\times\frac{122}{110}=115.35$
Payment in time 0 units	$4.1818\times\frac{112}{116}=4.0376$	$4.2909\times\frac{112}{119}=4.0385$	$115.35\times\frac{112}{125}=103.35$

The amounts in terms of time 0 money units are calculated using the inflation index values on the actual payment dates, with no 3 month lag.

The real yield is then found by solving for i the equation of value:

$$100=4.0376v+4.0385v^2+103.35v^3$$

We can solve this using trial and error. When $i=3\%$, the right-hand side gives 102.31, and when $i=4\%$, the right-hand side gives 99.49. Linearly interpolating, the real yield is:

$$i\approx 3\%+\frac{100-102.31}{99.49-102.31}\times(4\%-3\%)=3.8\%\ pa$$

We will look further at index-linked bonds in Section 4.

3.5 The effects of inflation

Consider the simplest situation, in which an investor can lend and borrow money at the same rate of interest i_1. In certain economic conditions the investor may assume that some or all elements of the future cashflows should incorporate allowances for inflation (*ie* increases in prices and wages). The extent to which the various items in the cashflow are subject to inflation may differ. For example, wages may increase more rapidly than the prices of certain goods, or vice versa, and some items (such as the income from rent-controlled property) may not rise at all, even in highly inflationary conditions.

The case when *all* items of cashflow are subject to the same rate of escalation j per time unit is of special interest. In this case we find or estimate c_t^j and $\rho^j(t)$, the net cashflow and the net rate of cashflow allowing for escalation at rate j per unit time, by the formulae:

$$c_t^j = (1+j)^t c_t$$

$$\rho^j(t) = (1+j)^t \rho(t)$$

where c_t and $\rho(t)$ are estimates of the net cashflow and the net rate of cashflow respectively at time t without any allowance for inflation. It follows that, with allowance for inflation at rate j per unit time, the net present value of an investment or business project at rate of interest i is:

$$NPV_j(i) = \sum c_t(1+j)^t(1+i)^{-t} + \int_0^\infty \rho(t)(1+j)^t(1+i)^{-t}\,dt$$

$$= \sum c_t(1+i_0)^{-t} + \int_0^\infty \rho(t)(1+i_0)^{-t}\,dt \qquad (3.1)$$

where $1+i_0 = \dfrac{1+i}{1+j}$, **or** $i_0 = \dfrac{i-j}{1+j}$. **(3.2)**

Here j is the rate of inflation, i is the money yield and i_0 is the real yield. Previously, i' was used for the real yield.

If j is not too large, one sometimes uses the approximation $i_0 \approx i - j$.

Question

A developer buys a property for £90,000, and immediately spends £10,000 to buy materials in order to develop it. The developer expects to make the improvements over the next year and then sell the property. An identical property, with the improvements already made, is worth £110,000 today.

Assuming that the rate of inflation is 5% *pa* for the next year, calculate:

(i) the annual money yield on the investment

(ii) the annual real yield on the investment.

Solution

(i) The developer spends $90,000+10,000=100,000$ at outset. In one year's time, the developer expects the property to be worth $1.05\times 110,000$. We can find the money yield on the investment, i, from:

$$100,000=\frac{1.05\times 110,000}{1+i} \quad \Rightarrow \quad i=15.5\%$$

(ii) In today's money, the developed property is worth 110,000, so we can find the real yield on the investment, i_0, from:

$$100,000=\frac{110,000}{1+i_0} \quad \Rightarrow \quad i_0=10\%$$

These results are of considerable practical importance, because projects which are apparently unprofitable when rates of interest are high may become highly profitable when even a modest allowance is made for inflation.

It is, however, true that in many ventures the positive cashflow generated in the early years of the venture is insufficient to pay bank interest, so recourse must be had to further borrowing (unless the investor has adequate funds of their own). This does not undermine the profitability of the project, but the investor would require the agreement of his lending institution before further loans could be obtained and this might cause difficulties in practice.

Question

An entrepreneur is considering a business project that will be financed by a flexible loan that can be increased or repaid at any time.

The only outlay required in the project is an initial cost of £80,000. The income from the project will be received annually in arrears. At the end of the first year, the income is expected to be £8,800 and this is expected to increase each year thereafter by 10% *pa* compound.

The entrepreneur may borrow and invest money at 12% *pa* interest. If the project is expected to last for twelve years, calculate:

(i) the largest loan amount outstanding over the term of the project

(ii) the net present value of the project at 12% *pa* effective.

Solution

(i) ***Largest loan amount***

The interest due at the end of the first year is $0.12\times 80,000=£9,600$. The income is only £8,800 and so the loan increases to:

$$80,000+9,600-8,800=£80,800$$

The interest due at the end of the second year is $0.12 \times 80,800 = £9,696$. The income is only $1.1 \times 8,800 = £9,680$ and so the loan increases to:

$$80,800 + 9,696 - 9,680 = £80,816$$

The interest due at the end of the third year is $0.12 \times 80,816 = £9,697.92$, whereas the income is now $1.1^2 \times 8,800 = £10,648$. As the income is more than enough to pay the interest, the loan will decrease.

So the maximum loan amount outstanding is £80,816.

(ii) ***Net present value***

The net present value equals:

$$NPV = -80,000 + 8,800v + 8,800 \times 1.1v^2 + \cdots + 8,800 \times 1.1^{11} v^{12}$$

$$= -80,000 + \frac{8,800v(1 - 1.1^{12} v^{12})}{1 - 1.1v} = £5,555$$

using the formula for the sum of 12 terms of a geometric progression with first term $8,800v$ and common ratio $1.1v$.

4 Index-linked bonds

Index-linked bond cashflows are described in Chapter 3. The coupon and redemption payments are increased according to an index of inflation.

We have also already looked at an example involving index-linked securities in the previous section of this chapter.

Given simple assumptions about the rate of future inflation, it is possible to estimate the future payments. Given these assumptions we may calculate the price or yield by solving the equation of value using the estimated cashflows.

For example, let the nominal annual coupon rate for an *n*-year index-linked bond be *D* per £1 nominal face value with coupons payable half-yearly, and let the nominal redemption price be *R* per £1 nominal face value. We assume that payments are inflated by reference to an index with base value $Q(0)$, such that the coupon due at time t years is:

$$\frac{D}{2}\frac{Q(t)}{Q(0)}$$

Then the equation of value, given an effective (money) yield of *i* per annum, and a present value or price *P* per £1 nominal at issue or immediately following a coupon payment, is:

$$P=\sum_{k=1}^{2n}\frac{D}{2}\frac{Q(k/2)}{Q(0)}v_i^{k/2}+R\frac{Q(n)}{Q(0)}v_i^n$$

We estimate the unknown value of $Q(t)$ using some assumption about future inflation and using the latest known value – which may be $Q(0)$. For example, assume inflation increases at rate j_t per annum in the year $t-1$ to t, then we have:

$$Q(½)=Q(0)(1+j_1)^{½}$$

$$Q(1)=Q(0)(1+j_1)$$

$$Q(1½)=Q(0)(1+j_1)(1+j_2)^{½}$$

$$Q(2)=Q(0)(1+j_1)(1+j_2)$$

etc

Question

An index-linked bond was issued on 1 April 2016. It pays half-yearly coupons and is redeemable on 1 April 2034. The nominal redemption rate is 100%. There is no time lag on the indexation. The coupon paid on 1 April 2018 was £2.10. A non-taxpayer buys £100 nominal of the bond on 1 April 2018, just after the payment of the coupon.

Assuming that past inflation has been 4% *pa* and future inflation is 5.25% *pa*, calculate the price this investor should pay in order to obtain a money rate of return of 10% *pa*.

Solution

To obtain the money rate of return we need to calculate the payments actually received by the investor.

The coupon payments are calculated allowing for inflation from the date of issue of the bond to the date of payment. Since the coupon payment of £2.10 on 1 April 2018 will include an allowance for past inflation, we can calculate the future coupons (*ie* those received by the investor) by increasing £2.10 at the rate of 5.25% *pa*.

The redemption payment is calculated allowing for inflation from the date of issue of the bond to the date of redemption – a period of 18 years. This will include 2 years of past inflation at 4% *pa* and 16 years of future inflation at 5.25% *pa*. Since the nominal redemption rate is 100%, the assumed redemption payment is $100 \times 1.04^2 \times 1.0525^{16}$.

The price the investor should pay is the present value of the payments:

$$
\begin{aligned}
P &= 2.10 \times 1.0525^{0.5} v^{0.5} + 2.10 \times 1.0525 v + 2.10 \times 1.0525^{1.5} v^{1.5} + \cdots \\
&\qquad + 2.10 \times 1.0525^{16} v^{16} + 100 \times 1.04^2 \times 1.0525^{16} v^{16} \\
&= 2.10 \times 1.0525^{0.5} v^{0.5} \frac{\left(1 - (1.0525^{0.5} v^{0.5})^{32}\right)}{1 - 1.0525^{0.5} v^{0.5}} + 100 \times 1.04^2 \times 1.0525^{16} v^{16} \\
&= 47.6642 + 53.3749 \\
&= £101.04
\end{aligned}
$$

Here we have calculated the present value of the coupon payments using the formula for the sum of a geometric progression consisting of 32 terms, with first term $2.10 \times 1.0525^{0.5} v^{0.5}$ and common ratio $1.0525^{0.5} v^{0.5}$.

As we have already mentioned, **it is important to bear in mind that the index used may not be the same as the actual inflation index value at time t that one would use, for example, to calculate the real (inflation-adjusted) yield. In the case of UK index-linked bonds, the payments are increased using the index values from three months before the payment date. Real yields would be calculated using the inflation index values *at* the payment date.**

Like equities, index-linked bonds (and fixed-interest bonds) may be offered for sale 'ex-dividend'. No new principles are involved in the valuation of ex-dividend index-linked bonds.

If a bond is offered for sale ex-dividend, then the purchaser will not receive the next coupon.

Question

An inflation index takes the following values:

1/1/16:	121.2	1/1/17:	123.9	1/1/18:	125.2
1/4/16:	122.8	1/4/17:	124.2	1/4/18:	126.0
1/7/16:	123.1	1/7/17:	124.4		
1/10/16:	123.6	1/10/17:	124.9		

A two-year index-linked bond is purchased at issue on 1/4/16 by a tax-exempt investor for a price of £101 per £100 nominal. The nominal redemption rates is 100%. All coupon and redemption payments are linked to the inflation index three months prior to the payment date. The coupons on the bond are of nominal amount 4% *pa* payable half-yearly in arrears and are payable on 1 April and 1 October every year.

Calculate the real yield obtained by the investor.

Solution

The nominal coupon rate is 4% *pa*, so the nominal coupon payment is £4 *pa* payable half-yearly in arrears. This means that, before indexation, the amount of each coupon payment is £2 per £100 nominal. Since the nominal redemption rate is 100%, the redemption payment before indexation is £100 per £100 nominal.

We first use the values of the inflation index to calculate the actual coupon and redemption payments per £100 nominal of the bond:

Coupon on 1/10/16:

$$2\times\frac{index_{1/7/16}}{index_{1/1/16}}=2\times\frac{123.1}{121.2}=2.031353$$

Coupon on 1/4/17:

$$2\times\frac{index_{1/1/17}}{index_{1/1/16}}=2\times\frac{123.9}{121.2}=2.044554$$

Coupon on 1/10/17:

$$2\times\frac{index_{1/7/17}}{index_{1/1/16}}=2\times\frac{124.4}{121.2}=2.052805$$

Coupon on 1/4/18:

$$2\times\frac{index_{1/1/18}}{index_{1/1/16}}=2\times\frac{125.2}{121.2}=2.066007$$

Redemption on 1/4/18:

$$100\times\frac{index_{1/1/18}}{index_{1/1/16}}=100\times\frac{125.2}{121.2}=103.30033$$

So the total payment made on 1/4/18 is:

$$2.066007+103.30033=105.366337$$

We can now set up the equation of value for the real yield using the actual coupon and redemption payments, and taking into account inflation. This time there is no lag in the inflation index:

$$101=2.031353\times\frac{122.8}{123.6}\times v^{0.5}+2.044554\times\frac{122.8}{124.2}\times v+2.052805\times\frac{122.8}{124.9}\times v^{1.5}$$

$$+105.366337\times\frac{122.8}{126.0}\times v^{2}$$

$$=2.018205v^{0.5}+2.021508v+2.018291v^{1.5}+102.690366v^{2}$$

The values multiplying the discount factors in this equation represent the real cashflows from the investor's point of view, ie the cashflows expressed in 1/4/16 prices.

As a first guess for the real yield, we can try the nominal coupon rate of 4%.

At 4%, the right-hand side is 100.7688, and at 3%, the right-hand side is 102.6775. Linearly interpolating, we find:

$$i\approx 3\%+\frac{101-102.6775}{100.7688-102.6775}(4\%-3\%)=3.9\%$$

So the real yield is approximately 3.9% *pa.*

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 13 Summary

The price of a fixed-interest security can be calculated as the present value of the coupon and redemption payments. When the proceeds are subject to income tax or capital gains tax, the net payments must be used. The formulae used for calculating the price are:

Ignoring tax: $P = Da_{\overline{n}|}^{(p)} + Rv^n$

Allowing for income tax: $P' = D(1-t_1)a_{\overline{n}|}^{(p)} + Rv^n$

Allowing for income and capital gains tax: $P'' = D(1-t_1)a_{\overline{n}|}^{(p)} + Rv^n - t_2(R-P'')v^n$

Capital gains tax is a tax levied on the difference between the sale or redemption price of a stock (or other asset) and the purchase price, if lower.

The capital gains test can be used to establish whether there is a capital gain or not:

- $i^{(p)} > \frac{D}{R}(1-t_1) \Rightarrow$ capital gain
- $i^{(p)} < \frac{D}{R}(1-t_1) \Rightarrow$ capital loss
- $i^{(p)} = \frac{D}{R}(1-t_1) \Rightarrow$ neither a capital gain nor a capital loss, so the price is equal to the redemption payment.

Based on a given purchase price for a fixed-interest security, the running yield and the redemption yield can be calculated, on either a gross or a net basis.

The running yield is the coupon divided by the price. The redemption yield is the rate of interest that equates the price with the present value of the coupon and redemption payments.

For a security with a redemption date that is selected at the option of the borrower, an investor who wishes to achieve a yield of at least i should value the security on the assumption that will give the lowest yield, *ie* the worst case. The worst case is the latest possible redemption date for a gain and the earliest possible redemption date for a loss.

Equities are usually valued by assuming that dividends increase at a constant rate and continue in perpetuity. For an equity that pays annual dividends that are expected to increase at a constant annual rate of g, the price P immediately after a dividend of D has just been paid is given by:

$$P = \sum_{k=1}^{\infty} D(1+g)^k v^k = \frac{D(1+g)}{i-g}$$

Property is valued in a similar way to equities although rents are generally fixed for a number of years at a time and some contracts are for a fixed term. The equation of value for property is:

$$P = \sum_{k=1}^{\infty} \tfrac{1}{m} D_{k/m} \times v^{k/m} \quad \text{where } \tfrac{1}{m} D_{k/m} \text{ is the rental income at time } k/m$$

The real rate of interest (i') on a transaction is the rate of interest after allowing for the effect of inflation (j) on a payment series:

$$i' = \frac{i-j}{1+j} \approx i - j$$

where i is the money rate of interest.

Index-linked bonds can be valued by allowing for the increases in the coupons and the redemption payment. The equation of value for an index-linked bond, with a nominal annual coupon of D payable half-yearly in arrears, and a nominal redemption payment of R , is:

$$P = \sum_{k=1}^{2n} \frac{D}{2} \frac{Q(k/2)}{Q(0)} v_i^{k/2} + R \frac{Q(n)}{Q(0)} v_i^n \quad \text{where } Q(t) \text{ is the index value at time } t$$

Chapter 13 Practice Questions

13.1 In a particular country, which uses dollars as its national currency, price inflation has been running at 20% *pa* for the last 20 years. Calculate the average annual real rate of return for each of the following investments:

(i) A set of gold coins purchased for \$14,000 on 1 January 2015 and sold for \$20,000 on 31 December 2017.

(ii) A painting purchased for \$3,000 on 1 March 2017 and sold for \$3,200 on 1 September 2017.

(iii) A diamond purchased for \$13,000 on 1 July 2016 and sold for \$10,000 on 1 July 2018.

(iv) A statuette purchased for \$7,500 on 1 November 2010 and sold for \$19,000 on 31 December 2017.

13.2 On 1 January 2015 an investor purchased £10,000 nominal of a stock that pays coupons half-yearly on 30 June and 31 December each year at the rate of 6% *pa* and is redeemable at par on 31 December 2027. The investor is liable for income tax at the rate of 40%, but is not liable for capital gains tax. Calculate the price paid by the investor in order to achieve a net redemption yield of 5% *pa* effective.

13.3 A fixed-interest security with a coupon rate of 6% *pa* payable half-yearly in arrears is purchased by an investor who is subject to capital gains tax at a rate of 30%. The fixed-interest security is redeemable at par after 15 years. The price paid is such that the investment provides a gross redemption yield of 10% *pa* effective. Calculate the amount of capital gains tax payable by the investor per £100 nominal.

13.4 A stock with a term of 9½ years has a coupon rate of 5% *pa* payable half-yearly in arrears and is redeemable at 105%. An investor who is not subject to tax purchases £100 nominal of the stock for £85. Calculate the yield obtained by the investor.

13.5 An investor purchases a bond 6 months after issue. The bond will be redeemed at 105% eight years after issue and pays coupons of 4% per annum annually in arrears. The investor pays tax of 25% on income and 15% on capital gains.

Exam style

(i) Calculate the purchase price of the bond per £100 nominal to provide the investor with a rate of return of 5% per annum effective. [6]

The real rate of return expected by the investor from the bond is 2% per annum effective.

(ii) Calculate the annual rate of inflation expected by the investor. [2]
[Total 8]

13.6 Exam style

An investor purchased a government bond on 1 January in a particular year. The bond pays coupons of 6% *pa* six monthly in arrears on 30 June and 31 December. The bond is due to be redeemed at 105% 11 years after the purchase date.

The investor pays income tax at the rate of 23% on 1 April on any coupon payments received in the previous year (1 April to 31 March), and also pays capital gains tax on that date at the rate of 40% on any capital gains realised in the previous year.

(i) Calculate the price paid for £100 nominal of the bond, given that the investor achieves a net yield of 5% *pa* effective interest. [7]

(ii) Without doing any further calculations, explain how and why your answer to (i) would alter if tax were collected on 1 October instead of 1 April each year. [2]
[Total 9]

13.7 Exam style

An investor purchases £100 nominal of a fixed-interest stock, which pays coupons of 7% *pa* half-yearly in arrears. The stock is redeemable at par and can be redeemed at the option of the borrower at any time between 5 and 10 years from the date of issue. The investor is subject to tax at the rate of 40% on income and 25% on capital gains.

(i) Calculate the maximum price that the investor should pay in order to obtain a net yield of at least 6% *pa*. [5]

(ii) Given that this was the price paid by the investor, calculate his net annual running yield, convertible half-yearly. [1]
[Total 6]

13.8 An equity pays half-yearly dividends. A dividend of d per share is due in exactly 3 months' time. Subsequent dividends are expected to grow at a compound rate of g per half-year forever.

(i) If i denotes the annual effective rate of return on the equity, show that P, the price per share, is given by:

$$P = \frac{d(1+i)^{1/4}}{(1+i)^{1/2} - (1+g)}$$

(ii) The current price of the share is £3.60, dividend growth is expected to be 2% per half-year and the next dividend payment in 3 months is expected to be 12p.

Calculate the expected annual effective rate of return for an investor who purchases the share.

13.9 Exam style

An ordinary share pays dividends on each 31 December. A dividend of 35p per share was paid on 31 December 2017. The dividend growth is expected to be 3% in 2018, and a further 5% in 2019. Thereafter, dividends are expected to grow at 6% per annum compound in perpetuity.

(i) Calculate the present value of the dividend stream described above at a rate of interest of 8% per annum effective for an investor holding 100 shares on 1 January 2018. [4]

An investor buys 100 shares for £17.20 each on 1 January 2018. He expects to sell the shares for £18 on 1 January 2021.

(ii) Calculate the investor's expected real rate of return.

You should assume that dividends grow as expected and use the following values of the inflation index:

Year:	2018	2019	2020	2021
Inflation index at start of year:	110.0	112.3	113.2	113.8

[5]
[Total 9]

13.10 An index-linked zero-coupon bond was issued on 1 January 2013 for redemption at par on 31 December 2017. The redemption payment was linked to a price inflation index with a 6-month time lag. The value of the price index on different dates is given below:

Date	Index	Date	Index
01.01.12	144	01.01.16	181
01.07.12	148	01.07.16	182
01.01.13	155	01.01.17	188
01.07.13	160	01.07.17	193
01.01.14	162	01.01.18	201
01.07.14	168		
01.01.15	175		
01.07.15	177		

Calculate the annual effective money and real rates of return obtained by an investor who purchased £10,000 nominal of the stock on 1 January 2015 for £10,250 and held it until redemption.

13.11 Exam style

On 25 October 2013 a certain government issued a 5-year index-linked stock. The stock had a nominal coupon rate of 3% per annum payable half-yearly in arrears and a nominal redemption price of 100%. The actual coupon and redemption payments were index-linked by reference to a retail price index as at the month of payment.

An investor, who was not subject to tax, bought £10,000 nominal of the stock on 26 October 2017. The investor held the stock until redemption.

You are given the following values of the retail price index:

	2013	-----	2017	2018
April	-----	-----	-----	171.4
October	149.2	-----	169.4	173.8

(i) Calculate the coupon payment that the investor received on 25 April 2018 and the coupon and redemption payments that the investor received on 25 October 2018. [3]

(ii) Calculate the purchase price that the investor paid on 25 October 2017 if the investor achieved an effective real yield of 3.5% per annum effective on the investment. [4]

[Total 7]

13.12 Exam style

On 15 May 2017 the government of a country issued an index-linked bond of term 10 years. Coupons are payable half-yearly in arrears, and the annual nominal coupon rate is 8%. The nominal redemption price is 102%.

Coupon and redemption payments are indexed by reference to the value of a retail price index with a time lag of 6 months. The retail price index value in November 2016 was 185 and in May 2017 was 190.

The issue price of the bond was such that, if the retail price index were to increase continuously at a rate of 2% *pa* from May 2017, a tax-exempt purchaser of the bond at the issue date would obtain a real yield of 3% *pa* convertible half-yearly.

Show that the issue price of the bond is £146.85 per £100 nominal. [9]

Chapter 13 Solutions

13.1 We can first calculate the average annual money rates of return, which are:

(i) for the set of gold coins: $\left(\dfrac{20,000}{14,000}\right)^{1/3} - 1 = 12.62\%$

(ii) for the painting: $\left(\dfrac{3,200}{3,000}\right)^{2} - 1 = 13.78\%$

(iii) for the diamond: $\left(\dfrac{10,000}{13,000}\right)^{1/2} - 1 = -12.29\%$

(iv) for the statuette: $\left(\dfrac{19,000}{7,500}\right)^{1/(7\frac{2}{12})} - 1 = 13.85\%$

The real rates of return, i', can then be calculated using:

$$1 + i' = \frac{1+i}{1.2} \quad \Rightarrow \quad i' = \frac{i - 0.2}{1.2}$$

where i is the money rate of return. So the answers are:

(i) –6.15%

(ii) –5.19%

(iii) –26.91%

(iv) –5.13%

13.2 The term of the bond is 13 years (from 1 January 2015 to 31 December 2027). Letting P denote the price per £100 nominal, we have:

$$P = 6(1-0.4)a_{\overline{13}|}^{(2)} + 100v^{13} \quad @5\%$$

$$= 3.6 \times 9.5096 + 100 \times 1.05^{-13} = £87.2666$$

So the price paid for £10,000 nominal is £8,726.66.

13.3 The *gross* redemption yield tells us the rate of return the investor earns *before* any tax is paid. So the price paid per £100 nominal is:

$$P = 6a_{\overline{15}|}^{(2)} + 100v^{15} \, @10\%$$

$$= 6 \times 7.7917 + 100 \times 1.10^{-15} = £70.69$$

Since the stock is redeemed at par, the capital gains tax payable is $0.3 \times (100 - 70.69) = £8.79$.

13.4 The yield is the interest rate that satisfies the equation:

$$85 = 5a^{(2)}_{\overline{9.5}|} + 105v^{9.5}$$

We can find the yield using trial and error.

For a rough guess at the yield, we can use the total payment at a 'typical' time (taken to be close to the end of the term, as the redemption payment is by far the largest cashflow):

$$85 = (9.5 \times 5 + 105)v^8 \quad \Rightarrow \quad i \approx 7.6\%$$

At 8%: $5a^{(2)}_{\overline{9.5}|} + 105v^{9.5} = 5 \times 6.6101 + 105 \times 1.08^{-9.5} = 83.5938$

At 7%: $5a^{(2)}_{\overline{9.5}|} + 105v^{9.5} = 5 \times 6.8902 + 105 \times 1.07^{-9.5} = 89.6645$

Linear interpolation gives the yield:

$$i \approx 7\% + \frac{85 - 89.6645}{83.5938 - 89.6645}(8\% - 7\%) = 7.8\%$$

13.5 (i) ***Purchase price of bond***

The capital gains test tells us that there is a capital gain if:

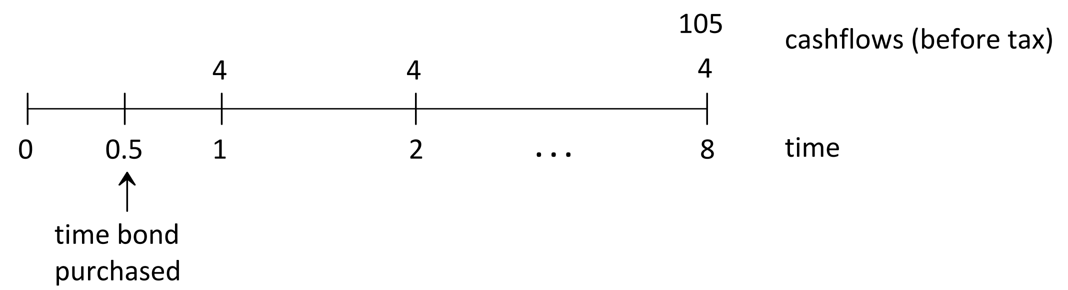
$$i^{(p)} > (1 - t_1)\frac{D}{R}$$

In this case:

$$i = 0.05 > (1 - 0.25)\frac{4}{105} = 0.0286$$

so there is a capital gain. [1]

A timeline for the cashflows is:



The present value of the net coupon payments at time 0 is:

$$0.75 \times 4a_{\overline{8}|}$$

So the present value of the net coupon payments at time $t = 0.5$ (the time of purchase) is:

$$1.05^{0.5} \times 0.75 \times 4a_{\overline{8}|} \qquad [1]$$

Letting P denote the price for £100 nominal of the bond:

$$P = 1.05^{0.5} \times 0.75 \times 4a_{\overline{8}|} + 105v^{7.5} - 0.15(105 - P)v^{7.5} \qquad [2]$$

$$= 1.05^{0.5} \times 3a_{\overline{8}|} + 89.25v^{7.5} + 0.15v^{7.5}P$$

Rearranging gives:

$$P(1 - 0.15v^{7.5}) = 1.05^{0.5} \times 3a_{\overline{8}|} + 89.25v^{7.5}$$

$$\Rightarrow \quad P = \frac{1.05^{0.5} \times 3 \times 6.4632 + 89.25 \times 1.05^{-7.5}}{1 - 0.15 \times 1.05^{-7.5}} = £91.26 \qquad [2]$$

[Total 6]

(ii) ***Annual rate of inflation***

To calculate the real rate of return, i', given a money rate of return, i, and constant inflation, j, we use:

$$1 + i' = \frac{1+i}{1+j} \qquad [1]$$

Therefore:

$$1 + j = \frac{1+i}{1+i'} \quad \Rightarrow \quad j = \frac{1.05}{1.02} - 1 = 2.94\% \ pa \qquad [1]$$

[Total 2]

13.6 (i) ***Price paid***

We will first check if there is a capital gain:

$$i^{(2)} @ 5\% = 2\left(1.05^{0.5} - 1\right) = 4.94\%$$

and: $$(1 - t_1)\frac{D}{R} = (1 - 0.23)\frac{6}{105} = 4.4\%$$

Since $i^{(2)} > (1 - t_1)\frac{D}{R}$, there is a capital gain. [1]

The capital gains test is not exact in this case as the income tax is not paid at the time each coupon is received.

To set up the equation of value for the bond, we need to consider the present value of all the relevant cashflows, including the tax payments.

The first payment of income tax is due on 1 April in the second year of ownership. The tax is payable on that date for both coupons in the previous 12 months. The amount of income tax payable on that date is $0.23 \times 6 = 1.38$ per £100 nominal of the bond owned. Income tax of 1.38 will also be paid on 1 April in each of the following 10 years.

The present value of the income tax payments on the date of purchase is $1.38 a_{\overline{11}|} v^{0.25}$. [1]

The capital gains tax is payable on 1 April after the bond is redeemed, *ie* 11.25 years after the purchase date. The present value of the capital gains tax payable is:

$$0.4(105 - P)v^{11.25} \qquad [1]$$

Let the price paid for £100 nominal be *P*. The equation of value is then:

$$P = 6a_{\overline{11}|}^{(2)} - 1.38a_{\overline{11}|}v^{0.25} + 105v^{11} - 0.4(105 - P)v^{11.25} \qquad [2]$$

Using an interest rate of 5% *pa* effective:

$$P = 6 \times 8.40898 - 1.38 \times 8.30641 \times 1.05^{-0.25} + 105 \times 1.05^{-11} - 0.4(105 - P) \times 1.05^{-11.25}$$

$$\Rightarrow \quad P(1 - 0.4 \times 1.05^{-11.25}) = 76.2625$$

$$\Rightarrow \quad P = £99.18$$

So the price for £100 nominal of the bond is £99.18. [2]

[Total 7]

(ii) ***Change in date of payment of tax***

The tax payment is being deferred from April to October, so the present value of the tax will reduce. [1]

As tax is outgo from the investor's point of view, reducing its present value means that the price the investor pays to achieve a given yield will increase. [1]

[Total 2]

13.7 (i) ***Price paid***

We first check whether there is a capital gain on redemption. To do this, we compare:

$$(1 - t_1)\frac{D}{R} = 0.6 \times \frac{7}{100} = 0.042$$

with:

$$i^{(2)} @ 6\% = 0.059126$$

As $i^{(2)} > (1 - t_1)\frac{D}{R}$, there is a capital gain. [1]

We need to look at the worst case scenario to ensure that the investor obtains a minimum yield of 6%. The worst case for an investor receiving a capital gain is to have the latest possible redemption date (*ie* the stock will be redeemed after 10 years). [1]

Let P denote the maximum price that the investor should pay in order to achieve a net yield of at least 6% *pa.* The equation of value is:

$$P = 0.6 \times 7a_{\overline{10}|}^{(2)} + 100v^{10} - 0.25(100 - P)v^{10} \quad [2]$$

$$\Rightarrow \quad P\left(1 - \frac{0.25}{1.06^{10}}\right) = 4.2 \times 7.46888 + \frac{75}{1.06^{10}}$$

$$\Rightarrow \quad P = £85.13 \quad [1]$$

[Total 5]

(ii) ***Net running yield***

The net running yield is defined to be the net coupon per £100 nominal divided by the price per £100 nominal. So, the investor's net annual running yield is:

$$\frac{7 \times 0.6}{85.13} = 4.93\% \quad [1]$$

Since the coupons are paid half-yearly, this running yield is convertible half-yearly.

[Total 1]

13.8 (i) ***Formula for share price***

The price of a share is equal to the present value of the dividends it pays:

$$P = dv^{1/4} + d(1+g)v^{3/4} + d(1+g)^2 v^{1\frac{1}{4}} + \cdots$$

$$= dv^{1/4}\left[1 + (1+g)v^{1/2} + (1+g)^2 v + \cdots\right]$$

The expression in square brackets is an infinite geometric progression with first term 1, and common ratio $(1+g)v^{1/2}$:

$$P = dv^{1/4} \times \frac{1}{1-(1+g)v^{1/2}} = \frac{dv^{1/4}}{1-(1+g)v^{1/2}}$$

Multiplying both the numerator and the denominator of the fraction by a factor of $(1+i)^{1/2}$:

$$P = \frac{d(1+i)^{1/4}}{(1+i)^{1/2} - (1+g)}$$

(ii) ***Annual effective rate of return***

Substituting the values into the formula from part (i) gives:

$$3.60 = \frac{0.12(1+i)^{1/4}}{(1+i)^{1/2} - 1.02}$$

$$\Rightarrow \quad 3.60(1+i)^{1/2} - 0.12(1+i)^{1/4} - 3.672 = 0$$

Solving this as a quadratic in $(1+i)^{1/4}$ gives:

$$(1+i)^{1/4} = \frac{0.12 \pm \sqrt{0.12^2 - 4 \times 3.6 \times -3.672}}{2 \times 3.6} = 1.02675$$

Hence the rate of return is:

$$i = 1.02675^4 - 1 = 11.1\% \ pa$$

13.9 *This question is Subject CT1, April 2012, Question 9 (with dates updated).*

(i) ***Present value of dividend stream***

The PV of the future dividends from 100 shares on 1 January 2018 is:

$$PV = 35(1.03)v + 35(1.03)(1.05)v^2 + 35(1.03)(1.05)(1.06)v^3 + 35(1.03)(1.05)(1.06)^2 v^4 + \cdots$$

$$= 35(1.03)v + 35(1.03)(1.05)v^2 + 35(1.03)(1.05)(1.06)v^3(1 + 1.06v + 1.06^2 v^2 + \cdots) \qquad [2]$$

Using the formula for an infinite geometric progression with first term 1 and common ratio $1.06v$ to evaluate the infinite summation gives:

$$PV = 35(1.03)v\left[1 + 1.05v + (1.05)(1.06)v^2 \times \frac{1}{1-1.06v}\right]$$

$$= 35 \times \frac{1.03}{1.08}\left[1 + \frac{1.05}{1.08} + \frac{1.05 \times 1.06}{1.08^2} \times \frac{1}{1 - \frac{1.06}{1.08}}\right]$$

$$= £1,785.81 \qquad [2]$$

[Total 4]

(ii) ***Expected real rate of return***

The following table shows the real and money cashflows per 100 shares.

Date	Money cashflows (£)	Inflation index	Real cashflows (£)
1/1/18	$-1,720.00$	110.0	$-1,720.00$
31/12/18	$+35(1.03)$ = +36.05	112.3	$36.05\times\dfrac{110}{112.3}=35.31167$
31/12/19	$+35(1.03)(1.05)$ = +37.8525	113.2	$37.8525\times\dfrac{110}{113.2}=36.78246$
31/12/20	+35(1.03)(1.05)(1.06) = +40.12365	113.8	$40.12365\times\dfrac{110}{113.8}=38.78384$
1/1/21	+1,800.00	113.8	$1,800.00\times\dfrac{110}{113.8}=1,739.89455$

The real rate of return, i', satisfies the following equation of value:

$$1,720=35.31167v+36.78246v^2+1,778.67840v^3 \qquad [3]$$

where $v=\dfrac{1}{1+i'}$.

Using $i'=2.5\%$, RHS = 1,721.14. [½]

Using $i'=3\%$, RHS = 1,696.70. [½]

Interpolating gives:

$$i'=2.5\%+\frac{1,720-1,721.14}{1,696.70-1,721.14}(3\%-2.5\%)=2.52\% \qquad [1]$$

[Total 5]

13.10 The amount of the redemption payment is calculated using the index values from 6 months before the date of issue of the bond, and 6 months before the date of the redemption payment:

$$10{,}000 \times \frac{Index(01.07.17)}{Index(01.07.12)} = 10{,}000 \times \frac{193}{148} = £13{,}040.54$$

So the investor's money rate of return is found from the equation:

$$10{,}250 = 13{,}040.54v^3 \quad \Rightarrow \quad (1+i)^3 = 1.27225 \quad \Rightarrow \quad i = 8.36\%$$

In terms of 01.01.15 prices, the redemption payment is:

$$13{,}040.54 \times \frac{Index(01.01.15)}{Index(01.01.18)} = 13{,}040.54 \times \frac{175}{201} = 11{,}353.70$$

The investor's real rate of return is then found from the equation:

$$10{,}250 = 11{,}353.70v'^3 \quad \Rightarrow \quad (1+i')^3 = 1.10768 \quad \Rightarrow \quad i' = 3.47\%$$

13.11 *This question is Subject CT1, April 2014, Question 5 (with dates updated).*

(i) ***Coupon and redemption payments***

Since the nominal coupon rate is 3% *pa*, the investor actually received 1.5% (before indexation) per half-year. So the coupon payment received on 25 April 2018 was:

$$0.015 \times \frac{171.4}{149.2} \times 10{,}000 = £172.319$$ [1]

The coupon payment received on 25 October 2018 was:

$$0.015 \times \frac{173.8}{149.2} \times 10{,}000 = £174.732$$ [1]

The redemption proceeds also received on 25 October 2018 were:

$$\frac{173.8}{149.2} \times 10{,}000 = £11{,}648.794$$ [1]

[Total 3]

(ii) ***Purchase price***

We first calculate the real values as at 25 October 2017 of the 2018 cashflows.

The April 2018 coupon has real value:

$$172.319 \times \frac{169.4}{171.4} = 170.308$$ [1]

The total of the final coupon and redemption proceeds (in real values) is:

$$(174.732+11{,}648.794)\times\frac{169.4}{173.8}=11{,}524.196 \qquad [1]$$

So the real equation of value, using the effective real yield of 3.5% *pa*, is:

$$P=170.308v^{0.5}+11{,}524.196v^{1} \qquad [1]$$

$$=170.308\times1.035^{-0.5}+11{,}524.196\times1.035^{-1}$$

$$=11{,}301.893$$

So the purchase price is £11,301.89. [1]

[Total 4]

13.12 The annual nominal coupon rate is 8% and coupons are payable half-yearly, so each nominal coupon payment is £4 per £100 nominal. The amount of each coupon payment is indexed with reference to the retail price index value with a lag of 6 months. So the actual coupon payment received in November 2017 is:

$$4\times\frac{\text{Index value May 2017}}{\text{Index value November 2016}}=4\times\frac{190}{185}$$

and the actual coupon payment received in May 2018 is:

$$4\times\frac{\text{Index value November 2017}}{\text{Index value November 2016}}=4\times\frac{190}{185}\times1.02^{0.5}$$

assuming that the index value grows at 2% *pa* from its May 2017 value.

The money and real values (as at May 2017) of the purchase price for £100 nominal, P, and the coupon and redemption payments are given in the table below:

Date	Money cashflow	Real cashflow
May 17	$-P$	$-P$
Nov 17	$4\times\frac{190}{185}$	$4\times\frac{190}{185}\times1.02^{-0.5}$
May 18	$4\times\frac{190}{185}\times1.02^{0.5}$	$4\times\frac{190}{185}\times1.02^{0.5}\times1.02^{-1}=4\times\frac{190}{185}\times1.02^{-0.5}$
Nov 18	$4\times\frac{190}{185}\times1.02$	$4\times\frac{190}{185}\times1.02\times1.02^{-1.5}=4\times\frac{190}{185}\times1.02^{-0.5}$
…	…	…
May 27	$(4+102)\times\frac{190}{185}\times1.02^{9.5}$	$(4+102)\times\frac{190}{185}\times1.02^{9.5}\times1.02^{-10}=(4+102)\times\frac{190}{185}\times1.02^{-0.5}$

[3 for money cashflows, 3 for real cashflows]

We need to calculate the price paid to obtain a real yield of 3% *pa* convertible half-yearly. This is equivalent to a real yield of 1.5% per half-year effective. So we will work in half-years:

$$P = 4 \times \frac{190}{185} \times 1.02^{-0.5}\left(v + v^2 + \cdots + v^{20}\right) + 102 \times \frac{190}{185} \times 1.02^{-0.5} v^{20}$$

$$= \frac{190}{185} \times 1.02^{-0.5}\left(4a_{\overline{20|}} + 102v^{20}\right) \quad @1.5\% \qquad [2]$$

Evaluating this gives:

$$P = \frac{190}{185} \times 1.02^{-0.5}\left(4 \times 17.16864 + 102 \times 1.015^{-20}\right) = £146.85 \qquad [1]$$

[Total 9]

End of Part 2

What next?

1. Briefly **review** the key areas of Part 2 and/or re-read the **summaries** at the end of Chapters 10 to 13.
2. Ensure you have attempted some of the **Practice Questions** at the end of each chapter in Part 2. If you don't have time to do them all, you could save the remainder for use as part of your revision.
3. Attempt **Assignment X2**.

14

Term structure of interest rates

Syllabus objectives

2.6 Show an understanding of the term structure of interest rates.

2.6.1 Describe the main factors influencing the term structure of interest rates.

2.6.2 Explain what is meant by, derive the relationships between, and evaluate:

- discrete spot rates and forward rates
- continuous spot rates and forward rates.

2.6.3 Explain what is meant by the par yield and yield to maturity.

2.7 Understanding duration, convexity and immunisation of cashflows.

2.7.1 Define the duration and convexity of a cashflow sequence, and illustrate how these may be used to estimate the sensitivity of the value of the cashflow sequence to a shift in interest rates.

2.7.2 Evaluate the duration and convexity of a cashflow sequence.

2.7.3 Explain how duration and convexity are used in the (Redington) immunisation of a portfolio of liabilities.

0 Introduction

So far, it has generally been assumed that the interest rate *i* or force of interest δ earned on an investment are independent of the term of that investment. In practice the interest rate offered on investments does usually vary according to the term of the investment. It is often important to take this variation into consideration.

In investigating this variation we make use of *unit zero-coupon bond* prices. A unit zero-coupon bond of term *n*, say, is an agreement to pay £1 at the end of *n* years. No coupon payments are paid. It is also called a *pure discount bond*.

We denote the price at issue of a unit zero-coupon bond maturing in *n* years by P_n.

Question

Calculate the yield achieved by investing in a 15-year unit zero-coupon bond if $P_{15} = 0.54$.

Solution

The yield is the value of i that solves the bond's equation of value:

$$0.54 = v^{15} \quad \Rightarrow \quad 1+i = 0.54^{-1/15} \quad \Rightarrow \quad i = 4.19\%$$

Sections 1 and 2 of this chapter look at ways of expressing interest rates that vary by term and Section 3 gives the reasons why interest rates vary by term. Section 4 then looks at some numerical measures that allow us to quantify the effect of a change in interest rates on series of cashflows, and techniques used to minimise the risk of interest rate changes.

1 Discrete-time rates

1.1 Discrete-time spot rates

The yield on a unit zero-coupon bond with term *n* years, y_n, is called the '*n*-year spot rate of interest'.

The *n*-year spot rate is a measure of the average interest rate over the period from now until *n* years' time. Sometimes s_n is used to denote the *n*-year spot rate instead of y_n.

Using the equation of value for the zero-coupon bond we find the yield on the bond, y_n, from:

$$P_n = \frac{1}{(1+y_n)^n} \quad \Rightarrow \quad (1+y_n) = P_n^{-\frac{1}{n}}$$

Question

The prices for £100 nominal of zero-coupon bonds of various terms are as follows:

1 year = £94 5 years = £70 10 years = £47 15 years = £30

Calculate the spot rates for these terms and sketch a graph of these rates as a function of the term.

Solution

The spot rates for the various terms can be found from the equations of value:

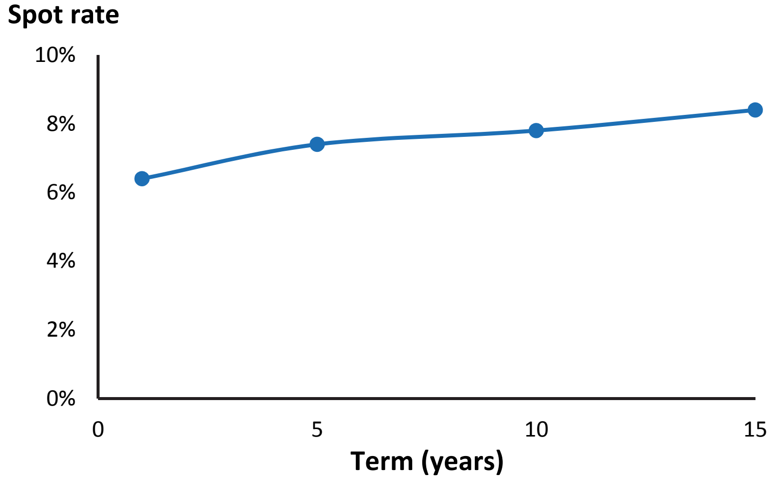
1 year: $100(1+y_1)^{-1} = 94 \quad \Rightarrow \quad y_1 = 6.4\%$

5 years: $100(1+y_5)^{-5} = 70 \quad \Rightarrow \quad y_5 = 7.4\%$

10 years: $100(1+y_{10})^{-10} = 47 \quad \Rightarrow \quad y_{10} = 7.8\%$

15 years: $100(1+y_{15})^{-15} = 30 \quad \Rightarrow \quad y_{15} = 8.4\%$

A graph of these spot rates is shown below:



Since rates of interest differ according to the term of the investment, in general $y_s \neq y_t$ for $s \neq t$. Every fixed-interest investment may be regarded as a combination of (perhaps notional) zero-coupon bonds. For example, a bond paying coupons of *D* every year for *n* years, with a final redemption payment of *R* at time *n* may be regarded as a combined investment of *n* zero-coupon bonds with maturity value *D*, with terms of 1 year, 2 years .., *n* years, plus a zero-coupon bond of nominal value *R* with term *n* years.

Defining $v_{y_t} = (1+y_t)^{-1}$, the price of the bond is:

$$A = D(P_1 + P_2 + \cdots + P_n) + RP_n$$
$$= D(v_{y_1} + v_{y_2}^2 + \cdots + v_{y_n}^n) + Rv_{y_n}^n$$

This is actually a consequence of 'no arbitrage'; the portfolio of zero-coupon bonds has the same payouts as the fixed-interest bond, and the prices must therefore be the same.

Arbitrage is the existence of risk-free profits. This is discussed in great detail in Subject CM2.

If the price of a ten-year fixed-interest security were greater than the price given by the above formula, then investors would spot the anomaly and seek to make risk-free profits by selling holdings of the fixed-interest security (at the high price) and buying an appropriate combination of zero-coupon bonds (which would cost less). The future cashflows of the investors would not change but they would make an immediate risk-free profit. This is an example of arbitrage.

In this scenario, the increased demand for the zero-coupon bonds relative to the fixed-interest security would result in the price of the zero-coupon bonds rising and the price anomaly (and arbitrage opportunity) being removed. In practice, these price changes can happen very quickly, so we assume that no arbitrage opportunities exist.

The key result of the 'no arbitrage' assumption is that if two investments provide identical cashflows in the future, they must have the same price now. This is called the 'Law of One Price'.

The variation by term of interest rates is often referred to as the *term structure of interest rates.* The curve of spot rates $\{y_t\}$ is an example of a *yield curve*.

The graph above is an example of a yield curve. It isn't always spot rates that are plotted in a yield curve. It might instead be redemption yields or forward rates (which we will meet shortly).

Question

The current annual term structure of interest rates is:

(7%, 7.25%, 7.5%, 7.75%, 8%, …)

ie $y_1 = 7\%$, $y_2 = 7.25\%$, $y_3 = 7.5\%$, $y_4 = 7.75\%$ and $y_5 = 8\%$.

A five-year fixed-interest security has just been issued. It pays coupons of 6% annually in arrears and is redeemable at par. Calculate:

(i) the price per £100 nominal of the security

(ii) the gross redemption yield of the security.

Solution

(i) The price per £100 nominal is given by:

$$P = 6(v_{7\%} + v_{7.25\%}^2 + v_{7.5\%}^3 + v_{7.75\%}^4 + v_{8\%}^5) + 100v_{8\%}^5$$
$$= £92.25$$

(ii) The gross redemption yield is the value of i that solves the equation:

$$92.25 = 6a_{\overline{5}|} + 100v^5$$

The gross redemption yield reflects the overall return from owning the security. It is a weighted average of the interest rates that apply over the term of the security, where the weights are the present values of the cashflows that occur at the different durations.

So the gross redemption yield must lie between 7% and 8% here (the range of spot rates over the five-year term), and will be closer to 8% as that is the spot rate that applies when the largest cashflow occurs, at time 5.

Trying 7.5%, we see that the right-hand side is £93.93, and trying 8%, the right-hand side is £92.01. Using linear interpolation, the gross redemption yield is:

$$i \approx 7.5\% + \frac{92.25 - 93.93}{92.01 - 93.93} \times (8\% - 7.5\%) = 7.94\%$$

1.2 Discrete-time forward rates

So far we have defined spot rates, which tell us about interest rates over a period that starts *now*. In this section we will consider *forward rates*, which tell us about interest rates over *future* periods that may start at a future time.

The discrete-time forward rate, $f_{t,r}$, is the annual interest rate agreed at time 0 for an investment made at time $t > 0$ for a period of *r* years.

That is, if an investor agrees at time 0 to invest £100 at time *t* for *r* years, the accumulated investment at time $t + r$ is:

$$100(1+f_{t,r})^r$$

The forward rate, $f_{t,r}$, is a measure of the average interest rate between times *t* and $t+r$.

Forward rates, spot rates and zero-coupon bond prices are all connected. The accumulation at time *t* of an investment of 1 at time 0 is $(1+y_t)^t$. If we agree at time 0 to invest the amount $(1+y_t)^t$ at time t for r years, we will earn an annual rate of $f_{t,r}$. So we know that £1 invested for $t+r$ years will accumulate to $(1+y_t)^t(1+f_{t,r})^r$. But we also know from the $(t+r)$ spot rates that £1 invested for $t+r$ years accumulates to $(1+y_{t+r})^{t+r}$, and we also know from the zero-coupon bond prices that £1 invested for $t+r$ years accumulates to P_{t+r}^{-1}. Hence we know that:

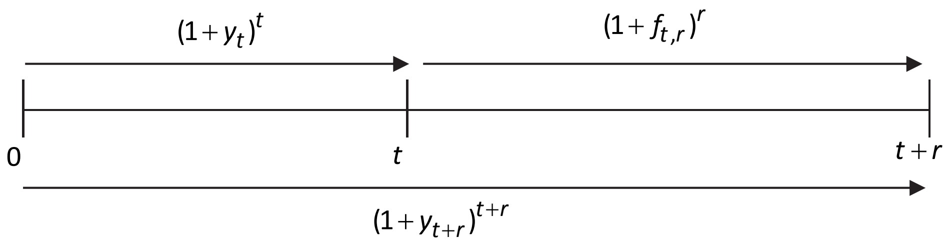
$$(1+y_t)^t(1+f_{t,r})^r = (1+y_{t+r})^{t+r} = P_{t+r}^{-1}$$

from which we find that:

$$(1+f_{t,r})^r = \frac{(1+y_{t+r})^{t+r}}{(1+y_t)^t} = \frac{P_t}{P_{t+r}}$$

so that the full term structure may be determined given the spot rates, the forward rates or the zero-coupon bond prices.

The connection between the spot rates and the forward rates can be represented on a timeline, as follows:



Accumulating payments from time 0 to time $t+r$ using the spot rate y_{t+r} is equivalent to first accumulating to time *t* using the spot rate y_t, and then accumulating from time *t* to time $t+r$ using the forward rate $f_{t,r}$.

One-period forward rates **are of particular interest. The one-period forward rate at time *t* (agreed at time 0) is denoted $f_t = f_{t,1}$. We define $f_0 = y_1$. Comparing an amount of £1 invested for *t* years at the spot rate y_t, and the same investment invested 1 year at a time with proceeds reinvested at the appropriate one-year forward rate, we have:**

$$(1+y_t)^t = (1+f_0)(1+f_1)(1+f_2)\cdots(1+f_{t-1})$$

The one-year forward rate, f_t, is therefore the rate of interest from time *t* to time $t+1$. It can be expressed in terms of spot rates:

$$(1+f_t) = \frac{(1+y_{t+1})^{t+1}}{(1+y_t)^t}$$

Question

The 3, 5 and 7-year spot rates are 6%, 5.7% and 5% *pa* respectively. The 3-year forward rate from time 4 is 5.2% *pa*. Calculate:

(i) f_3

(ii) $f_{5,2}$

(iii) y_4

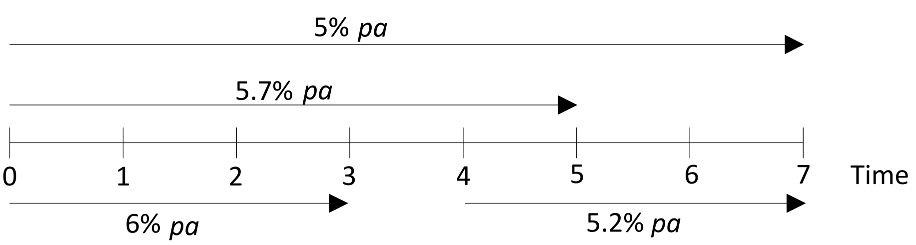
(iv) $f_{3,4}$

Solution

We are given:

$$y_3 = 6\%, \quad y_5 = 5.7\%, \quad y_7 = 5\%, \quad f_{4,3} = 5.2\%$$

We can present these rates on a timeline as follows:



(i) We have:

$$(1+y_3)^3(1+f_3)(1+f_{4,3})^3 = (1+y_7)^7$$

$$\Rightarrow \quad 1+f_3 = \frac{1.05^7}{1.06^3 \times 1.052^3} = 1.0148$$

$$\Rightarrow \quad f_3 = 1.48\%$$

(ii) We have:

$$(1+y_5)^5(1+f_{5,2})^2 = (1+y_7)^7$$

$$\Rightarrow \quad 1+f_{5,2} = \left(\frac{1.05^7}{1.057^5}\right)^{\frac{1}{2}} = \sqrt{1.0665}$$

$$\Rightarrow \quad f_{5,2} = 3.27\%$$

(iii) We have:

$$(1+y_4)^4(1+f_{4,3})^3 = (1+y_7)^7$$

$$\Rightarrow \quad 1+y_4 = \left(\frac{1.05^7}{1.052^3}\right)^{\frac{1}{4}} = 1.0485$$

$$\Rightarrow \quad y_4 = 4.85\%$$

(iv) We have:

$$(1+y_3)^3(1+f_{3,4})^4 = (1+y_7)^7$$

$$\Rightarrow \quad 1+f_{3,4} = \left(\frac{1.05^7}{1.06^3}\right)^{\frac{1}{4}} = 1.0426$$

$$\Rightarrow \quad f_{3,4} = 4.26\%$$

2 Continuous-time rates

2.1 Continuous-time spot rates

The continuous-time spot rate is the force of interest that is equivalent to the spot rate expressed as an effective rate of interest.

Let P_t be the price of a unit zero-coupon bond of term *t*. Then the *t*-year *spot force of interest* is Y_t where:

$$P_t = e^{-Y_t t} \quad \Rightarrow \quad Y_t = -\frac{1}{t}\log P_t$$

This is also called the continuously compounded spot rate of interest or the continuous-time spot rate. Y_t and its corresponding discrete annual rate y_t are connected in the same way as δ and *i*; an investment of £1 for *t* years at a discrete spot rate y_t accumulates to $(1+y_t)^t$; at the continuous-time rate it accumulates to $e^{Y_t t}$; these must be equal, so $y_t = e^{Y_t} - 1$.

2.2 Continuous-time forward rates

The continuous-time forward rate $F_{t,r}$ is the force of interest equivalent to the annual forward rate of interest $f_{t,r}$.

A £1 investment of duration *r* years, starting at time *t*, agreed at time $0 \le t$ accumulates using the annual forward rate of interest to $(1+f_{t,r})^r$ at time $t+r$.

Using the equivalent forward force of interest the same investment accumulates to $e^{F_{t,r} r}$.

Hence the annual rate and continuous-time rate are related as:

$$f_{t,r} = e^{F_{t,r}} - 1$$

The relationship between the continuous-time spot and forward rates may be derived by considering the accumulation of £1 at a continuous-time spot rate of Y_t for *t* years, followed by the continuous-time forward rate of $F_{t,r}$ for *r* years. Compare this with an investment of £1 at a continuous-time spot rate of Y_{t+r} for $t+r$ years. The two investments are equivalent, so the accumulated values must be the same. Hence:

$$e^{tY_t} e^{rF_{t,r}} = e^{(t+r)Y_{t+r}}$$

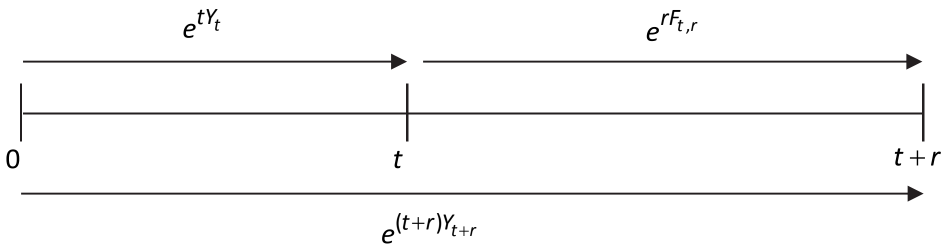
$$\Rightarrow \quad tY_t + rF_{t,r} = (t+r)Y_{t+r}$$

$$\Rightarrow \quad F_{t,r} = \frac{(t+r)Y_{t+r} - tY_t}{r}$$

Also, using $Y_t = -\frac{1}{t}\log P_t$, we have:

$$F_{t,r} = \frac{1}{r}\log\left(\frac{P_t}{P_{t+r}}\right)$$

Once again we can represent the connection between the continuous-time spot and forward rates on a timeline.



Question

The prices for £100 nominal of zero-coupon bonds of various terms are as follows:

1 year = £94 5 years = £70 10 years = £47 15 years = £30

Calculate:

(i) Y_{10}

(ii) $F_{5,10}$

Solution

(i) We can calculate Y_{10} using the price of the 10-year zero-coupon bond:

$$Y_{10} = -\frac{1}{10}\log P_{10} = -\frac{1}{10}\log 0.47 = 7.55\%$$

Instead of using the general formula, we could take a first principles approach and set up the equation of value:

$$47 = 100e^{-10Y_{10}}$$

ie the price paid (of £47) is equal to the present value of the redemption payment of £100 calculated using the 10-year continuous-time spot rate. This gives the same answer as before.

(ii) $F_{5,10}$ is the continuous-time forward rate applying from time 5 to time 15, so we can calculate this using the 5-year and 15-year zero-coupon bond prices:

$$F_{5,10} = \frac{1}{10}\log\left(\frac{P_5}{P_{15}}\right) = \frac{1}{10}\log\left(\frac{0.7}{0.3}\right) = 8.47\%$$

Again, we could take a first principles approach using the equation of value for the 15-year bond:

$$30 = 100e^{-5Y_5}e^{-10F_{5,15}}$$

ie the price paid (of £30) is equal to the present value of the redemption payment of £100, calculated by discounting it for 10 years using the continuous-time forward rate (between time 5 and time 15) and for 5 years using the continuous-time spot rate (between time 0 and time 5). Now, considering the 5-year bond:

$$70 = 100e^{-5Y_5}$$

so the equation of value for the 15-year bond becomes:

$$30 = 70e^{-10F_{5,15}}$$

Solving this gives the same answer as before.

2.3 Instantaneous forward rates

The instantaneous forward rate F_t is defined as:

$$F_t = \lim_{r \to 0} F_{t,r}$$

The instantaneous forward rate may broadly be thought of as the forward force of interest applying in the instant of time $t \to t + \Delta t$.

$$F_t = \lim_{r \to 0} \frac{1}{r} \log\left(\frac{P_t}{P_{t+r}}\right) \qquad (1)$$

$$= \lim_{r \to 0} \frac{\log P_t - \log P_{t+r}}{r}$$

$$= -\lim_{r \to 0} \frac{\log P_{t+r} - \log P_t}{r} \qquad (2)$$

$$= -\frac{d}{dt}\log P_t \qquad (3)$$

$$= -\frac{1}{P_t}\frac{d}{dt}P_t \qquad (4)$$

Step (3) above uses the definition of a derivative in terms of a limit:

$$\frac{d}{dt}f(t) = \lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$$

and step (4) uses the chain rule for differentiation.

We also find, by integrating both sides of (3) and using the fact that $P_0 = 1$ (as the price of a unit zero-coupon bond of term zero years must be 1), that:

$$\boldsymbol{P_t = e^{-\int_0^t F_s\,ds}}$$

This formula might look familiar. Earlier in the course we defined $v(t)$ to be the present value of 1 due at time *t* and we expressed $v(t)$ in terms of $\delta(t)$, the force of interest per unit time at time *t*, as follows:

$$v(t) = e^{-\int_0^t \delta(s)ds}$$

Comparing these last two equations, we see that P_t is equivalent to $v(t)$ and F_s is equivalent to $\delta(s)$.

Note

We have described in this chapter the *initial term structure*, where everything is fixed at time 0. In practice the term structure varies rapidly over time, and the 5-year spot rate tomorrow may be quite different from the 5-year spot rate today. In more sophisticated treatments we model the change in term structure over time.

In this case all the variables we have used, *ie*:

$$\boldsymbol{P_t \quad y_t \quad f_{t,r} \quad Y_t \quad F_{t,r}}$$

need another argument, *v*, say, to give the 'starting point'. For example, $y_{v,t}$ would be the *t*-year discrete spot rate of interest applying at time *v*; $F_{v,t,r}$ would be the force of interest agreed at time *v*, applying to an amount invested at time $v+t$ for the *r*-year period to time $v+t+r$.

Question

Using the above notation, suppose that $y_{0,5} = 6\%$, $y_{5,5} = 7.5\%$, $F_{0,5,5} = 7\%$ and $F_{5,5,5} = 8.25\%$.

Calculate the price for £100 nominal of a ten-year zero-coupon bond issued at:

(i) time 0

(ii) time 5.

Solution

(i) To calculate the price of the bond issued at time 0, we use the initial term structure of interest rates, *ie* $y_{0,5}$ and $F_{0,5,5}$, which are the 5-year spot rate and the 5-year force of interest from time 5 to time 10, as at time 0, respectively.

So the price for £100 nominal of the zero-coupon bond issued at time 0 is:

$$100(1+y_{0,5})^{-5}e^{-5F_{0,5,5}} = 100\times 1.06^{-5}\times e^{-5\times 0.07} = £52.66$$

(ii) In this case, we use the term structure of interest rates at time 5, *ie* $y_{5,5}$ and $F_{5,5,5}$, which are the 5-year spot rate and the 5-year force of interest from time 10 to time 15, as at time 5, respectively.

So the price for £100 nominal of the zero-coupon bond issued at time 5 is:

$$100(1+y_{5,5})^{-5}e^{-5F_{5,5,5}} = 100\times 1.075^{-5}\times e^{-5\times 0.0825} = £46.11$$

3 Theories of the term structure of interest rates

3.1 Why interest rates vary over time

The prevailing interest rates in investment markets usually vary depending on the time span of the investments to which they relate. This variation determines the *term structure* of interest rates.

The variation arises because the interest rates that lenders expect to receive and borrowers are prepared to pay are influenced by the following factors, which are not normally constant over time:

Supply and demand

Interest rates are determined by market forces, *ie* the interaction between borrowers and lenders. If cheap finance is easy to obtain or if there is little demand for finance, this will push interest rates down.

Base rates

In many countries there is a central bank that sets a base rate of interest. This provides a reference point for other interest rates. For example, an interest rate in the UK may be expressed as the Bank of England's base rate plus 4%. Investors will have a view on how this rate is likely to move in the future.

Interest rates in other countries

The interest rates in a particular country will also be influenced by the cost of borrowing in other countries because major investment institutions have the alternative of borrowing from abroad.

Expected future inflation

Lenders will expect the interest rates they obtain to outstrip inflation. So periods of high inflation tend to be associated with high interest rates.

Tax rates

If tax rates are high, interest rates may also be high, because investors will require a certain level of return after tax.

Risk associated with changes in interest rates

In general, rates of interest tend to increase as the term increases because the risk of loss due to a change in interest rates is greater for longer-term investments. This is the idea behind liquidity preference theory, which we meet in the next section.

3.2 The theories

Some examples of typical (spot rate) yield curves are given below.

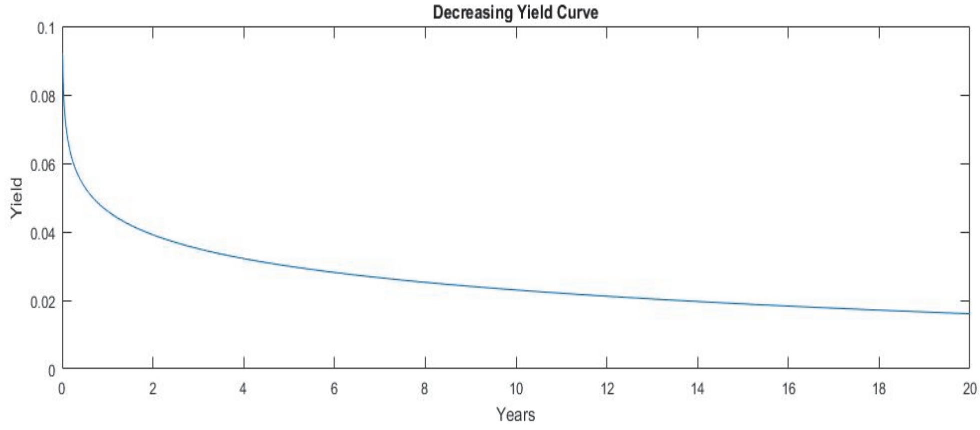


Figure 1: Decreasing yield curve (used with permission from Dominic Cortis)

In Figure 1 the long-term bond yields are lower than the short-term bonds. Since price is a decreasing function of yield, an interpretation is that long-term bonds are more expensive than short-term bonds. There are several possible explanations – for example it is possible that investors believe that they will get a higher overall return from long-term bonds, despite the lower current yields, and the higher demand for long-term bonds has pushed up the price, which is equivalent to pushing down the yield, compared with short-term bonds.

Other explanations for different yield curve shapes are given below.

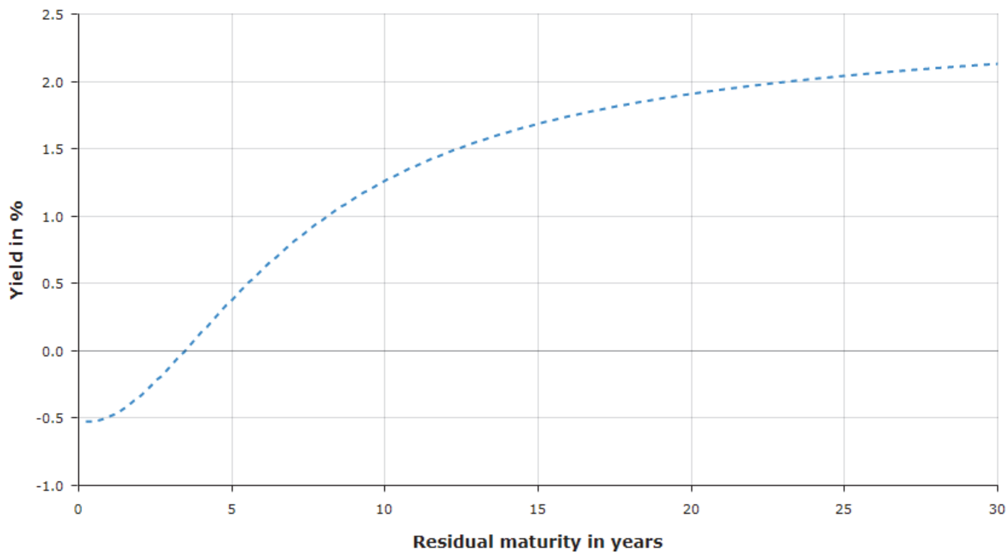


Figure 2: Increasing yield curve: Euro Area Yield Curve for all bonds (obtained from European Central Bank website on 7 February 2018)

In Figure 2 the long-term bonds are higher yielding (or cheaper) than the short-term bonds. This shows the case of an increasing yield curve.

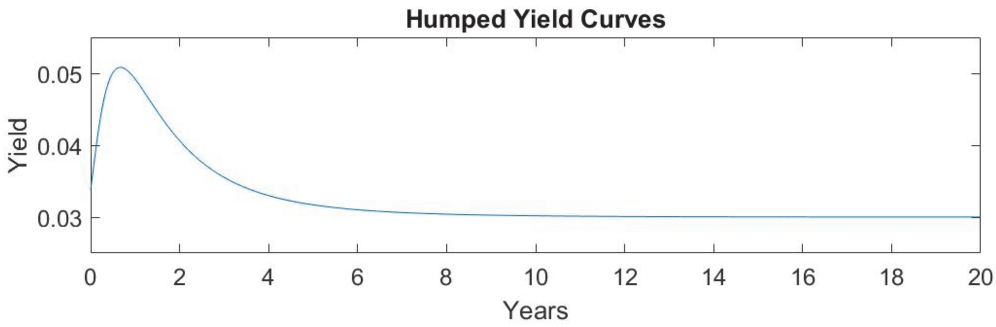


Figure 3: Humped yield curve (used with permission from Dominic Cortis)

In Figure 3 the short-term bonds are generally cheaper than the long bonds, but the very short rates (with terms less than one year) are lower than the one-year rates.

The three most popular explanations for the fact that interest rates vary according to the term of the investment are:

1. **Expectations theory**
2. **Liquidity preference**
3. **Market segmentation**

Expectations theory

The relative attraction of short and longer-term investments will vary according to expectations of future movements in interest rates. An expectation of a fall in interest rates will make short-term investments less attractive and longer-term investments more attractive. In these circumstances yields on short-term investments will rise and yields on long-term investments will fall. An expectation of a rise in interest rates will have the converse effect.

In Figure 1 it appears that the demand for long-term bonds may be greater than for short, implying an expectation that interest rates will fall. By buying long-term bonds investors can continue getting higher rates after a future fall in interest rates, for the duration of the long bond.

In Figure 2 the demand is higher for short-term bonds – perhaps indicating an expectation of a rise in interest rates.

If it is expected that interest rates will rise, investors will wish to avoid locking into the low current rate over the long term, and so will choose shorter-term investments. This increased demand for shorter-term investments increases the price of these bonds and reduces the yield, leading to an increasing yield curve, as in Figure 2.

Liquidity preference

Consider a ten-year and a twenty-year zero-coupon bond. If the spot rate for all terms is 5% *pa* then the prices for £100 nominal are $100\times1.05^{-10}=£61.39$ and $100\times1.05^{-20}=£37.69$ respectively.

If interest rates rise to 6%, then the price of both bonds will fall:

- the ten-year bond price falls to $100\times1.06^{-10}=£55.84$, a 9% drop
- the twenty-year bond price falls to $100\times1.06^{-20}=£31.18$, a 17% drop.

Longer-dated bonds are more sensitive to interest rate movements than short-dated bonds. It is assumed that risk averse investors will require compensation (in the form of higher yields) for the greater risk of loss on longer bonds. This might explain some of the excess return offered on long-term bonds over short-term bonds in Figure 2.

Later in this chapter we will look at measures that allow us to quantify the effects of changes in interest rates.

Market segmentation

Bonds of different terms are attractive to different investors, who will choose assets that are similar in term to their liabilities. The liabilities of banks, for example, are very short-term (investors may withdraw a large proportion of the funds at very short notice); hence banks invest in very short-term bonds. Many pension funds have liabilities that are very long-term, so pension funds are more interested in the longest-dated bonds. The demand for bonds will therefore differ for different terms.

The supply of bonds will also vary by term, as governments and companies' strategies may not correspond to the investors' requirements.

Remember that governments and companies issue bonds because they need to borrow money, not because they are kind-hearted and want to give investors something to invest in. More bonds will be supplied if more money needs to be borrowed. This will put downward pressure on prices.

The market segmentation hypothesis argues that the term structure emerges from these different forces of supply and demand.

These theories are covered in more detail in Subject CP1.

Question

Outline what would happen to yields of fixed-interest bonds if:

(i) bond prices fall

(ii) demand for fixed-interest bonds falls

(iii) the government issues many more bonds

(iv) institutional investors suddenly decide to invest less in equities and more in fixed-interest bonds.

Solution

(i) If bond prices fall, then bond yields will rise, as investors will now be paying less to receive the same coupon and redemption payments.

(ii) If demand for fixed-interest bonds falls, then the price of the bonds will fall, and bond yields will rise.

(iii) In this case, the supply of fixed-interest bonds will increase, and this puts downward pressure on prices. As bond prices fall, bond yields will rise.

(iv) This is the opposite scenario to (ii). Here, the demand for fixed-interest bonds increases, so the price of the bonds will increase, and bond yields will fall.

3.3 Yields to maturity

The yield to maturity for a coupon-paying bond (also called the redemption yield) has been defined as the effective rate of interest at which the discounted value of the proceeds of a bond equal the price. It is widely used, but has the disadvantage that it depends on the coupon rate of the bond, and therefore does not give a simple model of the relationship between term and yield.

In the UK, yield curves plotting the average (smoothed) yield to maturity of coupon-paying bonds are produced separately for 'low coupon', 'medium coupon' and 'high coupon' bonds.

Question

The current annual term structure of interest rates is:

(6%, 6%, 6%, 6%, 7%)

ie $y_1 = y_2 = y_3 = y_4 = 6\%$ and $y_5 = 7\%$.

Calculate the gross redemption yield of a five-year fixed-interest security that is redeemable at par if the annual coupon, payable in arrears, is:

(i) 2%

(ii) 4%.

Solution

(i) ***2% coupon rate***

The cashflows at times 1, 2, 3 and 4 are discounted using an interest rate of 6% *pa* (as the spot rate is the same at all these durations), and the cashflow at time 5 – made up of the final coupon payment and the redemption payment – is discounted using the 5-year spot rate of 7% *pa*.

Letting P denote the price for £100 nominal of the bond:

$$P = 2a_{\overline{4}|6\%} + 102v_{7\%}^{5} = 2\times 3.4651 + 102\times 1.07^{-5} = £79.65$$

The gross redemption yield is the interest rate, i, that satisfies the equation of value:

$$79.65 = 2a_{\overline{5}|} + 100v^{5}$$

The gross redemption yield is a weighted average of the interest rates over the term of the bond, where the weights are the present values of the cashflows that occur at the different durations. The gross redemption yield is likely to be close to 7% here, as that is the spot rate associated with the duration of the largest cashflow (the redemption payment).

At 7%, the right-hand side gives £79.50, and at 6.5%, the right-hand side gives £81.30. Linearly interpolating, we find the gross redemption yield to be:

$$i \approx 6.5\% + \frac{79.65 - 81.30}{79.50 - 81.30}\times(7\% - 6.5\%) = 6.96\%$$

(ii) ***4% coupon rate***

In this case, the price for £100 nominal of the bond is:

$$P = 4a_{\overline{4}|6\%} + 104v_{7\%}^{5} = 4\times 3.4651 + 104\times 1.07^{-5} = £88.01$$

The gross redemption yield is the interest rate, i, that satisfies the equation of value:

$$88.01 = 4a_{\overline{5}|} + 100v^{5}$$

At 7%, the right-hand side gives £87.70, and at 6.5% the right-hand side gives £89.61. Linearly interpolating, we find the gross redemption yield to be:

$$i \approx 6.5\% + \frac{88.01 - 89.61}{87.70 - 89.61}\times(7\% - 6.5\%) = 6.92\%$$

The gross redemption yield of the bond with the 4% coupon rate is lower than that for the bond with the 2% coupon rate. This is because the bond with the 4% coupon rate has higher cashflows at the earlier durations, so these gain more weighting in the calculation and, since the spot rates at the earlier durations are lower, the gross redemption yield is lower.

3.4 Par yields

We have already met the *yield to maturity* or the *redemption yield* for a fixed-interest investment. This is just the constant interest rate that satisfies the equation of value. For a zero-coupon bond, this is the same as the spot rate.

The *n-year par yield* represents the coupon per £1 nominal that would be payable on a bond with term *n* years, which would give the bond a current price under the current term structure of £1 per £1 nominal, assuming the bond is redeemed at par.

That is, if yc_n is the *n*-year par yield:

$$\mathbf{1 = (yc_n)(v_{y_1} + v_{y_2}^2 + v_{y_3}^3 + \cdots + v_{y_n}^n) + 1v_{y_n}^n}$$

The par yields give an alternative measure of the relationship between the yield and term of investments. The difference between the par yield rate and the spot rate is called the 'coupon bias'.

Question

Calculate the 5-year par yield if the annual term structure of interest rates is:

$$(6\%, 6.25\%, 6.5\%, 6.75\%, 7\%, \ldots)$$

ie $y_1 = 6\%$, $y_2 = 6.25\%$, $y_3 = 6.5\%$, $y_4 = 6.75\%$ and $y_5 = 7\%$.

Solution

The 5-year par yield yc_5 is found from the equation:

$$yc_5(v_{y_1} + v_{y_2}^2 + \cdots + v_{y_5}^5) + v_{y_5}^5 = 1$$

Using the spot rates given, this becomes:

$$yc_5(1.06^{-1} + 1.0625^{-2} + 1.065^{-3} + 1.0675^{-4} + 1.07^{-5}) + 1.07^{-5} = 1$$

ie: $yc_5 \times 4.1401 + 0.71299 = 1 \quad \Rightarrow \quad yc_5 = 6.93\%$

4 Duration, convexity and immunisation

One of the key concerns for the manager of a fixed-interest investment portfolio is how the portfolio would be affected if there were a change in interest rates and, in particular, whether such a movement might compromise the ability of the fund to meet its liabilities.

In this section we consider simple measures of vulnerability to interest rate movements.

We will also look at the technique of immunisation, which is a method of minimising the risks relating to interest rates.

For simplicity we assume a flat yield curve, and that when interest rates change, all change by the same amount, so that the curve stays flat. A flat yield curve implies that $y_t = f_{t,r} = i$ for all t, r and $Y_t = F_{t,r} = F_t = \delta$ for all t, r.

4.1 Interest rate risk

Suppose an institution holds assets of value V_A, to meet liabilities of value V_L. Since both V_A and V_L represent the discounted value of future cashflows, both are sensitive to the rate of interest. We assume that the institution is healthy at time 0 so that currently $V_A \geq V_L$.

If $V_A > V_L$, then we say that there is a surplus in the fund equal to $V_A - V_L$. If $V_A < V_L$, then the fund is in deficit.

If rates of interest fall, both V_A and V_L will increase. If rates of interest rise then both will decrease. We are concerned with the risk that following a downward movement in interest rates the value of assets increases by less than the value of liabilities, or that, following an upward movement in interest rates the value of assets decreases by more than the value of the liabilities.

In other words, for a fund currently in surplus, we are concerned that after a movement in interest rates the fund moves into deficit.

In order to examine the impact of interest rate movements on different cashflow sequences, we will use changes in the yield to maturity to represent changes in the underlying term structure. This is approximately (but not exactly) the same as assuming a constant movement of similar magnitude in the one-period forward rates.

Before we can look at a technique used to minimise this interest rate risk we must be familiar with the measures: effective duration, duration and convexity.

4.2 Effective duration

One measure of the sensitivity of a series of cashflows to movements in the interest rates is the *effective duration* (or volatility). Consider a series of cashflows $\{C_{t_k}\}$ for $k = 1, 2, \ldots, n$. Let A be the present value of the payments at rate (yield to maturity) i, so that:

$$A = \sum_{k=1}^{n} C_{t_k} v_i^{t_k}$$

Then the effective duration is defined to be:

$$\boldsymbol{\nu(i) = -\frac{1}{A}\frac{d}{di}A = -\frac{A'}{A}} \qquad \textbf{(4.1)}$$

$$\boldsymbol{= \frac{\sum_{k=1}^{n} C_{t_k}\, t_k\, v_i^{t_k+1}}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}}}$$

To obtain this last relationship, we need to differentiate A with respect to i. Since:

$$A = \sum_{k=1}^{n} C_{t_k} v^{t_k} = \sum_{k=1}^{n} C_{t_k} (1+i)^{-t_k}$$

it follows that:

$$A' = \sum_{k=1}^{n} C_{t_k} (-t_k)(1+i)^{-t_k-1} = (-1)\sum_{k=1}^{n} C_{t_k} t_k v^{t_k+1}$$

This is a measure of the rate of change of value of A with i, which is independent of the size of the present value. Equation (4.1) assumes that the cashflows do not depend on the rate of interest.

For a small movement ε in interest rates, from i to $i+\varepsilon$, the relative change in value of the present value is approximately $-\varepsilon\nu(i)$ so the new present value is approximately $A(1-\varepsilon\nu(i))$.

Effective duration is denoted by the Greek letter nu, ν.

Question

Consider a three-year fixed-interest bond that pays coupons annually in arrears at a rate of 5% *pa* and is redeemable at par.

Let $P(i)$ denote the price for £100 nominal of this bond based on a yield of i *pa*.

Calculate $\nu(0.05)$, the volatility of the bond at a yield of 5% *pa*, and use this to calculate the approximate price of the bond if the yield falls by 1% to 4% *pa.*

Solution

We have:

$$P(i) = 5v + 5v^2 + 105v^3 = 5(1+i)^{-1} + 5(1+i)^{-2} + 105(1+i)^{-3}$$

and:

$$P'(i) = 5(-1)(1+i)^{-2} + 5(-2)(1+i)^{-3} + 105(-3)(1+i)^{-4} = -5(1+i)^{-2} - 10(1+i)^{-3} - 315(1+i)^{-4}$$

So, the volatility at a yield of 5% *pa* is:

$$\nu(0.05) = -\frac{P'(0.05)}{P(0.05)} = -\left(\frac{-5(1.05)^{-2} - 10(1.05)^{-3} - 315(1.05)^{-4}}{5(1.05)^{-1} + 5(1.05)^{-2} + 105(1.05)^{-3}} \right) = -\left(\frac{-272.325}{100} \right) = 2.723$$

This value tells us that if interest rates change by 1%, the price of this bond will change by approximately 2.723% (in the opposite direction to the interest rate movement).

So, if the yield falls by 1% to 4% *pa*, the bond price will rise by approximately 2.723% to:

$$1.02723 \times P(0.05) = 1.02723 \times 100 = £102.72$$

For comparison, the exact value is:

$$P(0.04) = 5(1.04)^{-1} + 5(1.04)^{-2} + 105(1.04)^{-3} = £102.78$$

4.3 Duration

Another measure of interest rate sensitivity is the duration, also called Macaulay duration or discounted mean term (DMT). This is the mean term of the cashflows $\{C_{t_k}\}$, weighted by present value. That is, at rate *i*, the duration of the cashflow sequence $\{C_{t_k}\}$ is:

$$\tau = \frac{\sum_{k=1}^{n} t_k \, C_{t_k} v_i^{t_k}}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}}$$

The discounted mean term for a continuously payable payment stream (or a mixture of discrete and continuous payments) is calculated similarly, but with the summations replaced by integrals for the continuous payments.

Comparing this expression with the equation for the effective duration it is clear that:

$$\tau = (1+i)\nu(i)$$

Note the following points:

- The discounted mean term is dependent on the interest rate used to calculate the present values, as well as the amounts and timings of the cashflows.
- The discounted mean term is calculated as an average, so the DMT of a series of cashflows must take a value that is between the times of the first and last cashflows.

Question

Consider a 10-year fixed-interest bond that pays coupons annually in arrears at a rate of 8% *pa* and is redeemable at par.

Calculate the discounted mean term of this bond using an interest rates of 5% *pa* effective.

Solution

The discounted mean term, calculated at interest rate i, is:

$$DMT(i) = \frac{\sum_{t=1}^{10} t \times 8v^t + 10 \times 100v^{10}}{\sum_{t=1}^{10} 8v^t + 100v^{10}}$$

In terms of annuities, this can be written as:

$$DMT(i) = \frac{8(Ia)_{\overline{10|}} + 1{,}000v^{10}}{8a_{\overline{10|}} + 100v^{10}}$$

since $a_{\overline{n|}} = \sum_{t=1}^{n} v^t$ and $(Ia)_{\overline{n|}} = \sum_{t=1}^{n} tv^t$.

Evaluating this expression using an interest rate of 5% *pa*, we obtain:

$$DMT(0.05) = \frac{8 \times 39.3738 + 1{,}000 \times 1.05^{-10}}{8 \times 7.7217 + 100 \times 1.05^{-10}} = 7.54 \text{ years}$$

The first cashflow from the bond is at time 1 and the last cashflow is at time 10. As expected, the DMT is between 1 and 10. The DMT is closer to 10 as it is a weighted average of the payment times. The weights are the present values of the cashflows, and the largest cashflow occurs at time 10.

Another way of deriving the Macaulay duration is in terms of the force of interest, δ:

$$\tau = -\frac{1}{A}\frac{d}{d\delta}A = \frac{di}{d\delta}\nu(i)$$

$$i = e^{\delta} - 1 \quad \Rightarrow \quad \frac{di}{d\delta} = e^{\delta}$$

$$\Rightarrow \quad \tau = e^{\delta}\nu(i) = (1+i)\nu(i)$$

The equation for τ in terms of the cashflows C_{t_k} may be found by differentiating A with respect to δ, recalling that $v_i^{t_k} = e^{-\delta t_k}$.

The first line above can be obtained by starting with the definition of A written in terms of δ:

$$A = \sum_{k=1}^{n} C_{t_k} v^{t_k} = \sum_{k=1}^{n} C_{t_k} e^{-\delta t_k}$$

Differentiating this expression with respect to δ gives:

$$\frac{dA}{d\delta} = \sum_{k=1}^{n} (-t_k) C_{t_k} e^{-\delta t_k}$$

So, in terms of δ, the DMT can be written as:

$$\tau = \frac{\sum_{k=1}^{n} t_k\, C_{t_k}\, v_i^{t_k}}{\sum_{k=1}^{n} C_{t_k}\, v_i^{t_k}} = \frac{\sum_{k=1}^{n} t_k\, C_{t_k}\, e^{-\delta t_k}}{\sum_{k=1}^{n} C_{t_k}\, e^{-\delta t_k}} = -\frac{1}{A}\frac{dA}{d\delta}$$

Then using the chain rule for differentiation, we can write:

$$\tau = -\frac{1}{A}\frac{dA}{d\delta} = -\frac{1}{A}\frac{dA}{di}\frac{di}{d\delta} = \nu(i)\frac{di}{d\delta}$$

The duration of an *n*-year coupon-paying bond, with coupons of *D* payable annually, redeemed at *R*, is:

$$\tau = \frac{D(Ia)_{\overline{n|}} + Rnv^n}{Da_{\overline{n|}} + Rv^n}$$

This is identical to what we found in the previous question.

The duration of an *n*-year zero-coupon bond of nominal amount 100, say, is:

$$\tau = \frac{100nv^n}{100v^n} = n$$

This last result should be intuitively obvious. The average term of a series of cashflows that has only one payment must be the time of that cashflow.

Both the volatility and the discounted mean term provide a measure of the average 'life' of an investment. This is important when considering the effect of changes in interest rates on investment portfolios since an investment with a longer term will in general be more affected by a change in interest rates than an investment with a shorter term.

Question

Consider a fixed-interest security that pays coupons annually in arrears at a rate of 3% *pa* and is redeemable at par.

Determine the effect on the price of £100 nominal of this security if interest rates over all terms increase from 7% *pa* to 8% *pa*, assuming that the term of the security is:

(a) 5 years

(b) 25 years.

Solution

Letting P denote the price of £100 nominal of this security, and assuming a term of n years, we have:

$$P = 3a_{\overline{n|}} + 100v^n$$

(a) ***Term of* 5 *years***

When $i = 0.07$, $P = 3\times 4.1002 + 100\times 1.07^{-5} = £83.60$.

When $i = 0.08$, $P = 3\times 3.9927 + 100\times 1.08^{-5} = £80.04$.

This is a fall in price of 4.3%.

(b) ***Term of* 25 *years***

When $i = 0.07$, $P = 3\times 11.6536 + 100\times 1.07^{-25} = £53.39$.

When $i = 0.08$, $P = 3\times 10.6748 + 100\times 1.08^{-25} = £46.63$.

This is a fall in price of 12.7%.

So we see that the change in interest rate has a greater effect on the longer 25-year security, than it has on the shorter 5-year security.

Roughly speaking, a change in interest rates has the same effect on the present value of a cashflow series as it has on a zero-coupon bond with the same discounted mean term or volatility.

Note that another definition of duration exists: the *modified duration*. This can be expressed in terms of the Macaulay duration as:

$$\frac{\tau}{1 + \frac{i^{(p)}}{p}}$$

where $i^{(p)}$ and p are as defined earlier.

4.4 Convexity

To determine more precisely the effect of a change in the interest rate (which we will need to do to carry out immunisation calculations in the next section), we need another quantity called *convexity*.

The *convexity* of the cashflow series $\{C_{t_k}\}$ is defined as:

$$\boldsymbol{c(i) = \frac{1}{A}\frac{d^2}{di^2}A = \frac{A''}{A}}$$

$$\boldsymbol{= \left(\frac{1}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}}\right)\left(\sum_{k=1}^{n} C_{t_k}\ t_k\ (t_k + 1) v_i^{t_k+2}\right)}$$

Question

Consider a share that pays a dividend of D annually in arrears in perpetuity.

Derive an expression for the convexity of the share's cashflows, in terms of the annual effective interest rate i.

Solution

The present value of the dividends is:

$$P(i) = Da_{\overline{\infty|}} = \frac{D}{i}$$

Differentiating:

$$P'(i) = -\frac{D}{i^2} \quad \text{and} \quad P''(i) = \frac{2D}{i^3}$$

So the convexity is:

$$\frac{P''(i)}{P(i)} = \frac{2D}{i^3}\bigg/\frac{D}{i} = \frac{2D}{i^3} \times \frac{i}{D} = \frac{2}{i^2}$$

For series of cashflows with the same discounted mean term, a series consisting of payments paid close together will have a low convexity, whereas a series that is more spread out over time will have a higher convexity.

Question

Consider the following two assets:

- Asset A is an 11-year zero-coupon bond.
- Asset B will provide a lump sum payment of £9,663 in 5 years' time and a lump sum payment of £26,910 in 20 years' time.

For each asset, calculate the volatility and convexity, using an interest rate of 10% *pa* effective.

Solution

Asset A

The present value of £100 nominal of Asset A at interest rate i is:

$$P_A(i) = 100(1+i)^{-11}$$

The volatility and convexity are given by:

$$\nu_A(i) = -\frac{P_A'(i)}{P_A(i)} = -\frac{100(-11)(1+i)^{-12}}{100(1+i)^{-11}} = \frac{11}{1+i}$$

and: $$c_A(i) = \frac{P_A''(i)}{P_A(i)} = \frac{100(-11)(-12)(1+i)^{-13}}{100(1+i)^{-11}} = \frac{11\times 12}{(1+i)^2}$$

When $i = 10\%$, we have:

$$\nu_A(0.1) = \frac{11}{1.1} = 10$$

and: $$c_A(0.1) = \frac{11\times 12}{1.1^2} = 109.1$$

The volatility of 10 corresponds to a discounted mean term of:

$$\tau_A = 1.1\times \nu_A(0.1) = 1.1\times 10 = 11 \text{ years}$$

as expected for an 11-year zero-coupon bond.

Asset B

The present value of Asset B at interest rate i is:

$$P_B(i) = 9{,}663(1+i)^{-5} + 26{,}910(1+i)^{-20}$$

The volatility and convexity are given by:

$$\nu_B(i) = -\frac{P_B'(i)}{P_B(i)} = -\frac{9{,}663(-5)(1+i)^{-6} + 26{,}910(-20)(1+i)^{-21}}{9{,}663(1+i)^{-5} + 26{,}910(1+i)^{-20}}$$

and: $$c_B(i) = \frac{P_B''(i)}{P_B(i)} = \frac{9{,}663(-5)(-6)(1+i)^{-7} + 26{,}910(-20)(-21)(1+i)^{-22}}{9{,}663(1+i)^{-5} + 26{,}910(1+i)^{-20}}$$

When $i = 10\%$, we have:

$$\nu_B(0.1) = -\frac{9{,}663(-5)(1.1)^{-6} + 26{,}910(-20)(1.1)^{-21}}{9{,}663(1.1)^{-5} + 26{,}910(1.1)^{-20}} = 10$$

and: $$c_B(0.1) = \frac{9{,}663(-5)(-6)(1.1)^{-7} + 26{,}910(-20)(-21)(1.1)^{-22}}{9{,}663(1.1)^{-5} + 26{,}910(1.1)^{-20}} = 153.7$$

So, Asset A and Asset B have the same volatilities (and hence the same discounted mean term of 11 years), but Asset B has the higher convexity because it involves payments that are more spread out around the discounted mean term.

Combining convexity and duration gives a more accurate approximation to the change in *A* following a small change in interest rates. For small ε :

$$\frac{A(i+\varepsilon) - A(i)}{A} = \frac{\partial A}{\partial i} \times \frac{1}{A} \times \varepsilon + \tfrac{1}{2} \times \frac{\partial^2 A}{\partial i^2} \times \frac{1}{A} \times \varepsilon^2 + \cdots$$

$$\approx -\varepsilon \nu(i) + \varepsilon^2 \times \tfrac{1}{2} \times c(i)$$

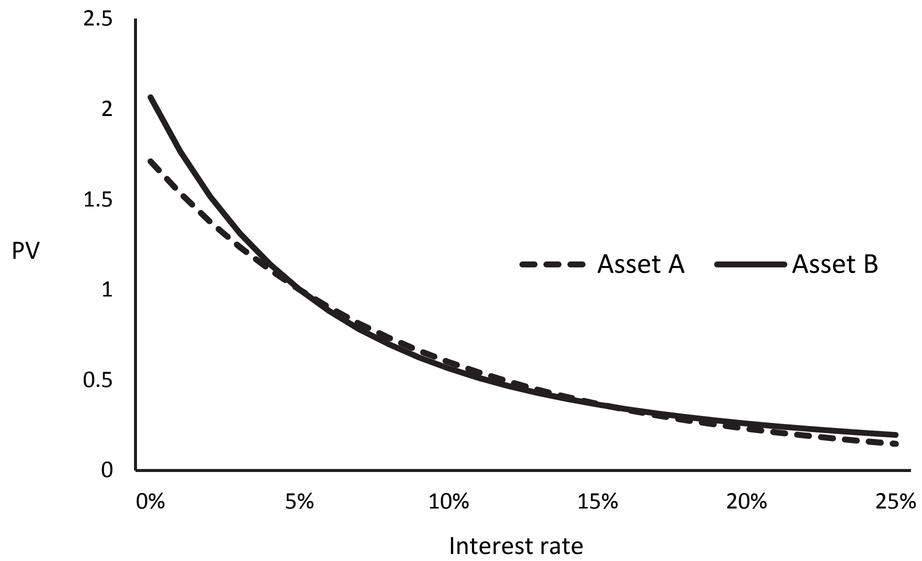
This last result comes from applying Taylor's formula, which appears on page 3 of the *Tables*:

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

Convexity gives a measure of the change in duration of a bond when the interest rate changes. Positive convexity implies that $\tau(i)$ is a decreasing function of i. This means, for example, that A increases more when there is a decrease in interest rates than it falls when there is an increase of the same magnitude in interest rates.

Why is it called 'convexity'?

'Convexity' refers to the shape of the graph of the present value as a function of the interest rate. The following graph shows the present value of the two assets in the previous question (scaled to have a present value of 1 unit at 5% interest). We can see that Asset B (which has the higher convexity) has a more 'curved' graph than Asset A (which has the lower convexity).



4.5 Immunisation

Suppose an organisation has liabilities that will require a known series of cashflows (which we will assume are all negative) and holds assets that will generate a known series of cashflows (which we will assume are all positive) to meet these liabilities.

If it were possible to select a portfolio of assets that generated cashflows that exactly matched the liabilities of the fund (in terms of timing and amount), then the fund would be completely protected against any changes in interest rates. However, this is an idealised scenario and, apart from in very simple cases, perfect matching of this kind cannot be achieved.

It may, however, be possible to choose an asset portfolio that offers the fund a milder form of protection. Suppose the present value of the fund's liabilities and assets, calculated at a valuation rate of interest i, which reflects the interest rate in the market, are $V_L(i)$ and $V_A(i)$, respectively. Then the fund has a *surplus* of $S(i)=V_A(i)-V_L(i)$. We can consider how this surplus would be affected by changes in the interest rate i. In particular, we would be concerned about the downside risk if a change in market interest rates causes the surplus to become negative, *ie* a deficit.

In simple cases, it is possible to select an asset portfolio that will protect this surplus against small changes in the interest rate. This is known as *immunisation*. In the 1950s, the actuary Frank Redington derived the three conditions that are required to achieve immunisation.

Consider a fund with asset cashflows $\{A_{t_k}\}$ and liability cashflows $\{L_{t_k}\}$. Let $V_A(i)$ be the present value of the assets at effective rate of interest i and let $V_L(i)$ be the present value of the liabilities at rate i; let $\nu_A(i)$ and $\nu_L(i)$ be the volatility of the asset and liability cashflows respectively, and let $c_A(i)$ and $c_L(i)$ be the convexity of the asset and liability cashflows respectively.

At rate of interest i_0 the fund is immunised against small movements in the rate of interest of ε if and only if $V_A(i_0)=V_L(i_0)$ and $V_A(i_0+\varepsilon)\geq V_L(i_0+\varepsilon)$.

In words, a fund is said to be *immunised* against small changes in the interest rate if:

- the surplus in the fund at the current interest rate is zero and
- any small change in the interest rate (in either direction) would lead to a positive surplus.

Redington's conditions

Then consider the surplus $S(i) = V_A(i) - V_L(i)$.

From Taylor's theorem:

$$S(i_0 + \varepsilon) = S(i_0) + \varepsilon S'(i_0) + \frac{\varepsilon^2}{2} S''(i_0) + \cdots$$

Consider the terms on the right hand side. We know that $S(i_0) = 0$.

This is because the definition of immunisation requires that the surplus is zero at the current interest rate *ie* $S(i_0) = V_A(i_0) - V_L(i_0) = 0$. In other words, $V_A(i_0) = V_L(i_0)$, *ie* the present value of the assets at the current interest rate is equal to the present value of the liabilities.

The second term, $\varepsilon S'(i_0)$, will be equal to zero for any values of ε (positive or negative) if and only if $S'(i_0) = 0$, that is if $V_A'(i_0) = V_L'(i_0)$.

Since we have already assumed that $V_A(i_0) = V_L(i_0)$, this is equivalent to:

$$-\frac{V_A'(i_0)}{V_A(i_0)} = -\frac{V_L'(i_0)}{V_L(i_0)}$$

In other words, the assets and the liabilities must have the same volatility.

This is equivalent to requiring that $\nu_A(i) = \nu_L(i)$ or (equivalently) that the durations of the two cashflow series are the same.

In the third term, $\frac{\varepsilon^2}{2}$ is always positive, regardless of the sign of ε. Thus, if we ensure that $S''(i_0) > 0$, then the third term will also always be positive.

This is equivalent to requiring that $V_A''(i_0) > V_L''(i_0)$, which is equivalent to requiring that $c_A(i) > c_L(i)$.

For small $|\varepsilon|$ the fourth and subsequent terms in the Taylor expansion will be very small. Hence, given the three conditions above, the fund is protected against small movements in interest rates. This result is known as *Redington's immunisation* after the British actuary who developed the theory.

The conditions for Redington's immunisation may be summarised as follows:

1. **$V_A(i_0) = V_L(i_0)$ – that is, the value of the assets at the starting rate of interest is equal to the value of the liabilities.**

2. **The volatilities of the asset and liability cashflow series are equal, that is, $\nu_A(i_0) = \nu_L(i_0)$.**

3. **The convexity of the asset cashflow series is greater than the convexity of the liability cashflow series – that is, $c_A(i_0) > c_L(i_0)$.**

As mentioned in the above reasoning, the second condition could be replaced by one of the following equivalent conditions:

- the discounted mean terms of the asset and liability cashflow series are equal
- $V_A'(i_0) = V_L'(i_0)$

The third condition could be replaced by the equivalent condition:

- $V_A''(i_0) > V_L''(i_0)$

Question

A fund must make payments of £50,000 at time 6 years and at time 8 years. Interest rates are currently 7% *pa* effective at all durations.

Show that immunisation against small changes in the interest rate can be achieved by purchasing appropriately chosen nominal amounts of a 5-year zero-coupon bond and a 10-year zero-coupon bond.

Solution

Let P denote the nominal amount purchased of the 5-year bond (so that a cashflow of P is received at time 5) and let Q denote the nominal amount purchased of the 10-year (so that a cashflow of Q is received at time 10). Then the present value of the assets is:

$$V_A(0.07) = Pv^5 + Qv^{10} \quad @7\%$$

By Redington's first condition, this must equal the present value of the liabilities, which is:

$$V_L(0.07) = 50,000(v^6 + v^8) \ @7\% = £62,418$$

This gives us our first equation:

$$Pv^5 + Qv^{10} = 62,418 \qquad (1)$$

The negative of the derivative (with respect to i) of the PV of the assets is given by:

$$-V_A'(0.07) = P \times 5v^6 + Q \times 10v^{11} \quad @7\%$$

So the volatility of the assets is:

$$-\frac{V_A'(0.07)}{V_A(0.07)}=\frac{5Pv^6+10Qv^{11}}{V_L(0.07)}=\frac{5Pv^6+10Qv^{11}}{62,418}$$

where we have used equation 1 to set $V_A(0.07)=V_L(0.07)=£62,418$.

By Redington's second condition, this must equal the volatility of the liabilities, which is:

$$-\frac{V_L'(0.07)}{V_L(0.07)}=\frac{50,000(6v^7+8v^9)}{62,418}=\frac{404,398}{62,418}$$

Since the denominators in the volatilities are the same, we have the second equation:

$$5Pv^6+10Qv^{11}=404,398 \qquad (2)$$

We can solve equations (1) and (2) simultaneously. Multiplying equation (1) by $5v$ and then subtracting this from equation (2) gives:

$$5Qv^{11}=404,398-5v\times 62,418 \quad \Rightarrow \quad Q=\frac{404,398-5\times 1.07^{-1}\times 62,418}{5\times 1.07^{-11}}=£47,454$$

Substituting this into equation 1 gives:

$$P=(62,418-Qv^{10})(1+i)^5=(62,418-47,454\times 1.07^{-10})(1.07)^5=£53,710$$

This determines the portfolio of assets we require. We now need to check Redington's third condition. With these values of P and Q, the convexity of the assets is:

$$\frac{V_A''(0.07)}{V_A(0.07)}=\frac{P(-5)(-6)v^7+Q(-10)(-11)v^{12}}{62,418}=\frac{3,321,152}{62,418}=53.21$$

The convexity of the liabilities is:

$$\frac{V_L''(0.07)}{V_L(0.07)}=\frac{50,000((-6)(-7)v^8+(-8)(-9)v^{10})}{62,418}=\frac{50,000\times 61.045}{62,418}=48.90$$

So the convexity of the assets exceeds the convexity of the liabilities. This is what we would expect, since the asset cashflows (at times 5 and 10) are more spread out around the discounted mean term than the liability cashflows (at times 6 and 8).

Since all three of Redington's conditions are satisfied, the fund is immunised against small changes in the interest rate around 7% *pa*.

We can verify numerically that this fund is indeed immunised by calculating the surplus for interest rates a small distance either side of 7% pa.

The surplus, calculated at interest rate i, is:

$$V_A(i)-V_L(i)=(53{,}710v^5+47{,}454v^{10})-50{,}000(v^6+v^8)$$

So: $V_A(0.065)-V_L(0.065)=64{,}482-64{,}478=4>0$

and: $V_A(0.075)-V_L(0.075)=60{,}437-60{,}433=4>0$

We see that a 0.5% movement in interest rates in either direction will result in a positive surplus, ie the fund is immunised.

In practice there are difficulties with implementing an immunisation strategy based on these principles. For example, the method requires continuous rebalancing of portfolios to keep the asset and liability volatilities equal.

The asset portfolio required to provide Redington immunisation normally depends on the initial interest rate. Once the interest rates have moved away from the initial rate, it may be necessary to 'rebalance' the portfolio so that it is once again immunised at the new rate. This makes the practical application of the technique quite laborious except in very simple situations.

Other limitations of immunisation include:

- **There may be options or other uncertainties in the assets or in the liabilities, making the assessment of the cashflows approximate rather than known.**
- **Assets may not exist to provide the necessary overall asset volatility to match the liability volatility.**
- The theory relies upon small changes in interest rates. The fund may not be protected against large changes.
- The theory assumes a flat yield curve and requires the same change in interest rates at all terms. In practice, this is rarely the case.
- Immunisation removes the likelihood of making large profits.

Despite these problems, immunisation theory remains an important consideration in the selection of assets.

In practice, actuaries making investment decisions are aware of Redington's theory in a general sense. For example, they are aware of the consequences of investing 'long' (*ie* holding assets with a higher DMT than the liabilities), but they would not normally apply the theory directly. A more open-ended technique called *asset-liability modelling* is often used instead, and this is covered in later subjects.

Chapter 14 Summary

The yield on a unit zero-coupon bond with term n years is called the n-year spot rate of interest, y_n.

The variation by term of interest rates is often referred to as the term structure of interest rates. The curve of spot rates is an example of a yield curve.

The discrete-time forward rate, $f_{t,r}$, is the annual interest rate agreed at time 0 for an investment made at time $t > 0$ for a period of r years.

There is a direct relationship between forward rates of interest and spot rates:

$$(1+f_{t,r})^r = \frac{(1+y_{t+r})^{t+r}}{(1+y_t)^t} = \frac{P_t}{P_{t+r}}$$

where P_t denotes the price at issue of a unit zero-coupon bond with term t years.

The three most popular explanations for the fact that interest rates vary according to the term of the investment are:

1. Expectations theory
2. Liquidity preference
3. Market segmentation

The performance of a fixed-interest investment can be assessed by its yield to maturity or redemption yield.

The n-year par yield represents the coupon per £1 nominal that would be payable on a bond with term n years, which would give the bond a price under the current term structure of £1 per £1 nominal, assuming the bond is redeemed at par.

The effects of changes in interest rates on the cashflows generated by an asset or required by a liability can be quantified by calculating the discounted mean term (duration), the volatility and the convexity. The discounted mean term and the volatility are related.

$$Volatility = \nu(i) = -\frac{A'}{A} = \sum_{k=1}^{n} C_k t_k v^{t_k+1} \bigg/ \sum_{k=1}^{n} C_k v^{t_k}$$

$$DMT = \tau(i) = \sum_{k=1}^{n} t_k C_k v^{t_k} \bigg/ \sum_{k=1}^{n} C_k v^{t_k} = (1+i) \times Volatility$$

$$Convexity = c(i) = \frac{A''}{A} = \sum_{k=1}^{n} C_k t_k (t_k+1) v^{t_k+2} \bigg/ \sum_{k=1}^{n} C_k v^{t_k}$$

The surplus in a fund can be immunised against small changes in interest rates if Redington's conditions can be met. These require the assets and liabilities to have the same present value and discounted mean term (or volatility), and for the convexity of the assets to exceed that of the liabilities. The conditions are:

1. $V_A(i_0) = V_L(i_0)$ *ie* $PV(Assets) = PV(Liabilities)$

2. $V'_A(i_0) = V'_L(i_0)$ *ie* $Volatility(Assets) = Volatility(Liabilities)$

   or $DMT(Assets) = DMT(Liabilities)$

3. $V''_A(i_0) > V''_L(i_0)$ *ie* $Convexity(Assets) > Convexity(Liabilities)$

Chapter 14 Practice Questions

14.1 The n-year spot rate is estimated using the function $y_n = 0.09 - 0.03e^{-0.1n}$. Calculate the one-year forward rate at time 10.

14.2 Let f_t denote the 1-year forward rate for a transaction beginning at time t. Calculate the 3-year par yield, given that:

$$f_0 = 6\%, \qquad f_1 = 6.5\%, \qquad f_2 = 7\%$$

14.3 Exam style

In a particular bond market, the two-year par yield at time $t = 0$ is 5.65% and the issue price at time $t = 0$ of a two-year fixed-interest stock, paying coupons of 7% annually in arrears and redeemed at 101, is £103.40 per £100 nominal.

Calculate:

(a) the one-year spot rate

(b) the two-year spot rate. [6]

14.4 Exam style

Three bonds each paying annual coupons in arrears of 6% and redeemable at £103 per £100 nominal reach their redemption dates in exactly one, two and three years' time, respectively. The price of each bond is £97 per £100 nominal.

(i) Calculate the gross redemption yield of the 3-year bond. [3]

(ii) Calculate the one-year and two-year spot rates implied by the information given. [3]

[Total 6]

14.5 (i) Explain what is meant by the following theories of the shape of the yield curve:

Exam style

(a) market segmentation theory

(b) liquidity preference theory. [4]

Short-term, one-year annual effective interest rates are currently 6%; they are expected to be 5% in one year's time; 4% in two years' time and 3% in three years' time.

(ii) Calculate the gross redemption yields from one-year, two-year, three-year and four-year zero-coupon bonds using the above expected interest rates. [4]

The price of a coupon-paying bond is calculated by discounting individual payments from the bond at the zero-coupon yields in part (ii).

(iii) Calculate the gross redemption yield of a bond that pays a coupon of 4% per annum annually in arrears and is redeemed at 110% in exactly four years. [5]

(iv) Explain why the gross redemption yield of a bond that pays a coupon of 8% per annum annually in arrears and is redeemed at par would be greater than that calculated in part (iii). [2]

The government introduces regulations that require banks to hold more government bonds with very short terms to redemption.

(v) Explain, with reference to market segmentation theory, the likely effect of this regulation on the pattern of spot rates calculated in part (ii). [2]

[Total 17]

14.6 Consider a fixed-interest security that pays coupons of 10% at the end of each year and is redeemable at par at the end of the third year.

Calculate, using an effective interest rate of 8% *pa*, the:

(i) volatility of the cashflows

(ii) discounted mean term of the cashflows

(iii) convexity of the cashflows.

14.7 Consider the three 20-year annuities described below:

(i) level payments of £1,000 payable annually in arrears

(ii) increasing payments made annually in arrears, where the first payment is £1,000 and the payments increase by 10% *pa* compound each year thereafter

(iii) continuous payments at the rate of £1,000 *pa* over the 20 years.

Calculate the discounted mean term of each annuity using an interest rate of 10% *pa* effective.

14.8 Exam style

A company has to pay £ $2{,}000(10-t)$ at the end of year t, for $t=5,6,7,8,9$. It values these liabilities assuming that there will be a constant effective annual rate of interest of 6% *pa*.

(i) Calculate the present value of the company's liabilities. [3]

The company wants to immunise its exposure to the liabilities by investing in two bonds:

- Bond A pays coupons of 5% *pa* annually in arrears and is redeemable at par in 15 years' time
- Bond B is a zero-coupon bond that is redeemable at par in 5 years' time.

The gross redemption yield on both stocks is the same as the interest rate used to value the liabilities.

(ii) (a) Determine the amount that the company should invest in each of the two bonds to ensure that the present value and discounted mean term of the assets are equal to those of the liabilities.

(b) State the third condition required for immunisation. [9]

[Total 12]

14.9 Exam style

An insurance company has liabilities of £10 million due in 10 years' time and £20 million due in 15 years' time. The company's assets consist of two zero-coupon bonds. One pays £7.404 million in 2 years' time and the other pays £31.834 million in 25 years' time. The current interest rate is 7% per annum effective.

(i) Show that Redington's first two conditions for immunisation against small changes in the rate of interest are satisfied for this insurance company. [6]

(ii) Calculate the present value of profit that the insurance company will make if the interest rate increases immediately to 7.5% per annum effective. [2]

(iii) Explain, without any further calculation, why the insurance company made a profit as a result of the change in the interest rate. [2]

[Total 10]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 14 Solutions

14.1 Using the formula given, the 10-year and 11-year spot rates are:

$$y_{10} = 0.09 - 0.03e^{-0.1\times10} = 0.07896$$

and:

$$y_{11} = 0.09 - 0.03e^{-0.1\times11} = 0.08001$$

Therefore:

$$1+f_{10} = \frac{(1+y_{11})^{11}}{(1+y_{10})^{10}} = 1.0906$$

ie the one-year forward rate at time 10 is 9.06%.

14.2 The par yield, c , is the annual coupon that gives a bond price equal to the par value based on the current term structure of interest rates.

Using the 1-year forward rates given, the 3-year par yield is found from the equation:

$$1 = c\left(\frac{1}{1.06} + \frac{1}{1.06\times1.065} + \frac{1}{1.06\times1.065\times1.07}\right) + \frac{1}{1.06\times1.065\times1.07}$$

ie $1 = c(0.94340 + 0.88582 + 0.82787) + 0.82787 \quad \Rightarrow \quad c = 0.06478$

The 3-year par yield is therefore 6.478%.

14.3 Let y_1 be the one-year spot rate and y_2 be the two-year spot rate.

Using the two-year par yield of 5.65%, and assuming that we have £100 nominal, gives:

$$100 = \frac{5.65}{1+y_1} + \frac{100+5.65}{(1+y_2)^2} = \frac{5.65}{1+y_1} + \frac{105.65}{(1+y_2)^2} \qquad [1]$$

Considering the two-year fixed-interest bond gives a second equation:

$$103.40 = \frac{7}{1+y_1} + \frac{101+7}{(1+y_2)^2} = \frac{7}{1+y_1} + \frac{108}{(1+y_2)^2} \qquad [1]$$

Rearranging both equations to get $\frac{1}{(1+y_2)^2}$ gives:

$$\frac{1}{(1+y_2)^2} = \frac{1}{105.65}\left(100 - \frac{5.65}{1+y_1}\right) = \frac{1}{108}\left(103.40 - \frac{7}{1+y_1}\right)$$

This can be solved for y_1 :

$$10{,}800-\frac{610.2}{1+y_1}=10{,}924.21-\frac{739.55}{1+y_1}$$

$$\Rightarrow \quad \frac{129.35}{1+y_1}=124.21$$

$$\Rightarrow \quad y_1=4.14\% \qquad [2]$$

Substituting this into the equation for the par yield:

$$100=\frac{5.65}{1.0414}+\frac{105.65}{(1+y_2)^2} \quad \Rightarrow \quad (1+y_2)^2=1.11711 \quad \Rightarrow \quad y_2=5.69\% \qquad [2]$$

So:

(a) the one-year spot rate is 4.14%

(b) the two-year spot rate is 5.69%.

[Total 6]

14.4 *This question is Subject CT1, April 2013, Question 3.*

(i) ***GRY of the three-year bond***

The equation of value for the three-year bond is:

$$97=6v+6v^2+109v^3 \qquad [1]$$

This equation can be solved by trial and error. To obtain a first guess for the gross redemption yield, we note that the coupons provide an annual return of about 6%, and there is a capital gain at redemption of 6 (the redemption amount minus the price paid), so the overall yield will be greater than 6%.

Using trial and error we get:

$$i=8\% \quad \Rightarrow \quad RHS=97.2273$$

$$i=8.5\% \quad \Rightarrow \quad RHS=95.9637 \qquad [1]$$

Interpolating between 8% and 8.5%, we have:

$$GRY \approx 8\%+\frac{97-97.2273}{95.9637-97.2273}\times(8.5\%-8\%)=8.09\%\ pa \qquad [1]$$

[Total 3]

(ii) ***Spot rates***

The equation of value for the one-year bond (with one-year spot rate y_1) is:

$$97 = 109 \times \frac{1}{1+y_1} \quad \Rightarrow \quad y_1 = 12.371\% \ pa \qquad [1]$$

The equation of value for the two-year bond (with two-year spot rate y_2) is:

$$97 = 6 \times \frac{1}{1+y_1} + 109 \times \frac{1}{(1+y_2)^2} \quad \Rightarrow \quad y_2 = 9.049\% \ pa \qquad [2]$$

[Total 3]

14.5 *This question is Subject CT1, September 2014, Question 8.*

(i)(a) ***Market segmentation theory***

Market segmentation theory says that the shape of the yield curve is determined by supply and demand at different terms. So, for example, yields at the short end of the curve will be determined by demand from investors who have a preference for short-dated stocks, *ie* those with short-dated liabilities, as well as by the supply of short-dated stock. Similarly, yields at the long end will be a function of demand from investors interested in buying long bonds to match long-dated liabilities, and of supply of long-dated stock. [2]

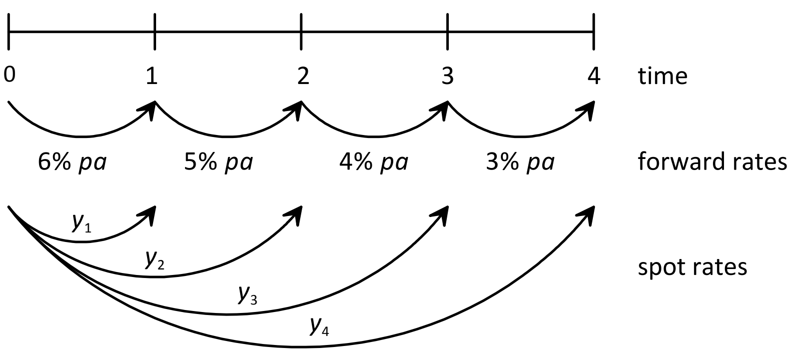
(i)(b) ***Liquidity preference theory***

Liquidity preference theory states that investors will in general prefer more liquid (*ie* shorter) stocks to less liquid ones, as short-dated stocks are less sensitive to changes in interest rates. Hence, investors purchasing long-dated bonds will require higher yields in order to compensate them for the greater volatility of the stock they are purchasing. [2]

[Total 4]

(ii) ***Gross redemption yields from zero-coupon bonds***

The diagram of the rates we are given in the question is:



The gross redemption yields for the zero-coupon bonds are the spot rates. Hence:

$$y_1 = 6\%\ pa \qquad [1]$$

$$(1+y_2)^2 = 1.06 \times 1.05 \quad \Rightarrow \quad y_2 = 5.499\%\ pa \qquad [1]$$

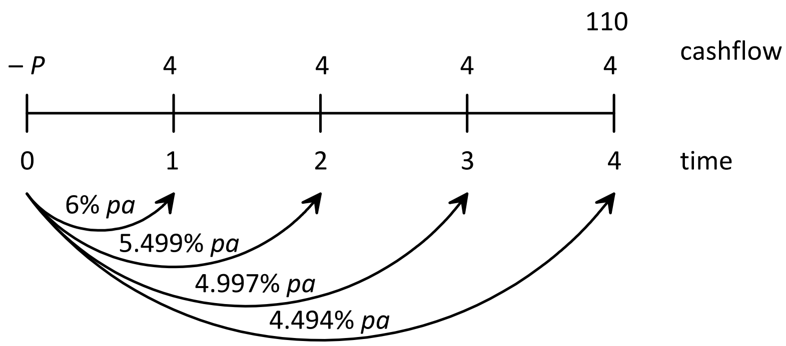
$$(1+y_3)^3 = 1.06 \times 1.05 \times 1.04 \quad \Rightarrow \quad y_3 = 4.997\%\ pa \qquad [1]$$

$$(1+y_4)^4 = 1.06 \times 1.05 \times 1.04 \times 1.03 \quad \Rightarrow \quad y_4 = 4.494\%\ pa \qquad [1]$$

[Total 4]

(iii) ***Gross redemption yield from coupon-paying bond***

The cashflow diagram is:



So the price of the bond is given by:

$$P = 4\left[1.06^{-1} + 1.05499^{-2} + 1.04997^{-3}\right] + 114 \times 1.04494^{-4} = 106.441 \qquad [1]$$

The equation for the gross redemption yield is:

$$106.441 = 4a_{\overline{4}|} + 110v^4 \qquad [1]$$

We will use trial and error to solve this for the GRY. Since we discounted the cashflows at spot rates of 6%, 5.499%, 4.997% and 4.494%, the GRY will lie between the smallest and largest of these values. Also, since most of the cashflow arises at the end of four years, it is likely that y_4 will give us a reasonable first approximation.

$$i = 4.5\% \quad \Rightarrow \quad 4a_{\overline{4}|} + 110v^4 = 106.592 \qquad [1]$$

$$i = 5\% \quad \Rightarrow \quad 4a_{\overline{4}|} + 110v^4 = 104.681 \qquad [1]$$

Using linear interpolation between these two values, we obtain a gross redemption yield of:

$$4.5\%+\frac{106.441-106.592}{104.681-106.592}\times(5\%-4.5\%)=4.54\%$$ [1]

[Total 5]

(iv) ***Explanation***

For the bond with the 8% coupon redeemed at par, a greater proportion of the proceeds are received earlier, in the coupon payments, rather than in the redemption proceeds. [1]

Since interest rates are higher earlier on, the GRY (which is a weighted average of the rates used to discount the payments) would be higher. [1]

[Total 2]

(v) ***Effect of regulation***

If banks are required to hold more short-dated bonds, demand for these types of bonds will increase. [1]

If the supply of these bonds remains the same, this will push up the price of these bonds, causing their gross redemption yields and hence spot rates to fall. [1]

[Total 2]

14.6 The present value of the cashflows for £100 nominal of this stock is:

$$P(i)=10v+10v^2+110v^3$$

Differentiating gives:

$$P'(i)=10(-1)v^2+10(-2)v^3+110(-3)v^4$$

$$P''(i)=10(-1)(-2)v^3+10(-2)(-3)v^4+110(-3)(-4)v^5$$

(i) Using the above expressions, the volatility is:

$$-\frac{P'(0.08)}{P(0.08)}=-\frac{10(-1)(1.08)^{-2}+10(-2)(1.08)^{-3}+110(-3)(1.08)^{-4}}{10(1.08)^{-1}+10(1.08)^{-2}+110(1.08)^{-3}}=\frac{267.01}{105.15}=2.54$$

(ii) The discounted mean term is the volatility multiplied by $1+i$:

$$2.54\times1.08=2.74 \text{ years}$$

(iii) The convexity is:

$$\frac{P''(0.08)}{P(0.08)}=\frac{10(-1)(-2)(1.08)^{-3}+10(-2)(-3)(1.08)^{-4}+110(-3)(-4)(1.08)^{-5}}{105.15}$$

$$=\frac{958.35}{105.15}=9.11$$

14.7 (i) For the level annuity, payable in arrears, we have:

$$DMT=\frac{\sum_{t=1}^{20} t\times 1{,}000v^t}{\sum_{t=1}^{20} 1{,}000v^t}=\frac{\sum_{t=1}^{20} tv^t}{\sum_{t=1}^{20} v^t}=\frac{(Ia)_{\overline{20|}}}{a_{\overline{20|}}}=\frac{63.9205}{8.5136}=7.51 \text{ years}$$

(ii) For the compound increasing annuity, payable in arrears, we have:

$$DMT=\frac{\sum_{t=1}^{20} t\times 1{,}000\times 1.1^{t-1}\times v^t}{\sum_{t=1}^{20} 1{,}000\times 1.1^{t-1}\times v^t}=\frac{\sum_{t=1}^{20} t\times 1.1^{t-1}\times v^t}{\sum_{t=1}^{20} 1.1^{t-1}\times v^t}$$

Here $i=0.1$, so $1.1^{t-1}v^{t-1}=1$, and this expression simplifies to:

$$DMT=\frac{\sum_{t=1}^{20} tv}{\sum_{t=1}^{20} v}=\frac{\sum_{t=1}^{20} t}{\sum_{t=1}^{20} 1}=\frac{\frac{1}{2}\times 20\times 21}{20}=10.5 \text{ years}$$

To evaluate the numerator in the expression above, we have used the formula for the sum of the first n integers:

$$\sum_{k=1}^{n} k=\frac{1}{2}n(n+1)$$

(iii) For the continuous annuity, we have:

$$\frac{\int_0^{20} t\times 1{,}000v^t\,dt}{\int_0^{20} 1{,}000v^t\,dt}=\frac{\int_0^{20} tv^t\,dt}{\int_0^{20} v^t\,dt}=\frac{\left(\bar{I}\bar{a}\right)_{\overline{20|}}}{\bar{a}_{\overline{20|}}}=\frac{62.5286}{8.9325}=7.00 \text{ years}$$

14.8 (i) ***Present value of liabilities***

The liabilities are illustrated on the timeline below:

Payment			10,000	8,000	6,000	4,000	2,000
Time (yrs)	0	...	5	6	7	8	9

The present value of the liabilities is:

$$PV_L=10{,}000v^5+8{,}000v^6+6{,}000v^7+4{,}000v^8+2{,}000v^9 \qquad [1]$$

If $i = 0.06$, then:

$$PV_L = 10{,}000(1.06)^{-5} + 8{,}000(1.06)^{-6} + 6{,}000(1.06)^{-7} + 4{,}000(1.06)^{-8} + 2{,}000(1.06)^{-9}$$ [2]

$$= £20{,}796$$

[Total 3]

Alternatively, we could calculate this as:

$$PV_L = 2{,}000v^4\left[6a_{\overline{5}|} - (Ia)_{\overline{5}|}\right] = 2{,}000(1.06)^{-4}(25.2742 - 12.1469)$$

(ii)(a) ***Amount to be invested in each bond***

Let A denote the amount invested in Bond A and B denote the amount invested in Bond B.

The present value of the assets is then:

$$PV_A = A + B$$

Setting this equal to the present value of the liabilities, we have:

(1) … $A + B = 20{,}796$ [1]

The discounted mean term of the liabilities is:

$$DMT_L = \frac{5\times 10{,}000v^5 + 6\times 8{,}000v^6 + 7\times 6{,}000v^7 + 8\times 4{,}000v^8 + 9\times 2{,}000v^9}{PV_L}$$

$$= \frac{129{,}865}{PV_L} = 6.245 \text{ years}$$ [1]

To calculate the discounted mean term of the assets, we first need to calculate the price of Bond A per £100 nominal. This is:

$$P = 5a_{\overline{15}|} + 100v^{15} = (5\times 9.7122) + \frac{100}{1.06^{15}} = 90.288$$ [1]

So an investment of A in Bond A buys $\dfrac{A}{90.288}$ lots of £100 nominal.

The discounted mean term of the assets is then:

$$DMT_A = \frac{\frac{A}{90.288}\left(1\times 5v + 2\times 5v^2 + \cdots + 15\times 5v^{15} + 15\times 100v^{15}\right) + 5B}{PV_A}$$

$$= \frac{\frac{A}{90.288}\left(5(Ia)_{\overline{15}|} + 1{,}500v^{15}\right) + 5B}{PV_A}$$

$$= \frac{\frac{A}{90.288}\left(5\times 67.2668 + \frac{1{,}500}{1.06^{15}}\right) + 5B}{PV_A}$$

$$= \frac{10.657A + 5B}{PV_A} \qquad [2]$$

Setting the discounted mean term of the assets equal to the discounted mean term of the liabilities we obtain:

(2) … $10.657A + 5B = 129{,}865$ [1]

Now multiplying (1) by 5, we get:

(3) … $5A + 5B = 103{,}980$

and subtracting (3) from (2):

$5.657A = 25{,}885 \quad \Rightarrow \quad A = £4{,}576$ [1]

Substituting this back into (1), we find that:

$B = 20{,}796 - 4{,}576 = £16{,}220$ [1]

Alternatively, defining A to be the nominal amount purchased of Bond A, and B to be the nominal amount purchased of Bond B, the present value of the assets is:

$$PV_A = A\left(0.05a_{\overline{15}|} + v^{15}\right) + Bv^5 = 0.90288A + 0.74726B$$

The DMT of the assets is:

$$DMT_A = \frac{A\left(0.05v + 2\times 0.05v^2 + \cdots + 15\times 0.05v^{15} + 15v^{15}\right) + 5Bv^5}{PV_A}$$

$$= \frac{A\left(0.05(Ia)_{\overline{15}|} + 15v^{15}\right) + 5Bv^5}{PV_A}$$

$$= \frac{9.6223A + 3.7363B}{PV_A}$$

These give us the simultaneous equations:

$$0.90288A + 0.74726B = 20{,}796$$

$$9.6223A + 3.7363B = 129{,}865$$

Solving these gives $A = 5{,}068$ *and* $B = 21{,}707$. *Converting these values into the amounts invested in the bonds (ie the present values) asked for in the question, gives the same answers as above.*

(ii)(b) ***Other condition for immunisation***

We also require that the convexity of the assets is greater than the convexity of the liabilities at the current rate of interest, *ie*:

$$\frac{d^2 PV_A}{di^2} \Big/ PV_A > \frac{d^2 PV_L}{di^2} \Big/ PV_L \qquad [1]$$

[Total 9]

14.9 (i) ***Two conditions for immunisation***

The present value of the assets and liabilities at 7% *pa* are:

$$PV_A = 7.404v^2 + 31.834v^{25} = 12.332 \qquad [½]$$

$$PV_L = 10v^{10} + 20v^{15} = 12.332 \qquad [½]$$

Since these are equal, the first condition for immunisation is satisfied. [1]

The discounted mean terms for the assets and liabilities at 7% *pa* are:

$$DMT_A = \frac{7.404v^2 \times 2 + 31.834v^{25} \times 25}{PV_A} = \frac{159.569}{PV_A} \qquad [1½]$$

$$DMT_L = \frac{10v^{10} \times 10 + 20v^{15} \times 15}{PV_L} = \frac{159.569}{PV_L} \qquad [1½]$$

Since we know that the denominators are equal and we can see that the numerators are equal, the DMTs are also equal. [1]

So the first two conditions for immunisation are both satisfied. [Total 6]

Instead of comparing the numerators here, we could have calculated the actual DMTs to be:

$$\frac{159.569}{12.332} = 12.939 \text{ years}$$

Alternatively, we could consider the volatilities:

$$vol_A = -\frac{PV_A{}'}{PV_A} = -\left[\frac{7.404v^3 \times (-2) + 31.834v^{26} \times (-25)}{PV_A}\right] = -\left[\frac{-149.130}{12.332}\right] = 12.093$$

$$vol_A = -\frac{PV_L{}'}{PV_L} = -\left[\frac{10v^{11} \times (-10) + 20v^{16} \times (-15)}{PV_L}\right] = -\left[\frac{-149.130}{12.332}\right] = 12.093$$

and conclude that the volatility of the assets is equal to the volatility of the liabilities (or the numerator of the volatility of the assets is equal to the numerator of the volatility of the liabilities).

(ii) ***Present value of profit***

The present value of the profit at the new interest rate of 7.5% *pa* is:

$$PV_{profit} = PV_A - PV_L = 7.404v^2 + 31.834v^{25} - 10v^{10} - 20v^{15} = 0.015773 \qquad [2]$$

This gives us a present value of profit of about £15,800. [Total 2]

(iii) ***Explanation of profit***

In this scenario, we can see that the spread of the times of the asset cashflows around the discounted mean term is greater than the corresponding spread of the liabilities. [1]

So the third condition for immunisation will also be satisfied, and we will be immunised against any losses arising from a small change in the interest rate. So, when the rate changes to 7.5% *pa*, we make a small profit. [1]

[Total 2]

15

The life table

Syllabus objectives

4.1 Define various assurance and annuity contracts.

4.1.1 Define the following terms:

– premium

– benefit

4.2 Develop formulae for the means and variances of the payments under various assurance and annuity contracts, assuming a constant deterministic interest rate.

4.2.1 Describe the life table functions l_x and d_x and their select equivalents $l_{[x]+r}$ and $d_{[x]+r}$.

4.2.2 Define the following probabilities: ${}_np_x$, ${}_nq_x$, ${}_{n|m}q_x$, ${}_{n|}q_x$ and their select equivalents ${}_np_{[x]+r}$, ${}_nq_{[x]+r}$, ${}_{n|m}q_{[x]+r}$, ${}_{n|}q_{[x]+r}$.

4.2.3 Express the probabilities defined in 4.2.2 in terms of life table functions defined in 4.2.1.

0 Introduction

So far in this course, we have concentrated on placing a value on cashflows that are certain to happen. For example, $a_{\overline{5}|}$ gives the present value of a payment of 1 at the end of each of the next 5 years, where each payment definitely occurs.

We now start to consider valuing uncertain future cashflows, such as those faced by a life insurance company. These cashflows may be dependent on the survival, or otherwise, of an individual, or on other uncertain future events, such as when an individual becomes sick, or chooses to retire.

Some different types of life insurance contracts, *eg* term assurances and pure endowments, were first introduced in Chapter 3.

Life insurance contracts (also called policies) are made between a life insurance company and one or more persons called the policyholders.

The policyholder(s) will agree to pay an amount or a series of amounts to the life insurance company, called premiums.

The premiums may be paid:

- on a regular basis (known as regular premiums), typically paid monthly, quarterly or annually, or
- as one single payment (known as a single premium).

In return the life insurance company agrees to pay an amount or amounts called the benefit(s), to the policyholder(s) on the occurrence of a specified event.

In this subject we first consider contracts with a single policyholder and then later show how to extend the theory to two policyholders.

The benefits payable under simple life insurance contracts are of two main types.

(a) The benefit may be payable on or following the death of the policyholder.

For example, under a term assurance contract, the insurance company will make a payment to the policyholder's estate if the policyholder dies during the term of the policy.

(b) The benefit(s) may be payable provided the life survives for a given term. An example of this type of contract is an annuity, under which amounts are payable at regular intervals as long as the policyholder is still alive.

Such annuities are called 'life annuities' and we will look at these in Chapter 17. Note that life annuities are different to the annuities we studied in the earlier part of this course, as each payment under a life annuity is not certain to occur.

More generally, the theory of this subject may be applied to 'near-life' contingencies – such as the state of health of a policyholder – or to 'non-life' contingencies – such as the cost of replacing a machine at the time of failure. However, in this Subject, we confine ourselves to cases where the payment is of known amount only.

1 Present values of payments under life insurance and annuity contracts

1.1 Equations of value

To calculate premiums for life insurance contracts, we can use a similar *equation of value* to the one we met in Chapter 10, and have used extensively since:

Present value of income = Present value of outgo

Now, because each payment is not certain to happen, we should include the probability of its occurrence and this leads us to consider *expected present values*, rather than present values. So, the equation of value becomes:

Expected present value of income = Expected present value of outgo

When we are applying this equation to calculate an insurance premium, we are considering the money into and out of the insurance company. So, the items that we will be considering are:

- premiums coming in
- expected payments to policyholders (*ie* benefits) going out
- expenses that are incurred by the company.

Much actuarial work is concerned with finding a fair price for a life insurance contract. In such calculations we must consider:

(a) the time value of money, and

(b) the uncertainty attached to payments to be made in the future, depending on the death or survival of a given life.

This requires us to bring together the topics covered earlier in this course, in particular compound interest, and the topics covered in Subject CS2, in particular the unknown future lifetime and its associated probabilities.

1.2 Allowance for investment income

Premiums are usually paid in advance (*eg* at the start of each month or year). This is to protect the insurance company against policyholders dying before paying any premium, and therefore receiving a benefit payment 'for free'. In addition, policyholders can lapse their contracts (*ie* stop paying their premiums), so if premiums were payable annually in arrears, policyholders might be tempted to lapse at the end of a year, thus receiving that year's insurance protection without paying the corresponding premium.

The benefits purchased by the premiums will be paid at a later time – sometimes much later, *eg* at the end of the term of the contract, or when the policyholder dies. This means that the insurer will need to allow for the time value of money when calculating the appropriate premium to charge.

In this and subsequent chapters we will usually assume that money can be invested or borrowed at some given rate of interest. We will always assume that the rate of interest is *known*, that is, deterministic, but we will not always assume that the rate of interest is *constant*. When the rate of interest *is* constant, we denote the effective compound rate of interest per annum by i and define $v=(1+i)^{-1}$, and we will use these without further comment.

In practice, life insurance companies may price products using either of the approaches mentioned above, *ie* they may:

- assume a constant interest rate for each future year
- use a deterministic approach where interest rates are assumed to change in a predetermined way.

Alternatively, they may use a stochastic approach, where future interest rates are random and follow a certain statistical distribution, although we will not consider this approach in this Subject.

1.3 Other assumptions

We will also assume knowledge of the basic probabilities introduced in CS2, namely $_tp_x$ and $_tq_x$ and their degenerate quantities, when $t=1$, p_x and q_x, and the force of mortality μ_x. Further, we assume knowledge of the key survival model formulae, the fundamental ones being that:

$$_tq_x=\int_0^t {}_sp_x\,\mu_{x+s}\,ds$$

and:

$$_tp_x=\exp\left\{-\int_0^t \mu_{x+s}\,ds\right\}$$

In the notation defined above, $_tp_x$ is the probability that a life aged x survives for at least another t years and $_tq_x$ is the probability that a life aged x dies within the next t years. In addition, p_x is the probability that a life aged x survives for at least another year and q_x is the probability that a life aged x dies within the next year. Note that $_tp_x+{}_tq_x=1$ and $p_x+q_x=1$.

The force of mortality at exact age $x+s$, μ_{x+s}, denotes the annual rate of transfer between alive and dead at exact age $x+s$, *ie* it is the annual rate at which people are dying at that exact age. It is probably most helpful to think of this as:

$\mu_{x+s}\,ds\approx$ probability of a life aged $x+s$ dying over the short time interval $(x+s, x+s+ds)$

For example:

$${}_s p_x \, \mu_{x+s} \, ds \approx$$ probabilty of a life aged x living for another s years and then dying in the next instant of time

so (remembering that an integral is just the continuous version of a summation):

$${}_t q_x = \int_0^t {}_s p_x \, \mu_{x+s} \, ds =$$ probability that a life aged x dies at any of the possible moments over the next t years

Using the two basic building blocks described in Section 1.1, and the assumptions made in this section and the last, we will develop formulae for the means and variances of the present value of contingent benefits.

We will do this in Chapter 16 for assurance contracts and Chapter 17 for annuity contracts.

We will consider ways of assigning probability values to the unknown future lifetime, so as to evaluate the formulae. Two different assumptions are typically used in practice when determining the probability values. The first is to assume that the underlying probability depends on age only. The second is to assume that the underlying probability depends on age plus duration since some specific event. For example, considering mortality, the assumption in the second case can allow for the likely lower level of mortality which might result from having to pass a medical test before the insurer agrees to issue a life insurance contract.

Under the first assumption, all lives aged x are assumed to have the same mortality, regardless of when they took out their policies. Under the second assumption, this is no longer the case. The mortality of the policyholder is now assumed also to depend on the time that has elapsed since the policy was issued.

The first assumption is described as assuming *ultimate mortality*, and the second as assuming *select mortality*. We will return later to discussing select mortality but assume for the moment that ultimate mortality applies.

2 The life table

2.1 Introduction

As mentioned above, to calculate the expected present value of a life insurance contract, we need to consider probabilities of survival and death, *ie* ${}_tp_x$ and ${}_tq_x$.

The calculation of the relevant probabilities can be simplified by use of *life tables*.

The life table is a device for calculating probabilities such as ${}_tp_x$ and ${}_tq_x$ using a one-dimensional array. The key to the definition of a life table is the relationship:

$${}_{t+s}p_x = {}_tp_x \times {}_sp_{x+t} = {}_sp_x \times {}_tp_{x+s}$$

This result is called the *principle of consistency* and is introduced in Subject CS2. This result holds because, for example, surviving for $t+s$ years from age x is the same as surviving for t years from age x (to reach age $x+t$) and then surviving a further s years to reach age $x+t+s$.

2.2 Constructing a life table

To construct a life table, we choose a starting age, which will be the lowest age in the table. We denote this lowest age α. This choice of α will often depend on the data which were available. For example, in studies of pensioners' mortality it is unusual to observe anyone younger than (say) 50, so 50 might be a suitable choice for α in a life table that is to represent pensioners' mortality.

We also need to choose the highest age in the table, ω, which is the age beyond which survival is assumed to be impossible. ω is referred to as the *limiting age* of the table.

We next choose an arbitrary positive number and denote it l_α. We call l_α the *radix* of the life table. It is convenient to interpret l_α as being the number of lives starting out at age α in a homogeneous population, but the mathematics does not depend on this interpretation.

As we will see, the choice of the radix (l_α) is not important. When setting the radix, it is common to choose a large round number such as 100,000. This is purely for presentational purposes.

For $\alpha \le x \le \omega$, define the function l_x by:

$$l_x = l_\alpha \times {}_{x-\alpha}p_\alpha$$

We assume that the probabilities ${}_{x-\alpha}p_\alpha$ are all known. By definition, $l_\omega = 0$.

Now we see that, for $\alpha \le x \le \omega$ and for $t \ge 0$:

$${}_tp_x = \frac{{}_{t+x-\alpha}p_\alpha}{{}_{x-\alpha}p_\alpha} = \frac{l_{x+t}}{l_\alpha} \times \frac{l_\alpha}{l_x} = \frac{l_{x+t}}{l_x}$$

Hence, if we know the function l_x for $\alpha \le x \le \omega$, we can find any probability ${}_tp_x$ or ${}_tq_x$.

The choice of radix is unimportant since the l_α terms cancel. The formula for ${}_tp_x$ involves a ratio of two entries from the life table, so the answer would be the same if the radix (and hence all the other figures in the life table) were multiplied by 2, say.

Question

Write down an expression for ${}_tq_x$ in terms of the function l_x .

Solution

The required expression is:

$${}_tq_x = 1 - {}_tp_x = 1 - \frac{l_{x+t}}{l_x} = \frac{l_x - l_{x+t}}{l_x}$$

The function l_x is called the *life table*. It depends on age only, so it is more easily tabulated than the probabilities ${}_tp_x$ or ${}_tq_x$, although the importance of this has diminished with the widespread use of computers.

The life table is an important tool that enables actuaries to calculate a wide range of useful figures from a single set of tabulated factors. A clear understanding of the underlying principles is still important today since life tables are at the heart of much actuarial valuation software.

2.3 The force of mortality

Earlier in this chapter, we introduced the force of mortality at age x , μ_x , as the annual rate of transfer between alive and dead at exact age x, where:

$$\mu_x h \approx \text{ probability of a life aged } x \text{ dying over the short time interval } (x, x+h)$$

ie $\mu_x h \approx {}_hq_x$

More formally, we write this as:

$$\mu_x = \lim_{h \to 0+} \frac{1}{h} \times {}_hq_x$$

In terms of life table functions, this is:

$$\mu_x = \lim_{h \to 0+} \frac{1}{h} \times \frac{l_x - l_{x+h}}{l_x} = -\frac{1}{l_x} \times \lim_{h \to 0+} \frac{l_{x+h} - l_x}{h}$$

The limit in this last expression matches the definition of a derivative. So we have:

$$\mu_x = -\frac{1}{l_x} \times \frac{d}{dx} l_x = -\frac{d}{dx} \ln l_x$$

where the final equality uses the chain rule for differentiation.

2.4 Interpretation

If we interpret l_α to be the number of lives known to be alive at age α (in which case it has to be an integer) then we can interpret l_x $(x > \alpha)$ as the *expected* number of those lives who survive to age x.

After all, l_x $(x > \alpha)$ is defined as l_α multiplied by ${}_{x-\alpha}p_\alpha$, the probability of a life aged α surviving to age x.

A life table is sometimes given a deterministic interpretation. That is, l_α is interpreted as above, and l_x $(x > \alpha)$ is interpreted as the number of lives who will survive to age x, as if this were a fixed quantity.

Then the symbol ${}_tp_x = l_{x+t}/l_x$ is taken to be the proportion of the l_x lives alive at age x who survive to age $x+t$. This is the so-called 'deterministic model of mortality'. It is not such a fruitful approach as the stochastic model that we have outlined, and we will not use it. In particular, while it is useful in computing quantities like premium rates, it is of no use when we need to analyse mortality data.

This deterministic model of mortality is too inflexible for our needs. After all, lives rarely survive to later ages in the exact proportions dictated by the life table. The stochastic approach allows us to model the inherent variability.

Another point to note is that the probabilities, ${}_tp_x$ *etc*, that are used to construct a life table can only be *estimates* of the true underlying probabilities. So the life table does not *define* the survival probabilities and mortality rates – it is merely one representation of them.

2.5 Using the life table

We now introduce a further life table function d_x. For $\alpha \le x \le \omega - 1$, define:

$$d_x = l_x - l_{x+1}$$

We interpret d_x as the expected number of lives who die between age x and age $x+1$, out of the l_α lives alive at age α.

Note that:

$$q_x = 1 - p_x = \frac{l_x}{l_x} - \frac{l_{x+1}}{l_x} = \frac{d_x}{l_x}$$

We can also see that:

$$d_x + d_{x+1} + \cdots + d_{x+n-1} = l_x - l_{x+n}$$

and that (if x and ω are integers):

$$d_x + d_{x+1} + \cdots + d_{\omega-1} = l_x$$

Question

Express in words the two results:

(i) $d_x + d_{x+1} + \cdots + d_{x+n-1} = l_x - l_{x+n}$

(ii) $d_x + d_{x+1} + \cdots + d_{\omega-1} = l_x$

Solution

(i) The difference between the number of lives expected to be alive at age *x* and the number of lives expected to be alive at age $x+n$ (*ie* $l_x - l_{x+n}$) is equal to the total of the number of lives expected to die between those two ages (*ie* $d_x + d_{x+1} + \cdots + d_{x+n-1}$).

(ii) The expected number of lives alive at age *x* (*ie* l_x) equals the sum of the expected number of deaths at each age from x onwards, *ie* all those lives alive at age x must die at some point in the future.

It is usual to tabulate values of l_x and d_x at integer ages, and often other functions such as μ_x, p_x or q_x as well. For an example, see the English Life Table No. 15 (Males) in the 'Formulae and Tables for Examinations'.

ELT15 (Males) appears in the *Tables* on pages 68 and 69.

The following is an extract from that table.

Age, x	l_x	d_x	q_x	μ_x
0	100,000	814	0.00814	
1	99,186	62	0.00062	0.00080
2	99,124	38	0.00038	0.00043
3	99,086	30	0.00030	0.00033
4	99,056	24	0.00024	0.00027
5	99,032	22	0.00022	0.00023

(No value is given for μ_0 because of the difficulty of calculating a reasonable estimate from observed data.) It is easy to check the relationships:

$$ {}_tp_x = \frac{l_{x+t}}{l_x} \qquad d_x = l_x - l_{x+1} \qquad q_x = \frac{d_x}{l_x} $$

Question

Using the extract from ELT15 (Males) given above, calculate the values of:

(i) p_2

(ii) ${}_2p_3$

(iii) ${}_4q_1$

(iv) l_6

Solution

(i) $p_2 = \dfrac{l_3}{l_2} = \dfrac{99{,}086}{99{,}124} = 0.99962$ or $p_2 = 1 - q_2 = 1 - 0.00038 = 0.99962$

(ii) ${}_2p_3 = \dfrac{l_5}{l_3} = \dfrac{99{,}032}{99{,}086} = 0.99946$

(iii) ${}_4q_1 = 1 - {}_4p_1 = 1 - \dfrac{l_5}{l_1} = 1 - \dfrac{99{,}032}{99{,}186} = 0.00155$

(iv) $l_6 = l_5 - d_5 = 99{,}032 - 22 = 99{,}010$

2.6 Lifetime random variables

Since the remaining lifetime of an individual is of unknown duration, we can model it using a random variable.

Recall from Subject CS2:

T_x = complete future lifetime of a life aged x

which has probability density function ${}_tp_x\,\mu_{x+t}$.

T_x represents the exact length of the remaining lifetime a life currently aged x, *ie* the exact length of time the life survives for after age x, before dying.

As we saw earlier in this chapter:

$${}_tq_x = \int_0^t {}_sp_x\,\mu_{x+s}\,ds = \text{probability that a life aged } x \text{ dies at any of the possible moments over the next } t \text{ years}$$

Expressing this in terms of the complete future lifetime random variable:

$$ {}_tq_x = P(T_x \le t) = \int_0^t {}_sp_x\, \mu_{x+s}\, ds $$

so we see that the probability is obtained by integrating the PDF over the range of possible values, as usual. The probability $P(T_x \le t)$ is the cumulative distribution function (or CDF) of the random variable T_x and is usually denoted $F_x(t)$.

Also recall from Subject CS2 that T_x has expected value:

$$ \boldsymbol{E[T_x] = \int_{t=0}^{\infty} t\, {}_tp_x\, \mu_{x+t}\, dt = \int_{t=0}^{\infty} {}_tp_x\, dt = \mathring{e}_x} $$

$\mathring{e}_x$ is called the *complete expected future lifetime* at age x or the *complete expectation of life* at age x, and its value is tabulated in some life tables. For example, based on ELT15 (Females) mortality, $\mathring{e}_{30} = 49.937$.

The first integral formula for $E[T_x]$ is based on the standard approach to calculating the expected value of a continuous random variable, *ie* taking the variable, multiplying by the PDF and then integrating over the range of values the variable can take.

The second integral formula for $E[T_x]$ is much easier to use and can be obtained from the first using integration by parts. Firstly, differentiating the CDF of a random variable gives the PDF, so we have:

$$ \frac{d}{dt} P(T_x \le t) = {}_tp_x\, \mu_{x+t} $$

We can also write this derivative as follows:

$$ \frac{d}{dt} P(T_x \le t) = \frac{d}{dt}\, {}_tq_x = \frac{d}{dt}(1 - {}_tp_x) = -\frac{d}{dt}\, {}_tp_x $$

Combining these gives:

$$ -\frac{d}{dt}\, {}_tp_x = {}_tp_x\, \mu_{x+t} \quad \Rightarrow \quad \frac{d}{dt}\, {}_tp_x = -{}_tp_x\, \mu_{x+t} $$

Now using the formula on page 3 of the *Tables*, with $u = t$ and $\dfrac{dv}{dt} = {}_tp_x\, \mu_{x+t}$:

$$ \int_{t=0}^{\infty} t\, {}_tp_x\, \mu_{x+t}\, dt = \left[t(-{}_tp_x) \right]_0^{\infty} - \int_{t=0}^{\infty} 1 \times (-{}_tp_x) dt = \int_{t=0}^{\infty} {}_tp_x\, dt $$

since ${}_{\infty}p_x = 0$.

Question

Suppose that in a particular life table:

$$l_x = 100 - x \qquad \text{for } 0 \le x \le 100$$

Calculate the complete expected future lifetime of a life currently aged 50.

Solution

Using the given expression for l_x :

$$_tp_{50} = \frac{l_{50+t}}{l_{50}} = \frac{50-t}{50} \qquad \text{for } 0 \le t \le 50$$

We only consider values of t up to 50, as there is a zero probability of surviving to age 100 in this life table.

So the complete expected future lifetime of a life currently aged 50 is:

$$\mathring{e}_{50} = \int_0^{50} {}_tp_{50}\,dt = \int_0^{50} \frac{50-t}{50}\,dt = \frac{1}{50}\left[50t - \frac{1}{2}t^2\right]_0^{50} = \frac{1}{50}\left(50^2 - \frac{1}{2}50^2\right) = 25$$

Also recall from Subject CS2:

$$\boldsymbol{K_x} \textbf{ = curtate future lifetime of a life aged } \boldsymbol{x} \textbf{ = int}\left[\boldsymbol{T_x}\right]$$

K_x is the integer part of T_x, and so represents the complete number of years a life currently aged x survives for, before dying.

Question

A person is currently exactly 40 years old. Suppose that this person dies aged 76 years and 197 days. State the values of T_{40} and K_{40} for this individual.

Solution

T_{40} is the exact length of time the person survives for from the current age (40 exact) to the age at death. This is therefore 36 years and 197 days.

K_{40} is the integer part of this, and therefore equals 36 years.

2.7 The pattern of human mortality

We now return to the extract from the ELT15 (Males) table:

Age, x	l_x	d_x	q_x	μ_x
0	100,000	814	0.00814	
1	99,186	62	0.00062	0.00080
2	99,124	38	0.00038	0.00043
3	99,086	30	0.00030	0.00033
4	99,056	24	0.00024	0.00027
5	99,032	22	0.00022	0.00023

Notice that $\mu_x > q_x$ at all ages in this part of the table. At some higher ages it is found that $\mu_x < q_x$. In fact, since:

$$q_x = \int_0^1 {}_tp_x\, \mu_{x+t}\, dt$$

we see that if ${}_tp_x\mu_{x+t}$ is increasing for $0 \le t \le 1$:

$$q_x = \int_0^1 {}_tp_x\, \mu_{x+t}\, dt \; > {}_0p_x\mu_{x+0} = \mu_x$$

while if ${}_tp_x\mu_{x+t}$ is decreasing for $0 \le t \le 1$:

$$q_x = \int_0^1 {}_tp_x\, \mu_{x+t}\, dt \; < {}_0p_x\mu_{x+0} = \mu_x$$

It is therefore of interest to note the behaviour of the function ${}_tp_x\mu_{x+t}$ for $0 \le t < \omega - x$.

As mentioned above, this function is the probability density function of T_x.

Figure 1 shows ${}_tp_0\mu_t$ (*ie* the density of $T = T_0$) for the English Life Table No. 15 (Males).

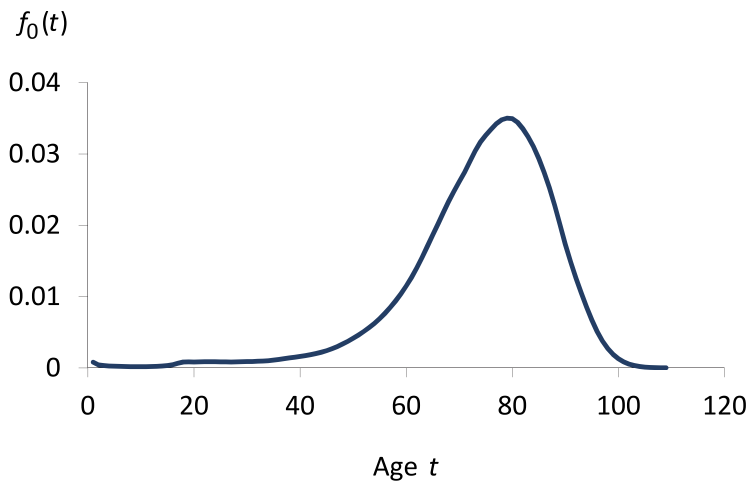


Figure 1

$f_0(t) = {}_tp_0\mu_t$ (ELT15 (Males) Mortality Table)

The graph has the following features, which are typical of life tables based on human mortality in modern times:

(1) Mortality just after birth ('infant mortality') is very high.

(2) Mortality falls during the first few years of life.

(3) There is a distinct 'hump' in the function at ages around 18–25. This is often attributed to a rise in accidental deaths during young adulthood, and is called the 'accident hump'.

(4) From middle age onwards there is a steep increase in mortality, reaching a peak at about age 80.

(5) The probability of death at higher ages falls again (even though μ_x continues to increase) since the probabilities of surviving to these ages are small.

2.8 More notation

We will now introduce some more actuarial notation for probabilities of death, and give formulae for them in terms of the life table l_x.

These definitions relate to the deferred probabilities of death.

Define:

$$_{n|m}q_x = P[\, n < T_x \le n + m \,]$$

In words, $_{n|m}q_x$ is the probability that a life age x will survive for n years but die during the subsequent m years.

It can be seen that:

$$\boldsymbol{{}_{n|m}q_x = \frac{l_{x+n} - l_{x+n+m}}{l_x}}$$

or alternatively that:

$$\boldsymbol{{}_{n|m}q_x = {}_n p_x \times {}_m q_{x+n}}$$

We can see that these two results are equivalent since:

$${}_n p_x \times {}_m q_{x+n} = \frac{l_{x+n}}{l_x} \times \frac{l_{x+n} - l_{x+n+m}}{l_{x+n}} = \frac{l_{x+n} - l_{x+n+m}}{l_x}$$

This probability can also be expressed in the following ways:

- ${}_{n|m}q_x = {}_n p_x \times {}_m q_{x+n} = {}_n p_x \times (1 - {}_m p_{x+n}) = {}_n p_x - {}_{n+m} p_x$, *ie* the probability that the life survives for another n years, but not for another $n+m$ years
- ${}_{n|m}q_x = {}_n p_x - {}_{n+m} p_x = (1 - {}_n q_x) - (1 - {}_{n+m} q_x) = {}_{n+m} q_x - {}_n q_x$, *ie* the probability that the life dies within the next $n+m$ years, but does not die within the next n years.

Question

Using ELT15 (Males) mortality, calculate the probability of a 37-year old dying between age 65 and age 75.

Solution

The required probability is:

$${}_{28|10}q_{37} = \frac{l_{65} - l_{75}}{l_{37}}$$

Looking up the values in the *Tables* gives:

$${}_{28|10}q_{37} = \frac{79{,}293 - 53{,}266}{96{,}933} = 0.26851$$

An important special case for actuarial calculations is $m = 1$, since we often use probabilities of death over one year of age. By convention, we drop the '*m*' and write:

$$\boldsymbol{{}_{n|1}q_x = {}_{n|}q_x}$$

In words, ${}_{n|}q_x$ is the probability that a life aged x will survive for n years but die during the subsequent year, *ie* die between ages $x+n$ and $x+n+1$.

So:

$$ {}_{n|}q_x = {}_n p_x \times q_{x+n} = \frac{l_{x+n} - l_{x+n+1}}{l_x} = \frac{d_{x+n}}{l_x} $$

Recall that K_x, the curtate future lifetime random variable, is the integer part of T_x, and represents the complete number of years a life currently aged x survives for, before dying.

We now see that the probability function of K_x can be written:

$$ P[K_x = k] = {}_{k|}q_x $$

$P[K_x = k]$ is the probability that a life aged x dies between time k years and time $k+1$ years in the future, *ie* between age $x+k$ and age $x+k+1$.

Question

Using ELT15 (Males) mortality, calculate:

(i) $P(K_{30} = 40)$

(ii) $P(T_{30} > 40)$

Solution

(i) $P(K_{30} = 40)$ is the probability that the curtate future lifetime of a 30-year-old is equal to 40 years, *ie* it is the probability that a life aged exactly 30 survives for 40 years and then dies in the following year (between age 70 and age 71):

$$ P(K_{30} = 40) = {}_{40|}q_{30} = \frac{d_{70}}{l_{30}} = \frac{2,674}{97,645} = 0.02738 $$

(ii) $P(T_{30} > 40)$ is the probability that the complete future lifetime of a 30-year-old exceeds 40 years, *ie* it is the probability that a life aged exactly 30 survives for at least another 40 more years:

$$ P(T_{30} > 40) = {}_{40}p_{30} = \frac{l_{70}}{l_{30}} = \frac{68,055}{97,645} = 0.69696 $$

Also:

$$ E[K_x] = \sum_{k=0}^{\infty} k \, {}_{k|}q_x = \sum_{k=1}^{\infty} {}_k p_x = e_x \approx \mathring{e}_x - ½ $$

e_x is called the *curtate expected future lifetime* at age x or the *curtate expectation of life* at age x, and its value is tabulated in some life tables. For example, based on AM92 mortality, $e_{20} = 58.447$.

The first summation formula for $E[K_x]$ is based on the standard approach to calculating the expected value of a discrete random variable, *ie* taking each value the variable can take, multiplying by the probability that the random variable takes that value and then summing over all possible values.

The second summation formula for $E[K_x]$ is much easier to use and can be obtained from the first as follows:

$$\begin{aligned}\sum_{k=0}^{\infty} k\ {}_{k|}q_x &= 1\times {}_{1|}q_x + 2\times {}_{2|}q_x + 3\times {}_{3|}q_x + \cdots \\ &= {}_{1|}q_x + {}_{2|}q_x + {}_{3|}q_x + \cdots \\ &\qquad\quad + {}_{2|}q_x + {}_{3|}q_x + \cdots \\ &\qquad\qquad\qquad + {}_{3|}q_x + \cdots\end{aligned}$$

The first row in the summation above, *ie*:

$${}_{1|}q_x + {}_{2|}q_x + {}_{3|}q_x + \cdots$$

is the probability that the life dies between age $x+1$ and age $x+2$, plus the probability that the life dies between age $x+2$ and age $x+3$, plus the probability that the life dies between age $x+3$ and age $x+4$, and so on. This is the probability that the life dies at some time after age $x+1$, which is equal to the probability that the life survives until at least age $x+1$, p_x.

Similarly, ${}_{2|}q_x + {}_{3|}q_x + \cdots$ represents the probability that the life dies at some time after age $x+2$, which is equal to the probability that the life survives until at least age $x+2$, ${}_2p_x$.

So the summation becomes:

$$E[K_x] = \sum_{k=0}^{\infty} k\ {}_{k|}q_x = p_x + {}_2p_x + {}_3p_x + \cdots = \sum_{k=1}^{\infty} {}_kp_x$$

Question

Suppose that in a particular life table:

$$l_x = 100 - x \qquad \text{for } 0 \le x \le 100$$

Calculate the curtate expected future lifetime of a newborn life.

Solution

The curtate expected future lifetime of a newborn life is:

$$e_0 = p_0 + {}_2p_0 + \cdots + {}_{99}p_0 = \frac{l_1}{l_0} + \frac{l_2}{l_0} + \cdots + \frac{l_{99}}{l_0}$$

We do not include terms from ${}_{100}p_0$ onwards in the summation, as there is a zero probability of surviving to age 100 in this life table.

Using the given expression for l_x, we have:

$$e_0 = \frac{99}{100} + \frac{98}{100} + \cdots + \frac{1}{100} = \frac{99 + 98 + \cdots + 1}{100}$$

The numerator can be evaluated using the sum of the first *n* positive integers, $\frac{1}{2}n(n+1)$, so:

$$e_0 = \frac{\frac{1}{2} \times 99 \times 100}{100} = \frac{4{,}950}{100} = 49.5$$

The approximate relationship between the complete and curtate expectations of life:

$$e_x \approx \overset{\circ}{e}_x - ½$$

is based on the assumption that deaths occur halfway between birthdays. Under this assumption, the total length of time a life survives for will be half a year longer than the number of complete years it survives for, *ie*:

$$K_x = T_x - \tfrac{1}{2}$$

Taking the expectation of both sides of this leads to the relationship:

$$e_x = \overset{\circ}{e}_x - \tfrac{1}{2}$$

This relationship is exact if the assumption holds, but in practice it is just an approximation.

3 Life table functions at non-integer ages

3.1 Introduction

Life table functions such as l_x, q_x or μ_x are usually tabulated at integer ages only, but sometimes we need to compute probabilities involving non-integer ages or durations, such as ${}_{2.5}p_{37.5}$. We can do so using approximate methods. We will show two methods.

In both cases, we suppose that we split up the required probability so that we need only approximate over single years of age.

Whilst the underlying force of mortality can vary greatly at different ages, it should not change significantly within a single year of age, so by taking this approach our approximations should be reasonably accurate.

For example, we would write:

$${}_3p_{55.5} \text{ as } {}_{0.5}p_{55.5} \times {}_2p_{56} \times {}_{0.5}p_{58}$$

The middle factor can be found from the life table. To approximate the other two factors we need only consider single years of age.

3.2 Method 1 – uniform distribution of deaths (UDD)

The first method is based on the assumption that, for integer x and $0 \le t \le 1$, the function ${}_tp_x\mu_{x+t}$ is a constant.

Since this is the density (PDF) **of the time to death from age x, it is seen that this assumption is equivalent to a uniform distribution of the time to death, conditional on death falling between these two ages. Hence it is called the *Uniform Distribution of Deaths* (or UDD) assumption.**

In other words, for an individual aged exactly x, the probability of dying on one particular day over the next year is the same as that of dying on any other day over the next year.

The UDD assumption implicitly assumes that μ_x increases over the year of age. This follows because the quantity ${}_tp_x$ is a decreasing function of t, so if the function ${}_tp_x\,\mu_{x+t}$ is constant, μ_{x+t} must be an increasing function of t.

Since ${}_sq_x = \int_0^s {}_tp_x\mu_{x+t}\,dt$, by putting $s = 1$ we must have:

$${}_tp_x\mu_{x+t} = q_x \quad (0 \le t \le 1)$$

Remember that we are assuming that ${}_tp_x\mu_{x+t}$ is constant over the year.

Therefore:

$${}_sq_x = \int_0^s q_x\,dt = s\,q_x$$

This is sometimes taken as the definition of the UDD assumption.

Since q_x can be found from the life table, we can use this to approximate any $_sq_x$ or $_sp_x$ $(0 \le s \le 1)$.

Question

Calculate the value of $_{0.5}p_{58}$ using ELT15 (Females) mortality, assuming a uniform distribution of deaths between integer ages.

Solution

Using the UDD assumption, we have:

$$_{0.5}p_{58} = 1 - {}_{0.5}q_{58} = 1 - 0.5 \times q_{58} = 1 - 0.5 \times 0.00660 = 0.99670$$

Note that we must have an integer age x in the above formula, so (in our example) we can now estimate $_{0.5}p_{58}$ but not $_{0.5}p_{55.5}$.

However, using $_tp_x = {}_sp_x \times {}_{t-s}p_{x+s}$, it can be shown that for integer age x and $0 \le s < t \le 1$:

$$_{t-s}q_{x+s} = \frac{(t-s)q_x}{1 - sq_x}$$

Rearranging $_tp_x = {}_sp_x \times {}_{t-s}p_{x+s}$ gives:

$$_{t-s}p_{x+s} = \frac{_tp_x}{_sp_x}$$

and using the earlier result that under the UDD assumption $_sq_x = sq_x$, we can derive the above result as follows:

$$_{t-s}q_{x+s} = 1 - {}_{t-s}p_{x+s} = 1 - \frac{_tp_x}{_sp_x} = 1 - \frac{1 - {}_tq_x}{1 - {}_sq_x} = 1 - \frac{1 - tq_x}{1 - sq_x}$$

$$= \frac{(1 - sq_x) - (1 - tq_x)}{1 - sq_x} = \frac{(t-s)q_x}{1 - sq_x}$$

This result, with *s* = 0.5, *t* = 1, can now be used to estimate $_{0.5}p_{55.5}$, for example.

Question

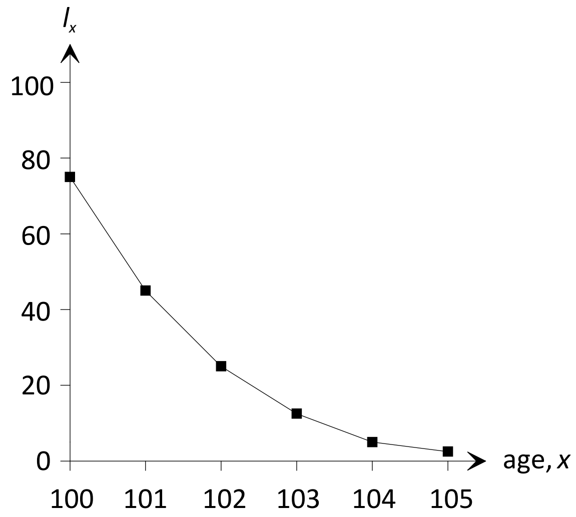
Calculate the value of $_{0.5}p_{55.5}$ using ELT15 (Females) mortality, assuming a uniform distribution of deaths between integer ages.

Solution

Using the UDD assumption, we have:

$${}_{0.5}p_{55.5} = 1 - {}_{0.5}q_{55.5} = 1 - \frac{0.5q_{55}}{1 - 0.5q_{55}} = 1 - \frac{0.5 \times 0.00475}{1 - 0.5 \times 0.00475} = 0.99762$$

Under the UDD assumption l_x is made up of straight-line segments between integer ages, as shown in the following graph of l_x between ages 100 and 105:



The following question justifies why this is the case.

Question

Show that, under the UDD assumption over each year of age:

(i) $l_{x+t} = l_x - t\,d_x$

(ii) $l_{x+t} = (1-t)l_x + t\,l_{x+1}$

for $x = 0, 1, 2, ..., \omega - 1$ and for $0 \le t \le 1$.

Solution

(i) We have:

$$l_{x+t} = l_x \times {}_tp_x = l_x(1 - {}_tq_x) = l_x(1 - t\,q_x) = l_x - t(l_x\,q_x) = l_x - t\,d_x$$

(ii) Using part (i):

$$l_{x+t} = l_x - t\,d_x = l_x - t(l_x - l_{x+1}) = (1-t)l_x + t\,l_{x+1}$$

The result in part (ii) means that the values of l_{x+t} for $0<t<1$ can be found using linear interpolation, and provides an alternative approach to using the formulae:

$${}_sq_x = sq_x \quad \text{and} \quad {}_{t-s}q_{x+s} = \frac{(t-s)q_x}{1-sq_x}$$

Question

Calculate the value of ${}_{0.5}p_{55.5}$ using ELT15 (Females) mortality and linear interpolation, based on the assumption of a uniform distribution of deaths between integer ages.

Solution

Using the UDD assumption, we have:

$${}_{0.5}p_{55.5} = \frac{l_{56}}{l_{55.5}} = \frac{l_{56}}{0.5l_{55}+0.5l_{56}} = \frac{94{,}082}{0.5\times 94{,}532+0.5\times 94{,}082} = \frac{94{,}082}{94{,}307} = 0.99761$$

This answer is slightly different from the answer obtained previously for this probability, due to the fact that the values shown in the *Tables* are rounded.

3.3 Method 2 – constant force of mortality (CFM)

The second method of approximation is based on the assumption of a *constant force of mortality*. That is, for integer x and $0 \le t < 1$, we suppose that:

$$\mu_{x+t} = \mu = \text{constant}$$

Then for $0 \le s < t < 1$ the formula:

$${}_{t-s}p_{x+s} = \exp\left\{-\int_s^t \mu_{x+r}\,dr\right\} = e^{-(t-s)\mu}$$

can be used to find the required probabilities. We do this by first noting that, under this assumption, $p_x = e^{-\mu}$, so we can simply write:

$${}_{t-s}p_{x+s} = \left(p_x\right)^{t-s}$$

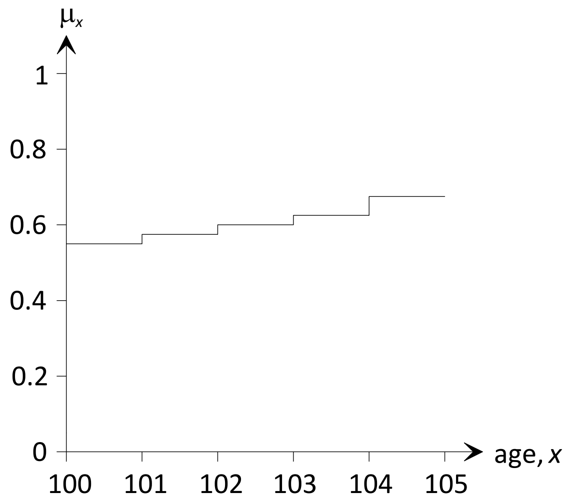
where p_x can be found from the life table. Hence we can easily calculate any required probability.

Usually, the value of q_x will be tabulated and $p_x = 1-q_x$, meaning we don't need to work out the value of the constant force of mortality to calculate the probabilities.

Also, note that when $s=0$ the formula simplifies a little to:

$${}_tp_x = \left(p_x\right)^t$$

Under this assumption μ_x has a stepped shape, as shown in the following graph of μ_x between ages 100 and 105:



Question

Calculate ${}_3p_{62.5}$ based on PFA92C20 mortality using:

(i) the CFM assumption

(ii) the UDD assumption.

Solution

First of all, we can split up the probability at integer ages as follows:

$${}_3p_{62.5} = {}_{0.5}p_{62.5} \times {}_2p_{63} \times {}_{0.5}p_{65}$$

Looking up values from PFA92C20 in the *Tables*:

$${}_2p_{63} = \frac{l_{65}}{l_{63}} = \frac{9,703.708}{9,775.888} = 0.992617$$

(i) Under the constant force of mortality assumption:

$${}_{0.5}p_{62.5} = \left(p_{62}\right)^{0.5} = \left(1-q_{62}\right)^{0.5} = \left(1-0.002885\right)^{0.5} = 0.998556$$

Also:

$${}_{0.5}p_{65} = \left(p_{65}\right)^{0.5} = \left(1-q_{65}\right)^{0.5} = \left(1-0.004681\right)^{0.5} = 0.997657$$

So, overall:

$${}_3p_{62.5} = 0.998556 \times 0.992617 \times 0.997657 = 0.988861$$

(ii) Under the UDD assumption:

$$ {}_{0.5}p_{62.5} = 1 - {}_{0.5}q_{62.5} = 1 - \frac{0.5q_{62}}{1-0.5q_{62}} = 1 - \frac{0.5 \times 0.002885}{1-0.5\times 0.002885} = 0.998555 $$

and: ${}_{0.5}p_{65} = 1 - {}_{0.5}q_{65} = 1 - 0.5q_{65} = 1 - 0.5 \times 0.004681 = 0.997660$

So overall:

$$ {}_{3}p_{62.5} = 0.998555 \times 0.992617 \times 0.997660 = 0.988863 $$

Alternatively, assuming UDD, we could take the following approach:

$$ {}_{3}p_{62.5} = \frac{l_{65.5}}{l_{62.5}} = \frac{0.5l_{65} + 0.5l_{66}}{0.5l_{62} + 0.5l_{63}} $$

$$ = \frac{0.5(9,703.708 + 9,658.285)}{0.5(9,804.173 + 9,775.888)} = \frac{9,680.9965}{9,790.0305} = 0.988863 $$

4 Evaluating probabilities without use of the life table

The main alternative to using a life table is to postulate a formula and/or parameter values for the probability $_tp_x$ and then evaluate the expressions directly.

This means that we hypothesise a formula for $_tp_x$, estimate the values of the parameters in the formula, and then calculate the probabilities for different values of x and t.

Equivalently a formula or values for $_tq_x$ or μ_{x+t} could be postulated.

The difficulty of adopting this approach is that the postulation would need to be valid across the whole age range for which the formulae might be applied. As can be seen from the discussion in Section 2.7 above, the shape of human mortality may make a simple postulation difficult. Simple formulae may, however, be more appropriate and expedient for non-life contingencies.

The following example shows how mortality and survival probabilities can be evaluated without using a life table.

Here, we assume that the formula for μ_{x+t} is a very simple one – simply that it takes a constant value at all ages. (We are also assuming that there is no upper limit to age.)

Question

In a certain population, the force of mortality equals 0.025 at all ages.

Calculate:

(i) the probability that a new-born baby will survive to age 5

(ii) the probability that a life aged exactly 10 will die before age 12

(iii) the probability that a life aged exactly 5 will die between ages 10 and 12.

Solution

(i) $$_5p_0 = \exp\left(-\int_0^5 0.025\,dt\right) = e^{-0.125} = 0.88250$$

(ii) $$_2q_{10} = 1 - {}_2p_{10} = 1 - \exp\left(-\int_0^2 0.025\,dt\right) = 1 - e^{-0.05} = 0.04877$$

(iii) $$_{5|2}q_5 = {}_5p_5 \times {}_2q_{10} = e^{-0.125} \times 0.04877 = 0.88250 \times 0.04877 = 0.04304$$

In general, where the force of mortality is constant over the whole of an n-year period:

$$_np_x = e^{-n\mu} \quad \text{and} \quad _nq_x = 1 - e^{-n\mu}$$

5 Select mortality

5.1 Introduction

So far, we have made an assumption of ultimate mortality, that is, that mortality varies by age only. In real life, many other factors other than just age might affect observed mortality rates. In practice, therefore, the evaluation of assurance and annuity benefits is often modified to allow for factors other than just age, which affect the survival probabilities.

Examples of factors (other than age) that might affect observed mortality rates are sex, smoker status and occupation.

Many factors can be allowed for by segregating the population, the assumption being that an age pattern of mortality can be discerned in the sub-population. For example, a population may well be segregated by sex and then sex-specific mortality rates (and mortality tables) can be used directly by using the techniques so far described.

Where, however, the pattern of mortality is assumed to depend not just on age, consequently slightly more complicated survival probabilities are employed. The most important, in the case of human mortality, is where the mortality rates depend upon duration as well as age, called *select rates*.

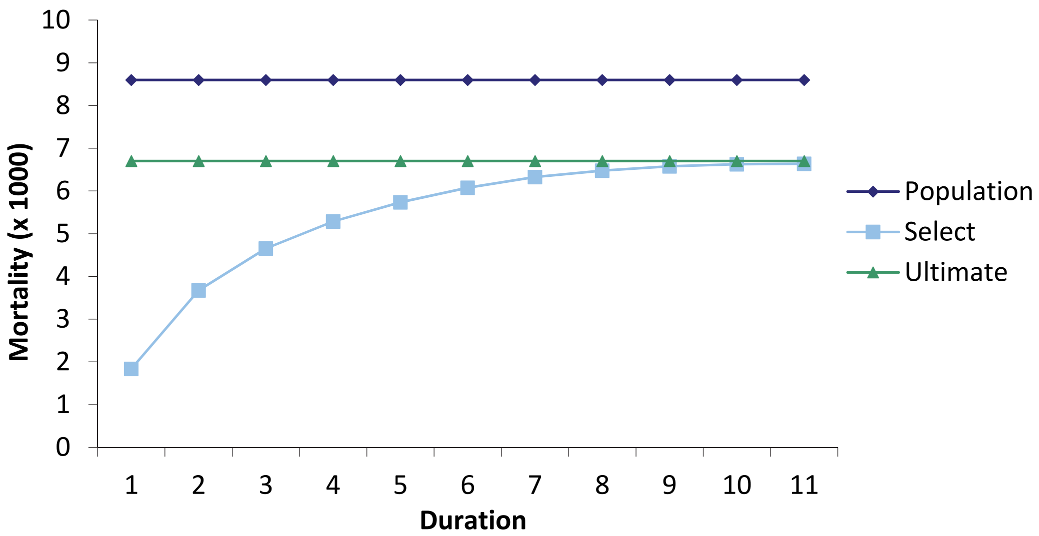
To understand why the concept of duration in the population is important, we will consider life assurance policyholders (*ie* those who hold policies that make a payment on death).

In order to take out their policies, policyholders will have been subject to some *medical underwriting*. This means that they provide evidence about their recent state of health, answering simple questions on the policy proposal form, and perhaps also attending a medical examination if the sum insured is very high. The aim of this underwriting is to allow the company to screen out very bad risks, and to charge appropriately higher premiums for worse than average risks.

So policyholders with duration of one year, say, will have recently satisfied the company about their state of health. We would therefore expect their mortality to be better than that of policyholders of the same age with duration of, say, 3 years who passed the medical underwriting hurdle several years ago. Moreover, we would expect both these sets of policyholders to display better mortality than policyholders of the same age with duration 10 or 20 years, for instance. The mortality of the recently joined policyholders is called *select* mortality, and we expect it to be better than that of longer duration policyholders, whose mortality we call *ultimate* mortality.

However, it is important to realise that even ultimate mortality may be good when compared with the mortality of the whole population of a country. This is because those taking out life insurance contracts tend to be the more affluent in society, who can afford to purchase insurance products, and financially better-off individuals tend to have lower mortality (*eg* they may be able to access better health care).

If we were to plot mortality for any given age x by varying duration, and compare it also against population mortality, we might expect something like:



When creating a life table based on select mortality, we adopt a pragmatic approach and look for the duration beyond which there is no significant change in mortality with further increases in duration. This is called the *select period*, and we examine the choice of the select period in Section 5.3.

We take this approach because, intuitively, we would not expect a difference in mortality between someone who answered some health questions on a policy proposal form 10 years ago and someone else (of the same age) who answered such questions 20 years ago. In addition, in practice, it would be very difficult to measure mortality rates if we want to split the population up by every year of duration as well as by age and sex because in each group we might see only 50 to 100 policyholders, even for a very large company.

Above, we have discussed select mortality in the context of life assurance policyholders. In fact, the effect of selection is also found with annuity business: the mortality of policyholders who have recently bought an annuity policy is lighter than that of policyholders who took out their policy some time ago.

5.2 Mortality rates that depend on both age and duration

Select rates are usually studied by modelling the force of mortality μ as a function of the age at joining the population and the duration since joining the population. The usual notation is:

$[x]+r$	**age at date of transition**
$[x]$	**age at date of joining population**
r	**duration from date of joining the population until date of transition**

$\mu_{[x]+r}$	**the transition rate (force of mortality) at exact duration r having joined the population at age $[x]$**
$l_{[x]+r}$	**expected number of lives alive at duration r having joined the population at age $[x]$, based on some assumed radix**

The important principle to grasp with this notation is that the term in square brackets [ ] denotes the age at joining the population, so the age 'now' (*ie* the moment of transition, or when exposed to possible transition) will be $x+r$. In some circumstances, we may be given the age 'now' of y and the duration r; in this case the transition rate 'now' would be expressed as $\mu_{[y-r]+r}$.

In effect, a model showing how μ varies with r is constructed for each value of $[x]$. Instead of the single life age-specific life table described earlier we have a series of life tables, one for each value of $[x]$.

5.3 Displaying select rates

Once select rates have been estimated, it is conventional to display estimated rates for each age at entry into the population, $[x]$, by age attained at the date of transition *ie* $[x]+r$. This can be done in an array:

$$\begin{array}{ccccc}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\mu_{[x]} & \mu_{[x-1]+1} & \mu_{[x-2]+2} & \mu_{[x-3]+3} & \cdots \\
\mu_{[x+1]} & \mu_{[x]+1} & \mu_{[x-1]+2} & \mu_{[x-2]+3} & \cdots \\
\mu_{[x+2]} & \mu_{[x+1]+1} & \mu_{[x]+2} & \mu_{[x-1]+3} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Question

Assuming that the force of mortality increases with both age and policy duration, comment on the expected relationship between the following pairs of values:

(i) $\mu_{[x+1]}$ and $\mu_{[x]+1}$

(ii) $\mu_{[x+2]+2}$ and $\mu_{[x+4]}$

(iii) $\mu_{[x+4]}$ and $\mu_{[x+1]+2}$

Solution

(i) We would expect to see $\mu_{[x+1]} < \mu_{[x]+1}$, because the rates both relate to the same age, $x+1$, but $\mu_{[x]+1}$ relates to a later duration (1 instead of 0).

(ii) Here, we would expect to see $\mu_{[x+2]+2} > \mu_{[x+4]}$, again because they relate to the same age, but the first term is for duration 2 rather than 0.

We might also expect the difference to be higher than in part (i), because:

- it involves twice the duration difference (2 instead of 1)
- the current age is three years older ($x+4$ instead of $x+1$), which should increase the size of the effect of duration differences on mortality.

(iii) How $\mu_{[x+4]}$ compares with $\mu_{[x+1]+2}$ will depend on how duration and age affect mortality, because the current age has decreased (reducing mortality) while duration has increased (increasing mortality). It is therefore impossible to give a general answer in this case.

Each diagonal (↘) of the array represents a model of how rates vary with duration since joining the population for a particular age at the date of joining, *ie* each is a set of life table mortality rates.

The diagonals referred to here are those going from top left to bottom right.

The rates displayed on the rows of the array are rates for lives that have a common age attained at the time of transition, but different ages at the date of joining the population.

For example, the second row in the table above contains $\mu_{[x+1]}$ and $\mu_{[x]+1}$. These both apply to lives aged $x+1$, but with differing durations. Similarly, the next row relates to lives age $x+2$ but with differing durations, and so on.

If the rates did not depend on the duration since the date of joining the population, then apart from sampling error the rates on each row would be equal.

Usually it is the case that rates are assumed to depend on duration until duration s, and after s they are assumed to be independent of duration. This phenomenon is termed *temporary initial selection* and s is called the length of the select period. In any investigation s is determined empirically by considering the statistical significance of the differences in transition rates along each row and the substantive impact of the different possible values of s.

We examine rows to ensure that we consider different durations but equal ages.

Typical select periods seen in life company investigations range from one to five years. For instance, the AM92 tables (based on UK assured lives data gathered over the period 1991-1994) in the *Tables* are based on a two-year select period.

The UK 1980 assured lives table AM80 uses a five-year select period, although durations 2 to 4 are grouped together. This is because, when the underlying data values were analysed, no significant differences in mortality rates at these durations were apparent. So a typical row shows the rates:

Duration 0	Duration 1	Duration 2-4	Duration 5+
$\mu_{[34]}$	$\mu_{[33]+1}$	$\mu_{[32]+2} = \mu_{[31]+3} = \mu_{[30]+4}$	μ_{34}

Once a value of s has been determined, the estimates of the rates for duration $\geq s$ are pooled to obtain a common estimated value that is used in all the life tables in which it is needed. The array can be written:

$$\begin{array}{llllll}
\cdots & \cdots & \cdots & \cdots\ \cdots & \cdots & \\
\mu_{[x]} & \mu_{[x-1]+1} & \mu_{[x-2]+2} & \cdots\ \mu_{[x-s+1]+s-1} & & \mu_x \\
\mu_{[x+1]} & \mu_{[x]+1} & \mu_{[x-1]+2} & \cdots\ \mu_{[x-s+2]+s-1} & & \mu_{x+1} \\
\mu_{[x+2]} & \mu_{[x+1]+1} & \mu_{[x]+2} & \cdots\ \mu_{[x-s+3]+s-1} & & \mu_{x+2} \\
\cdots & \cdots & \cdots & \cdots\ \cdots & \cdots &
\end{array}$$

The right hand column of the above array represents a set of rates that are common to all the constituent life tables in the model of rates by age and duration. It is called an ultimate table.

In the *Tables*, the AM92 Select mortality rates are tabulated in the way shown above.

The initial select probabilities can be displayed in a similar way.

'Initial probabilities' refers to q_x values, as q_x is sometimes called 'the initial rate of mortality'.

5.4 Constructing select and ultimate life tables

The first step in constructing life tables is to refine the crude estimated rates (μ or q) into a smooth set of rates that statistically represent the true underlying mortality rates. This refinement process, called 'graduation', is dealt with in CS2, and here we assume that we have a set of suitable graduated mortality rates available.

Knowledge of the graduation process is not required in this subject.

Using the graduated values of the initial probabilities displayed in the array:

$$\begin{array}{llllll}
\cdots & \cdots & \cdots & \cdots\ \cdots & \cdots & \\
q_{[x]} & q_{[x-1]+1} & q_{[x-2]+2} & \cdots\ q_{[x-s+1]+s-1} & & q_x \\
q_{[x+1]} & q_{[x]+1} & q_{[x-1]+2} & \cdots\ q_{[x-s+2]+s-1} & & q_{x+1} \\
q_{[x+2]} & q_{[x+1]+1} & q_{[x]+2} & \cdots\ q_{[x-s+3]+s-1} & & q_{x+2} \\
\cdots & \cdots & \cdots & \cdots\ \cdots & \cdots &
\end{array}$$

a table representing the select and ultimate experience can be constructed.

This is achieved by firstly constructing the ultimate life table based on the final column of the array and the formula described in Section 2.2 above:

- **choose the starting age of the table, α**
- **choose an arbitrary radix for the table, l_α**
- **recursively calculate the values of l_x using $l_{x+1} = l_x(1-q_x)$.**

Beginning with the appropriate ultimate value in the final column, the select life table functions for each row of the array are then determined. This is achieved by 'working backwards' up each diagonal using:

$$l_{[x]+t} = \frac{l_{[x]+t+1}}{(1 - q_{[x]+t})}$$

for $t = s-1, s-2, \ldots, 1, 0$, and noting in the first iteration that $l_{[x]+s-1+1} = l_{x+s}$.

Question

Given the following select and ultimate q_x values, calculate the values of $l_{[x]}$ and $l_{[x]+1}$ for $x = 45$ and $x = 46$, assuming that $l_{47} = 1{,}000$.

Age	Duration 0	Duration 1	Duration 2+ (*ie* Ultimate)
45	0.000838		
46	0.000924	0.001158	
47	0.001018	0.001284	0.001415
48		0.001423	0.001564
49			0.001729

Solution

Starting from l_{47}, we work back diagonally upwards to the left to $l_{[45]}$:

$$l_{[45]+1} = \frac{l_{47}}{\left(1 - q_{[45]+1}\right)} = \frac{1{,}000}{\left(1 - 0.001158\right)} = 1{,}001.16$$

$$l_{[45]} = \frac{l_{[45]+1}}{\left(1 - q_{[45]}\right)} = \frac{1{,}001.16}{\left(1 - 0.000838\right)} = 1{,}002.00$$

Next, working down the column from l_{47} to l_{48}:

$$l_{48} = l_{47}\left(1 - q_{47}\right) = 1{,}000\left(1 - 0.001415\right) = 998.59$$

This then allows us to calculate $l_{[46]}$ and $l_{[46]+1}$:

$$l_{[46]+1} = \frac{l_{48}}{\left(1 - q_{[46]+1}\right)} = \frac{998.59}{\left(1 - 0.001284\right)} = 999.87$$

$$l_{[46]} = \frac{l_{[46]+1}}{\left(1 - q_{[46]}\right)} = \frac{999.87}{\left(1 - 0.000924\right)} = 1{,}000.79$$

5.5 Using tabulated select life table functions

Some probabilities are particularly useful for life contingencies calculations. We have already defined:

$$_{n|m}q_x = \frac{l_{x+n} - l_{x+n+m}}{l_x}$$

$$_{n|}q_x = \frac{l_{x+n} - l_{x+n+1}}{l_x}$$

representing the m-year and 1-year probabilities of transition when the event of transition is deferred for n years.

Similar probabilities can be defined for each select mortality table:

$$_{n|m}q_{[x]+r} = \frac{l_{[x]+r+n} - l_{[x]+r+n+m}}{l_{[x]+r}}$$

$$_{n|}q_{[x]+r} = \frac{l_{[x]+r+n} - l_{[x]+r+n+1}}{l_{[x]+r}}$$

with the special case of $n = 0$ and $m = n$ being of particular interest:

$$_{n}q_{[x]+r} = \frac{l_{[x]+r} - l_{[x]+r+n}}{l_{[x]+r}}$$

and the complement of this n year transition probability, the n year survival probability, is:

$$_{n}p_{[x]+r} = \frac{l_{[x]+r+n}}{l_{[x]+r}}$$

The above probabilities may also be expressed in terms of the expected number of deaths in the select mortality table by defining:

$$d_{[x]+r} = l_{[x]+r} - l_{[x]+r+1}$$

Question

Calculate the following probabilities using AM92 mortality:

(i) $_2p_{[42]}$

(ii) $_2p_{42}$

(iii) $_3q_{[40]+1}$

(iv) $_{2|}q_{[41]+1}$

Solution

(i) We have:

$${}_2p_{[42]} = \frac{l_{44}}{l_{[42]}} = \frac{9,814.3359}{9,834.7030} = 0.997929$$

As the select period of the AM92 life table is 2 years, $l_{[x]+k} = l_{x+k}$ for $k \geq 2$, so $l_{[42]+2} = l_{44}$.

We can alternatively calculate this as:

$${}_2p_{[42]} = p_{[42]} \times p_{[42]+1} = \left(1 - q_{[42]}\right)\left(1 - q_{[42]+1}\right)$$

Looking up the relevant values in the *Tables*, we have:

$${}_2p_{[42]} = (1 - 0.000922)(1 - 0.001150) = 0.997929$$

as before.

(ii) This is a probability based solely on ultimate mortality:

$${}_2p_{42} = \frac{l_{44}}{l_{42}} = \frac{9,814.3359}{9,837.0661} = 0.997689$$

This could alternatively be calculated as:

$${}_2p_{42} = p_{42} \times p_{43} = \left(1 - q_{42}\right)\left(1 - q_{43}\right)$$

to obtain the same answer.

The value obtained here is lower than the value of ${}_2p_{[42]}$ from part (i), because the select life, $[42]$, will have undergone some medical underwriting at that age, and so should be in generally better health, and less likely to die within the next two years, than a member of the ultimate population.

(iii) We have:

$${}_3q_{[40]+1} = \frac{l_{[40]+1} - l_{44}}{l_{[40]+1}} = \frac{9,846.5384 - 9,814.3359}{9,846.5384} = 0.003270$$

(iv) We have:

$${}_{2|}q_{[41]+1} = \frac{l_{44} - l_{45}}{l_{[41]+1}} = \frac{d_{44}}{l_{[41]+1}} = \frac{13.0236}{9,836.5245} = 0.001324$$

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 15 Summary

Modelling mortality

We can model mortality by assuming that the complete future lifetime of a life aged x is a continuous random variable, T_x. Assuming some limiting age for the population, ω, T_x can take values between 0 and $\omega - x$.

We also model the curtate future lifetime of a life aged x using K_x, which is the integer part of T_x. The expected values of these lifetime random variables are given by:

$$E[T_x] = \mathring{e}_x = \int_{t=0}^{\infty} {}_tp_x \, dt \qquad E[K_x] = e_x = \sum_{k=1}^{\infty} {}_kp_x \approx \mathring{e}_x - ½$$

From this starting point, we can calculate probabilities of survival $({}_tp_x)$ and death $({}_tq_x)$ for an individual aged x over a period of t years.

Definitions of probabilities of death and survival

$${}_tq_x = F_x(t) = P[T_x \le t]$$

$${}_tp_x = 1 - {}_tq_x = 1 - F_x(t) = P[T_x > t]$$

$${}_{t+s}p_x = {}_tp_x \times {}_sp_{x+t} = {}_sp_x \times {}_tp_{x+s}$$

Force of mortality

The force of mortality μ_x is the instantaneous rate of mortality at age x. It is defined as:

$$\mu_x = \lim_{h \to 0+} \frac{1}{h} \times {}_hq_x \quad \text{or, equivalently} \quad \mu_x = -\frac{1}{l_x} \times \frac{d}{dx} l_x = -\frac{d}{dx} \ln l_x$$

In addition:

$${}_tq_x = \int_0^t {}_sp_x \, \mu_{x+s} \, ds \qquad \text{and} \qquad {}_tp_x = \exp\left\{ -\int_0^t \mu_{x+s} \, ds \right\}$$

Using a life table

We can tabulate probabilities and other quantities for each year of age in a life table. Simple formulae allow us to use the entries in the life table to calculate other useful quantities:

$$l_x = l_\alpha \times {}_{x-\alpha}p_\alpha \qquad {}_tp_x = \frac{l_{x+t}}{l_x} \qquad d_x = l_x - l_{x+1} \qquad q_x = \frac{d_x}{l_x}$$

$${}_{n|m}q_x = P[n < T_x \le n+m] = {}_np_x \times {}_mq_{x+n} = \frac{l_{x+n} - l_{x+n+m}}{l_x}$$

Dealing with non-integer ages

By making an assumption about mortality within a year of age, we can calculate the probability of survival for individuals at non-integer ages and for periods other than a whole number of years.

Uniform distribution of deaths

Assumption: ${}_tp_x\mu_{x+t}$ is a constant for integer x and $0 \le t \le 1$

$${}_sq_x = sq_x \quad \text{and} \quad {}_{t-s}q_{x+s} = \frac{(t-s)q_x}{1-sq_x} \quad (0 \le s \le t \le 1)$$

$$l_{x+t} = (1-t)l_x + t\,l_{x+1} \quad (0 \le t \le 1)$$

Constant force of mortality

Assumption: μ_{x+t} is a constant for integer x and $0 \le t \le 1$

$${}_tp_x = e^{-t\mu} = (p_x)^t \quad \text{and} \quad {}_{t-s}p_{x+s} = e^{-(t-s)\mu} = (p_x)^{t-s} \quad (0 \le s \le t \le 1)$$

Select mortality

Lives who are 'selected' from a larger group of people, in some non-random way with respect to their mortality, will experience different mortality at any given age from the group as a whole. They are then referred to as having select mortality.

An example occurs when applicants for life assurance are underwritten, so that only the 'better' risks are accepted for cover at the insurer's standard premium rates.

Lives subject to select mortality are denoted by $[x]+r$, where x is the age at selection and r is the number of years since selection, meaning that $x+r$ is the current age.

A select mortality table (such as AM92) has select functions (*eg* $q_{[x]+r}$, $l_{[x]+r}$) for $r = 0, 1, ..., s-1$, where s is the select period of the table. The select period is the number of years since selection during which mortality rates are assumed to be dependent upon the duration since selection as well as on current age.

For $r \ge s$, mortality is assumed to be a function of age only (called ultimate mortality), *eg*:

$$q_x = q_{[x-r]+r} \quad \text{for } r \ge s$$

The version of the AM92 table quoted in the *Tables* has a 2-year select period. This means that, for example, the expected progression of survivors through the table is:

$$l_{[x]} \to l_{[x]+1} \to l_{x+2} \to l_{x+3} \to l_{x+4} \to \cdots$$

Chapter 15 Practice Questions

15.1 Calculate the following probabilities assuming AM92 mortality applies:

(i) ${}_{10}p_{90}$

(ii) ${}_{5}p_{[79]+1}$

(iii) ${}_{2}q_{[70]}$

(iv) ${}_{5|}q_{[60]}$

(v) ${}_{10|15}q_{50}$

15.2 The mortality of a certain population is governed by the life table function $l_x = 100 - x$, $0 \leq x \leq 100$. Calculate the values of the following expressions:

(i) ${}_{10}p_{30}$

(ii) μ_{30}

(iii) $P(T_{30} < 20)$

(iv) $P(K_{30} = 20)$

15.3 A population is subject to a constant force of mortality of 0.015 *pa*. Calculate:

(i) the probability that a life aged exactly 20 dies before age 21.25.

(ii) the probability that a life aged exactly 22.5 dies between the ages of 25 and 27

(iii) the complete expectation of life for a life aged exactly 28.

15.4 Examine the column of d_x shown in the English Life Table No.15 (Males) in the Formulae and Tables for Examinations (pages 68-69).

Exam style

Describe the key characteristics of this mortality table using the data to illustrate your points. [6]

15.5 The AM92 table is based on the mortality of assured male lives in the UK.

(i) As there is no corresponding mortality table for female lives, in order to calculate survival probabilities for female policyholders, an actuary decides to use the AM92 table, but with an 'age rating' of 4 years applied, *ie* a female aged x is considered to experience mortality equivalent to a male aged $x-4$.

(a) Explain the rationale underlying this approach.

(b) Calculate the probability that a female policyholder aged 62 survives for at least the next 10 years.

(ii) A male policyholder aged 65 is known to be in poor health, and it has been determined that his mortality is 200% of AM92 Ultimate, *ie* he is subject to q_x values equal to twice those of the AM92 Ultimate table.

Calculate the probability that this policyholder will die before age 67.

15.6 A select life table is to be constructed with a select period of two years added to the ELT15 (Males) table, which is to be treated as the ultimate table. Select rates are to be derived by applying the same ratios select : ultimate seen in the AM92 table, *ie*:

$$q_{[x]} = \frac{q'_{[x]}}{q'_x} q_x \text{ and } q_{[x]+1} = \frac{q'_{[x]+1}}{q'_{x+1}} q_{x+1}$$

where the dash notation q'_x refers to AM92 mortality.

Calculate the value of $l_{[60]}$.

15.7 The table below is part of a mortality table used by a life insurance company to calculate probabilities for a special type of life insurance policy.

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	l_{x+4}
51	1,537	1,517	1,502	1,492	1,483
52	1,532	1,512	1,497	1,487	1,477
53	1,525	1,505	1,490	1,480	1,470
54	1,517	1,499	1,484	1,474	1,462
55	1,512	1,492	1,477	1,467	1,453

(i) Calculate the probability that a policyholder who was accepted for insurance exactly 2 years ago and is now aged exactly 55 will die between age 56 and age 57.

(ii) Calculate the corresponding probability for an individual of the same age who has been a policyholder for many years.

(iii) Comment on your answers to (i) and (ii).

15.8 Given that $p_{80} = 0.988$, calculate ${}_{0.5}p_{80}$ assuming:

(i) a uniform distribution of deaths between integer ages

(ii) a constant force of mortality between integer ages.

15.9 Calculate the value of ${}_{1.75}p_{45.5}$ using AM92 Ultimate mortality and assuming that:

Exam style

(i) deaths are uniformly distributed between integer ages. [3]

(ii) the force of mortality is constant between integer ages. [3]

[Total 6]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 15 Solutions

15.1 (i) $${}_{10}p_{90} = \frac{l_{100}}{l_{90}} = \frac{95.8476}{1,658.5545} = 0.057790$$

(ii) $${}_{5}p_{[79]+1} = \frac{l_{85}}{l_{[79]+1}} = \frac{3,385.2479}{5,176.2224} = 0.654000$$

(iii) $${}_{2}q_{[70]} = 1 - {}_{2}p_{[70]} = 1 - \frac{l_{72}}{l_{[70]}} = 1 - \frac{7,637.6208}{7,960.9776} = 0.040618$$

(iv) $${}_{5|}q_{[60]} = \frac{d_{65}}{l_{[60]}} = \frac{125.6412}{9,263.1422} = 0.013564$$

(v) $${}_{10|15}q_{50} = \frac{l_{60} - l_{75}}{l_{50}} = \frac{9,287.2164 - 6,879.1673}{9,712.0728} = 0.247944$$

15.2 (i) $${}_{10}p_{30} = \frac{l_{40}}{l_{30}} = \frac{60}{70} = \frac{6}{7}$$

(ii) We have:

$$\mu_x = -\frac{d}{dx}\ln l_x = -\frac{d}{dx}\ln(100 - x) = \frac{1}{100 - x}$$

So:

$$\mu_{30} = \frac{1}{70}$$

(iii) $$P(T_{30} < 20) = {}_{20}q_{30} = 1 - \frac{l_{50}}{l_{30}} = 1 - \frac{50}{70} = \frac{2}{7}$$

(iv) $$P(K_{30} = 20) = {}_{20}p_{30} \times q_{50} = \frac{l_{50} - l_{51}}{l_{30}} = \frac{50 - 49}{70} = \frac{1}{70}$$

15.3 (i) $${}_{1.25}q_{20} = 1 - {}_{1.25}p_{20} = 1 - e^{-1.25\mu} = 1 - e^{-0.01875} = 0.018575$$

(ii) $${}_{2.5|2}q_{22.5} = {}_{2.5}p_{22.5}\ {}_{2}q_{25} = e^{-2.5\mu}\left(1 - e^{-2\mu}\right) = e^{-0.0375}\left(1 - e^{-0.03}\right) = 0.028467$$

(iii) $$\mathring{e}_{28} = \int_0^\infty {}_{t}p_{28}\, dt = \int_0^\infty e^{-t\mu}\, dt = \left[-\frac{1}{\mu}e^{-t\mu}\right]_0^\infty = \frac{1}{\mu} = \frac{1}{0.015} = 66.67 \text{ years}$$

15.4 *This question is Subject CT5, September 2012, Question 8.*

- d_x is the number of lives expected to die between exact ages x and $x+1$ in the mortality table, and is equal to $l_x q_x$ at each age. [½]
- In ELT15 (Males), $d_0 = 814$. This relatively high figure reflects the high mortality occurring in the first few weeks of life. [1]
- For ages from one to fourteen, d_x takes low values, reflecting the low mortality in childhood, because of the protected environment in which most children live. [1]
- In the late teens, d_x increases. This reflects the higher mortality rate at these ages, primarily as a consequence of car accidents and other accidental deaths. [1]
- d_x remains fairly constant in the twenties. From $x = 30$ onwards, d_x starts to rise, reflecting the increasing underlying mortality with age. [½]
- d_x peaks at age 79, which is the most likely age at which a newborn life will die. [½]
- At ages beyond 79, values of d_x start to decrease. The underlying mortality rates are still rising steeply at this point, but since fewer lives survive to these higher ages the number of deaths falls off, since the population size is smaller. [1]
- d_x falls to zero by age 110, which is the age beyond which survival is assumed to be impossible for this group of lives. [½]

[Total 6]

15.5 (i)(a) The rationale is that females experience lower mortality than males. If the pattern of mortality is similar for the two sexes, but the average age at death of females exceeds that of males by 4 years, then we can approximate the mortality of a female by that of a male 4 years younger.

(i)(b) The probability that a female policyholder aged 62 survives for at least the next 10 years is the same as the probability that a male policyholder aged 58 survives for at least the next 10 years:

$$_{10}p_{58} = \frac{l_{68}}{l_{58}} = \frac{8,404.4916}{9,413.8004} = 0.892784$$

(ii) The adjusted annual probabilities of death are:

$$q'_{65} = 2q_{65} = 2 \times 0.014243 = 0.028486$$

and: $$q'_{66} = 2q_{66} = 2 \times 0.015940 = 0.031880$$

The probability of surviving for 2 years can then be calculated as:

$$_2p'_{65} = p'_{65} \times p'_{66} = (1-0.028486)(1-0.031880) = 0.940542$$

So the probability of dying within 2 years is:

$${}_2q'_{65} = 1 - 0.94054 = 0.059458$$

15.6 We can derive $l_{[60]}$ from:

$$l_{[60]}\left(1 - q_{[60]}\right) = l_{[60]+1}$$

and:

$$l_{[60]+1}\left(1 - q_{[60]+1}\right) = l_{62}$$

Now:

$$q_{[60]+1} = \frac{q'_{[60]+1}}{q'_{61}} \times q_{61} = \frac{0.008680}{0.009009} \times 0.01560 = 0.01503$$

and:

$$q_{[60]} = \frac{q'_{[60]}}{q'_{60}} \times q_{60} = \frac{0.005774}{0.008022} \times 0.01392 = 0.01002$$

So we have:

$$l_{[60]+1} = \frac{l_{62}}{1 - q_{[60]+1}} = \frac{84{,}173}{1 - 0.01503} = 85{,}457.5$$

and then:

$$l_{[60]} = \frac{l_{[60]+1}}{1 - q_{[60]}} = \frac{85{,}457.5}{1 - 0.01002} = 86{,}322.3$$

15.7 (i) *The table in the question is not laid out in the same way as AM92 in the Tables.*

The policyholder is currently aged $[53]+2$. So the probability of dying between ages 56 and 57 is:

$$\frac{l_{[53]+3} - l_{57}}{l_{[53]+2}} = \frac{1{,}480 - 1{,}470}{1{,}490} = 0.00671$$

(ii) The corresponding probability for an ultimate policyholder is:

$$\frac{l_{56} - l_{57}}{l_{55}} = \frac{1{,}477 - 1{,}470}{1{,}483} = 0.00472$$

(iii) For the usual types of policies (life assurance and annuities), policyholders experience lighter mortality during the select period.

Here, however, the mortality rate is higher in (i) than in (ii), so these policyholders experience heavier mortality during the select period. This could occur, for example, if the policy was sold to individuals who had recently had a particular form of medical treatment that increased mortality rates during the first few years.

15.8 (i) Assuming a uniform distribution of deaths:

$$_{0.5}p_{80} = 1 - {}_{0.5}q_{80} = 1 - 0.5q_{80} = 1 - 0.5(1 - p_{80}) = 1 - 0.5 \times 0.012 = 0.99400$$

(ii) Assuming a constant force of morality:

$$_{0.5}p_{80} = (p_{80})^{0.5} = 0.988^{0.5} = 0.99398$$

15.9 (i) ***Uniform distribution of deaths***

To calculate the value of $_{1.75}p_{45.5}$, we first split the age range up into single years of age:

$$_{1.75}p_{45.5} = {}_{0.5}p_{45.5} \times p_{46} \times {}_{0.25}p_{47}$$ [½]

Now:

$$p_{46} = 1 - q_{46} = 1 - 0.001622 = 0.998378$$ [½]

$$_{0.25}p_{47} = 1 - {}_{0.25}q_{47} = 1 - 0.25q_{47} = 1 - 0.25 \times 0.001802 = 0.999550$$ [½]

and:

$$_{0.5}p_{45.5} = 1 - {}_{0.5}q_{45.5} = 1 - \frac{0.5q_{45}}{1 - 0.5q_{45}} = 1 - \frac{0.5 \times 0.001465}{1 - 0.5 \times 0.001465} = 0.999267$$ [½]

So we have:

$$_{1.75}p_{45.5} = 0.999267 \times 0.998378 \times 0.999550 = 0.997197$$ [1]

[Total 3]

Alternatively, we could calculate this as follows:

$$_{1.75}p_{45.5} = \frac{l_{47.25}}{l_{45.5}} = \frac{0.75l_{47} + 0.25l_{48}}{0.5l_{45} + 0.5l_{46}}$$

$$= \frac{0.75 \times 9{,}771.0789 + 0.25 \times 9{,}753.4714}{0.5 \times 9{,}801.3123 + 0.5 \times 9{,}786.9534}$$

$$= \frac{9{,}766.677025}{9{,}794.13285} = 0.997197$$

(ii) ***Constant force of mortality***

To calculate the value of ${}_{1.75}p_{45.5}$, we first split the age range up into single years of age:

$$ {}_{1.75}p_{45.5} = {}_{0.5}p_{45.5} \times p_{46} \times {}_{0.25}p_{47} $$ [½]

Now:

$$ p_{46} = 1 - q_{46} = 1 - 0.001622 = 0.998378 $$ [½]

$$ {}_{0.25}p_{47} = (p_{47})^{0.25} = (1 - q_{47})^{0.25} = (1 - 0.001802)^{0.25} = 0.999549 $$ [½]

and:

$$ {}_{0.5}p_{45.5} = (p_{45})^{0.5} = (1 - q_{45})^{0.5} = (1 - 0.001465)^{0.5} = 0.999267 $$ [½]

So we have:

$$ {}_{1.75}p_{45.5} = 0.999267 \times 0.998378 \times 0.999549 = 0.997197 $$ [1]

[Total 3]

16

Life assurance contracts

Syllabus objectives

4.1 Define various assurance contracts.

4.1.1 Define the following terms:

– whole life assurance

– term assurance

– pure endowment

– endowment assurance

including assurance contracts where the benefits are deferred.

4.2 Develop formulae for the means and variances of the payments under various assurance contracts, assuming a constant deterministic interest rate.

4.2.4 Define the assurance factors and their select and continuous equivalents.

4.2.7 Obtain expressions in the form of sums/integrals for the mean and variance of the present value of benefit payments under each contract defined in 4.1.1, in terms of the (curtate) random future lifetime, assuming (constant) contingent benefits are payable at the middle or end of the year of contingent event or continuously. Where appropriate, simplify the above expressions into a form suitable for evaluation by table look-up or other means.

0 Introduction

In the previous chapter, we looked at how life tables can be used to calculate probabilities of survival and death for an individual.

We shall now define the present values of all the main life insurance and annuity contracts, and show how the means and variances of these present values can be calculated.

We will look at assurance contracts in this chapter, and annuity contracts in the next.

The simplest life insurance contract is the *whole life assurance*. The benefit under such a contract is an amount, called the *sum assured*, which will be paid on the policyholder's death.

A *term assurance* contract is a contract to pay a sum assured on or after death, provided death occurs during a specified period, called the term of the contract.

A *pure endowment* contract provides a sum assured at the end of a fixed term, provided the policyholder is alive.

An *endowment assurance* is a combination of:

(i) a term assurance, and

(ii) a pure endowment.

That is, a sum assured is payable either on death during the term or on survival to the end of the term. The sums assured payable on death or survival need not be the same, although they often are.

1 Whole life assurance contracts

We begin by looking at the simplest assurance contract, the whole life assurance, which pays the sum assured on the policyholder's death. For the moment we ignore the premiums which the policyholder might pay.

We will use this simple contract to introduce important concepts, in particular the expected present value (EPV) of a payment contingent on an uncertain future event. We will then apply these concepts to other types of life insurance contracts.

We briefly introduced the idea of an expected present value in the previous chapter, where we set out the basic equation of value for an insurance contract.

The present value at time 0 of a certain payment of 1 to be made at time t is v^t.

In this case, we know precisely when the payment will be made, *ie* at time *t.*

Suppose, however, that the time of payment is not certain but is a random variable, say H. Then the present value of the payment is v^H, which is also a random variable. A whole life assurance benefit is a payment of this type.

For a whole life assurance, the time to payment is a random variable, as we do not know in advance when the policyholder will die.

1.1 Present value random variable

For the moment we will introduce two conventions, which can be relaxed later on.

Convention 1: **we suppose that we are considering a benefit payable to a life who is currently aged x, where x is an integer.**

Convention 2: **we suppose that the sum assured is payable, not *on* death, but at the end of the year of death (based on policy year).**

These will not always hold in practice, of course, but they simplify the application of life table functions.

Under these conventions we see that the whole life sum assured will be paid at time $K_x + 1$.

We will often use the notational convention '(x)' to indicate the phrase 'a life who is currently aged *x*'.

Recall that K_x is the number of complete years (x) survives for, before dying. For example, if K_x takes the value 5, then (x) dies between time 5 years and time 6 years. So we see that, for our whole life assurance, the duration $K_x + 1$ takes us to the *end* of the year in which (x) dies.

If the sum assured is S, then the present value of the benefit is Sv^{K_x+1}, a random variable. We now wish to consider the expected value and the variance of Sv^{K_x+1}.

1.2 The expected present value

Since K_x is a discrete random variable that must take a non-negative integer value:

$$E(K_x) = \sum_{k=0}^{\infty} k\,P(K_x = k)$$

and:

$$E\left[g(K_x)\right] = \sum_{k=0}^{\infty} g(k)P(K_x = k)$$

for any function g (assuming that the sum exists, *ie* is finite).

In particular, for $g(k) = v^{k+1}$, **we have:**

$$\boldsymbol{E\left[v^{K_x+1}\right] = \sum_{k=0}^{\infty} v^{k+1} P(K_x = k) = \sum_{k=0}^{\infty} v^{k+1}\ {}_{k|}q_x}$$

The deferred probability of death ${}_{k|}q_x$ was introduced in the previous chapter. It is the probability that a life aged x survives for k years, but dies before time $k+1$, and is the probability function of the random variable K_x.

To construct the above formula, we start with the present value of the benefit assuming that K_x takes the value k, which is v^{k+1}. We then multiply this by $P(K_x = k)$, which is the probability that death occurs between time k and time $k+1$, and then sum over all possible values of k. As death could occur in *any* future year, this involves summing to infinity.

This is the standard way to calculate the expected value of a random variable, and so gives us an expression for the expectation of the present value of the benefit (or just 'expected present value' for the benefits, for short).

Question

Calculate $E\left(v^{K_x+1}\right)$ at an interest rate of 0%.

Solution

If $i = 0\%$, then $v = 1$, so:

$$E\left(v^{K_x+1}\right) = E\left(1^{K_x+1}\right) = \sum_{k=0}^{\infty} 1^{k+1} P(K_x = k) = \sum_{k=0}^{\infty} P(K_x = k) = P(K_x \ge 0) = 1$$

ie it is the probability that the life will die eventually.

Actuarial notation for the expected present value

$E[v^{K_x+1}]$ is the *expected present value* (EPV) of a sum assured of 1, payable at the end of the year of death. Such functions play a central role in life insurance mathematics and are included in the standard actuarial notation. We define:

$$A_x = E[v^{K_x+1}] = \sum_{k=0}^{\infty} v^{k+1} {}_{k|}q_x$$

[Note that, for brevity, we have written the sum as $\sum_{k=0}^{\infty}$ instead of $\sum_{k=0}^{\omega-x-1}$. This should cause no confusion, since ${}_k p_x = 0$ for $k \geq \omega - x$.]

As we saw when considering life tables, we use the symbol ω to indicate the maximum possible age for a population. So the probability of surviving from any age x to age ω (and beyond) is 0.

In standard actuarial notation, a capital A function denotes the expected present value of a single payment of 1 unit. The subscript x (in the A_x symbol) indicates that the payment is contingent (*ie* depends on) the status of 'x' in some way.

The status 'x' is called a *life status*, and relates to a person who is currently x years old exactly. The status remains in existence (*ie* remains 'active') for as long as the person stays alive into the future. The status ceases to exist (or 'fails') at the point in the future when this person, currently aged x, dies. So, in summary, the life status 'x':

- remains active for as long as (x) remains alive in the future
- fails at the moment at which (x) dies in the future.

So, returning to A_x, we can now say that this symbol represents the EPV of a single payment of 1 unit paid *at the end of the year* in which the status x fails or, in other words, the payment is made at the end of the year in which a person, currently aged x, dies.

We will see later in this chapter how we vary the notation when the payment is made immediately on death, rather than at the end of the year.

Also $E\left[Sv^{K_x+1}\right] = S\,E\left[v^{K_x+1}\right]$, so if the sum assured is S, then the EPV of the benefit is $S\,A_x$.

Values of A_x at various rates of interest are tabulated in (for example) the AM92 tables, which can be found in 'Formulae and Tables for Examinations'.

This is the yellow *Tables* book.

Question

Find A_{40} (AM92 at 6%).

Solution

0.12313

(This appears on page 102 of the *Tables*.)

For comparison purposes, using the same mortality and interest rate (AM92, 6%), we find:

$$A_{30} = 0.07328$$

and: $A_{70} = 0.48265$

We see that the values increase with age. This is because the 70-year-old is more likely to die in the near future, so the benefit has a higher expected present value (since we are discounting for a shorter period).

Alternatively, the values of:

$$\sum_{k=0}^{\infty} v^{k+1} {}_{k|}q_x \; (= A_x)$$

can be obtained by direct computation from a life table, most conveniently using a computer. This has the advantage that any rate of interest can be used, including the possibility of having an interest rate that varies over time, or with duration.

Question

A whole life assurance pays a benefit of $50,000 at the end of the policy year of death of a life now aged exactly 90. Mortality is assumed to follow the life table given below:

Age, x	l_x	d_x
90	100	25
91	75	35
92	40	40
93	0	0

Calculate the expected present value of this benefit using an effective rate of interest of 5% *pa*.

Solution

The expected present value is:

$$
\begin{aligned}
50{,}000A_{90} &= 50{,}000\sum_{k=0}^{2} v^{k+1}\ {}_{k|}q_{90} \\
&= 50{,}000\sum_{k=0}^{2} v^{k+1}\frac{d_{90+k}}{l_{90}} \\
&= 50{,}000\left(1.05^{-1}\times\frac{25}{100}+1.05^{-2}\times\frac{35}{100}+1.05^{-3}\times\frac{40}{100}\right) \\
&= \$45{,}054.53
\end{aligned}
$$

The summation here contains only three terms, because under this mortality table the life will definitely have died before age 93.

1.3 Variance of the present value random variable

Recall that:

$$\text{var}[X]=E\left[X^2\right]-\left[E(X)\right]^2$$

for any random variable X, and:

$$\text{var}\left[g(X)\right]=E\left[\left(g(X)\right)^2\right]-\left[E\left(g(X)\right)\right]^2$$

for any random variable X and any function g.

Turning now to the variance of v^{K_x+1}, we have:

$$\text{var}\left[v^{K_x+1}\right]=E\left[\left(v^{K_x+1}\right)^2\right]-\left[E\left(v^{K_x+1}\right)\right]^2$$

This can also be written as:

$$\mathbf{var[}v^{K_x+1}\mathbf{]}=\sum_{k=0}^{\infty}(v^{k+1})^2\ {}_{k|}q_x-\left(A_x\right)^2$$

But since $\left(v^{k+1}\right)^2=\left(v^2\right)^{k+1}$, the first term is just ${}^2A_x$ where the '2' prefix denotes an EPV calculated at a rate of interest $(1+i)^2-1$.

So:

$$\text{var}\left[v^{K_x+1}\right] = {}^2A_x - \left(A_x\right)^2$$

The 'trick' used here is to notice that $(v^2)^{k+1}$ is the same as v^{k+1}, except that we have replaced v^2 by v. So we are effectively using a new interest rate (i^*, say) for which $v^* = v^2$, *ie:*

$$\frac{1}{1+i^*} = \left(\frac{1}{1+i}\right)^2 \quad \Rightarrow \quad i^* = (1+i)^2 - 1$$

Alternatively, since $v = e^{-\delta}$, we could say that we are using a new force of interest (δ^*, say) for which $v^* = v^2$, *ie*:

$$e^{-\delta^*} = (e^{-\delta})^2 = e^{-2\delta} \quad \Rightarrow \quad \delta^* = 2\delta$$

So, we could say that the '2' prefix on the symbol ${}^2A_x$ indicates that it is evaluated at twice the original force of interest.

So provided we can calculate EPVs at any rates of interest, it is easy to find the variance of a whole life benefit.

Note that:

$$\textbf{var}\left[Sv^{K_x+1}\right] = S^2\ \textbf{var}\left[v^{K_x+1}\right]$$

Values of ${}^2A_x$ at various rates of interest are tabulated in AM92 'Formulae and Tables for Examinations', but of course any rate of interest can be assumed when a computer is used.

Question

Claire, aged exactly 30, buys a whole life assurance with a sum assured of £50,000 payable at the end of the year of her death.

Calculate the standard deviation of the present value of this benefit using AM92 Ultimate mortality and 6% *pa* interest.

Solution

The standard deviation of the present value is:

$$\sqrt{50{,}000^2\left[{}^2A_{30} - \left(A_{30}\right)^2\right]} = 50{,}000\sqrt{0.01210 - 0.07328^2} = £4{,}102$$

2 Term assurance contracts

A term assurance contract is a contract to pay a sum assured on or after death, provided death occurs during a specified period, called the term of the contract.

If the life survives to the end of the term of the contract there will be no payment made by the life insurer to the policyholder.

2.1 Present value random variable

Consider such a contract, which is to pay a sum assured at the end of the year of death of a life aged x, provided this occurs during the next n years. We assume that n is an integer.

Let F denote the present value of this payment. F is a random variable.

We are just using the letter F here for notational convenience. It is not standard notation.

If the policyholder dies within the n-year term, then $F = v^{K_x+1}$. If the policyholder is still alive at the end of the n-year term, then $F = 0$. We can express this more mathematically as follows:

$$F = \begin{cases} v^{K_x+1} & \text{if } K_x < n \\ 0 & \text{if } K_x \geq n \end{cases}$$

2.2 Expected present value

The expected present value of the term assurance is:

$$E[F] = \sum_{k=0}^{n-1} v^{k+1} P(K_x = k) + 0 \times P(K_x \geq n)$$

Writing this using actuarial notation:

$$\boldsymbol{E[F] = \sum_{k=0}^{n-1} v^{k+1}\, {}_{k|}q_x + 0 \times {}_n p_x}$$

$$\boldsymbol{= \sum_{k=0}^{n-1} v^{k+1}\, {}_{k|}q_x}$$

Actuarial notation for the expected present value

Before we look at this, we need to introduce a second kind of status, the duration status $\overline{n|}$. This is easily distinguished from the life status because the number (n) is enclosed by the corner symbol ($\overline{|}$), which we used extensively in earlier chapters.

The status $\overline{n|}$ remains active for as long as the duration of time from the valuation date does not exceed n years. The status fails at the moment the elapsed time reaches n years in duration.

Question

Describe in words the difference in meaning between $A_{\overline{10}|}$ and A_{10}.

Solution

$A_{\overline{10}|}$ is the present value of 1 unit paid in exactly 10 years' time (*ie* the payment is made when the 10-year duration status fails). Since this is a certain payment, we don't use the phrase 'expected present value' for this. In addition, since $A_{\overline{10}|} = v^{10}$, we don't usually have much call to use the $A_{\overline{n}|}$ notation, as it is just as easy to write v^n every time we need it.

A_{10}, on the other hand, is the expected present value of a payment of 1 unit, paid at the end of the year of death of a person currently aged exactly 10 years old.

In actuarial notation, we define:

$$A^{1}_{x:\overline{n}|} = E[F] = \sum_{k=0}^{n-1} v^{k+1}{}_{k|}q_x$$

to be the EPV of a term assurance benefit of 1, payable at the end of the year of death of a life x, provided this occurs during the next n years.

The summation formula here is of the same form as that for the whole life assurance, except that under a term assurance the benefit is only paid if the life dies within *n* years, so the summation is only for *n* years.

Now let's explain the more complex notation used here ($A^{1}_{x:\overline{n}|}$).

This still represents the EPV of a single payment, because it is a 'big A' symbol. However, the payment is now contingent on what happens to *two* statuses in some way (multiple statuses are shown in the subscript separated by a colon).

The exact condition for payment is identified by the number that is placed above the statuses (*ie* the 1) and where it is placed:

- the number is positioned above the life status x: this indicates that the payment is made *only* when the life status x fails (*ie* dies)
- the number over the x is 1: this tells us that the life status x has to fail *first* out of the two statuses involved, in order for the payment to be made.

As the $\overline{n}|$ status will fail after n years, then (x) has to die within n years for the payment to be made. In other words, $A^{1}_{x:\overline{n}|}$ is the EPV of 1 unit paid only on the death of (x), provided that occurs within n years.

As before, the symbol also indicates that the payment is made at the end of the year of death. This aspect of the notation will become clearer later in the chapter.

2.3 Variance of the present value random variable

Along the same lines as for the whole life assurance:

$$\text{var}[F] = {}^2A^{1}_{x:\overline{n}|} - \left(A^{1}_{x:\overline{n}|}\right)^2$$

where the '2' prefix means that the EPV is calculated at rate of interest $(1+i)^2 - 1$, as before.

3 Pure endowment contracts

A pure endowment contract provides a sum assured at the end of a fixed term, provided the policyholder is then alive.

3.1 Present value random variable

Consider a pure endowment contract to pay a sum assured of 1 after n years, provided a life aged x is still alive. We assume that n is an integer.

Let G denote the present value of the payment.

Again, we are using the letter G for notational convenience. This is not standard notation.

The payment is equal to zero if the policyholder dies within n years, but 1 if the policyholder survives to the end of the term. So, if the policyholder dies within the n-year term, then $G=0$. On the other hand, if the policyholder is still alive at the end of the n-year term, then $G=v^n$.

This can be written mathematically as:

$$G=\begin{cases} 0 & \text{if } K_x < n \\ v^n & \text{if } K_x \geq n \end{cases}$$

3.2 Expected present value

The expected present value of the pure endowment is:

$$E[G]=0\times P(K_x<n)+v^n P(K_x\geq n)$$

In actuarial notation (and reversing the order of the two terms), this is:

$$\boldsymbol{E[G]=v^n\,{}_np_x+(0\times{}_nq_x)}$$

Question

At a certain company, the probability of each employee leaving during any given year is 5%, independent of the other employees. Those who remain with the company for 25 years are given \$1,000,000.

Calculate the expected present value of this payment to a new starter, assuming an effective interest rate of 7% *pa* and ignoring the possibility of death.

Solution

The probability of remaining with the company in any given year is 0.95, so the probability of remaining with the company for 25 years is 0.95^{25}. Therefore, the expected present value of the benefit is equal to:

$$1{,}000{,}000\times\frac{1}{1.07^{25}}\times 0.95^{25}=\$51{,}109$$

Actuarial notation for the expected present value

In actuarial notation, we define:

$$\boldsymbol{A_{\overset{\quad 1}{x:\overline{n}|}}=E[G]=v^n\,{}_np_x}$$

to be the EPV of a pure endowment benefit of 1, payable after $\boldsymbol{n}$ years to a life aged $\boldsymbol{x}$.

The notation here is $A_{\overset{\quad 1}{x:\overline{n}|}}$ because:

- the benefit is paid *only* when the term of *n* years ends, *ie* at the moment at which the $\overline{n}|$ status fails, so the number (whatever it may be) needs to be placed above the $\overline{n}|$
- the benefit will only be paid (at time *n*) if the person is still alive at that time: this requires the status $\overline{n}|$ to be the *first* of the two statuses to fail, and hence we need the number to be a '1'.

3.3 Variance of the present value random variable

Following the same lines as before:

$$\boldsymbol{\text{var}[G]={}^2A_{\overset{\quad 1}{x:\overline{n}|}}-\left(A_{\overset{\quad 1}{x:\overline{n}|}}\right)^2}$$

where, as usual, the '2' prefix denotes an EPV calculated at a rate of interest of $(1+i)^2-1$.

Question

Calculate the standard deviation of the present value of the payment described in the previous question.

Solution

In the previous question, we calculated the expected present value of the payment to be:

$$E(PV)=1{,}000{,}000\times\frac{1}{1.07^{25}}\times 0.95^{25}=\$51{,}109$$

To calculate the standard deviation, we will also need to calculate $E(PV^2)$. This is:

$$E(PV^2) = \left(1{,}000{,}000 \times \frac{1}{1.07^{25}}\right)^2 \times 0.95^{25} = 1{,}000{,}000^2 \times \frac{1}{1.07^{50}} \times 0.95^{25} = \$9{,}416{,}754{,}493$$

The standard deviation is:

$$\begin{aligned} sd(PV) &= \sqrt{E(PV^2) - \left[E(PV)\right]^2} \\ &= \sqrt{9{,}416{,}754{,}493 - 51{,}109^2} \\ &= \$82{,}490 \end{aligned}$$

4 Endowment assurance contracts

An endowment assurance is a combination of:

- a term assurance and
- a pure endowment.

That is, a sum assured is payable either on death during the term or on survival to the end of the term. The sums assured payable on death or survival need not be the same, although they often are.

4.1 Present value random variable

Consider an endowment assurance contract to pay a sum assured of 1 to a life now aged x at the end of the year of death, if death occurs during the next n years, or after n years if the life is then alive. We assume that n is an integer.

Let H be the present value of this payment.

Again, we are using the letter H for notational convenience. This is not standard notation.

The benefit is paid on death or survival, so its value must be the sum of the values of a benefit paid on death and a benefit paid on survival.

In terms of the present values already defined, $H = F + G$.

Hence:

$$H = \begin{cases} v^{K_x+1} & \text{if } K_x < n \\ v^n & \text{if } K_x \geq n \end{cases}$$

This can also be written as:

$$H = v^{\min\{K_x+1, n\}}$$

4.2 Expected present value

The expected present value of the endowment assurance is:

$$E[H] = \sum_{k=0}^{n-1} v^{k+1} P(K_x = k) + v^n P(K_x \geq n)$$

This can also be written as:

$$\boldsymbol{E[H] = E[F] + E[G]}$$

$$\boldsymbol{= \sum_{k=0}^{n-1} v^{k+1}\ {}_{k|}q_x + v^n\ {}_n p_x}$$

$$\boldsymbol{= \sum_{k=0}^{n-2} v^{k+1}\ {}_{k|}q_x + v^n\ {}_{n-1} p_x}$$

The last expression holds because payment at time n is certain if the life survives to age $x+n-1$.

Actuarial notation for the expected present value

In actuarial notation we define:

$$\boldsymbol{A_{x:\overline{n|}} = E[H]}$$

$$\boldsymbol{= E[F] + E[G]}$$

$$\boldsymbol{= A^{1}_{x:\overline{n|}} + A_{x:\overline{n|}}^{\ \ 1}}$$

The interpretation of the $A_{x:\overline{n|}}$ symbol is that where we have no number above either status, it implies that the payment is made on the *first* of the two statuses to fail, regardless of order. So $A_{x:\overline{n|}}$ is the EPV of 1 unit, paid after *n* years, or at the end of the year of death of (*x*), whichever occurs first.

Values of $A_{x:\overline{n|}}$ are tabulated where $x+n=60$ or $x+n=65$, for AM92 mortality at both 4% and 6% interest.

Question

Allan, aged exactly 40, has just bought a 20-year endowment assurance policy. The sum assured is £100,000 and this is payable on survival to age 60 or at the end of the year of earlier death.

Calculate the expected present value of the benefit paid to Allan, assuming AM92 Ultimate mortality and 4% *pa* interest.

Solution

The expected present value is:

$$100{,}000 A_{40:\overline{20|}} = 100{,}000 \times 0.46433 = £46{,}433$$

The value of $A_{40:\overline{20|}}$ (AM92 Ultimate mortality, 4% interest) is taken from page 100 of the Tables.

4.3 Variance of the present value random variable

Note that F and G are not independent random variables (one must be zero and the other non-zero).

This is because the life will either survive to the end of the n-year period or die during it.

Therefore:

$$\text{var}[H] \neq \text{var}[F] + \text{var}[G]$$

We must find $\text{var}[H]$ from first principles.

As before, we find that:

$$\text{var}[H] = {}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2$$

where the '2' prefix denotes an EPV calculated at rate of interest $(1+i)^2 - 1$.

Question

Derive the formula above for $\text{var}[H]$ from first principles.

Solution

We can use the expression $H = v^{\min\{K_x+1,\,n\}}$, and the same approach as we've used elsewhere in this chapter:

$$\begin{aligned}
\text{var}[H] &= \text{var}\left[v^{\min\{K_x+1,\,n\}}\right] \\
&= E\left[\left(v^{\min\{K_x+1,\,n\}}\right)^2\right] - \left(E\left[v^{\min\{K_x+1,\,n\}}\right]\right)^2 \\
&= E\left[\left(v^2\right)^{\min\{K_x+1,\,n\}}\right] - \left(A_{x:\overline{n}|}\right)^2 \\
&= {}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2
\end{aligned}$$

where ${}^2A_{x:\overline{n}|}$ is calculated at rate of interest $(1+i)^2 - 1$.

It is necessary to express our random variable using a single term if we wish to derive a variance like this, ie as $H = v^{\min\{K_x+1,\,n\}}$ *rather than* $H = \begin{cases} v^{K_x+1} & \text{if } K_x < n \\ v^n & \text{if } K_x \geq n \end{cases}$.

5 Deferred assurance benefits

Although not as common as deferred annuities, deferred assurance benefits can be defined in a similar way.

Deferred annuities will be introduced in Chapter 17.

5.1 Present value random variable

For example, a whole life assurance with sum assured 1, payable to a life aged x but deferred n years is a contract to pay a death benefit of 1 provided death occurs after age $x+n$.

If we let J denote the present value of this benefit, then:

$$J=\begin{cases}0 & \text{if } K_x<n \\ v^{K_x+1} & \text{if } K_x\geq n\end{cases}$$

5.2 Expected present value

If the benefit is payable at the end of the year of death (if at all), the EPV of this assurance is denoted $_{n|}A_x$.

As usual, the subscript of $n|$ to the left of the symbol indicates that the event is deferred for n years. The subscript x to the right of the symbol means that the payment will be made on the failure of the life status (but now we have to 'wait' at least n years before anything can be paid).

It can be shown that:

$$_{n|}A_x = A_x - A^{1}_{x:\overline{n}|} = v^n\,{}_np_x\,A_{x+n}$$

Note the appearance of $v^n\,{}_np_x$. The factor $v^n\,{}_np_x$ is important and useful in developing EPVs. It plays the role of the pure interest discount factor v^n, where now the payment or present value being discounted depends on the survival of a life aged x.

Question

Prove that $_{n|}A_x = A_x - A^{1}_{x:\overline{n}|} = v^n\,{}_np_x\,A_{x+n}$.

Solution

The first result is proved as follows:

$$_{n|}A_x = E(J) \qquad \text{where:} \qquad J=\begin{cases}0 & \text{if } K_x<n \\ v^{K_x+1} & \text{if } K_x\geq n\end{cases}$$

Now $A_x = E(X)$, say, where:

$$X = v^{K_x+1}$$

and $A^{1}_{x:\overline{n}|} = E(Y)$, say, where:

$$Y = \begin{cases} v^{K_x+1} & \text{if } K_x < n \\ 0 & \text{if } K_x \geq n \end{cases}$$

So:

$$X - Y = \begin{cases} v^{K_x+1} - v^{K_x+1} \quad = 0 & \text{if } K_x < n \\ v^{K_x+1} - 0 \quad = v^{K_x+1} & \text{if } K_x \geq n \end{cases}$$

$$= J$$

Therefore:

$$E(J) = E(X - Y) = E(X) - E(Y) = A_x - A^{1}_{x:\overline{n}|}$$

Furthermore:

$$A_x - A^{1}_{x:\overline{n}|} = \sum_{k=0}^{\infty} v^{k+1} P(K_x = k) - \sum_{k=0}^{n-1} v^{k+1} P(K_x = k)$$

$$= \sum_{k=n}^{\infty} v^{k+1} P(K_x = k)$$

$$= \sum_{k=n}^{\infty} v^{k+1} {}_k p_x \, q_{x+k}$$

$$= v^n \, {}_n p_x \sum_{k=n}^{\infty} v^{k+1-n} \, {}_{k-n} p_{x+n} \, q_{x+k}$$

since ${}_k p_x = {}_n p_x \, {}_{k-n} p_{x+n}$ (for $k \geq n$).

If we let $j=k-n$ in the summation, we can write:

$$A_x - A^{1}_{x:\overline{n}|} = v^n \, {}_n p_x \sum_{j=0}^{\infty} v^{j+1} \, {}_j p_{x+n} \, q_{x+n+j}$$

$$= v^n \, {}_n p_x \sum_{j=0}^{\infty} v^{j+1} P\left(K_{x+n} = j\right)$$

$$= v^n \, {}_n p_x \, A_{x+n}$$

as required.

We can evaluate the expected present value of deferred assurances using values from the *Tables*.

Question

A life assurance policy pays a benefit of £20,000 at the end of the policy year of death of a life now aged exactly 55, provided that death occurs after exact age 60.

Calculate the expected present value of this benefit assuming that the effective annual rate of interest is 4% and mortality follows the AM92 Ultimate table.

Solution

The expected present value is:

$$20{,}000 \, {}_{5|}A_{55} = 20{,}000 v^5 \, {}_5 p_{55} \, A_{60}$$

$$= 20{,}000(1+i)^{-5} \times \frac{l_{60}}{l_{55}} \times A_{60}$$

$$= 20{,}000 \times 1.04^{-5} \times \frac{9{,}287.2164}{9{,}557.8179} \times 0.45640$$

$$= £7{,}290$$

5.3 Variance of the present value random variable

One useful feature of deferred assurances is that it is easier to find their variances directly than is the case for (deferred) **annuities. For example, let X be the present value of a whole life assurance and Y the present value of a temporary assurance with term n years, both for a sum assured of 1 payable at the end of the year of death of a life aged x. Then $E[X] = A_x$, $E[Y] = A^{1}_{x:\overline{n}|}$ and:**

$$E[X-Y] = {}_{n|}A_x = A_x - A^{1}_{x:\overline{n}|}$$

We took this approach in the previous question.

Moreover:

$$\text{var}[X-Y]=\text{var}[X]+\text{var}[Y]-2\text{cov}[X,Y]$$

and it can be shown by considering the distributions of X and Y, that

$$\text{cov}[X,Y]={}^2A^1_{x:\overline{n}|}-A_x\,A^1_{x:\overline{n}|}$$

where the '2' superscript has its usual meaning.

Question

Show that $\text{cov}(X,Y)={}^2A^1_{x:\overline{n}|}-A_xA^1_{x:\overline{n}|}$.

Solution

From Subject CS1, the covariance of two random variables X and Y is defined by the equation:

$$\text{cov}(X,Y)=E(XY)-E(X)E(Y)$$

For a whole life assurance with a sum assured of 1 payable at the end of the year of death of a life currently aged x, the present value random variable is:

$$X=v^{K_x+1}$$

and: $E(X)=A_x$

Furthermore, for an n-year term assurance issued to the same life, the present value random variable is:

$$Y=\begin{cases}v^{K_x+1} & \text{if } K_x<n\\ 0 & \text{otherwise}\end{cases}$$

and: $E(Y)=A^1_{x:\overline{n}|}$

Multiplying X and Y gives:

$$XY=\begin{cases}v^{2(K_x+1)} & \text{if } K_x<n\\ 0 & \text{otherwise}\end{cases}$$

So we see that XY is just the same as Y, except for the fact that v has been replaced by v^2. Hence $E(XY)={}^2A^1_{x:\overline{n}|}$, where, as usual, the superscript of 2 to the left of the assurance symbol indicates that v has been replaced by v^2, or in other words that we are valuing the benefit using an interest rate of $(1+i)^2-1$ (or, equivalently, twice the force of interest).

So we have:

$$\text{cov}(X,Y) = E(XY) - E(X)E(Y) = {}^2A^1_{x:\overline{n}|} - A_x A^1_{x:\overline{n}|}$$

This covariance result can now be used in the formula:

$$\text{var}[X-Y] = \text{var}[X] + \text{var}[Y] - 2\text{cov}[X,Y]$$

Hence:

$$\mathbf{var}[\boldsymbol{X}-\boldsymbol{Y}] = {}^2\boldsymbol{A_x} - (\boldsymbol{A_x})^2 + {}^2\boldsymbol{A^1_{x:\overline{n}|}} - \left(\boldsymbol{A^1_{x:\overline{n}|}}\right)^2 - 2\left({}^2\boldsymbol{A^1_{x:\overline{n}|}} - \boldsymbol{A_x}\,\boldsymbol{A^1_{x:\overline{n}|}}\right)$$

$$= {}^2\boldsymbol{A_x} - \left(\boldsymbol{A_x} - \boldsymbol{A^1_{x:\overline{n}|}}\right)^2 - {}^2\boldsymbol{A^1_{x:\overline{n}|}}$$

$$= {}^2\boldsymbol{A_x} - \left({}_{n|}\boldsymbol{A_x}\right)^2 - {}^2\boldsymbol{A^1_{x:\overline{n}|}}$$

$$= {}_{n|}{}^2\boldsymbol{A_x} - \left({}_{n|}\boldsymbol{A_x}\right)^2$$

This result for the variance of a deferred whole life assurance can alternatively be obtained by considering the present value random variable:

$$J = \begin{cases} 0 & \text{if } K_x < n \\ v^{K_x+1} & \text{if } K_x \geq n \end{cases}$$

Now:

$$E[J] = \sum_{k=n}^{\infty} v^{k+1}\,{}_{k|}q_x = {}_{n|}A_x$$

so:

$$\text{var}[J] = E\left[J^2\right] - \left(E[J]\right)^2$$

$$= \sum_{k=n}^{\infty}\left(v^{k+1}\right)^2 {}_{k|}q_x - \left({}_{n|}A_x\right)^2$$

$$= \sum_{k=n}^{\infty}\left(v^2\right)^{k+1} {}_{k|}q_x - \left({}_{n|}A_x\right)^2$$

$$= {}_{n|}{}^2A_x - \left({}_{n|}A_x\right)^2$$

as before.

5.4 Deferred term assurance

So far in this section, we have looked at a deferred whole life assurance.

We will now consider a deferred term assurance, where a benefit of 1 is payable at the end of the year of death of a life currently aged x, provided that death occurs between age $x+m$ and age $x+m+n$.

Present value random variable

If we let M denote the present value of this benefit, then:

$$M=\begin{cases}0 & \text{if } K_x<m \text{ or } K_x\geq m+n \\ v^{K_x+1} & \text{if } m\leq K_x<m+n\end{cases}$$

Expected present value

The expected present value of the benefit is denoted ${}_{m|}A^{1}_{x:\overline{n}|}$. Similar to the deferred whole life assurance above, it can be shown that:

$${}_{m|}A^{1}_{x:\overline{n}|}=A^{1}_{x:\overline{m+n}|}-A^{1}_{x:\overline{m}|}=v^m\,{}_mp_x\,A^{1}_{x+m:\overline{n}|}$$

Variance of the present value random variable

This is given by the formula:

$$\text{var}\left[M\right]={}_{m|}{}^{2}A^{1}_{x:\overline{n}|}-\left({}_{m|}A^{1}_{x:\overline{n}|}\right)^2$$

6 Benefits payable immediately on death

So far we have assumed that assurance death benefits have been paid at the end of the year of death. In practice, assurance death benefits are paid a short time after death, as soon as the validity of the claim can be verified.

Assuming a delay until the end of the year of death is therefore not a prudent approximation, but assuming that there is no delay and that the sum assured is paid immediately on death is a prudent approximation.

This is due to the fact that if we delay payment until the end of the year, the 'expected present value of the benefits' part of the equation of value will be smaller. Thus the premium charged for those benefits will be lower. Likewise, assuming that claims are paid immediately on death would give a higher (*ie* more prudent) premium.

The present value of death benefits payable immediately on the death of the policyholder can be expressed in terms of the policyholder's complete future lifetime, T_x. This random variable was introduced in Chapter 15, along with its PDF ${}_tp_x\,\mu_{x+t}$.

Related to such assurance benefits are annuities under which payment is made in a continuous stream instead of at discrete intervals. Of course, this does not happen in practice, but such an assumption is reasonable if payments are very frequent, say weekly or daily. Later in the course we will consider annuities with a payment frequency between continuous and annual.

We will look at continuously payable annuities in Chapter 17, and those with a payment frequency between continuous and annual in Chapter 18.

6.1 Whole life assurance

Consider a whole life assurance with sum assured 1, payable immediately on the death of a life aged *x*.

Present value random variable

The payments will be made exactly T_x years from now (as T_x is the time from now until the exact moment of death for a life currently aged *x*). So:

The present value of this benefit is v^{T_x}.

Expected present value

As shown in Subject CS1, for any continuous random variable *Y*, with probability density function $f_Y(y)$, the expectation is:

$$E(Y)=\int_{-\infty}^{\infty} y\, f_Y(y)\,dy$$

Also, for any function g:

$$E\left(g(Y)\right)=\int_{-\infty}^{\infty} g(y)\,f_Y(y)\,dy$$

Since the density function of T_x is ${}_tp_x\,\mu_{x+t}$, the EPV of the benefit, denoted by $\bar{A}_x$ is:

$$\bar{A}_x = E\left[v^{T_x}\right]=\int_0^\infty v^t\,{}_tp_x\mu_{x+t}\,dt$$

and its variance can be shown to be:

$$\text{var}\left[v^{T_x}\right]={}^2\bar{A}_x-(\bar{A}_x)^2$$

Intuitively, the integral expression for the expectation can be built up as follows. The expression ${}_tp_x\mu_{x+t}\,dt$ is the probability that the life, currently aged x, survives to time t and then dies during the short interval $(t,t+dt)$. The factor v^t gives the present value of the payment made if the life dies in this interval, and the integral sums this over all future time periods. As usual, this sum of all the possible values multiplied by their probabilities gives us the expectation.

The variance formula given above is derived from the general formula $\text{var}[X]=E[X^2]-E[X]^2$, as follows:

$$\begin{aligned}\text{var}\left[v^{T_x}\right]&=E\left[\left(v^{T_x}\right)^2\right]-\left(E\left[v^{T_x}\right]\right)^2\\&=E\left[\left(v^2\right)^{T_x}\right]-\left(\bar{A}_x\right)^2\\&=\int_0^\infty\left(v^2\right)^t\,{}_tp_x\,\mu_{x+t}\,dt-\left(\bar{A}_x\right)^2\\&={}^2\bar{A}_x-\left(\bar{A}_x\right)^2\end{aligned}$$

where ${}^2\bar{A}_x$ is calculated at rate of interest $(1+i)^2-1$

The standard actuarial notation for the EPVs of assurances with a death benefit payable immediately on death, or of annuities payable continuously, is a bar added above the symbol for the EPV of an assurance with a death benefit payable at the end of the year of death, or an immediate annuity with annual payments, respectively.

6.2 Term assurance

Term assurance contracts with a death benefit payable immediately on death can be defined in a similar way, with the obvious notation for their EPVs and deferred assurance benefits likewise.

Consider a term assurance with a sum assured of 1 payable immediately upon the death of a life now aged x, provided that this life dies within n years.

Present value random variable

Let $\bar{F}$ denote the present value of this benefit. Then:

$$\bar{F} = \begin{cases} v^{T_x} & \text{if } T_x < n \\ 0 & \text{if } T_x \geq n \end{cases}$$

Expected present value

The expected present value is:

$$E\left(\bar{F}\right) = \int_0^n v^t f_{T_x}(t)\,dt = \int_0^n v^t \,{}_t p_x \,\mu_{x+t}\,dt$$

where $f_{T_x}(t) = {}_t p_x \,\mu_{x+t}$ is the probability density function of T_x, as mentioned earlier.

Actuarial notation for the expected present value

The EPV of the benefit is denoted $\bar{A}^{1}_{x:\overline{n|}}$.

Variance of the present value random variable

Its variance is ${}^2\bar{A}^{1}_{x:\overline{n|}} - (\bar{A}^{1}_{x:\overline{n|}})^2$.

6.3 Endowment assurance

Consider an endowment assurance with a sum assured of 1 payable after n years or immediately upon the earlier death of a life now aged x.

Present value random variable

Let $\bar{H}$ denote the present value of this benefit. Then:

$$\bar{H} = \bar{F} + G = \begin{cases} v^{T_x} & \text{if } T_x < n \\ v^n & \text{if } T_x \geq n \end{cases}$$

Recall that the random variable G is equal to the present value of the pure endowment. As this can never be paid earlier than the maturity date, there is no need for a random variable $\bar{G}$.

Expected present value

The expected present value is:

$$E(\bar{H}) = \int_0^n v^t f_{T_x}(t)dt + \int_n^\infty v^n f_{T_x}(t)dt$$

$$= \int_0^n v^t \, {}_tp_x \, \mu_{x+t} \, dt + v^n \int_n^\infty f_{T_x}(t)dt$$

$$= \int_0^n v^t \, {}_tp_x \, \mu_{x+t} \, dt + v^n P(T_x > n)$$

$$= \int_0^n v^t \, {}_tp_x \, \mu_{x+t} \, dt + v^n \, {}_np_x$$

Actuarial notation for the expected present value

The EPV of the benefit is denoted $\bar{A}_{x:\overline{n}|}$.

Variance of the present value random variable

Its variance is ${}^2\bar{A}_{x:\overline{n}|} - (\bar{A}_{x:\overline{n}|})^2$.

6.4 Other relationships

We leave the reader to supply the definitions and proofs of the following:

$$\bar{A}_x = \bar{A}^1_{x:\overline{n}|} + {}_{n|}\bar{A}_x$$

$$\bar{A}_{x:\overline{n}|} = \bar{A}^1_{x:\overline{n}|} + A_{x:\overset{\ \ 1}{\overline{n}|}}$$

$${}_{n|}\bar{A}_x = v^n \, {}_np_x \, \bar{A}_{x+n}$$

Note in the second of these that it is only death benefits that are affected by the changed time of payment. Survival benefits such as a pure endowment are not affected.

Question

Explain each of the formulae shown above by general reasoning.

Solution

In the first equation, a whole life assurance is equal to a term assurance for *n* years (which pays out on death provided this occurs within *n* years) plus a whole life assurance that is deferred for *n* years (which pays out on death provided this occurs after *n* years). All benefits are paid immediately on death.

In the second equation, an endowment assurance, with a benefit paid immediately on death within *n* years or on survival to the end of the *n*-year term, is equal to an *n*-year term assurance with a benefit paid immediately on death plus a pure endowment benefit paid if the policyholder survives for the *n*-year period. Note that the pure endowment symbol does not have a bar over it – a bar represents a payment made immediately on death, and a pure endowment incorporates no death benefit whatsoever.

In the third equation, we have a deferred whole life assurance, under which the benefit is paid immediately on death, but only if death happens after *n* years. This is equal to a whole life assurance from age $x+n$, allowing for discounting for interest and the fact that the life aged x must survive for *n* years in order for the benefit to be paid.

6.5 Claims acceleration approximation

It is convenient to be able to estimate $\bar{A}_x$, $\bar{A}_{x:\overline{n|}}$ and so on in terms of commonly tabulated functions. One simple approximation is *claims acceleration*. Of deaths occurring between ages $x+k$ and $x+k+1$, say, ($k=0,1,2,\ldots$) roughly speaking the average age at death will be $x+k+½$. Under this assumption claims are paid on average 6 months before the end of the year of death.

So, for example:

$$\begin{aligned}\bar{A}_x &\approx v^{½}q_x + v^{1½}\,p_x\,q_{x+1} + v^{2½}\,{}_2p_x\,q_{x+2} + \cdots \\ &= (1+i)^{½}\left(v\,q_x + v^2\,p_x\,q_{x+1} + v^3\,{}_2p_x\,q_{x+2} + \cdots\right) \\ &= (1+i)^{½}A_x\end{aligned}$$

and a similar result holds for term assurances.

Therefore we obtain the approximate EPVs:

$$\bar{A}_x \cong (1+i)^{½}A_x$$

$$\bar{A}^{1}_{x:\overline{n|}} \cong (1+i)^{½}A^{1}_{x:\overline{n|}}$$

$$\bar{A}_{x:\overline{n|}} \cong (1+i)^{½}A^{1}_{x:\overline{n|}} + A_{x:\overset{\;\;1}{\overline{n|}}}$$

Note again that, in the case of the endowment assurance, only the death benefit is affected by the claims acceleration.

6.6 Further approximation

A second approximation is obtained by considering a whole life or term assurance to be a sum of deferred term assurances, each for a term of one year. Then, taking the whole life case as an example:

$$\bar{A}_x = {}_{0|}\bar{A}^{1}_{x:\overline{1}|} + {}_{1|}\bar{A}^{1}_{x:\overline{1}|} + {}_{2|}\bar{A}^{1}_{x:\overline{1}|} + \dots$$

$$= \bar{A}^{1}_{x:\overline{1}|} + vp_x\bar{A}^{1}_{x+1:\overline{1}|} + v^2{}_2p_x\bar{A}^{1}_{x+2:\overline{1}|} + \dots$$

Now:

$$\bar{A}^{1}_{x+k:\overline{1}|} = \int_0^1 v^t\, {}_tp_{x+k}\,\mu_{x+k+t}\,dt$$

If we now make the assumption that deaths are uniformly distributed between integer ages, then as we saw in Chapter 15, the PDF of the complete future lifetime random variable is constant between integer ages. So for an integer age $x+k$, we have:

$$f_{T_{x+k}}(t) = {}_tp_{x+k}\,\mu_{x+k+t} = \text{ constant} \qquad (0 \le t \le 1)$$

Now, using the formula for ${}_tq_x$ introduced in Section 1.3 of Chapter 15:

$$q_{x+k} = \int_0^1 {}_tp_{x+k}\,\mu_{x+k+t}\,dt = {}_tp_{x+k}\,\mu_{x+k+t} \qquad (0 \le t \le 1)$$

If we assume that deaths are uniformly distributed between integer ages, such that:

$$ {}_tp_{x+k}\,\mu_{x+k+t} = q_{x+k} \qquad (0 \le t < 1)$$

then:

$$\bar{A}^{1}_{x+k:\overline{1}|} \cong q_{x+k}\int_0^1 v^t\,dt = q_{x+k}\,\frac{iv}{\delta}$$

Here, the integration has been carried out as follows:

$$\int_0^1 v^t\,dt = \int_0^1 e^{-\delta t}\,dt = \left[-\frac{e^{-\delta t}}{\delta}\right]_0^1 = \frac{1-e^{-\delta}}{\delta} = \frac{1-v}{\delta} = \frac{d}{\delta} = \frac{iv}{\delta}$$

Hence:

$$\bar{A}_x \cong \frac{i}{\delta}\left(vq_x + v^2p_x\,q_{x+1} + v^3{}_2p_x\,q_{x+2} + \dots\right)$$

$$= \frac{i}{\delta}A_x$$

Similarly:

$$\bar{A}^{1}_{x:\overline{n|}} \cong \frac{i}{\delta} A^{1}_{x:\overline{n|}}$$

Question

Evaluate A_{40} and $\bar{A}_{40}$ based on AM92 Ultimate mortality at 4% *pa* interest.

Solution

We can look up the value of A_{40} in the *Tables*:

$$A_{40} = 0.23056$$

Then we can approximate $\bar{A}_{40}$ as:

$$\bar{A}_{40} \approx (1+i)^{½} A_{40} = 1.04^{½} \times 0.23056 = 0.23513$$

or: $$\bar{A}_{40} \approx \frac{i}{\delta} A_{40} = \frac{0.04}{\ln 1.04} \times 0.23056 = 0.23514$$

Despite the difference in the fifth decimal place, both of these approximations would be acceptable in the exam.

7 Evaluating means and variances using select mortality

Corresponding to the assurances defined earlier in this chapter are select equivalents defined as before, but assumed to be issued to a select life denoted $[x]$ rather than x.

So, for example, $A_{[x]}=\sum_{k=0}^{k=\infty} v^{k+1}\,{}_{k|}q_{[x]}$ can be used to calculate the EPV of benefits of a whole life assurance issued to a select life aged $[x]$ at entry.

The variance formulae established earlier also apply replacing x with $[x]$.

Question

A whole life assurance contract, under which the sum assured of £40,000 is payable immediately on death, is issued to a life aged exactly 35.

Using AM92 Select mortality and an interest rate of 6% *pa* effective, calculate:

(i) the expected present value of the benefits

(ii) the variance of the present value of the benefits.

Solution

(i) The EPV is given by:

$$40{,}000\bar{A}_{[35]}\approx 40{,}000\times 1.06^{0.5}A_{[35]}=40{,}000\times 1.06^{0.5}\times 0.09475=£3{,}902$$

(ii) The variance is given by:

$$40{,}000^2\left({}^2\bar{A}_{[35]}-\left(\bar{A}_{[35]}\right)^2\right)$$

where ${}^2\bar{A}_{[35]}$ is evaluated using an interest rate of $1.06^2-1=12.36\%$. So the variance is:

$$40{,}000^2\left(1.1236^{0.5}\times{}^2A_{[35]}-\left(1.06^{0.5}\times A_{[35]}\right)^2\right)$$

$$=40{,}000^2\left(1.1236^{0.5}\times 0.01765-\left(1.06^{0.5}\times 0.09475\right)^2\right)$$

$$=(£3{,}835)^2$$

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 16 Summary

Types of contracts

Assurance contracts are contracts where the insurer makes a payment on death. These might be term assurances, whole life assurances or endowment assurances. The benefits can be payable at the end of the year of death or immediately on death.

Pure endowment contracts make a payment on survival to a given age.

For each type of contract we can write down expressions for:

- the present value of the benefits, which is a random variable
- the expected present value of the benefits
- the variance of the present value of the benefits.

For each of the following benefits we assume that the sum assured is 1 and the policyholder is aged x at the outset.

Whole life assurance with benefit payable at the end of the year of death

Present value: v^{K_x+1}

Expected present value: $E\left[v^{K_x+1}\right]=A_x$

Variance of present value: $\text{var}\left[v^{K_x+1}\right]={}^2A_x-\left(A_x\right)^2$

Whole life assurance with benefit payable immediately on death

Present value: v^{T_x}

Expected present value: $E\left[v^{T_x}\right]=\bar{A}_x$

Variance of present value: $\text{var}\left[v^{T_x}\right]={}^2\bar{A}_x-\left(\bar{A}_x\right)^2$

Term assurance (n-year term) with benefit payable at the end of the year of death

Present value: $F = \begin{cases} v^{K_x+1} & \text{if } K_x < n \\ 0 & \text{if } K_x \geq n \end{cases}$

Expected present value: $E[F] = A^1_{x:\overline{n}|}$

Variance of present value: $\text{var}[F] = {}^2A^1_{x:\overline{n}|} - \left(A^1_{x:\overline{n}|}\right)^2$

Term assurance (n-year term) with benefit payable immediately on death

Present value: $\bar{F} = \begin{cases} v^{T_x} & \text{if } T_x < n \\ 0 & \text{if } T_x \geq n \end{cases}$

Expected present value: $E\left[\bar{F}\right] = \bar{A}^1_{x:\overline{n}|}$

Variance of present value: $\text{var}\left[\bar{F}\right] = {}^2\bar{A}^1_{x:\overline{n}|} - \left(\bar{A}^1_{x:\overline{n}|}\right)^2$

Pure endowment (n-year term)

Present value: $G = \begin{cases} 0 & \text{if } K_x < n \\ v^n & \text{if } K_x \geq n \end{cases}$

Expected present value: $E[G] = A_{x:\overset{1}{\overline{n}|}}$

Variance of present value: $\text{var}[G] = {}^2A_{x:\overset{1}{\overline{n}|}} - \left(A_{x:\overset{1}{\overline{n}|}}\right)^2$

Endowment assurance (n-year term) with benefit payable on survival to the maturity date or at the end of the year of earlier death

Present value: $H = F + G = \begin{cases} v^{K_x+1} & \text{if } K_x < n \\ v^n & \text{if } K_x \geq n \end{cases}$

Expected present value: $E[H] = A_{x:\overline{n}|}$

Variance of present value: $\text{var}[H] = {}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2$

Endowment assurance (n-year term) with benefit payable on survival to the maturity date or immediately on earlier death

Present value: $\bar{H} = \bar{F} + G = \begin{cases} v^{T_x} & \text{if } T_x < n \\ v^n & \text{if } T_x \geq n \end{cases}$

Expected present value: $E\left[\bar{H}\right] = \bar{A}_{x:\overline{n|}}$

Variance of present value: $\text{var}\left[\bar{H}\right] = {}^2\bar{A}_{x:\overline{n|}} - \left(\bar{A}_{x:\overline{n|}}\right)^2$

Deferred whole life assurance with benefit paid at end of year of death

Present value: $J = v^{K_x+1} - F = \begin{cases} 0 & \text{if } K_x < n \\ v^{K_x+1} & \text{if } K_x \geq n \end{cases}$

Expected present value: $E\left[J\right] = {}_{n|}A_x = A_x - A^{1}_{x:\overline{n|}}$

Variance of present value: $\text{var}\left[J\right] = {}^2_{n|}A_x - \left({}_{n|}A_x\right)^2$

The letters F, G, H and J have simply been used for notational convenience. They do not represent standard notation.

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 16 Practice Questions

16.1 If T_x and K_x are random variables measuring the complete and curtate future lifetimes, respectively, of a life aged x, write down an expression for each of the following symbols as the expectation of a random variable:

(i) A_x

(ii) $\bar{A}^{1}_{x:\overline{n}|}$

(iii) $A_{x:\overline{n}|}^{\ \ \ 1}$

16.2 The benefit payable under a special assurance policy has present value random variable:

$$W = v^{\max\{K_x+1,\, n\}}$$

where K_x is the curtate future lifetime of a person currently aged exactly x.

(i) Describe the benefit paid under this policy.

(ii) Express $E[W]$ in terms of standard actuarial notation.

(iii) Express $\text{var}[W]$ in terms of standard actuarial notation.

16.3 Calculate the expectation and standard deviation of the present value of the benefits from each of the following contracts issued to a life aged exactly 40, assuming that the annual effective interest rate is 4% and AM92 Ultimate mortality applies:

(i) a 20-year pure endowment, with a benefit of £10,000

(ii) a deferred whole life assurance with a deferred period of 20 years, under which the death benefit of £20,000 is paid at the end of the year of death, as long as this occurs after the deferred period has elapsed.

16.4 Using an interest rate of 6% *pa* effective and AM92 Ultimate mortality, calculate:

(i) $A_{50:\overline{15}|}^{\ \ \ \ 1}$

(ii) $\bar{A}_{50:\overline{15}|}$

16.5 If $l_{40} = 1{,}000$ and $l_{40+t} = l_{40} - 5t$ for $t = 1, 2, \ldots, 10$, calculate the value of $A_{40:\overline{10}|}$ at 6% *pa* interest.

16.6 A life insurance company issues a 3-year term assurance contract to a life aged exactly 42. The sum assured of 10,000 is payable at the end of the policy year of death.

Calculate the expected present value of these benefits assuming AM92 Select mortality and an interest rate of 5% *pa* effective.

16.7 A whole life assurance policy pays £10,000 immediately on death of a policyholder currently aged 50 exact, but only if death occurs after the age of 60.

(i) Write down an expression for the present value random variable of the benefit payable under this policy.

(ii) Determine an expression, in the form of an integral, for the expected present value of the benefit payment, and express your answer using standard actuarial notation.

(iii) Determine an expression for the variance of the present value of the benefit payment, expressing your answer using standard actuarial notation.

16.8 An endowment assurance contract with a term of 10 years pays a sum assured of £100,000 immediately on death and a sum of £50,000 on survival for 10 years.

Exam style

Calculate the expected present value and variance of this contract.

Basis:

Mortality: $\mu_x = 0.03$ throughout

Rate of interest: 5% per annum [8]

Chapter 16 Solutions

16.1 (i) $A_x = E\left(v^{K_x+1}\right)$

(ii) $\bar{A}^{1}_{x:\overline{n}|} = E\left[g\left(T_x\right)\right]$ where $g\left(T_x\right) = \begin{cases} v^{T_x} & \text{if } T_x < n \\ 0 & \text{if } T_x \ge n \end{cases}$

(iii) $A_{x:\overline{n}|}^{\ \ 1} = E\left[h\left(K_x\right)\right]$ where $h\left(K_x\right) = \begin{cases} 0 & \text{if } K_x < n \\ v^n & \text{if } K_x \ge n \end{cases}$

Alternatively, we could replace K_x with T_x here.

16.2 (i) ***Description of benefit***

This assurance policy pays 1 in n years' time if the life dies before time n years, or it pays 1 at the end of the year of death of the life if this occurs after time n years.

(ii) $\boldsymbol{E[W]}$

We can write:

$$W = \begin{cases} v^n & \text{if } K_x < n \\ v^{K_x+1} & \text{if } K_x \ge n \end{cases}$$

So:

$$E[W] = v^n P(K_x < n) + \sum_{k=n}^{\infty} v^{k+1} P(K_x = k)$$

$$= v^n \, {}_nq_x + \sum_{k=n}^{\infty} v^{k+1} \, {}_{k|}q_x$$

$$= v^n \, {}_nq_x + {}_{n|}A_x$$

$$= v^n \, {}_nq_x + v^n \, {}_np_x \, A_{x+n}$$

(iii) **var$\boldsymbol{[W]}$**

Now:

$$\text{var}[W] = E\left[W^2\right] - \left(E[W]\right)^2$$

We have:

$$E\left[W^2\right]=\left(v^n\right)^2 P(K_x<n)+\sum_{k=n}^{\infty}\left(v^{k+1}\right)^2 P(K_x=k)$$

$$=v^{2n}\,{}_nq_x+\sum_{k=n}^{\infty}\left(v^2\right)^{k+1}{}_{k|}q_x$$

$$=v^{2n}\,{}_nq_x+{}_{n|}^{2}A_x$$

$$=v^{2n}\,{}_nq_x+v^{2n}\,{}_np_x\,{}^2A_{x+n}$$

where the pre-superscript of 2 indicates a function evaluated at the interest rate $(1+i)^2-1$.

So:

$$\text{var}\left[W\right]=v^{2n}\,{}_nq_x+v^{2n}\,{}_np_x\,{}^2A_{x+n}-\left(v^n\,{}_nq_x+v^n\,{}_np_x\,A_{x+n}\right)^2$$

16.3 (i) ***Pure endowment***

The expected value of the benefits is:

$$10{,}000A_{40:\overline{20|}}^{\;\;\;\;\;1}=10{,}000v^{20}\,{}_{20}p_{40}$$

Now:

$${}_{20}p_{40}=\frac{l_{60}}{l_{40}}=\frac{9{,}287.2164}{9{,}856.2863}=0.94226$$

so the expected present value is:

$$10{,}000v^{20}\,{}_{20}p_{40}=10{,}000\times1.04^{-20}\times0.94226=£4{,}300.37$$

The standard deviation of the benefits is:

$$\sqrt{10{,}000^2\left({}^2A_{40:\overline{20|}}^{\;\;\;\;\;1}-\left(A_{40:\overline{20|}}^{\;\;\;\;\;1}\right)^2\right)}$$

where the function with the pre-superscript of 2 is evaluated at the interest rate $1.04^2-1=8.16\%$.

So, the standard deviation is:

$$10{,}000\sqrt{1.0816^{-20}\ {}_{20}p_{40}-\left(1.04^{-20}\ {}_{20}p_{40}\right)^2}$$

$$=10{,}000\sqrt{1.0816^{-20}\times 0.94226-\left(1.04^{-20}\times 0.94226\right)^2}$$

$$=£1{,}064.50$$

(ii) ***Deferred whole life assurance***

The expected value of the benefits is:

$$20{,}000\ {}_{20|}A_{40}=20{,}000v^{20}\ {}_{20}p_{40}\ A_{60}=20{,}000\times 1.04^{-20}\times 0.94226\times 0.45640=£3{,}925.37$$

The standard deviation of the benefits is:

$$\sqrt{20{,}000^2\left({}_{20|}^{\ 2}A_{40}-\left({}_{20|}A_{40}\right)^2\right)}$$

where the function with the pre-superscript of 2 is evaluated at the interest rate $1.04^2-1=8.16\%$.

So, the standard deviation is:

$$20{,}000\sqrt{1.0816^{-20}\ {}_{20}p_{40}\ {}^2A_{60}-\left(1.04^{-20}\ {}_{20}p_{40}\ A_{60}\right)^2}$$

$$=20{,}000\sqrt{1.0816^{-20}\times 0.94226\times 0.23723-\left(1.04^{-20}\times 0.94226\times 0.45640\right)^2}$$

$$=£1{,}793.11$$

16.4 (i) $A_{50:\overline{15|}}^{\ \ \ \ \ \ 1}=v^{15}\ {}_{15}p_{50}=v^{15}\times\dfrac{l_{65}}{l_{50}}=1.06^{-15}\times\dfrac{8{,}821.2612}{9{,}712.0728}=0.37899$

(ii) We have:

$$\bar{A}_{50:\overline{15|}}=\bar{A}_{50:\overline{15|}}^{1}+A_{50:\overline{15|}}^{\ \ \ \ \ \ 1}$$

where:

$$\bar{A}_{50:\overline{15|}}^{1}\approx 1.06^{0.5}A_{50:\overline{15|}}^{1}=1.06^{0.5}\left(A_{50:\overline{15|}}-A_{50:\overline{15|}}^{\ \ \ \ \ \ 1}\right)$$

So, using the *Tables* and the result of (i):

$$\bar{A}_{50:\overline{15|}}\approx 1.06^{0.5}(0.43181-0.37899)+0.37899=0.43337$$

16.5 We have:

$$A_{40:\overline{10}|} = \sum_{k=0}^{9} v^{k+1} {}_{k|}q_{40} + v^{10} {}_{10}p_{40} = \sum_{k=0}^{9} v^{k+1} \frac{d_{40+k}}{l_{40}} + v^{10} \frac{l_{50}}{l_{40}}$$

Using the definition of l_x given, we see that:

$$d_{40+k} = l_{40+k} - l_{40+k+1} = l_{40} - 5k - \left(l_{40} - 5(k+1)\right) = 5 \quad \text{for} \quad k = 0,1,2,\ldots,9$$

and: $l_{50} = l_{40} - 5 \times 10 = 1{,}000 - 50 = 950$

So the value of the endowment assurance is:

$$A_{40:\overline{10}|} = \frac{5}{1{,}000} \sum_{k=0}^{9} v^{k+1} + v^{10} \times \frac{950}{1{,}000}$$

$$= \frac{5a_{\overline{10}|} + 950v^{10}}{1{,}000} @6\%$$

$$= \frac{5 \times 7.3601 + 950 \times 1.06^{-10}}{1{,}000} = 0.5673$$

16.6 The expected present value of the benefits is:

$$10{,}000 A^{\;1}_{[42]:\overline{3}|} = 10{,}000 \sum_{k=0}^{2} v^{k+1} {}_{k|}q_{[42]}$$

$$= 10{,}000 \sum_{k=0}^{2} v^{k+1} \frac{d_{[42]+k}}{l_{[42]}}$$

$$= 10{,}000 \left(v \frac{d_{[42]}}{l_{[42]}} + v^2 \frac{d_{[42]+1}}{l_{[42]}} + v^3 \frac{d_{44}}{l_{[42]}} \right)$$

$$= \frac{10{,}000}{9{,}834.7030} \left(1.05^{-1} \times 9.0676 + 1.05^{-2} \times 11.2995 + 1.05^{-3} \times 13.0236\right)$$

$$= 30.64$$

16.7 (i) ***Present value random variable***

We can write this as:

$$X = \begin{cases} 0 & \text{if } T_{50} \le 10 \\ 10{,}000 v^{T_{50}} & \text{if } T_{50} > 10 \end{cases}$$

(ii) ***Expected present value***

Now:

$$E[X]=\int_0^{10} 0\times f_{T_{50}}(t)\,dt+\int_{10}^{\infty}10{,}000v^t f_{T_{50}}(t)\,dt$$

$$=0+10{,}000\int_{10}^{\infty} v^t\,{}_tp_{50}\,\mu_{50+t}\,dt$$

$$=10{,}000\times {}_{10|}\bar{A}_{50}$$

(iii) ***Variance***

We need:

$$\text{var}[X]=E\left[X^2\right]-\left(E[X]\right)^2$$

$$=\int_{10}^{\infty}10{,}000^2(v^t)^2\,{}_tp_{50}\,\mu_{50+t}\,dt-\left(10{,}000\,{}_{10|}\bar{A}_{50}\right)^2$$

Now:

$$\int_{10}^{\infty}10{,}000^2(v^t)^2\,{}_tp_{50}\,\mu_{50+t}\,dt=10{,}000^2\int_{10}^{\infty}(v^2)^t\,{}_tp_{50}\,\mu_{50+t}\,dt$$

$$=10{,}000^2\,{}^2_{10|}\bar{A}_{50}$$

where ${}^2\bar{A}$ is calculated at rate of interest $(1+i)^2-1$.

So:

$$\text{var}[X]=10{,}000^2\left[{}^2_{10|}\bar{A}_{50}-\left({}_{10|}\bar{A}_{50}\right)^2\right]$$

16.8 *This question is Subject CT5, April 2012, Question 12.*

Expected present value

Consider the present value random variable for this endowment assurance contract. This is:

$$PV=\begin{cases}100{,}000v^{T_x} & T_x<10\\ 50{,}000v^{10} & T_x\geq 10\end{cases} \qquad [1]$$

The probability density function for T_x is ${}_tp_x\,\mu_{x+t}$. Here the force of mortality is constant, so μ_{x+t} is just μ, and ${}_tp_x$ is equal to $e^{-\mu t}$.

The expected present value of the benefits is:

$$E(PV) = 100{,}000 \int_0^{10} {}_tp_x\, \mu_{x+t}\, v^t\, dt + 50{,}000 v^{10} \times {}_{10}p_x$$

$$= 100{,}000 \int_0^{10} {}_tp_x\, \mu_{x+t}\, e^{-\delta t}\, dt + 50{,}000 e^{-10\delta} \times {}_{10}p_x$$

$$= 100{,}000 \int_0^{10} \mu e^{-(\mu+\delta)t}\, dt + 50{,}000 e^{-10(\mu+\delta)}$$

$$= 100{,}000 \left[\frac{\mu e^{-(\mu+\delta)t}}{-(\mu+\delta)} \right]_0^{10} + 50{,}000 e^{-10(\mu+\delta)}$$

$$= 100{,}000 \left[\frac{\mu\left(1 - e^{-10(\mu+\delta)}\right)}{\mu+\delta} \right] + 50{,}000 e^{-10(\mu+\delta)} \qquad [2]$$

Substituting in $\mu = 0.03$ and $\delta = \ln 1.05$, we find that:

$$E(PV) = 100{,}000 \left[\frac{0.03\left(1 - e^{-10(0.03+\ln 1.05)}\right)}{0.03 + \ln 1.05} \right] + 50{,}000 e^{-10(0.03+\ln 1.05)}$$

$$= 20{,}759.008 + 22{,}739.906 = 43{,}498.914$$

So the expected present value of the contract is about £43,499. [1]

Variance of the present value

For the variance, we first need to calculate the second moment:

$$E\left(PV^2\right) = 100{,}000^2 \int_0^{10} {}_tp_x\, \mu_{x+t}\, e^{-2\delta t}\, dt + 50{,}000^2 e^{-20\delta} \times {}_{10}p_x$$

$$= 100{,}000^2 \int_0^{10} \mu e^{-(\mu+2\delta)t}\, dt + 50{,}000^2 e^{-10(\mu+2\delta)}$$

$$= 100{,}000^2 \left[\frac{\mu e^{-(\mu+2\delta)t}}{-(\mu+2\delta)} \right]_0^{10} + 50{,}000^2 e^{-10(\mu+2\delta)}$$

$$= 100{,}000^2 \left[\frac{\mu\left(1 - e^{-10(\mu+2\delta)}\right)}{\mu+2\delta} \right] + 50{,}000^2 e^{-10(\mu+2\delta)} \qquad [2]$$

Again, substituting in $\mu = 0.03$ and $\delta = \ln 1.05$, we find that:

$$E\left(PV^2\right) = 100{,}000^2 \left[\frac{0.03\left(1 - e^{-10(0.03+2\ln 1.05)}\right)}{0.03 + 2\ln 1.05}\right] + 50{,}000^2 e^{-10(0.03+2\ln 1.05)}$$

$$= 1{,}694{,}916{,}638 + 698{,}016{,}490 = 2{,}392{,}933{,}128$$ [1]

So the variance of the present value of the contract is:

$$Var\left(PV\right) = 2{,}392{,}933{,}128 - 43{,}498.914^2 = 500{,}777{,}609$$

or about $\left(£22{,}378\right)^2$. [1]

[Total 8]

17

Life annuity contracts

Syllabus objectives

4.1 Define various annuity contracts.

4.1.1 Define the following terms:

- whole life level annuity
- temporary level annuity
- guaranteed level annuity

including annuity contracts where the benefits are deferred.

4.2 Develop formulae for the means and variances of the payments under various annuity contracts, assuming a constant deterministic interest rate.

4.2.4 Define the annuity factors and their continuous equivalents.

4.2.5 Understand and use the relations between annuities payable in advance and in arrears, and between temporary, deferred and whole life annuities.

4.2.7 Obtain expressions in the form of sums/integrals for the mean and variance of the present value of benefit payments under each contract defined in 4.1.1, in terms of the (curtate) random future lifetime, assuming:

- annuities are paid in advance, in arrears or continuously, and the amount is constant
- premiums are payable in advance, in arrears or continuously; and for the full policy term or for a limited period.

Where appropriate, simplify the above expressions into a form suitable for evaluation by table look-up or other means.

1 Life annuity contracts

In the previous chapter, we described the first broad type of contract sold by a life insurance company: assurances. In this chapter we describe the other main type: annuities.

A *life annuity contract* provides payments of amounts, which might be level or variable, at stated times, provided a life is still then alive.

Here we consider four varieties of life annuity contract:

(1) Annuities under which payments are made for the whole of life, with level payments, called a *whole life level annuity* or, more usually, an immediate annuity.

(2) Annuities under which level payments are made only during a limited term, called a *temporary level annuity* or, more usually, just a temporary annuity.

(3) Annuities under which the start of payment is deferred for a given term, called a *deferred annuity*.

(4) Annuities under which payments are made for the whole of life, or for a given term if longer, called a *guaranteed annuity*.

The income from a life annuity may be:

- level, *eg* $£X$ per annum
- increasing, *eg* starting at $£Y$, but increasing at 5% per annum
- paid for the whole of a person's life, *ie* until the policyholder dies
- paid for a limited term, *eg* for at most 5 years
- paid for a minimum term, *eg* for at least 5 years
- deferred, *eg* $£Z$ per annum paid from the policyholder's 60th birthday

or a combination of the above.

Further, we consider the possibilities that payments are made in advance or in arrears. For the moment we consider only contracts under which level payments are made at yearly intervals.

Increasing life annuities will be dealt with in Chapter 19.

2 Whole life annuities payable annually in arrears

An *immediate* annuity is one under which the first payment is made within the first year. For the purposes of this Section we will assume that payments are made in arrears.

The word *immediate* is used to distinguish these from *deferred* annuities, where the payments start some years in the future (*eg* on reaching age 65).

2.1 Present value random variable

Consider an annuity contract to pay 1 at the end of each future year, provided a life now aged x is then alive.

For example, this could be a pension of £1 per annum paid annually in arrears to a life until death.

If the life dies between ages $x+k$ and $x+k+1$ ($k=0,\ldots,\omega-x-1$) which is to say, $K_x=k$, the present value at time 0 of the annuity payments which are made is $a_{\overline{k}|}$. (We define $a_{\overline{0}|}=0$.) Therefore the present value at time 0 of the annuity payments is $a_{\overline{K_x}|}$.

For example, suppose the person dies in the third year after taking out the annuity, *ie* death occurs between time 2 and time 3.

In this case, the life has survived for two complete years before dying in the third year, so the curtate future lifetime, K_x, takes the value 2. Since payments are made at the end of each year but only as long as the person is living (as shown in the diagram), exactly 2 payments are made here, and so the present value is $a_{\overline{2}|}$.

That is, when $K_x=2$, $PV=a_{\overline{2}|}$. Or, in general:

$$PV=a_{\overline{K_x}|}$$

Since we know the distribution of K_x, we can compute moments of $a_{\overline{K_x}|}$.

2.2 Expected present value

The expected value of $a_{\overline{K_x}|}$ is:

$$E\left[a_{\overline{K_x}|}\right]=\sum_{k=0}^{\infty}a_{\overline{k}|}P\left(K_x=k\right)$$

where we are again using the general formula for expectation:

$$E\left[g(x)\right]=\sum_{x}g(x)P\left(X=x\right)$$

Actuarial notation for the expected present value

We met the symbol $a_{\overline{n}|}$ earlier in the course. It can be interpreted as:

$a_{\overline{n}|}=$ *PV* of an immediate annuity of 1 unit *pa* paid in arrears *for as long as the status* $\overline{n}|$ *remains active*

In other words, this implies an annuity paid annually in arrears until the failure of the $\overline{n}|$ status, that is, until n years have elapsed.

We now introduce a_x, which has a similar meaning except that it relates to the life status x. So:

$a_x=$ *EPV* of an immediate annuity of 1 unit *pa* paid in arrears for as long as (x) remains alive in the future

So the payments stop at the moment that (x) dies. We describe a_x as an expected present value ('EPV'), not a present value ('PV'), as the payments are dependent on the life's survival and so are not certain to occur. The word 'expected' really means that we're making an allowance for the probability of payment.

The expectation of $a_{\overline{K_x}|}$ defines the expected present value (EPV) a_x, and:

$$\boldsymbol{a_x=E[a_{\overline{K_x}|}]=\sum_{k=0}^{\infty}a_{\overline{k}|}\;{}_{k|}q_x}$$

We can write this in a form that is easier to calculate.

We begin by writing:

$$\boldsymbol{a_x=\sum_{k=0}^{\infty}a_{\overline{k}|}\;{}_{k|}q_x=\sum_{k=1}^{\infty}\left(\sum_{j=0}^{k-1}v^{j+1}\right){}_{k|}q_x}$$

This result holds since $a_{\overline{0}|}=0$ and $a_{\overline{k}|}=v+v^2+\cdots+v^k=\sum_{j=0}^{k-1}v^{j+1}$.

If we write out the sum more fully:

$$\begin{aligned}a_x &= v \times {}_{1|}q_x \\ &\quad + \left(v+v^2\right) \times {}_{2|}q_x \\ &\quad + \left(v+v^2+v^3\right) \times {}_{3|}q_x \\ &\quad + \ldots\end{aligned}$$

Now, reversing the order of summation:

$$\begin{aligned}a_x &= v\left[{}_{1|}q_x + {}_{2|}q_x + \ldots\right] + v^2\left[{}_{2|}q_x + {}_{3|}q_x + \ldots\right] + \ldots \\ &= v\sum_{k=1}^{\infty} {}_{k|}q_x + v^2 \sum_{k=2}^{\infty} {}_{k|}q_x + \ldots\end{aligned}$$

So:

$$\boldsymbol{a_x = \sum_{j=0}^{\infty}\left(\sum_{k=j+1}^{\infty} {}_{k|}q_x\right) v^{j+1}}$$

Now, for example, $\sum_{k=1}^{\infty} {}_{k|}q_x$ is the probability that the life dies at some point after time 1 year, which is equal to the probability that the life is still alive at time 1 year, ${}_1p_x$, and in general, $\sum_{k=j+1}^{\infty} {}_{k|}q_x$ is equal to ${}_{j+1}p_x$. Hence:

$$\boldsymbol{a_x = \sum_{j=0}^{\infty} {}_{j+1}p_x \, v^{j+1} = \sum_{j=1}^{\infty} {}_jp_x \, v^j}$$

We can interpret the final formula for a_x above as follows.

Each payment is conditional on whether the policyholder is alive or not at the time the payment is due. The present value of the annuity payment made at time j is v^j. The expected present value of this payment is v^j multiplied by the probability that the policyholder is alive at this time, ${}_jp_x$. Summing over all future years gives the expected present value of all the future annuity payments.

Although the result $a_x = \sum_{j=1}^{\infty} {}_jp_x \, v^j$ is very useful, it is important to realise that it is *not* the *definition* of a_x. The definition is $a_x = \sum_{k=0}^{\infty} a_{\overline{k|}} \, {}_{k|}q_x$.

Question

A whole life annuity of 1 *pa* is payable annually in arrears to a life aged 90. The effective annual rate of interest is 5%. Mortality is assumed to follow the life table given below:

Age, x	l_x	d_x
90	100	25
91	75	35
92	40	40
93	0	0

Calculate the expected present value of this annuity.

Solution

Using the formula $a_x = \sum_{j=1}^{\infty} v^j \, {}_jp_x$, the expected present value is:

$$a_{90} = v\,p_{90} + v^2 \, {}_2p_{90} = v\frac{l_{91}}{l_{90}} + v^2 \frac{l_{92}}{l_{90}}$$

There are only 2 non-zero terms in the summation since ${}_jp_{90} = 0$ *for* $j \geq 3$.

Using the values from the table above, we have:

$$a_{90} = 1.05^{-1} \times \frac{75}{100} + 1.05^{-2} \times \frac{40}{100} = 1.07710$$

2.3 Variance of the present value random variable

The relationships derived in the previous chapter (for assurances) **provide the easiest approach to finding the variances of the present values of annuity benefits.**

Recall that:

$$a_{\overline{n|}} = \frac{1-v^n}{i}$$

Using this result and the properties of variance, we can write:

$$\text{var}\left[a_{\overline{K_x|}}\right] = \text{var}\left[\frac{1-v^{K_x}}{i}\right] = \frac{1}{i^2}\text{var}\left[v^{K_x}\right]$$

In Chapter 16 we showed that:

$$\text{var}\left[v^{K_x+1}\right] = {}^2A_x - (A_x)^2$$

where the 2 to the top left-hand corner of the first A indicates that the function is valued using the interest rate $(1+i)^2 - 1$.

So:

$$\text{var}\left[a_{\overline{K_x}|}\right] = \frac{1}{i^2}\text{var}\left[\frac{v^{K_x+1}}{v}\right]$$

$$= \frac{1}{i^2v^2}\text{var}\left[v^{K_x+1}\right]$$

$$= \frac{1}{d^2}\left[{}^2A_x - (A_x)^2\right]$$

Question

A whole life annuity of 100 *pa* is payable annually in arrears to a life aged 65.

Calculate the standard deviation of the benefits from this annuity, assuming AM92 Ultimate mortality and an annual effective rate of interest of 4%.

Solution

The variance of the benefits from this annuity is:

$$\text{var}\left[100a_{\overline{K_{65}}|}\right] = \frac{100^2}{d^2}\left[{}^2A_{65} - (A_{65})^2\right]$$

Using values from the *Tables*, and recalling that $d = iv$, this gives the variance as:

$$\frac{100^2}{(0.04/1.04)^2}\left[0.30855 - (0.52786)^2\right] = (449.69)^2$$

So the standard deviation is 449.69.

3 Whole life annuities payable annually in advance

An *annuity-due* is one under which payments are made in advance.

3.1 Present value random variable

Consider an annuity contract to pay 1 at the start of each future year, provided a life now aged x is then alive. By similar reasoning to that above, we see that the present value of these payments is $\ddot{a}_{\overline{K_x+1}|}$.

The annuity-due provides the following payments:

- 1 at age x (*ie* at the start of the first year), which is certain to be paid, and
- a further payment of 1 at the start of each subsequent policy year as long as the policyholder is still alive.

The second of these alone results in the same number of payments as under the whole life annuity paid in arrears, *ie* K_x payments. For the annuity in advance, therefore, there will be one additional payment at the start of the contract, so there are K_x+1 payments altogether.

Actuarial notation for the expected present value

In actuarial notation we denote the EPV, $E\left[\ddot{a}_{\overline{K_x+1}|}\right]$, by $\ddot{a}_x$.

$\ddot{a}_x$ has identical meaning to a_x – in that payments continue for as long as (x) survives – except that the payments are made in advance.

We can again write down $\ddot{a}_x$ in a form that is simple to calculate:

$$\begin{aligned}
\ddot{a}_x &= E\left[\ddot{a}_{\overline{K_x+1}|}\right] \\
&= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} v^j\right){}_{k|}q_x \\
&= \sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty} {}_{k|}q_x\right)v^j \\
&= \sum_{j=0}^{\infty} {}_jp_x v^j
\end{aligned}$$

Question

Develop a simple relationship between $\ddot{a}_x$ and a_x, and explain the result by general reasoning.

Solution

Using the simplified formulae we have obtained for $\ddot{a}_x$ and a_x, we see:

$$\ddot{a}_x - a_x = \sum_{j=0}^{\infty} {}_jp_x v^j - \sum_{j=1}^{\infty} {}_jp_x v^j = {}_0p_x v^0 = 1 \quad \Rightarrow \quad \ddot{a}_x = a_x + 1$$

Intuitively, the annuity-due is the same as the annuity payable in arrears, except for the additional payment of 1 unit made at outset. This payment has present value 1, and will definitely be paid, so its expected present value must also be 1.

As for assurances, a selection of the above functions are tabulated in the Formulae and Tables for Examinations.

Question

Find:

(i) $\ddot{a}_{30}$ **(AM92 at 4%)**

(ii) $\ddot{a}_{75}$ **(PMA92C20 at 4%)**

Solution

(i) **21.834** (This value appears on page 96 of the *Tables*.)

(ii) **9.456** (This value appears on page 114 of the *Tables*.)

3.2 Variance of the present value random variable

The variance is:

$$\text{var}\left[\ddot{a}_{\overline{K_x+1}|}\right] = \text{var}\left[\frac{1-v^{K_x+1}}{d}\right]$$

$$= \frac{1}{d^2}\text{var}\left[v^{K_x+1}\right]$$

$$= \frac{1}{d^2}\left[{}^2A_x - (A_x)^2\right]$$

As usual, the '2' superscript denotes an assurance function calculated at a rate of interest of $(1+i)^2 - 1$.

It is simple to prove the formula for the variance of an immediate whole life annuity payable annually in arrears starting from this result.

As shown earlier, **for an immediate annuity payable annually in arrears, we obtain:**

$$\text{var}\left[a_{\overline{K_x}|}\right] = \text{var}\left[\ddot{a}_{\overline{K_x+1}|} - 1\right]$$

$$= \text{var}\left[\ddot{a}_{\overline{K_x+1}|}\right]$$

$$= \frac{1}{d^2}\left[{}^2A_x - (A_x)^2\right]$$

4 Temporary annuities payable annually in arrears

A *temporary immediate annuity* differs from a whole life immediate annuity in that the payments are limited to a specified term.

4.1 Present value random variable

Consider a temporary immediate annuity contract to pay 1 at the end of each of the next n years, provided a life now aged x is then alive.

This contract differs from a whole life annuity since the payments stop after n years, even if the life is still alive.

If we let X denote the present value of the temporary annuity payable annually in arrears, then:

$$X = \begin{cases} a_{\overline{K_x}|} & \text{if } K_x < n \\ a_{\overline{n}|} & \text{if } K_x \geq n \end{cases}$$

From this we can see that the number of payments made is K_x or n, whichever is the smaller. So, alternatively we can write:

The present value of this benefit is $a_{\overline{\min\{K_x,n\}}|}$.

4.2 Expected present value

The expected present value is:

$$E\left[a_{\overline{\min\{K_x,n\}}|}\right] = \sum_{k=0}^{\infty} a_{\overline{\min\{k,n\}}|} P(K_x = k)$$

$$= \sum_{k=0}^{n-1} a_{\overline{k}|} P(K_x = k) + a_{\overline{n}|} \sum_{k=n}^{\infty} P(K_x = k)$$

$$= \sum_{k=0}^{n-1} a_{\overline{k}|} P(K_x = k) + a_{\overline{n}|} P(K_x \geq n)$$

$$= \sum_{k=0}^{n-1} a_{\overline{k}|}\, {}_{k|}q_x + a_{\overline{n}|}\, {}_{n}p_x$$

Looking at the formula above, we see that the benefit paid is:

- 0 if death occurs in the first year
- a payment of 1 (with present value $a_{\overline{1}|} = v$) if death occurs in the second year
- two payments of 1 (with present value $a_{\overline{2}|} = v + v^2$) if death occurs in the third year
- and so on, up until a payment of 1 every year for *n* years (present value $a_{\overline{n}|}$) if the life survives for *n* years, with no further payments.

Actuarial notation for the expected present value

In actuarial notation, $E\left[a_{\overline{\min\{K_x,n\}}|}\right]$ is denoted $a_{x:\overline{n}|}$.

Compare this symbol $a_{x:\overline{n}|}$ with the symbol $A_{x:\overline{n}|}$ we described in the previous chapter. Where $A_{x:\overline{n}|}$ relates to a *single* payment made *when* the first out of x and $\overline{n}|$ fails, $a_{x:\overline{n}|}$ relates to a *series* of payments that continues *until* the first out of x and $\overline{n}|$ fails.

As for a whole life annuity, we would like a formula to simplify the calculation of the expected present value. Writing the EPV as a summation, we have:

$$\begin{aligned} a_{x:\overline{n}|} &= E\left[a_{\overline{\min\{K_x,n\}}|}\right] \\ &= \sum_{k=1}^{n-1} a_{\overline{k}|}\ {}_{k|}q_x + a_{\overline{n}|}\ {}_n p_x \\ &= \sum_{k=1}^{n-1}\left(\sum_{j=0}^{k-1} v^{j+1}\right){}_{k|}q_x + \left(\sum_{j=0}^{n-1} v^{j+1}\right){}_n p_x \end{aligned}$$

Reversing the order of summation then gives:

$$a_{x:\overline{n}|} = \sum_{j=0}^{n-2}\left(\sum_{k=j+1}^{n-1} {}_{k|}q_x\right)v^{j+1} + \left(\sum_{j=0}^{n-1} v^{j+1}\right){}_n p_x$$

Note that ${}_n p_x = {}_{n|}q_x + {}_{n+1|}q_x + \cdots$, so:

$$\sum_{k=j+1}^{n-1} {}_{k|}q_x = {}_{j+1|}q_x + {}_{j+2|}q_x + \cdots + {}_{n-1|}q_x = {}_{j+1}p_x - {}_n p_x$$

Substituting this into the previous equation gives:

$$a_{x:\overline{n}|} = \sum_{j=0}^{n-2} \left({}_{j+1}p_x - {}_np_x \right) v^{j+1} + \left(\sum_{j=0}^{n-1} v^{j+1} \right) {}_np_x$$

$$= \sum_{j=0}^{n-2} v^{j+1} {}_{j+1}p_x + {}_np_x \left(\sum_{j=0}^{n-1} v^{j+1} - \sum_{j=0}^{n-2} v^{j+1} \right)$$

$$= \sum_{j=0}^{n-2} v^{j+1} {}_{j+1}p_x + v^n {}_np_x$$

$$= \sum_{j=0}^{n-1} v^{j+1} {}_{j+1}p_x$$

So, to calculate $a_{x:\overline{n}|}$, we use the following:

$$\boldsymbol{a_{x:\overline{n}|} = \sum_{j=0}^{n-1} {}_{j+1}p_x v^{j+1} = \sum_{j=1}^{n} {}_jp_x v^j}$$

This expression seems logical given that $a_x = \sum_{j=1}^{\infty} {}_jp_x v^j$. $a_{x:\overline{n}|}$ is the same summation but allows only for the first n payments. It is this final result for $a_{x:\overline{n}|}$ that is most useful when calculating expected present values in practice.

Question

Calculate the value of $a_{40:\overline{3}|}$ when the effective annual rate of interest is 6% and:

$l_x = 100 - x$ at all ages $x \le 100$

Solution

Using the given formula, we have:

$$a_{40:\overline{3}|} = \sum_{j=1}^{3} v^j {}_jp_{40} = v\frac{l_{41}}{l_{40}} + v^2 \frac{l_{42}}{l_{40}} + v^3 \frac{l_{43}}{l_{40}}$$

$$= 1.06^{-1} \times \frac{59}{60} + 1.06^{-2} \times \frac{58}{60} + 1.06^{-3} \times \frac{57}{60}$$

$$= 2.58564$$

4.3 Variance of the present value random variable

For a temporary immediate annuity payable annually in arrears we have:

$$\text{var}\left[a_{\overline{\min\{K_x,n\}}|}\right]=\frac{1}{d^2}\left[\,{}^2A_{x:\overline{n+1}|}-(A_{x:\overline{n+1}|})^2\right]$$

The easiest way to prove this formula is to use the result for a temporary annuity-due. So we will defer the proof until the end of the next section.

5 Temporary annuities payable annually in advance

A *temporary immediate annuity-due* has payments that are made in advance and are limited to a specified term.

5.1 Present value random variable

Consider a temporary immediate annuity-due contract to pay 1 at the start of each of the next n years, provided a life now aged x is then alive.

The present value of the benefit is $\ddot{a}_{\overline{\min[K_x+1,n]}|}$.

Another way to write this is as follows. If we let Y denote the present value of the temporary annuity-due, then:

$$Y = \begin{cases} \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x < n \\ \ddot{a}_{\overline{n}|} & \text{if } K_x \geq n \end{cases}$$

and so we can see that the number of payments is the smaller of $K_x + 1$ and n.

5.2 Expected present value

The expected present value is:

$$E\left[\ddot{a}_{\overline{\min\{K_x+1,n\}}|}\right] = \sum_{k=0}^{\infty} \ddot{a}_{\overline{\min\{k+1,n\}}|} P(K_x = k)$$

$$= \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} P(K_x = k) + \ddot{a}_{\overline{n}|} \sum_{k=n}^{\infty} P(K_x = k)$$

$$= \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\ {}_{k|}q_x + \ddot{a}_{\overline{n}|}\ {}_n p_x$$

Actuarial notation for the expected present value

In actuarial notation, $E\left[\ddot{a}_{\overline{\min[K_x+1,n]}|}\right]$ is denoted $\ddot{a}_{x:\overline{n}|}$.

Then:

$$\ddot{a}_{x:\overline{n}|} = E\left[\ddot{a}_{\overline{\min[K_x+1,n]}|}\right]$$

$$= \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\, {}_{k|}q_x + \ddot{a}_{\overline{n}|}\, {}_n p_x$$

$$= \sum_{k=0}^{n-1}\left(\sum_{j=0}^{k} v^j\right) {}_{k|}q_x + \left(\sum_{j=0}^{n-1} v^j\right) {}_n p_x$$

$$= \sum_{j=0}^{n-1}\left(\sum_{k=j}^{n-1} {}_{k|}q_x + {}_n p_x\right) v^j$$

$$= \sum_{j=0}^{n-1} {}_j p_x v^j$$

Again, this seems logical. It is similar to a whole life annuity-due, but with payments continuing only up to time $n-1$ (making n payments in total).

Question

Explain why $\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|} \neq 1$.

Solution

The identity $\ddot{a}_x - a_x = 1$ works for whole life annuities, but when considering temporary annuities $\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|} \neq 1$.

In terms of summations, we have:

$$a_{x:\overline{n}|} = v\, p_x + v^2\, {}_2p_x + \cdots + v^n\, {}_n p_x$$

and : $$\ddot{a}_{x:\overline{n}|} = 1 + v\, p_x + \cdots + v^{n-1}\, {}_{n-1}p_x$$

Comparing these, we see that the first and the last terms are different. In fact:

$$\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|} = 1 - v^n\, {}_n p_x$$

Following on from this, we have:

$$a_{x:\overline{n}|} + 1 = \ddot{a}_{x:\overline{n}|} + v^n\, {}_n p_x = 1 + v\, p_x + v^2\, {}_2p_x + \ldots + v^{n-1}\, {}_{n-1}p_x + v^n\, {}_n p_x = \ddot{a}_{x:\overline{n+1}|}$$

In other words, $\ddot{a}_{x:\overline{n+1}|} - a_{x:\overline{n}|} = 1$.

For functions dependent upon age as well as term, tabulations are restricted in order to save space. For example, in AM92, $\ddot{a}_{x:\overline{n}|}$ is tabulated for $x+n=60$ and for $x+n=65$.

Question

A 35-year-old purchases an endowment assurance with a term of 30 years. The premiums for the policy are payable annually in advance while the policy is in force, and each premium is £2,500.

Calculate the expected present value of the premiums paid, using AM92 Ultimate mortality, and an interest rate of 4% *pa* effective.

Solution

The endowment assurance will remain in force while the policyholder is alive, for at most 30 years. So the expected present value of the premiums payable can be calculated using a temporary annuity-due, with a term of 30 years:

$$2{,}500\ddot{a}_{35:\overline{30}|} = 2{,}500 \times 17.629 = £44{,}072.50$$

The value of $\ddot{a}_{35:\overline{30}|}$ (AM92 Ultimate mortality, 4% interest) appears on page 101 of the Tables.

5.3 Variance of the present value random variable

For a temporary immediate annuity-due:

$$\text{var}\left[\ddot{a}_{\overline{\min[K_x+1,n]}|}\right] = \frac{1}{d^2}\left[{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2\right]$$

This is proved in the same way as the variance formula for the whole life annuity-due:

$$\begin{aligned}
\text{var}\left[\ddot{a}_{\overline{\min\{K_x+1,n\}}|}\right] &= \text{var}\left[\frac{1-v^{\min\{K_x+1,n\}}}{d}\right] \\
&= \frac{1}{d^2}\text{var}\left[v^{\min\{K_x+1,n\}}\right] \\
&= \frac{1}{d^2}\left({}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2\right)
\end{aligned}$$

The last line above uses the result for the variance of an endowment assurance from the previous chapter.

We can now use this to prove the corresponding result for a temporary immediate annuity payable in arrears.

For a temporary immediate annuity (payable annually in arrears) we have:

$$\mathbf{var}\left[a_{\overline{\min[K_x,n]}|} \right] = \mathbf{var}\left[\ddot{a}_{\overline{\min[K_x+1,n+1]}|} - 1 \right]$$

$$= \mathbf{var}\left[\ddot{a}_{\overline{\min[K_x+1,n+1]}|} \right]$$

$$= \frac{1}{d^2}\left[{}^2A_{x:\overline{n+1}|} - (A_{x:\overline{n+1}|})^2 \right]$$

This proof first of all uses the relationship $a_{\overline{m}|} = \ddot{a}_{\overline{m+1}|} - 1$, with m replaced by $\min\{K_x, n\}$. The second line above follows because subtracting 1 from a random variable does not alter its variance. We then use the formula for the variance of a temporary annuity-due, derived earlier in this section, with n replaced by $n+1$, to give the required result.

6 Deferred annuities

Deferred annuities are annuities under which payment does not begin immediately but is deferred for one or more years.

6.1 Present value random variable

Consider, for example, an annuity of 1 per annum payable annually in arrears to a life now aged x, deferred for n years. Payment will be at ages $x+n+1$, $x+n+2$, .. , provided that the life survives to these ages, instead of at ages $x+1$, $x+2$, .. .

Let X represent the (random) present value of the annuity.

Here are three different ways of representing X:

$$X = \begin{cases} 0 & \text{if } K_x \le n \\ v^n a_{\overline{K_x - n}|} & \text{if } K_x > n \end{cases}$$

$$= \begin{cases} 0 & \text{if } K_x \le n \\ a_{\overline{K_x}|} - a_{\overline{n}|} & \text{if } K_x > n \end{cases}$$

$$= v^n a_{\overline{\max(K_x - n, 0)}|}$$

6.2 Expected present value

Then, by considering the distribution of *X*, and noting that $v^n a_{\overline{k-n}|} = {}_{n|}a_{\overline{k-n}|}$, **we have that:**

$$E(X) = \sum_{k=0}^{n} 0 \times P(K_x = k) + \sum_{k=n+1}^{\infty} {}_{n|}a_{\overline{k-n}|} \times P(K_x = k)$$

Now adding in the terms $\sum_{k=0}^{n} a_{\overline{k}|} P(K_x = k)$ and $a_{\overline{n}|} P(K_x > n)$ and subtracting them again, we can write:

$$E(X) = \sum_{k=0}^{n} a_{\overline{k}|} P(K_x = k) + a_{\overline{n}|} P(K_x > n) - \sum_{k=0}^{n} a_{\overline{k}|} P(K_x = k)$$

$$- a_{\overline{n}|} P(K_x > n) + \sum_{k=n+1}^{\infty} {}_{n|}a_{\overline{k-n}|} P(K_x = k)$$

We can combine the first, second and last terms in the line above as follows:

$$\sum_{k=0}^{n} a_{\overline{k}|} P(K_x = k) + a_{\overline{n}|} P(K_x > n) + \sum_{k=n+1}^{\infty} {}_{n|}a_{\overline{k-n}|} P(K_x = k)$$

$$= \sum_{k=0}^{n} a_{\overline{k}|} P(K_x = k) + \sum_{k=n+1}^{\infty} \left(a_{\overline{n}|} + v^n a_{\overline{k-n}|}\right) P(K_x = k)$$

$$= \sum_{k=0}^{n} a_{\overline{k}|} P(K_x = k) + \sum_{k=n+1}^{\infty} a_{\overline{k}|} P(K_x = k)$$

$$= \sum_{k=0}^{\infty} a_{\overline{k}|} P(K_x = k)$$

Hence:

$$\boldsymbol{E(X) = \sum_{k=0}^{\infty} a_{\overline{k}|} P(K_x = k) - \sum_{k=0}^{n} a_{\overline{k}|} P(K_x = k) - a_{\overline{n}|} P(K_x > n)}$$

$$\boldsymbol{= a_x - a_{x:\overline{n}|}}$$

Alternatively, we could obtain the same result by considering the random variables. If we let:

$$Y = a_{\overline{K_x}|} = \begin{cases} a_{\overline{K_x}|} & \text{if } K_x \le n \\ a_{\overline{K_x}|} & \text{if } K_x > n \end{cases} \qquad \text{so that } E(Y) = a_x$$

and: $$Z = \begin{cases} a_{\overline{K_x}|} & \text{if } K_x \le n \\ a_{\overline{n}|} & \text{if } K_x > n \end{cases} \qquad \text{so that } E(Z) = a_{x:\overline{n}|}$$

it is then easy to see that:

$$Y - Z = \begin{cases} a_{\overline{K_x}|} - a_{\overline{K_x}|} = 0 & \text{if } K_x \le n \\ a_{\overline{K_x}|} - a_{\overline{n}|} & \text{if } K_x > n \end{cases}$$

$$= X$$

Therefore:

$$E(X) = E(Y - Z) = E(Y) - E(Z) = a_x - a_{x:\overline{n}|}$$

This formula $E(X) = a_x - a_{x:\overline{n}|}$ is intuitively correct: the value of the deferred annuity is equal to a series of payments paid for the whole of life, less the value of the payments that will not be made for the first n years.

Actuarial notation for the expected present value

In actuarial notation, the EPV of this deferred annuity is denoted $_{n|}a_x$, so:

$$_{n|}a_x = a_x - a_{x:\overline{n}|}$$

The notation $_{n|}a_x$ represents the expected present value of an annuity of 1 unit *pa* payable in arrears until the failure (death) of life status *x*, with a waiting period of *n* years before payments can begin. The subscript to the right of the symbol always denotes the *current* age, *not* the age of the policyholder when payments begin.

Similarly, expressions can be derived for $_{m|}a_{x:\overline{n}|}$, the expected present value of an n-year temporary annuity deferred for m years (assuming survival to that point).

Alternative approach to evaluating the expected present value

An alternative way to evaluate $_{n|}a_x$ follows from:

$$\begin{aligned} _{n|}a_x &= a_x - a_{x:\overline{n}|} \\ &= \sum_{k=1}^{\infty} v^k \, _kp_x - \sum_{k=1}^{n} v^k \, _kp_x \\ &= \sum_{k=n+1}^{\infty} v^k \, _kp_x \end{aligned}$$

Letting $j = k - n$:

$$_{n|}a_x = \sum_{j=1}^{\infty} v^{j+n} \, _{j+n}p_x = v^n \, _np_x \sum_{j=1}^{\infty} v^j \, _jp_{x+n}$$

where $_{j+n}p_x = {_np_x} \, _jp_{x+n}$, using the principle of consistency. Renaming the variable as 'k' gives:

$$_{n|}a_x = v^n \, _np_x \sum_{k=1}^{\infty} v^k \, _kp_{x+n} = v^n \, _np_x \, a_{x+n}$$

Again, this final result is intuitively obvious. The expected present value of the deferred annuity benefit is equal to the expected present value of a life annuity for a *survivor* to age $x+n$, discounted back for n years to allow for interest, multiplied by the probability that the policyholder survives to age $x+n$.

Note the appearance once more of the 'discount factor' $v^n \, _np_x$.

7 Deferred annuities-due

Deferred annuities-due can be defined similarly, with the corresponding formulae such as:

$$_{n|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n}|} = v^n \, {}_np_x \ddot{a}_{x+n}$$

Note that $a_x = {}_{1|}\ddot{a}_x$.

Question

A 50-year-old woman purchases a deferred annuity to provide herself with an income of £15,000 *pa*, paid annually in advance from age 70 until death.

Calculate the expected present value of the benefits from this deferred annuity, using PFA92C20 mortality, and an interest rate of 4% *pa* effective.

Solution

The expected present value of the benefits is:

$$\begin{aligned} 15{,}000 \, {}_{20|}\ddot{a}_{50} &= 15{,}000 v^{20} \, {}_{20}p_{50} \, \ddot{a}_{70} \\ &= 15{,}000 v^{20} \frac{l_{70}}{l_{50}} \ddot{a}_{70} \\ &= 15{,}000 \times 1.04^{-20} \times \frac{9{,}392.621}{9{,}952.697} \times 12.934 \\ &= £83{,}561 \end{aligned}$$

In Chapter 16 we showed that the variance of a deferred whole life assurance is:

$$_{n|}{}^2A_x - ({}_{n|}A_x)^2$$

This can be used to find the variance of the corresponding deferred annuity-due.

However, it is easier to proceed using a first principles approach.

Question

Write down a single term expression to represent the present value of a deferred annuity-due of 1 *pa* payable to a life now aged x, with a deferment period of n years.

Hence derive a formula for the variance of the present value of this contract.

Solution

The present value random variable can be written as:

$$v^n \ddot{a}_{\overline{\max\{K_x+1-n,0\}}|}$$

We can check this as follows:

- If $K_x < n$, the present value is $v^n \ddot{a}_{\overline{0}|} = 0$.
- If $K_x = n$, the present value is $v^n \ddot{a}_{\overline{1}|} = v^n$, as it should be.
- If $K_x = n+1$, the present value is $v^n \ddot{a}_{\overline{2}|} = v^n(1+v) = v^n + v^{n+1}$, again as it should be.

The formula for larger values of K_x can be checked in a similar way.

It is always necessary to express our random variable using a single term if we wish to derive a variance.

The variance is:

$$\begin{aligned}
\operatorname{var}\left(v^n \ddot{a}_{\overline{\max\{K_x+1-n,0\}}|}\right) &= v^{2n} \operatorname{var}\left(\ddot{a}_{\overline{\max\{K_x+1-n,0\}}|}\right) \\
&= v^{2n} \operatorname{var}\left(\frac{1-v^{\max\{K_x+1-n,0\}}}{d}\right) \\
&= \frac{v^{2n}}{d^2} \operatorname{var}\left(v^{\max\{K_x+1-n,0\}}\right) \\
&= \frac{v^{2n}}{d^2} \operatorname{var}\left(\frac{v^{\max\{K_x+1,n\}}}{v^n}\right) \\
&= \frac{1}{d^2} \operatorname{var}\left(v^{\max\{K_x+1,n\}}\right) \\
&= \frac{1}{d^2}\left[E\left(v^{2\max\{K_x+1,n\}}\right) - \left(E\left(v^{\max\{K_x+1,n\}}\right)\right)^2\right]
\end{aligned}$$

Now:

$$v^{\max\{K_x+1,n\}} = \begin{cases} v^n & \text{if } K_x < n \\ v^{K_x+1} & \text{if } K_x \ge n \end{cases}$$

In other words, 1 is paid at time n if the life dies before time n, or 1 is paid at the end of the year of death if the life dies after time n.

The payment at time n if the life dies before time n is not a form of contract we have met, but a payment at the end of the year of death if the life dies after time n is a deferred whole life assurance. So:

$$E\left(v^{\max\{K_x+1,n\}}\right) = v^n \, {}_nq_x + {}_{n|}A_x$$

and:

$$E\left(v^{2\max\{K_x+1,n\}}\right) = v^{2n} \, {}_nq_x + {}_{n|}^{2}A_x$$

Hence, the variance of the present value of the deferred annuity is:

$$\operatorname{var}\left(v^n \ddot{a}_{\overline{\max\{K_x+1-n,0\}}|}\right) = \frac{1}{d^2}\left[v^{2n} \, {}_nq_x + {}_{n|}^{2}A_x - \left(v^n \, {}_nq_x + {}_{n|}A_x\right)^2\right]$$

8 Guaranteed annuities payable annually in advance

A *guaranteed annuity* differs from a whole life annuity in that the payments have a minimum specified term.

A *guaranteed annuity-due* has payments that are made in advance and have a minimum specified term.

Question

Suggest a reason why guaranteed annuities are commonplace.

Solution

When people buy annuities they are often investing large amounts of their life savings. Should they die soon after purchasing the annuity, and the annuity is not guaranteed, they would effectively lose nearly all of their life savings. Relatives of the deceased are likely to find this distressing, in addition to the emotional distress they would be experiencing at the time. The insurer could also experience bad publicity and suffer reputational damage as a result.

Issuing annuities with guaranteed payment periods (typically of five or ten years) reduces the financial loss on early death and so goes a long way in mitigating these problems.

8.1 Present value random variable

Consider a guaranteed annuity contract to pay 1 at the start of each future year for the next n years, and at the start of each subsequent future year provided a life now aged x is then alive.

The present value of this benefit is $\ddot{a}_{\overline{\max[K_x+1,n]}|}$.

If (x) dies within n years, then exactly n payments will be made. If (x) lives for longer than n years, then K_x+1 payments will be made. So the present value can also be written as:

$$\begin{cases} \ddot{a}_{\overline{n}|} & \text{if } K_x < n \\ \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x \geq n \end{cases}$$

8.2 Expected present value

In actuarial notation, $E[\ddot{a}_{\overline{\max[K_x+1,n]}|}]$ is denoted $\ddot{a}_{\overline{\overline{x:n}|}}$.

The combined status $\overline{u:v}$ (*ie* with a bar) means a status that is active while *either or both* of the individual statuses u and v are active. It is known as the *last survivor* status, and we will meet it again when we study multiple lives in a later chapter. When applied to an annuity, $\ddot{a}_{\overline{u:v}}$ implies that payments continue until the last surviving status fails, so in the case of $\ddot{a}_{\overline{x:\overline{n|}}}$ this means payments continue until the *later* of the death of (x), or the expiry of n years.

To calculate $\ddot{a}_{\overline{x:\overline{n|}}}$, we use the following:

$$\ddot{a}_{\overline{x:\overline{n|}}} = E[\ddot{a}_{\overline{\max[K_x+1,n]|}}] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{n|}}\ {}_{k|}q_x + \sum_{k=n}^{\infty} \ddot{a}_{\overline{k+1|}}\ {}_{k|}q_x$$

This is because the present value is $\ddot{a}_{\overline{n|}}$ if (x) dies in any of the first n years (*ie* for $0 \le k < n$), and $\ddot{a}_{\overline{K_x+1|}}$ if (x) dies in any year thereafter (*ie* for $K_x \ge n$).

Using $\ddot{a}_{\overline{t|}} = v^0 + v^1 + \cdots + v^{t-1}$, we obtain:

$$\ddot{a}_{\overline{x:\overline{n|}}} = \sum_{k=0}^{n-1}\left(\sum_{j=0}^{n-1} v^j\right) {}_{k|}q_x + \sum_{k=n}^{\infty}\left(\sum_{j=0}^{k} v^j\right) {}_{k|}q_x$$

$$= \left(v^0 + v^1 + \cdots + v^{n-1}\right) \times {}_{0|}q_x + \left(v^0 + v^1 + \cdots + v^{n-1}\right) \times {}_{1|}q_x + \cdots + \left(v^0 + v^1 + \cdots + v^{n-1}\right) \times {}_{n-1|}q_x$$

$$+ \left(v^0 + v^1 + \cdots + v^{n-1} + v^n\right) \times {}_{n|}q_x$$

$$+ \left(v^0 + v^1 + \cdots + v^{n-1} + v^n + v^{n+1}\right) \times {}_{n+1|}q_x$$

$$+ \cdots$$

$$= v^0 \sum_{k=0}^{\infty} {}_{k|}q_x + v^1 \sum_{k=0}^{\infty} {}_{k|}q_x + \cdots + v^{n-1} \sum_{k=0}^{\infty} {}_{k|}q_x$$

$$+ v^n \sum_{k=n}^{\infty} {}_{k|}q_x + v^{n+1} \sum_{k=n+1}^{\infty} {}_{k|}q_x + \cdots$$

$$= \sum_{j=0}^{n-1}\left(\sum_{k=0}^{\infty} {}_{k|}q_x\right) v^j + \sum_{j=n}^{\infty}\left(\sum_{k=j}^{\infty} {}_{k|}q_x\right) v^j$$

Now $\sum_{k=0}^{\infty} {}_{k|}q_x = 1$ and $\sum_{k=j}^{\infty} {}_{k|}q_x = {}_j p_x$, and so:

$$\ddot{a}_{x:\overline{n}|} = \sum_{j=0}^{n-1} 1.v^j + \sum_{j=n}^{\infty} {}_j p_x \, v^j$$

$$= \ddot{a}_{\overline{n}|} + \sum_{j=n}^{\infty} {}_j p_x \, v^j$$

$$= \ddot{a}_{\overline{n}|} + {}_{n|}\ddot{a}_x$$

Question

Calculate $\ddot{a}_{60:\overline{10}|}$.

Basis:

Mortality: AM92 Ultimate

Interest: 6% *pa* effective

Solution

We can evaluate this as follows:

$$\ddot{a}_{60:\overline{10}|} = \ddot{a}_{\overline{10}|} + {}_{10|}\ddot{a}_{60}$$

$$= \ddot{a}_{\overline{10}|} + v^{10} \; {}_{10}p_{60} \, \ddot{a}_{70}$$

$$= \left(\frac{1-v^{10}}{d}\right) + v^{10} \frac{l_{70}}{l_{60}} \ddot{a}_{70}$$

$$= \left(\frac{1-1.06^{-10}}{0.06/1.06}\right) + 1.06^{-10} \times \frac{8{,}054.0544}{9{,}287.2164} \times 9.140$$

$$= 7.80169 + 4.42605$$

$$= 12.2277$$

8.3 Variance of the present value random variable

The variance of this benefit is:

$$\text{var}[\ddot{a}_{\overline{\max[K_x+1,n]}|}] = \text{var}\left[\frac{1-v^{\max[K_x+1,n]}}{d}\right]$$

$$= \frac{1}{d^2}\text{var}\left[v^{\max[K_x+1,n]}\right]$$

$$= \frac{1}{d^2}\left(E\left[\left(v^{\max[K_x+1,n]}\right)^2\right] - \left(E\left[v^{\max[K_x+1,n]}\right]\right)^2\right)$$

$$= \frac{1}{d^2}\left(v^{2n}\,{}_nq_x + {}_{n|}^{2}A_x - \left(v^n\,{}_nq_x + {}_{n|}A_x\right)^2\right)$$

The last step above uses the same approach that we used when deriving the formula for the variance of a deferred annuity-due in the question at the end of Section 7.

In fact, we see that the variance of the deferred annuity-due and the variance of the guaranteed annuity-due are the same. This is because all the uncertainty in these contracts stems from the payments made if the policyholder is alive after time n, and these are identical for the two annuities. In both cases, the payments before time n have no uncertainty associated with them (the payments are 0 for the deferred annuity, and are 1 every year for the guaranteed annuity), and so contribute nothing to the variance.

9 Guaranteed annuities payable annually in arrears

9.1 Present value random variable

Consider a guaranteed annuity contract to pay 1 at the end of each future year for the next n years, and at the end of each subsequent future year provided a life now aged x is then alive.

The present value of this benefit is $a_{\overline{\max[K_x,n]}|}$.

This can alternatively be written:

$$\begin{cases} a_{\overline{n}|} & \text{if } K_x \le n \\ a_{\overline{K_x}|} & \text{if } K_x > n \end{cases} = \begin{cases} a_{\overline{n}|} & \text{if } K_x < n \\ a_{\overline{K_x}|} & \text{if } K_x \ge n \end{cases}$$

9.2 Expected present value

In actuarial notation, $E[a_{\overline{\max[K_x,n]}|}]$ is denoted $a_{\overline{x:\overline{n}|}}$.

To obtain a formula for this, we follow similar logic to Section 8.2.

To calculate $a_{\overline{x:\overline{n}|}}$, we use the following:

$$\begin{aligned} a_{\overline{x:\overline{n}|}} &= E[a_{\overline{\max[K_x,n]}|}] \\ &= \sum_{k=0}^{n-1} a_{\overline{n}|}\ {}_{k|}q_x + \sum_{k=n}^{\infty} a_{\overline{k}|}\ {}_{k|}q_x \\ &= \sum_{k=0}^{n-1} \left(\sum_{j=1}^{n} v^j \right) {}_{k|}q_x + \sum_{k=n}^{\infty} \left(\sum_{j=1}^{k} v^j \right) {}_{k|}q_x \\ &= \sum_{j=1}^{n} \left(\sum_{k=0}^{\infty} {}_{k|}q_x \right) v^j + \sum_{j=n+1}^{\infty} \left(\sum_{k=j}^{\infty} {}_{k|}q_x \right) v^j \\ &= a_{\overline{n}|} + \sum_{j=n+1}^{\infty} {}_jp_x\, v^j \\ &= a_{\overline{n}|} + {}_{n|}a_x \end{aligned}$$

9.3 Variance of the present value random variable

The variance of this benefit is:

$$\begin{aligned}\text{var}[a_{\overline{\max[K_x,n]}|}] &= \text{var}[\ddot{a}_{\overline{\max[K_x+1,n+1]}|} - 1] \\ &= \text{var}[\ddot{a}_{\overline{\max[K_x+1,n+1]}|}] \\ &= \frac{1}{d^2}\left(v^{2(n+1)}\ {}_{n+1}q_x + {}_{n+1|}^{\ \ 2}A_x - \left(v^{n+1}\ {}_{n+1}q_x + {}_{n+1|}A_x\right)^2\right)\end{aligned}$$

10 Continuous annuities

So far we have concentrated on annuities payable annually.

We will now consider annuities that are payable continuously. In practice, these may be used to approximate annuities under which payments are very frequent, *eg* weekly or daily.

10.1 Immediate annuity

Consider an immediate annuity of 1 per annum payable continuously during the lifetime of a life now aged x.

Present value random variable

The present value of this annuity is $\bar{a}_{\overline{T_x}|}$.

Expected present value

Recall that T_x is a continuous random variable with PDF $f_{T_x}(t) = {}_tp_x\,\mu_{x+t}$.

The EPV, denoted $\bar{a}_x$, is:

$$\bar{a}_x = E\left[\bar{a}_{\overline{T_x}|}\right] = \int_0^\infty \bar{a}_{\overline{t}|}\,{}_tp_x\,\mu_{x+t}dt$$

This is saying that the expected present value is the present value of an annuity paid to time t, multiplied by the probability of surviving to t and dying in the next instant after time t. The integral is 'summing' this over all future instants at which death could occur.

We can derive a more useful formula than that above.

Note that $\bar{a}_{\overline{t}|} = \int_0^t e^{-\delta s}ds$, so that $\dfrac{d}{dt}\bar{a}_{\overline{t}|} = e^{-\delta t} = v^t$, and then integrate by parts.

The formula for integrating by parts (given on page 3 of the *Tables*) is:

$$\int_0^\infty u\frac{dw}{dt}dt = \left[uw\right]_0^\infty - \int_0^\infty w\frac{du}{dt}dt$$

In this case, we set:

$$u = \bar{a}_{\overline{t}|} \quad \text{and} \quad \frac{dw}{dt} = {}_tp_x\,\mu_{x+t}$$

so that $\dfrac{du}{dt} = v^t$ and $w = -{}_tp_x$. The result for w follows because $f_{T_x}(t)$ is the PDF of T_x, and is therefore equal to the derivative of the CDF of T_x, giving:

$$f_{T_x}(t) = {}_tp_x\,\mu_{x+t} = \frac{d}{dt}F_{T_x}(t) = \frac{d}{dt}P(T_x \le t) = \frac{d}{dt}{}_tq_x = \frac{d}{dt}(1 - {}_tp_x) = \frac{d}{dt}(-{}_tp_x)$$

Putting this together gives:

$$\boldsymbol{\bar{a}_x = -\left[\bar{a}_{\overline{t|}}\ {}_tp_x\right]_0^\infty + \int_0^\infty v^t\ {}_tp_x dt = \int_0^\infty v^t\ {}_tp_x dt}$$

Again this final result follows by general reasoning. We take the present value of the payment made at time t, multiply by the probability that the life is alive at that time to receive the payment, and then integrate (*ie* sum) over all future times t.

Another way to prove this result, which is analogous to the proof of the discrete annuity result, is to write $\bar{a}_{\overline{t|}} = \int_0^t v^s ds$ and reverse the order of integration. Then:

$$\bar{a}_x = \int_0^\infty \bar{a}_{\overline{t|}} f_{T_x}(t)dt = \int_0^\infty \left(\int_0^t v^s\, ds\right) f_{T_x}(t)dt$$

$$= \int_0^\infty \left(\int_s^\infty f_{T_x}(t)dt\right) v^s\, ds = \int_0^\infty v^s P(T_x > s)ds$$

$$= \int_0^\infty v^s\ {}_sp_x\, ds$$

Variance of the present value random variable

The variance of $\bar{a}_{\overline{T_x|}}$ is:

$$\text{var}\left(\bar{a}_{\overline{T_x|}}\right) = \frac{1}{\delta^2}\left[{}^2\bar{A}_x - \left(\bar{A}_x\right)^2\right]$$

Question

Prove this result.

Solution

Using the formula for continuous annuities-certain:

$$\text{var}\left(\bar{a}_{\overline{T_x|}}\right) = \text{var}\left(\frac{1-v^{T_x}}{\delta}\right)$$

and using the properties of variance:

$$\text{var}\left(\bar{a}_{\overline{T_x|}}\right) = \frac{1}{\delta^2}\text{var}\left(v^{T_x}\right)$$

In Chapter 16 we saw that:

$$\text{var}\left(v^{T_x}\right) = {}^2\bar{A}_x - \left(\bar{A}_x\right)^2$$

Hence:

$$\text{var}\left(\bar{a}_{\overline{T_x}|}\right) = \frac{1}{\delta^2}\left[{}^2\bar{A}_x - \left(\bar{A}_x\right)^2 \right]$$

10.2 Other annuities

Temporary, deferred and guaranteed continuous annuities can be defined, and their EPVs calculated, in a similar way. Using the obvious notation, for example:

$$\bar{a}_{x:\overline{n}|} = E\left[\bar{a}_{\overline{\min[T_x,n]}|}\right] = \int_0^n \bar{a}_{\overline{t}|}\, {}_tp_x \mu_{x+t} dt + \bar{a}_{\overline{n}|}\, {}_np_x = \int_0^n v^t\, {}_tp_x dt$$

$$\bar{a}_x = \bar{a}_{x:\overline{n}|} + {}_{n|}\bar{a}_x$$

$$\bar{a}_{\overline{\overline{x:n}}|} = \bar{a}_{\overline{n}|} + {}_{n|}\bar{a}_x$$

$${}_{n|}\bar{a}_x = v^n\, {}_np_x\, \bar{a}_{x+n}$$

10.3 Approximations

To evaluate these annuities, use the approximation:

$$\bar{a}_x \approx \ddot{a}_x - ½$$

or: $$\bar{a}_x \approx a_x + ½$$

The rationale here is that $\ddot{a}_x$ and a_x represent two extremes, in which the payments are made at the beginning and at the end of each year, respectively. With $\bar{a}_x$, the payments are spread uniformly over the year, so we might expect its value to lie roughly midway between $\ddot{a}_x$ and a_x. Since $\ddot{a}_x$ differs from a_x by 1, $\bar{a}_x$ differs from them both by ½.

Question

A level annuity of £1,000 *pa* is to be paid continuously to a 40-year-old male for the rest of his life.

On the basis of 4% *pa* interest and AM92 Ultimate mortality, calculate the expected present value of this annuity.

Solution

The expected present value of the annuity is:

$$1{,}000\bar{a}_{40} \approx 1{,}000\left(\ddot{a}_{40} - ½\right) = 1{,}000\left(20.005 - ½\right) = £19{,}505$$

For temporary annuities:

$$\bar{a}_{x:\overline{n|}} \approx \ddot{a}_{x:\overline{n|}} - ½(1 - v^n \, {}_np_x)$$

We can prove this using the result $\bar{a}_{x:\overline{n|}} = \bar{a}_x - {}_{n|}\bar{a}_x = \bar{a}_x - v^n \, {}_np_x \, \bar{a}_{x+n}$ from Section 10.2, and the approximations for continuously payable whole life annuities we have just met. This gives:

$$\begin{aligned} \bar{a}_{x:\overline{n|}} &= \bar{a}_x - v^n \, {}_np_x \, \bar{a}_{x+n} \\ &\approx \ddot{a}_x - ½ - v^n \, {}_np_x \, (\ddot{a}_{x+n} - ½) \\ &= \ddot{a}_x - v^n \, {}_np_x \, \ddot{a}_{x+n} - ½(1 - v^n \, {}_np_x) \end{aligned}$$

But $\ddot{a}_x - v^n \, {}_np_x \, \ddot{a}_{x+n} = \ddot{a}_x - {}_{n|}\ddot{a}_x = \ddot{a}_{x:\overline{n|}}$. So we have:

$$\bar{a}_{x:\overline{n|}} \approx \ddot{a}_{x:\overline{n|}} - ½(1 - v^n \, {}_np_x)$$

11 Evaluating means and variances using select mortality

Corresponding to the annuities defined earlier in this chapter are select equivalents defined as before, but assumed to be issued to a select life denoted $[x]$ rather than x.

So, for example, $\ddot{a}_{[x]} = \sum_{k=0}^{k=\infty} {}_k p_{[x]} v^k$ can be used to calculate the EPV of benefits of a whole life annuity-due, with level annual payments, issued to a select life aged $[x]$ at entry.

The variance formulae established earlier also apply replacing x with $[x]$.

Question

A continuously payable temporary annuity is sold to a life aged exactly 40. The annuity makes payments at a rate of 5,000 *pa* until age 60 or the policyholder's earlier death.

Calculate the expected present value of the annuity payments, using AM92 Select mortality and an interest rate of 4% *pa* effective.

Solution

The expected present value of the annuity payments is:

$$5{,}000\bar{a}_{[40]:\overline{20|}} \approx 5{,}000\left(\ddot{a}_{[40]:\overline{20|}} - \frac{1}{2}\left(1 - v^{20}\ {}_{20}p_{[40]}\right)\right)$$

$$= 5{,}000\left(\ddot{a}_{[40]:\overline{20|}} - \frac{1}{2}\left(1 - v^{20}\frac{l_{60}}{l_{[40]}}\right)\right)$$

$$= 5{,}000\left(13.930 - \frac{1}{2}\left(1 - 1.04^{-20} \times \frac{9{,}287.2164}{9{,}854.3036}\right)\right)$$

$$= 68{,}225$$

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 17 Summary

Annuities

Annuity contracts pay a regular income to the policyholder. The income might be deferred to a future date and could be paid in advance, in arrears or continuously.

For each type of contract we can write down expressions for:

- the present value of the benefits, which is a random variable
- the expected present value of the benefits
- the variance of the present value of the benefits.

Whole life immediate annuity in arrears

Present value: $a_{\overline{K_x}|}$

Expected present value: $E\left(a_{\overline{K_x}|}\right)=a_x$

Variance of present value: $\text{var}\left(a_{\overline{K_x}|}\right)=\dfrac{1}{d^2}\left[{}^2A_x-\left(A_x\right)^2\right]$

Whole life immediate annuity-due

Present value: $\ddot{a}_{\overline{K_x+1}|}$

Expected present value: $E\left(\ddot{a}_{\overline{K_x+1}|}\right)=\ddot{a}_x$

Variance of present value: $\text{var}\left(\ddot{a}_{\overline{K_x+1}|}\right)=\dfrac{1}{d^2}\left[{}^2A_x-\left(A_x\right)^2\right]$

Continuously payable whole life annuity

Present value: $\bar{a}_{\overline{T_x}|}$

Expected present value: $E\left(\bar{a}_{\overline{T_x}|}\right)=\bar{a}_x$

Variance of present value: $\text{var}\left(\bar{a}_{\overline{T_x}|}\right)=\dfrac{1}{\delta^2}\left[{}^2\bar{A}_x-\left(\bar{A}_x\right)^2\right]$

Temporary immediate annuity in arrears

Present value: $a_{\overline{\min\{K_x,n\}}|}$

Expected present value: $E\left(a_{\overline{\min\{K_x,n\}}|}\right) = a_{x:\overline{n}|}$

Variance of present value: $\text{var}\left(a_{\overline{\min\{K_x,n\}}|}\right) = \dfrac{1}{d^2}\left[{}^2A_{x:\overline{n+1}|} - \left(A_{x:\overline{n+1}|}\right)^2\right]$

Temporary immediate annuity-due

Present value: $\ddot{a}_{\overline{\min\{K_x+1,n\}}|}$

Expected present value: $E\left(\ddot{a}_{\overline{\min\{K_x+1,n\}}|}\right) = \ddot{a}_{x:\overline{n}|}$

Variance of present value: $\text{var}\left(\ddot{a}_{\overline{\min\{K_x+1,n\}}|}\right) = \dfrac{1}{d^2}\left[{}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2\right]$

Deferred annuity-due

Present value: $Y = v^n\, \ddot{a}_{\overline{\max\{K_x+1-n,0\}}|}$

Expected present value: $E(Y) = {}_{n|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n}|} = v^n\, {}_np_x\, \ddot{a}_{x+n}$

Variance of present value: $\dfrac{1}{d^2}\left[v^{2n}\, {}_nq_x + {}_{n|}^{2}A_x - \left(v^n\, {}_nq_x + {}_{n|}A_x\right)^2\right]$

Guaranteed annuity-due

Present value: $\ddot{a}_{\overline{\max\{K_x+1,n\}}|}$

Expected present value: $E\left(\ddot{a}_{\overline{\max\{K_x+1,n\}}|}\right) = \ddot{a}_{\overline{x:\overline{n}|}}$

Variance of present value: As for the deferred annuity-due

Chapter 17 Practice Questions

17.1 If T_x and K_x are random variables measuring the complete and curtate future lifetimes, respectively, of a life aged x, write down an expression for each of the following symbols as the expectation of a random variable:

(i) $\bar{a}_x$

(ii) $\ddot{a}_{x:\overline{n|}}$

(iii) $\bar{a}_{\overline{x:\overline{n|}}}$

17.2 Calculate the expectation of the present value of the benefits from each of the following contracts issued to a life aged exactly 45, assuming that the annual effective interest rate is 4%, and AM92 Select mortality applies:

(i) a deferred whole life annuity-due, with a deferred period of 15 years, under which payments of £5,000 are made annually in advance while the policyholder is alive after the deferred period has elapsed

(ii) a guaranteed annuity, with a guarantee period of 15 years, under which payments of £5,000 are made annually in arrears for a minimum of 15 years and for life thereafter.

17.3 (i) Let Z be a random variable representing the present value of the benefits payable under an immediate life annuity that pays 1 per year in advance, issued to a life aged x.

Exam style

Show that $\text{var}(Z) = \dfrac{1}{d^2}\left({}^2A_x - (A_x)^2\right)$, where ${}^2A_x$ is an assurance calculated at a rate of interest which you should specify. [4]

(ii) A life office issues such a policy to a life aged exactly 65. The benefit is £275 per annum. Calculate the standard deviation of the annuity.

Basis: Mortality: AM92 Ultimate

Interest: 6% per annum throughout [3]

[Total 7]

17.4 A special 25-year life insurance policy is issued to a life aged x and provides the following benefits:

- a lump sum of £75,000 (payable at the end of the policy year) if death occurs during the first 10 years
- a dependants' pension (payable in the form of an annuity certain) of £5,000 *pa* payable on each remaining policy anniversary during the term (including the 25th anniversary) if death occurs after 10 years but before the end of the term of the policy
- a pension of £7,500 *pa* commencing on the day after the term of the policy expires and with payments on each subsequent policy anniversary while the policyholder is still alive.

Write down an expression for the present value random variable of the benefits under this policy.

17.5 A life currently aged x is subject to a constant force of mortality of 0.02 *pa*. The constant force of interest is 0.03 *pa*. Calculate:

(i) a_x

(ii) $\ddot{a}_x$

17.6 An annuity is payable continuously throughout the lifetime of a person now aged exactly 60, but for at most 10 years. The rate of payment at all times t during the first 5 years is £10,000 *pa*, and thereafter it is £12,000 *pa*.

Exam style

The force of mortality of this life is 0.03 *pa* between the ages of 60 and 65, and 0.04 *pa* between the ages of 65 and 70.

Calculate the expected present value of this annuity assuming a force of interest of 0.05 *pa*. [5]

Chapter 17 Solutions

17.1 (i) $\bar{a}_x = E\left[\bar{a}_{\overline{T_x}|}\right]$

(ii) $\ddot{a}_{x:\overline{n}|} = E\left[f(K_x)\right]$ where $f(K_x) = \begin{cases} \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x < n \\ \ddot{a}_{\overline{n}|} & \text{if } K_x \geq n \end{cases}$

Alternatively, we can write $\ddot{a}_{x:\overline{n}|} = E\left[\ddot{a}_{\overline{\min\{K_x+1,n\}}|}\right]$.

(iii) $\bar{a}_{\overline{x:\overline{n}|}} = E\left[g(T_x)\right]$ where $g(T_x) = \begin{cases} \bar{a}_{\overline{n}|} & \text{if } T_x < n \\ \bar{a}_{\overline{T_x}|} & \text{if } T_x \geq n \end{cases}$

Alternatively, we can write $\bar{a}_{\overline{x:\overline{n}|}} = E\left[\bar{a}_{\overline{\max\{T_x,n\}}|}\right]$.

17.2 (i) ***Deferred whole life annuity-due***

The expected present value of the benefits is:

$$5{,}000\ {}_{15|}\ddot{a}_{[45]} = 5{,}000 v^{15}\ {}_{15}p_{[45]}\ \ddot{a}_{60}$$

Now:

$${}_{15}p_{[45]} = \frac{l_{60}}{l_{[45]}} = \frac{9{,}287.2164}{9{,}798.0837} = 0.94786$$

So:

$$5{,}000 v^{15}\ {}_{15}p_{[45]}\ \ddot{a}_{60} = 5{,}000 \times 1.04^{-15} \times 0.94786 \times 14.134 = £37{,}195$$

(ii) ***Guaranteed annuity in arrears***

The expected present value of the benefits is:

$$5{,}000 a_{\overline{[45]:\overline{15}|}} = 5{,}000\left(a_{\overline{15}|} + v^{15}\ {}_{15}p_{[45]}\ a_{60}\right)$$

$$= 5{,}000\left(\left(\frac{1-1.04^{-15}}{0.04}\right) + 1.04^{-15}\ {}_{15}p_{[45]}\left(\ddot{a}_{60}-1\right)\right)$$

$$= 5{,}000\left(11.11839 + 1.04^{-15} \times 0.94786(14.134-1)\right)$$

$$= £90{,}155$$

17.3 (i) ***Proof***

The random variable Z is defined as:

$$Z = \ddot{a}_{\overline{K_x+1}|} \qquad [½]$$

where K_x is the curtate future lifetime random variable of a life currently aged x.

So:

$$\text{var}(Z) = \text{var}\left(\ddot{a}_{\overline{K_x+1}|}\right) = \text{var}\left(\frac{1-v^{K_x+1}}{d}\right) = \frac{1}{d^2}\text{var}\left(v^{K_x+1}\right) \qquad [1]$$

Expressing the variance in terms of expectations, we have:

$$\text{var}(Z) = \frac{1}{d^2}\left[E\left(v^{2(K_x+1)}\right) - \left(E\left(v^{K_x+1}\right)\right)^2\right] \qquad [1]$$

By definition:

$$A_x = E\left(v^{K_x+1}\right) \qquad [½]$$

and:

$$^2A_x = E\left(v^{2(K_x+1)}\right) \qquad [½]$$

So:

$$\text{var}(Z) = \frac{1}{d^2}\left[{}^2A_x - \left(A_x\right)^2\right]$$

where ^2A_x is evaluated at the rate of interest $i' = (1+i)^2 - 1$. [½]

[Total 4]

(ii) ***Standard deviation of the annuity***

The variance of the present value of this annuity is:

$$\frac{275^2}{d^2}\left[{}^2A_{65} - \left(A_{65}\right)^2\right] = \frac{275^2}{(0.06/1.06)^2}\left[0.19985 - 0.40177^2\right] = (952.42)^2 \qquad [2]$$

So the standard deviation of the present value is £952.42. [1]

[Total 3]

17.4 The present value random variable is:

$$\begin{cases} 75{,}000v^{K_x+1} & \text{if } 0 \le K_x \le 9 \\ 5{,}000v^{K_x+1}\ddot{a}_{\overline{25-K_x}|} & \text{if } 10 \le K_x \le 24 \\ 7{,}500v^{25}\ddot{a}_{\overline{K_x-24}|} & \text{if } K_x \ge 25 \end{cases}$$

A possible alternative solution is:

$$\begin{cases} 75{,}000v^{K_x+1} & \text{if } 0 \le K_x \le 9 \\ 5{,}000\left(a_{\overline{25}|} - a_{\overline{K_x}|}\right) & \text{if } 10 \le K_x \le 24 \\ 7{,}500\left(a_{\overline{K_x}|} - a_{\overline{24}|}\right) & \text{if } K_x \ge 25 \end{cases}$$

17.5 (i) We have:

$$a_x = \sum_{j=1}^{\infty} {}_jp_x\, v^j$$

where ${}_jp_x = e^{-0.02j}$ and $v^j = e^{-\delta j} = e^{-0.03j}$. So:

$$a_x = \sum_{j=1}^{\infty} e^{-0.03j} \times e^{-0.02j} = \sum_{j=1}^{\infty} e^{-0.05j}$$

This is the sum to infinity of a geometric progression with first term $a = e^{-0.05}$ and common ratio $r = e^{-0.05}$. So, using the formula:

$$S_\infty = \frac{a}{1-r}$$

gives:

$$a_x = \frac{e^{-0.05}}{1-e^{-0.05}} = 19.5042$$

(ii) $\ddot{a}_x = a_x + 1 = 19.5042 + 1 = 20.5042$

17.6 The expected present value of this annuity is:

$$10{,}000\,\bar{a}_{60:\overline{5}|} + 12{,}000\,{}_{5|}\bar{a}_{60:\overline{5}|} = 10{,}000\,\bar{a}_{60:\overline{5}|} + 12{,}000 v^5\,{}_5p_{60}\,\bar{a}_{65:\overline{5}|} \qquad [1]$$

Since the force of mortality is constant between age 60 and age 65:

$$v^5\,{}_5p_{60} = e^{-5\delta}e^{-5\mu} = e^{-5(0.05+0.03)} = e^{-0.4} = 0.67032 \qquad [1]$$

Also, using the constant force of mortality of 0.03 between the ages of 60 and 65:

$$\bar{a}_{60:\overline{5}|} = \int_0^5 v^t\,{}_tp_{60}\,dt = \int_0^5 e^{-(0.05+0.03)t}\,dt = \left[\frac{e^{-0.08t}}{-0.08}\right]_0^5 = \frac{1}{0.08}\left(1-e^{-0.4}\right) = 4.12100 \qquad [1]$$

and similarly using the constant force of mortality of 0.04 between the ages of 65 and 70:

$$\bar{a}_{65:\overline{5}|} = \int_0^5 v^t\,{}_tp_{65}\,dt = \int_0^5 e^{-(0.05+0.04)t}\,dt = \left[\frac{e^{-0.09t}}{-0.09}\right]_0^5 = \frac{1}{0.09}\left(1-e^{-0.45}\right) = 4.02635 \qquad [1]$$

So the expected present value of the annuity is:

$$(10{,}000 \times 4.12100) + (12{,}000 \times 0.67032 \times 4.02635) = £73{,}597 \qquad [1]$$

[Total 5]

18

Evaluation of assurances and annuities

Syllabus objectives

4.2 Develop formulae for the means and variances of the payments under various assurance and annuity contracts, assuming a constant deterministic interest rate.

- 4.2.4 Define the assurance and annuity factors and their select and continuous equivalents. Extend the annuity factors to allow for the possibility that payments are more frequent than annual but less frequent than continuous.
- 4.2.5 Understand and use the relations between annuities payable in advance and in arrears, and between temporary, deferred and whole life annuities.
- 4.2.6 Understand and use the relations between assurance and annuity factors using equation of value, and their select and continuous equivalents.

Syllabus objectives continued

4.2.7 Obtain expressions in the form of sums/integrals for the mean and variance of the present value of benefit payments under each contract defined in 4.1.1, in terms of the (curtate) random future lifetime, assuming:

- (constant) contingent benefits are payable at the middle or end of the year of contingent event or continuously
- annuities are paid in advance, in arrears or continuously, and the amount is constant
- premiums are payable in advance, in arrears or continuously; and for the full policy term or for a limited period.

Where appropriate, simplify the above expressions into a form suitable for evaluation by table look-up or other means.

0 Introduction

In Chapters 16 and 17, we introduced the basic functions of life insurance mathematics – expected present values of assurance and annuity contracts. The next step is to explore useful relationships between these EPVs. We can then apply the same ideas to other types of life insurance contracts.

The formulae we have derived for EPVs can be interpreted in a simple way, which is often useful in practice. Consider, for example:

$$A_x = \sum_{k=0}^{\infty} v^{k+1}\,{}_{k|}q_x \qquad \text{or} \qquad \ddot{a}_x = \sum_{k=0}^{\infty} v^k\,{}_k p_x$$

Each term of these sums can be interpreted as:

an amount payable at time k

$\times$ a discount factor for k years

$\times$ the probability that a payment will be made at time k.

The first term in each case is just 1, but it should be easy to see that this interpretation can be applied to *any* benefit, level or not, payable on death or survival. This makes it easy to write down formulae for EPVs.

For example, consider an annuity-due, under which an amount k will be payable at the start of the k th year provided a life aged x is then alive (an increasing annuity-due). With this interpretation of EPVs, we can write down the EPV of this benefit (which is denoted $(I\ddot{a})_x$):

$$(I\ddot{a})_x = \sum_{k=0}^{\infty} (k+1)v^k\,{}_k p_x$$

Increasing benefits will be covered in Chapter 19.

1 Evaluating assurance benefits

It is important to become familiar with the *Tables* and know which annuity and assurance functions are included in them.

As we have seen, the AM92 table, for example, contains the values of A_x at 4% and 6% *pa* interest. It also contains the values of $A_{x:\overline{n}|}$ (at 4% and 6% *pa* interest) for ages x and terms n such that $x+n=60$ and $x+n=65$. However, we need to be able to calculate the values of assurance functions, *eg* $A_{30:\overline{25}|}$ and $A^1_{40:\overline{25}|}$, that are not listed in the *Tables.*

There are, in fact, several ways to proceed. One possibility is to use the relationships that exist between the assurance functions in order to write the required function in terms of functions that are listed in the *Tables*.

We saw in Chapter 16 that:

$${}_{n|}A_x = A_x - A^1_{x:\overline{n}|} = v^n \, {}_n p_x \, A_{x+n}$$

Rearranging this gives:

$$A^1_{x:\overline{n}|} = A_x - v^n \, {}_n p_x \, A_{x+n}$$

So we can calculate the value of a term assurance by writing it in terms of whole life assurances.

Question

Calculate the values of:

(i) $A^1_{40:\overline{25}|}$

(ii) $A_{30:\overline{25}|}$

(iii) $\bar{A}_{30:\overline{25}|}$

using AM92 mortality and 4% *pa* interest.

Solution

(i) The value of the term assurance is:

$$A^1_{40:\overline{25}|} = A_{40} - v^{25} \, {}_{25}p_{40} \, A_{65}$$

$$= 0.23056 - \frac{1}{1.04^{25}} \times \frac{8,821.2612}{9,856.2863} \times 0.52786$$

$$= 0.05334$$

Alternatively, we could note that because 40 and 25 sum to 65, the value of the endowment assurance $A_{40:\overline{25|}}$ is tabulated. Recalling that an endowment assurance is the sum of a term assurance and a pure endowment, we can say:

$$A^{1}_{40:\overline{25|}} = A_{40:\overline{25|}} - A_{40:\overline{25|}}^{\ \ \ 1}$$

$$= A_{40:\overline{25|}} - v^{25}\,{}_{25}p_{40}$$

$$= 0.38907 - \frac{1}{1.04^{25}} \times \frac{8{,}821.2612}{9{,}856.2863}$$

$$= 0.05334$$

(ii) The value of the endowment assurance is:

$$A_{30:\overline{25|}} = A^{1}_{30:\overline{25|}} + A_{30:\overline{25|}}^{\ \ \ 1}$$

$$= A_{30} - v^{25}\,{}_{25}p_{30}\,A_{55} + v^{25}\,{}_{25}p_{30}$$

$$= 0.16023 - \frac{1}{1.04^{25}} \times \frac{9{,}557.8179}{9{,}925.2094} \times 0.38950 + \frac{1}{1.04^{25}} \times \frac{9{,}557.8179}{9{,}925.2094}$$

$$= 0.38076$$

(iii) The value of the endowment assurance under which the death benefit is payable immediately on death is:

$$\bar{A}_{30:\overline{25|}} = \bar{A}^{1}_{30:\overline{25|}} + A_{30:\overline{25|}}^{\ \ \ 1}$$

$$\approx 1.04^{½} A^{1}_{30:\overline{25|}} + v^{25}\,{}_{25}p_{30}$$

$$= 1.04^{½}\left(0.16023 - \frac{1}{1.04^{25}} \times \frac{9{,}557.8179}{9{,}925.2094} \times 0.38950\right) + \frac{1}{1.04^{25}} \times \frac{9{,}557.8179}{9{,}925.2094}$$

$$= 0.38115$$

Remember that it is only the death benefit part that gets multiplied by the acceleration factor, $(1+i)^{½}$.

We can use the same techniques, along with the formulae developed in earlier chapters, to calculate the variance of the present value of the benefits payable. Recall that this involves functions with a pre-superscript of 2 to indicate that they are evaluated at the interest rate $(1+i)^2 - 1$.

Question

A life office has just sold a 25-year term assurance policy to a life aged 40. The sum assured is £50,000 and is payable at the end of the year of death.

Calculate the variance of the present value of this benefit, assuming AM92 Ultimate mortality and 4% *pa* interest.

Solution

The variance of the present value is:

$$50{,}000^2\left[{}^2A^{1}_{40:\overline{25}|}-\left(A^{1}_{40:\overline{25}|}\right)^2\right]$$

From part (i) of the preceding question, we know that $A^{1}_{40:\overline{25}|}=0.05334$.

The function ${}^2A^{1}_{40:\overline{25}|}$ is evaluated using an interest rate of $1.04^2-1=8.16\%$. This gives:

$$\begin{aligned}{}^2A^{1}_{40:\overline{25}|}&={}^2A_{40}-\frac{1}{1.0816^{25}}\times{}_{25}p_{40}\times{}^2A_{65}\\&=0.06792-\frac{1}{1.0816^{25}}\times\frac{8{,}821.2612}{9{,}856.2863}\times0.30855\\&=0.02906\end{aligned}$$

So the variance of the present value is:

$$50{,}000^2\left[0.02906-0.05334^2\right]=(£8{,}096)^2$$

2 Evaluating annuity benefits

The following relationships are easy to prove:

$$\ddot{a}_x = 1 + a_x \qquad \ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}$$

$$a_x = v\,p_x\,\ddot{a}_{x+1} \qquad a_{x:\overline{n}|} = vp_x\,\ddot{a}_{x+1:\overline{n}|}$$

The question below illustrates the first of the formulae above, which we introduced in Chapter 17.

Question

Find a_{65} (PFA92C20 at 4%).

Solution

$$a_{65} = \ddot{a}_{65} - 1 = 13.871$$

We also obtained the result $\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}$ in Chapter 17.

We derive the relationship $a_{x:\overline{n}|} = vp_x\,\ddot{a}_{x+1:\overline{n}|}$ below. The relationship $a_x = vp_x\,\ddot{a}_{x+1}$ can be derived in the same way.

Question

Prove that $a_{x:\overline{n}|} = vp_x\,\ddot{a}_{x+1:\overline{n}|}$.

Solution

Starting with the definition of $a_{x:\overline{n}|}$, we have:

$$a_{x:\overline{n}|} = v\,p_x + v^2\,{}_2p_x + \ldots + v^n\,{}_np_x$$

Now, using the principle of consistency ${}_kp_x = {}_np_x\;{}_{k-n}p_{x+n}$ (for $k \geq n$) with $n = 1$, we are able to take out the factor $v\,p_x$, giving:

$$\begin{aligned} a_{x:\overline{n}|} &= v\,p_x\left(1 + v\,p_{x+1} + v^2\,{}_2p_{x+1} + \ldots + v^{n-1}\,{}_{n-1}p_{x+1}\right) \\ &= v\,p_x\,\ddot{a}_{x+1:\overline{n}|} \end{aligned}$$

As we have seen, the AM92 table contains the values of $\ddot{a}_x$ at 4% and 6% *pa* interest. It also contains the values of $\ddot{a}_{x:\overline{n}|}$ (at 4% and 6% *pa* interest) for ages x and terms n such that $x + n = 60$ and $x + n = 65$.

Other values can be calculated using formulae such as:

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - v^n \, {}_np_x \, \ddot{a}_{x+n}$$

This formula enables us to calculate the value of a temporary annuity-due in terms of whole life functions. It is a rearrangement of a relationship we met in Chapter 17 for a deferred annuity-due:

$${}_{n|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n}|} = v^n \, {}_np_x \, \ddot{a}_{x+n}$$

Similar formulae hold for temporary annuities payable annually in arrears and temporary annuities payable continuously:

$$a_{x:\overline{n}|} = a_x - v^n \, {}_np_x \, a_{x+n}$$

$$\bar{a}_{x:\overline{n}|} = \bar{a}_x - v^n \, {}_np_x \, \bar{a}_{x+n}$$

Question

Calculate the values of $\ddot{a}_{60:\overline{10}|}$ and $\bar{a}_{60:\overline{10}|}$ using AM92 mortality and 6% *pa* interest.

Solution

The value of the temporary annuity-due is:

$$\ddot{a}_{60:\overline{10}|} = \ddot{a}_{60} - v^{10} \, {}_{10}p_{60} \, \ddot{a}_{70}$$

$$= 11.891 - \frac{1}{1.06^{10}} \times \frac{8,054.0544}{9,287.2164} \times 9.140$$

$$= 7.465$$

The value of the temporary continuously-payable annuity can be written as:

$$\bar{a}_{60:\overline{10}|} = \bar{a}_{60} - v^{10} \, {}_{10}p_{60} \, \bar{a}_{70}$$

$$= \left(\ddot{a}_{60} - ½\right) - v^{10} \, {}_{10}p_{60} \left(\ddot{a}_{70} - ½\right)$$

$$= \ddot{a}_{60:\overline{10}|} - ½\left(1 - v^{10} \, {}_{10}p_{60}\right)$$

So, using the value of the temporary annuity-due, we have:

$$\bar{a}_{60:\overline{10}|} = 7.465 - ½\left(1 - \frac{1}{1.06^{10}} \times \frac{8,054.0544}{9,287.2164}\right) = 7.207$$

For the temporary continuously payable annuity in the question above, we used the relationship:

$$\bar{a}_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - ½\left(1 - v^n \, {}_np_x\right)$$

In Chapter 17, we also developed the corresponding formula for a temporary annuity in arrears:

$$a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - \left(1 - v^n \, {}_np_x\right)$$

These two results can be particularly useful when $x+n=60$ or $x+n=65$, as the value of $\ddot{a}_{x:\overline{n}|}$ is tabulated (at 4% and 6% *pa* interest) for AM92 mortality.

Question

Calculate the value of $a_{[40]:\overline{25}|}$ using AM92 mortality and 4% *pa* interest.

Solution

We can write:

$$\begin{aligned} a_{[40]:\overline{25}|} &= \ddot{a}_{[40]:\overline{25}|} - \left(1 - v^{25} \, {}_{25}p_{[40]}\right) \\ &= \ddot{a}_{[40]:\overline{25}|} - 1 + v^{25} \frac{l_{65}}{l_{[40]}} \\ &= 15.887 - 1 + 1.04^{-25} \times \frac{8,821.2612}{9,854.3036} \\ &= 15.223 \end{aligned}$$

Another way to write the factor $v^n \, {}_np_x$ is as $\frac{D_{x+n}}{D_x}$. D_x is an example of a commutation function, and is defined as follows:

$$D_x = v^x \, l_x$$

so that:

$$\frac{D_{x+n}}{D_x} = \frac{v^{x+n} \, l_{x+n}}{v^x \, l_x} = v^n \frac{l_{x+n}}{l_x} = v^n \, {}_np_x$$

In the past, commutation functions were widely used in the calculation of annuities and assurances. Values of the functions D_x, N_x, S_x, C_x, M_x and R_x are listed for AM92 mortality at 4% *pa* interest, but the only one of these that we will use in this course is D_x. We use it purely because it is quicker to calculate $\dfrac{D_{x+n}}{D_x}$ than $v^n\,{}_np_x$. However, if we are not using AM92 mortality and an assumed rate of interest of 4% *pa*, D_x is not available and we have to use $v^n\,{}_np_x$.

Question

Calculate a_{30} and ${}_{10|}a_{30}$, and also a_{70} and ${}_{10|}a_{70}$, based on AM92 mortality and 4% *pa* interest. Comment on your results.

Solution

For age 30 we have:

$$a_{30} = \ddot{a}_{30} - 1 = 21.834 - 1 = 20.834$$

$${}_{10|}a_{30} = v^{10}\,{}_{10}p_{30}\,a_{40} = \frac{D_{40}}{D_{30}}\left(\ddot{a}_{40} - 1\right) = \frac{2{,}052.96}{3{,}060.13}\left(20.005 - 1\right) = 12.750$$

Similarly for age 70 we have:

$$a_{70} = \ddot{a}_{70} - 1 = 10.375 - 1 = 9.375$$

$${}_{10|}a_{70} = v^{10}\,{}_{10}p_{70}\,a_{80} = \frac{D_{80}}{D_{70}}\left(\ddot{a}_{80} - 1\right) = \frac{228.48}{517.23}\left(6.818 - 1\right) = 2.570$$

a_{30} is much bigger than a_{70} since the payments are expected to continue for a much longer period.

${}_{10|}a_{30}$ is lower than a_{30}, because no payments are made during the first 10 years, and since these payments are very likely to made (and are discounted least), the difference is quite close to 10.

${}_{10|}a_{70}$ is lower than a_{70} for the same reason. The difference is less than is seen in the age 30 functions, however, since the payments during the first 10 years are less likely to be made (as the life is more likely to die between ages 70 and 80 than between ages 30 and 40).

It's important to be aware that sometimes the assumptions made about mortality or interest will be different in different time periods. In the following question, the mortality assumption is different pre- and post-retirement.

Question

A male pension policyholder is currently aged 50 and he will retire at age 65, from which age a pension of £5,000 *pa* will be paid annually in advance. Before retirement he is assumed to experience mortality in line with AM92 Ultimate and after retirement in line with PMA92C20.

Calculate the expected present value of the benefits assuming interest of 4% *pa*.

Solution

The expected present value of the benefits is:

$$EPV = 5{,}000 \times \frac{D'_{65}}{D'_{50}} \times \ddot{a}''_{65}$$

where $\frac{D'_{65}}{D'_{50}}$ is calculated using AM92 Ultimate mortality, and $\ddot{a}''_{65}$ uses PMA92C20 mortality.

Therefore:

$$EPV = 5{,}000 \times \frac{689.23}{1{,}366.61} \times 13.666 = £34{,}461$$

3 Premium conversion formulae

There are both discrete and continuous versions of the premium conversion formulae.

3.1 Discrete version

There is a simple and very useful relationship between the EPVs of certain assurance contracts and the EPVs of annuities-due:

$$\ddot{a}_x = E\left[\ddot{a}_{\overline{K_x+1}|}\right] = E\left[\frac{1-v^{K_x+1}}{d}\right] = \frac{1-E\left[v^{K_x+1}\right]}{d} = \frac{1-A_x}{d}$$

Hence $A_x = 1 - d\ddot{a}_x$ **.**

Question

Verify that $A_{65} = 1 - d\ddot{a}_{65}$ using AM92 mortality and 4% *pa* interest.

Solution

From the AM92 table with 4% *pa* interest:

$$\ddot{a}_{65} = 12.276$$

So using the premium conversion formula:

$$A_{65} = 1 - d\ddot{a}_{65} = 1 - \frac{0.04}{1.04} \times 12.276 = 0.52785$$

The slight difference between this and the *Tables* value of $A_{65} = 0.52786$ is due to rounding.

Along similar lines, we find that:

$$A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}$$

and as we shall see, similar relationships hold for all of the whole life and endowment assurance contracts that we consider.

These relationships also apply replacing x **with** $[x]$**,** ***ie*****:**

$$A_{[x]} = 1 - d\ddot{a}_{[x]}$$

and:

$$A_{[x]:\overline{n}|} = 1 - d\ddot{a}_{[x]:\overline{n}|}$$

3.2 Continuous version

Similar relationships hold between level annuities payable continuously and assurance contracts with death benefits payable immediately on death. In these, we use δ instead of d.

For whole life benefits:

$$\bar{a}_x = E\left[\bar{a}_{\overline{T_x}|}\right] = E\left[\frac{1-v^{T_x}}{\delta}\right] = \frac{1}{\delta}(1-\bar{A}_x)$$

Hence $\bar{A}_x = 1-\delta\,\bar{a}_x$.

For temporary benefits:

$$\bar{a}_{x:\overline{n}|} = E\left[\bar{a}_{\overline{\min[T_x,n]}|}\right] = E\left[\frac{1-v^{\min[T_x,n]}}{\delta}\right] = \frac{1}{\delta}(1-\bar{A}_{x:\overline{n}|})$$

Hence $\bar{A}_{x:\overline{n}|} = 1-\delta\,\bar{a}_{x:\overline{n}|}$.

These formulae also apply replacing x with $[x]$.

The premium conversion formulae are given on page 37 of the *Tables*.

3.3 Variance of benefits

We can use a similar approach to express the variances of annuities payable continuously. For example:

$$\text{var}\left[\bar{a}_{\overline{T_x}|}\right] = \text{var}\left[\frac{1-v^{T_x}}{\delta}\right] = \frac{1}{\delta^2}\text{var}\left[v^{T_x}\right] = \frac{1}{\delta^2}\left({}^2\bar{A}_x - (\bar{A}_x)^2\right)$$

We have already seen this result in Chapter 17.

4 Expected present values of annuities payable *m* times each year

Very often, premiums are not paid annually but are instead paid with some other frequency, *eg* every quarter, or every month. Likewise, we may want to value annuity benefits payable more frequently than annually.

We now consider the question of how annuities, with payments made more than once each year but less frequently than continuously, may be evaluated.

We define the expected present value of an immediate annuity of 1 per annum, payable *m* times each year in arrears **to a life aged *x*, as $a_x^{(m)}$. This comprises payments, each of $\frac{1}{m}$, at ages:**

$$x+\frac{1}{m},\ x+\frac{2}{m},\ x+\frac{3}{m} \text{ and so on.}$$

The expected present value may be written:

$$a_x^{(m)} = \frac{1}{m}\sum_{t=1}^{\infty} v^{t/m}\frac{l_{x+t/m}}{l_x}$$

If a mathematical formula for l_x is known, this expression may be evaluated directly.

So, for example, if $l_x = 1{,}000 - 10x$ for $90 \le x \le 100$, we have:

$$a_{90}^{(4)} = \frac{1}{4}\sum_{t=1}^{\infty} v^{t/4}\frac{l_{90+t/4}}{l_{90}} = \frac{1}{4}\sum_{t=1}^{\infty} v^{t/4}\left(\frac{100-2.5t}{100}\right)$$

More often, an approximation will be needed to evaluate the expression.

We now consider an annuity-due.

An expression for $\ddot{a}_x^{(m)}$ can be valued as a series of deferred annuities with annual payments of $\frac{1}{m}$ and deferred period of $\frac{t}{m}$; $t = 0, 1, 2, \ldots, m-1$:

$$\sum_{t=0}^{m-1}\frac{1}{m}\ {}_{\frac{t}{m}|}\ddot{a}_x$$

Using the approximation that a sum of 1 payable a proportion k $(0<k<1)$ through the year is equivalent to an amount of $(1-k)$ paid at the start of the year and an amount of k paid at the end of the year, we can write:

$${}_{\frac{t}{m}|}\ddot{a}_x \approx \ddot{a}_x - \frac{t}{m}$$

To explain this result, let's think about an example. Suppose that $t=3$ and $m=4$. Then the annuity-due is deferred for ¾ of a year. So there are payments of £1 at times ¾, 1¾, 2¾, *etc*. In symbols we have:

$$_{¾|}\ddot{a}_x = v^{¾}\,_{¾}p_x + v^{1¾}\,_{1¾}p_x + v^{2¾}\,_{2¾}p_x + \cdots$$

Now the EPV of £1 at time ¾ is very similar to the EPV of £¼ paid at time 0, plus the EPV of ¾ paid at time 1, *ie*:

$$v^{¾}\,_{¾}p_x \approx ¼ + ¾\,v\,p_x$$

Similarly, the EPV of £1 at time 1¾ is very similar to the EPV of £¼ paid at time 1, plus the EPV of ¾ paid at time 2, *ie*:

$$v^{1¾}\,_{1¾}p_x \approx ¼\,v\,p_x + ¾\,v^2\,_2p_x$$

Continuing similarly and adding up all these payments we get:

$$_{¾|}\ddot{a}_x \approx \left(¼ + ¾\,v\,p_x\right) + \left(¼\,v\,p_x + ¾\,v^2\,_2p_x\right) + \cdots$$

$$= ¼ + v\,p_x + v^2\,_2p_x + \cdots$$

$$= \ddot{a}_x - ¾$$

In general, we have:

$$_{\frac{t}{m}|}\ddot{a}_x \cong \ddot{a}_x - \frac{t}{m}$$

We now return to our expression for $\ddot{a}_x^{(m)}$ and substitute in this result for the deferred annuity.

So the expected present value of the *m*thly annuity is approximately:

$$\boldsymbol{\sum_{t=0}^{t=m-1} \frac{1}{m}\left(\ddot{a}_x - \frac{t}{m}\right) = m.\frac{1}{m}\ddot{a}_x - \frac{1}{m}.\frac{1}{2}\frac{(m-1)m}{m}}$$

This uses the result that the sum of the first n integers is $½n(n+1)$, so:

$$\frac{1}{m}\sum_{t=0}^{m-1}\frac{t}{m} = \frac{1}{m}\times\frac{1+2+\cdots+(m-1)}{m} = \frac{1}{m}\times\frac{½(m-1)m}{m}$$

That is:

$$\boldsymbol{\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{(m-1)}{2m}}$$

This formula is given on page 36 of the *Tables.*

The corresponding expression for $a_x^{(m)}$ then follows from the relationship:

$$\ddot{a}_x^{(m)} = \frac{1}{m} + a_x^{(m)}$$

that is:

$$a_x^{(m)} \approx a_x + \frac{m-1}{2m}$$

(Note that, letting $m \to \infty$, we obtain the expression $\bar{a}_x \approx \ddot{a}_x - ½$, as referred to in Chapter 17.)

The relationship:

$$\ddot{a}_x^{(m)} = \frac{1}{m} + a_x^{(m)}$$

holds because each payment under the annuities $a_x^{(m)}$ and $\ddot{a}_x^{(m)}$ is for amount $\frac{1}{m}$, and so the only difference between them is the payment of $\frac{1}{m}$ made at time 0 under the annuity-due that is not made under the annuity in arrears.

These approximations may be used to develop equivalent expressions for temporary and deferred annuities.

For example:

$$\begin{aligned}\ddot{a}_{x:\overline{n}|}^{(m)} &= \ddot{a}_x^{(m)} - v^n \, {}_np_x \, \ddot{a}_{x+n}^{(m)} \\ &= \left(\ddot{a}_x - \frac{m-1}{2m}\right) - v^n \, {}_np_x \left(\ddot{a}_{x+n} - \frac{m-1}{2m}\right) \\ &= \ddot{a}_x - v^n \, {}_np_x \, \ddot{a}_{x+n} - \frac{m-1}{2m}\left(1 - v^n \, {}_np_x\right) \\ &= \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}\left(1 - \frac{D_{x+n}}{D_x}\right)\end{aligned}$$

This formula is also given on page 36 of the *Tables.*

Question

Using AM92 mortality and 4% *pa* interest, calculate:

(i) $\ddot{a}_{60}^{(2)}$

(ii) $a_{60}^{(12)}$

(iii) $\ddot{a}_{50:\overline{15}|}^{(4)}$

Solution

(i) $\ddot{a}_{60}^{(2)} \approx \ddot{a}_{60} - \frac{1}{4} = 14.134 - 0.25 = 13.884$

(ii) $a_{60}^{(12)} \approx a_{60} + \frac{11}{24} = \ddot{a}_{60} - 1 + \frac{11}{24} = 14.134 - 1 + \frac{11}{24} = 13.592$

(iii) $\ddot{a}_{50:\overline{15}|}^{(4)} \approx \ddot{a}_{50:\overline{15}|} - \frac{3}{8}\left(1 - \frac{D_{65}}{D_{50}}\right) = 11.253 - \frac{3}{8}\left(1 - \frac{689.23}{1{,}366.61}\right) = 11.067$

5 Expected present values under a constant force of mortality

So far in this chapter we have evaluated assurance and annuity functions assuming that the underlying mortality is given by a mortality table (often AM92 Ultimate mortality).

We can also calculate these functions in the (admittedly somewhat artificial) situation where a life is subject to a constant force of mortality.

In this particular situation, it will usually be easier to calculate what we might call 'continuous time' mortality functions – for example, assurances payable immediately on death, or annuities payable continuously. We do this by first writing the expression we need as an integral, and then evaluating it.

We can also calculate the corresponding 'discrete time' functions – although these will often involve a summation of a series, usually a geometric progression.

Question

A life is subject to a constant force of mortality of 0.008 *pa* at all ages above 50. The constant force of interest of 4% *pa.* Calculate the exact values of:

(i) $\bar{A}_{50}$

(ii) A_{50}

Solution

(i) Since the PDF of the complete future lifetime random variable T_x is ${}_tp_x\ \mu_{x+t}$, we can calculate $\bar{A}_{50}$ as follows:

$$\bar{A}_{50} = E\left(v^{T_{50}}\right) = \int_0^\infty v^t\ {}_tp_{50}\ \mu_{50+t}\ dt$$

We know that $\mu_{50+t} = 0.008$ for all t, so that ${}_tp_{50} = e^{-0.008t}$. Writing $v^t = e^{-\delta t}$, we have:

$$\bar{A}_{50} = \int_0^\infty e^{-0.04t} \times e^{-0.008t} \times 0.008\ dt = \int_0^\infty 0.008e^{-0.048t}\ dt = \left[\frac{0.008e^{-0.048t}}{-0.048}\right]_0^\infty$$

$$= \frac{0.008}{0.048} = 0.16667$$

(ii) We now need to calculate:

$$A_{50} = E\left(v^{K_{50}+1}\right) = v\,q_{50} + v^2 p_{50}\,q_{51} + v^3\ {}_2p_{50}\,q_{52} + \cdots$$

Since $p_{50+t} = e^{-0.008}$ for all t, $q_{50+t} = 1 - e^{-0.008}$. Also using $v^t = e^{-0.04t}$, we find that:

$$A_{50} = e^{-0.04}\left(1-e^{-0.008}\right) + e^{-0.08}e^{-0.008}\left(1-e^{-0.008}\right) + e^{-0.12}e^{-0.016}\left(1-e^{-0.008}\right) + \cdots$$

This is a geometric progression with first term $a = e^{-0.04}\left(1-e^{-0.008}\right)$ and common ratio $r = e^{-0.048}$. So the sum to infinity is:

$$A_{50} = \frac{a}{1-r} = \frac{e^{-0.04}\left(1-e^{-0.008}\right)}{1-e^{-0.048}} = 0.16335$$

The answers obtained here are consistent with the use of the acceleration factor $(1+i)^{0.5}$. *Since* $\delta = 0.04$, $1+i = e^{0.04}$ *and:*

$$(1+i)^{0.5} A_{50} = e^{0.02} \times 0.16335 = 0.16665 \approx \bar{A}_{50}$$

We can use a similar approach to calculate the expected present value of annuity functions, where the life is subject to a constant force of mortality.

Question

A life is subject to a constant force of mortality of 0.02 *pa* at all ages above 40. The constant force of interest of 5% *pa.* Calculate the exact values of:

(i) $\bar{a}_{40:\overline{10}|}$

(ii) $\bar{a}_{\overline{40:\overline{10}|}}$

Solution

(i) Here we have:

$$\bar{a}_{40:\overline{10}|} = \int_0^{10} e^{-\delta t}\, {}_tp_{40}\, dt = \int_0^{10} e^{-0.05t} e^{-0.02t}\, dt = \left[\frac{e^{-0.07t}}{-0.07}\right]_0^{10} = \frac{1-e^{-0.7}}{0.07} = 7.192$$

(ii) Using the standard formula for a guaranteed annuity, we have:

$$\bar{a}_{\overline{40:\overline{10}|}} = \bar{a}_{\overline{10}|} + {}_{10}p_{40}\, e^{-10\delta}\, \bar{a}_{50}$$

The whole life annuity at age 50 has expected present value:

$$\bar{a}_{50} = \int_0^\infty e^{-0.05t} e^{-0.02t} \, dt = \left[\frac{e^{-0.07t}}{-0.07} \right]_0^\infty = \frac{1}{0.07} = 14.28571$$

The present value of the continuously payable annuity-certain is:

$$\bar{a}_{\overline{10}|} = \frac{1 - e^{-0.05\times10}}{0.05} = 7.86939$$

So the expected present value of the guaranteed annuity is:

$$\bar{a}_{\overline{40:\overline{10}|}} = 7.86939 + e^{-0.02\times10} e^{-0.05\times10} \times 14.28571 = 14.963$$

Chapter 18 Summary

Relationships between assurances

$$A^{1}_{x:\overline{n}|} = A_x - v^n \, {}_n p_x \, A_{x+n}$$

$$_{n|}A_x = A_x - A^{1}_{x:\overline{n}|} = v^n \, {}_n p_x \, A_{x+n}$$

Relationships between annuities

$$a_x = \ddot{a}_x - 1 \qquad \bar{a}_x \approx \ddot{a}_x - ½$$

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - v^n \, {}_n p_x \, \ddot{a}_{x+n} \qquad a_{x:\overline{n}|} = a_x - v^n \, {}_n p_x \, a_{x+n}$$

$$a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + v^n \, {}_n p_x \qquad \ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}$$

$$a_x = v \, p_x \, \ddot{a}_{x+1} \qquad a_{x:\overline{n}|} = v \, p_x \, \ddot{a}_{x+1:\overline{n}|}$$

Premium conversion formulae

$$A_x = 1 - d\ddot{a}_x \qquad A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}$$

$$\bar{A}_x = 1 - \delta \bar{a}_x \qquad \bar{A}_{x:\overline{n}|} = 1 - \delta \bar{a}_{x:\overline{n}|}$$

Annuities payable *m* times a year

The expected present value of a life annuity of 1 *pa*, payable in arrears m times a year to a life aged x is:

$$a_x^{(m)} = \frac{1}{m} \sum_{t=1}^{\infty} v^{t/m} \, {}_{t/m}p_x \approx a_x + \frac{m-1}{2m}$$

The corresponding annuity-due has expected present value:

$$\ddot{a}_x^{(m)} = \frac{1}{m} \sum_{t=0}^{\infty} v^{t/m} \, {}_{t/m}p_x = \frac{1}{m} + a_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m}$$

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 18 Practice Questions

18.1 Calculate the value of each of the following functions, using the given basis:

(i) $\ddot{a}_{65:\overline{20}|}$ AM92 mortality and 4% *pa* interest

(ii) $A_{68:\overline{2}|}$ AM92 mortality and 6% *pa* interest

(iii) $a_{\overline{62:5}|}$ PFA92C20 mortality and 4% *pa* interest

(iv) $\ddot{a}_{[55]+1:\overline{4}|}$ AM92 mortality and 4% *pa* interest

(v) $A_{[60]+1}$ AM92 mortality and 4% *pa* interest

(vi) $A^{\ \ 1}_{[40]:\overline{10}|}$ AM92 mortality and 6% *pa* interest

18.2 Calculate the values of:

(a) $\ddot{a}^{(12)}_{50:\overline{15}|}$

(b) $\ddot{a}^{(12)}_{\overline{50:\overline{15}|}}$

using AM92 mortality and 4% *pa* interest.

18.3 Calculate the expected present value of a payment of £2,000 made 6 months after the death of a life now aged exactly 60, assuming AM92 Select mortality and 6% *pa* interest.

18.4 (i) Calculate $\bar{A}_{30:\overline{25}|}$ and $\bar{a}_{30:\overline{25}|}$ independently, assuming AM92 mortality and 6% *pa* interest.

(iii) Hence verify that the usual premium conversion relationship holds approximately between these two functions.

18.5 Let K be a random variable representing the curtate future lifetime of a life aged 40, and let $g(K)$ be the function defined as:

$$g(K) = \begin{cases} 0 & \text{if } 0 \le K < 10 \\ v^{10}\,\ddot{a}_{\overline{K-9}|} & \text{if } 10 \le K < 35 \\ v^{10}\,\ddot{a}_{\overline{25}|} & \text{if } 35 \le K \end{cases}$$

Calculate $E[g(K)]$, assuming mortality follows AM92 Ultimate and interest is 4% *pa*.

18.6 A 10-year temporary annuity is payable continuously to a life now aged x. The rate of payment is 100 *pa* for the first 5 years and 150 *pa* for the next 5 years.

Exam style

(i) Write down an expression in terms of T_x for the present value of this annuity. [3]

(ii) Calculate the expected present value of the annuity assuming a constant force of interest of 0.04 *pa* and a constant force of mortality of 0.01 *pa*. [5]

[Total 8]

18.7 An impaired life aged 40 experiences 5 times the force of mortality of a life of the same age subject to standard mortality. A two-year term assurance policy is sold to this impaired life, and another two-year term assurance is sold to a standard life aged 40. Both policies have a sum assured of £10,000 payable at the end of the year of death.

Calculate the expected present value of the benefits payable to each life assuming that standard mortality is AM92 Ultimate and interest is 4% *pa*.

18.8 Assuming that the force of mortality between consecutive integer ages is constant in the AM92 Ultimate table, calculate the exact value of $\bar{A}_{50:\overline{2}|}$ using a rate of interest of 4% *pa*. [6]

Exam style

18.9 In a mortality table with a one-year select period $q_{[x]} = a.q_x$ for all $x \geq 0$ and for a certain constant $0 < a < 1$.

Exam style

(i) Let:

$K_{[x]+t}$ = curtate future lifetime of a person aged $x+t$ who became select t years ago

K_x = curtate future lifetime of a person aged x whose mortality is reflected by the ultimate part of the mortality table.

Explain whether the expected value of $K_{[x]+1}$ is less than, greater than, or equal to the expected value of K_{x+1}. [1]

(ii) Calculate $\bar{A}_{[45]:\overline{20}|}$ based on the following assumptions:

- $a = 0.9$
- Interest: 4% *pa*
- Ultimate mortality: ELT15 (Males)

Financial functions using ELT15 (Males) mortality are given on page 136 of the Tables. [6]

[Total 7]

Chapter 18 Solutions

18.1 (i) We can calculate the value of this temporary annuity-due as:

$$\ddot{a}_{65:\overline{20|}} = \ddot{a}_{65} - \frac{D_{85}}{D_{65}}\ddot{a}_{85} = 12.276 - \frac{120.71}{689.23} \times 5.333 = 11.342$$

(ii) The value of this endowment assurance is not tabulated, but since the term is 2 years only, we can calculate its value using a summation from first principles.

If death occurs during the first year of the contract, the payment will be made at the end of the first year. Otherwise, it will be made at the end of the second year. So the value is:

$$A_{68:\overline{2|}} = vq_{68} + v^2 p_{68}$$

$$= 1.06^{-1} \times 0.019913 + 1.06^{-2} \times (1 - 0.019913) = 0.89106$$

(iii) This is the EPV of a guaranteed annuity, payable annually in arrears for a minimum of 5 years and for the remaining lifetime of a person currently aged 62 exact. We can write:

$$a_{\overline{62:\overline{5|}}} = a_{\overline{5|}} + {}_{5|}a_{62}$$

$$= \frac{1-v^5}{i} + v^5 \frac{l_{67}}{l_{62}}\left(\ddot{a}_{67} - 1\right)$$

$$= \frac{1-1.04^{-5}}{0.04} + 1.04^{-5} \times \frac{9,605.483}{9,804.173} \times (14.111 - 1)$$

$$= 15.010$$

(iv) The value of $\ddot{a}_{[55]+1:\overline{4|}}$ is not tabulated, so to calculate its value we express it in terms of $\ddot{a}_{57:\overline{3|}}$, which is tabulated, as follows:

$$\ddot{a}_{[55]+1:\overline{4|}} = 1 + v\, p_{[55]+1}\, \ddot{a}_{57:\overline{3|}} = 1 + \frac{1}{1.04}(1 - 0.004903) \times 2.870 = 3.746$$

(v) As in part (iv), the value of $A_{[60]+1}$ is not tabulated, so to calculate its value we express it in terms of A_{62}, which is tabulated, as follows:

$$A_{[60]+1} = v\, q_{[60]+1} + v\, p_{[60]+1}\, A_{62}$$

$$= 1.04^{-1}\left(0.008680 + (1 - 0.008680) \times 0.48458\right)$$

$$= 0.47024$$

(vi) We can calculate the value of this term assurance as:

$$A^{\;1}_{[40]:\overline{10}|} = A_{[40]} - v^{10}\frac{l_{50}}{l_{[40]}}A_{50}$$

$$= 0.12296 - \frac{1}{1.06^{10}} \times \frac{9,712.0728}{9,854.3036} \times 0.20508$$

$$= 0.01010$$

18.2 (a) We can calculate the value of this temporary annuity-due as:

$$\ddot{a}^{(12)}_{50:\overline{15}|} = \ddot{a}_{50:\overline{15}|} - \frac{11}{24}\left(1 - \frac{D_{65}}{D_{50}}\right) = 11.253 - \frac{11}{24}\left(1 - \frac{689.23}{1,366.61}\right) = 11.026$$

(b) We can calculate the value of this guaranteed annuity-due as:

$$\ddot{a}^{(12)}_{\overline{50:\overline{15}|}} = \ddot{a}^{(12)}_{\overline{15}|} + \frac{D_{65}}{D_{50}}\ddot{a}^{(12)}_{65}$$

$$= \frac{1-v^{15}}{d^{(12)}} + \frac{D_{65}}{D_{50}}\left(\ddot{a}_{65} - \frac{11}{24}\right)$$

$$= \frac{1-1.04^{-15}}{0.039157} + \frac{689.23}{1,366.61} \times \left(12.276 - \frac{11}{24}\right)$$

$$= 17.318$$

18.3 The EPV of a payment of £2,000 made immediately on the death of a select life now aged 60 is:

$$2,000\,\bar{A}_{[60]}$$

So the EPV of a payment of £2,000 made 6 months after the death of this life is:

$$2,000\,v^{0.5}\,\bar{A}_{[60]} \approx 2,000\,v^{0.5}(1+i)^{0.5}\,A_{[60]} = 2,000A_{[60]} = 2,000 \times 0.32533 = £650.66$$

18.4 (i) ***Calculations***

The endowment assurance is calculated as:

$$\bar{A}_{30:\overline{25}|} = \bar{A}^{\,1}_{30:\overline{25}|} + A_{30:\overline{25}|}^{\;\;\;\;\;\;1} \approx 1.06^{0.5}A^{\,1}_{30:\overline{25}|} + v^{25}\,{}_{25}p_{30}$$

Remember that only the death benefit is accelerated.

Now:

$$v^{25}\,{}_{25}p_{30} = v^{25}\frac{l_{55}}{l_{30}} = 1.06^{-25} \times \frac{9,557.8179}{9,925.2094} = 0.22437$$

and:

$$A^{\;1}_{30:\overline{25|}} = A_{30} - v^{25}\,{}_{25}p_{30}\,A_{55} = 0.07328 - 0.22437 \times 0.26092 = 0.01474$$

So:

$$\bar{A}_{30:\overline{25|}} \approx 1.06^{0.5} \times 0.01474 + 0.22437 = 0.23955$$

The temporary annuity is calculated as:

$$\bar{a}_{30:\overline{25|}} = \bar{a}_{30} - v^{25}\,{}_{25}p_{30}\,\bar{a}_{55} \approx (16.372 - 0.5) - 0.22437 \times (13.057 - 0.5) = 13.055$$

(ii) ***Verification of premium conversion formula***

We would expect the following premium conversion relationship to hold:

$$\bar{A}_{30:\overline{25|}} = 1 - \delta\bar{a}_{30:\overline{25|}}$$

In part (i), we calculated $\bar{A}_{30:\overline{25|}}$ to be 0.23955. Calculating this using the right-hand side of the premium conversion relationship gives:

$$1 - \ln 1.06 \times 13.055 = 0.23933$$

So, the answers are approximately equal.

The slight discrepancy is due to the different assumptions underlying the two calculations in part (i).

18.5 The function $g(K)$ corresponds to a benefit that is deferred for 10 years, and then makes payments annually in advance during the remaining lifetime of a life now aged 40, up to a maximum of 25 payments, *ie* it is a deferred temporary annuity-due.

In terms of actuarial symbols, the expected present value is:

$$E[g(K)] = \frac{D_{50}}{D_{40}}\ddot{a}_{50:\overline{25|}} = \frac{D_{50}}{D_{40}}\left(\ddot{a}_{50} - \frac{D_{75}}{D_{50}}\ddot{a}_{75}\right)$$

So:

$$E[g(K)] = \frac{1,366.61}{2,052.96}\left(17.444 - \frac{363.11}{1,366.61} \times 8.524\right) = 10.104$$

18.6 (i) The present value of the annuity is $g(T_x)$, where:

$$g(T_x)=\begin{cases}100\bar{a}_{\overline{T_x}|} & \text{if } T_x<5\\ 150\bar{a}_{\overline{T_x}|}-50\bar{a}_{\overline{5}|} & \text{if } 5\le T_x<10\\ 150\bar{a}_{\overline{10}|}-50\bar{a}_{\overline{5}|} & \text{if } T_x\ge 10\end{cases}$$

[Total 3]

This could also be written as:

$$g(T_x)=\begin{cases}100\bar{a}_{\overline{T_x}|} & \text{if } T_x<5\\ 100\bar{a}_{\overline{5}|}+150v^5\,\bar{a}_{\overline{T_x-5}|} & \text{if } 5\le T_x<10\\ 100\bar{a}_{\overline{5}|}+150v^5\,\bar{a}_{\overline{5}|} & \text{if } T_x\ge 10\end{cases}$$

or as:

$$g(T_x)=\int_0^{\min\{T_x,10\}}\rho(s)v^s\,ds \text{ where } \rho(s)=\begin{cases}100 & \text{for } 0\le s<5\\ 150 & \text{for } 5\le s<10\end{cases}$$

(ii) In terms of actuarial notation, the expected present value of the annuity is:

$$100\bar{a}_{x:\overline{10}|}+50\,{}_{5|}\bar{a}_{x:\overline{5}|}=100\bar{a}_{x:\overline{10}|}+50v^5\,{}_5p_x\,\bar{a}_{x+5:\overline{5}|}$$ [1]

We could alternatively express the EPV of the annuity as:

$$100\bar{a}_{x:\overline{5}|}+150\,{}_{5|}\bar{a}_{x:\overline{5}|}=100\bar{a}_{x:\overline{5}|}+150v^5\,{}_5p_x\,\bar{a}_{x+5:\overline{5}|}$$

Since we have a constant force of interest of 0.04 *pa*:

$$v^t=e^{-\delta t}=e^{-0.04t}$$ [½]

and since we have a constant force of mortality of 0.01 *pa*:

$${}_tp_x=e^{-\mu t}=e^{-0.01t}$$ [½]

So:

$$\bar{a}_{x:\overline{10}|}=\int_0^{10}v^t\,{}_tp_x\,dt=\int_0^{10}e^{-0.05t}\,dt=\left[\frac{e^{-0.05t}}{-0.05}\right]_0^{10}=\frac{1}{0.05}\left(1-e^{-0.5}\right)$$ [1]

and:

$$\bar{a}_{x+5:\overline{5}|}=\int_0^{5}v^t\,{}_tp_{x+5}\,dt=\int_0^{5}e^{-0.05t}\,dt=\left[\frac{e^{-0.05t}}{-0.05}\right]_0^{5}=\frac{1}{0.05}\left(1-e^{-0.25}\right)$$ [1]

Therefore, the EPV of the annuity is:

$$100\bar{a}_{x:\overline{10}|}+50v^5\,{}_5p_x\,\bar{a}_{x+5:\overline{5}|}$$

$$=100\times\frac{1}{0.05}\left(1-e^{-0.5}\right)+50e^{-0.04\times 5}e^{-0.01\times 5}\times\frac{1}{0.05}\left(1-e^{-0.25}\right)$$

$$=959.21$$ [1]

[Total 5]

18.7 *Standard life*

The expected present value of the benefit is:

$$10{,}000A^{1}_{40:\overline{2}|}=10{,}000\left(A_{40}-\frac{D_{42}}{D_{40}}A_{42}\right)=10{,}000\left(0.23056-\frac{1{,}894.37}{2{,}052.96}\times 0.24787\right)=£18.38$$

Impaired life

From first principles, we can write the expected present value of the term assurance as:

$$10{,}000\left(vq^*_{40}+v^2p^*_{40}q^*_{41}\right)$$

where * denotes impaired mortality. Now using the formula for ${}_tp_x$ from page 32 of the *Tables*:

$$p^*_{40}=\exp\left(-\int_0^1\mu^*_{40+t}\,dt\right)=\exp\left(-5\int_0^1\mu_{40+t}\,dt\right)=\left(\exp\left(-\int_0^1\mu_{40+t}\,dt\right)\right)^5=\left(p_{40}\right)^5$$

using $e^{AB}=\left(e^A\right)^B$. Similarly:

$$q^*_{41}=1-p^*_{41}=1-\left(p_{41}\right)^5$$

Now:

$$p_{40}=1-q_{40}=1-0.000937=0.999063$$

and: $p_{41}=1-q_{41}=1-0.001014=0.998986$

So the expected present value of the benefit payable to the impaired life is:

$$10{,}000\left(v\left[1-\left(p_{40}\right)^5\right]+v^2\left(p_{40}\right)^5\left[1-\left(p_{41}\right)^5\right]\right)$$

$$=10{,}000\left(\frac{1}{1.04}\times\left(1-0.999063^5\right)+\frac{1}{1.04^2}\times\left(0.999063^5\right)\left(1-0.998986^5\right)\right)$$

$$=£91.53$$

18.8 We can write:

$$\bar{A}_{50:\overline{2}|} = \bar{A}^{\,1}_{50:\overline{2}|} + A_{50:\overline{2}|}$$

$$= \int_0^2 v^t \, {}_tp_{50}\mu_{50+t}\,dt + v^2 \, {}_2p_{50}$$

$$= \int_0^1 v^t \, {}_tp_{50}\mu_{\overline{50}}\,dt + vp_{50}\int_0^1 v^t \, {}_tp_{51}\mu_{\overline{51}}\,dt + v^2 \, {}_2p_{50} \qquad [2]$$

where we have written $\mu_{\overline{x}}$ to indicate the assumed constant force of mortality operating between integer ages x and $x+1$.

Now, since $p_x = e^{-\mu_{\overline{x}}}$:

$$\mu_{\overline{50}} = -\ln p_{50} = -\ln(1-q_{50}) = -\ln 0.997492 = 0.00251115 \qquad [½]$$

$$\mu_{\overline{51}} = -\ln p_{51} = -\ln(1-q_{51}) = -\ln 0.997191 = 0.00281295 \qquad [½]$$

So, using $v^t = e^{-\delta t}$ where $\delta = \ln 1.04$, and ${}_tp_x = e^{-\mu_{\overline{x}}t}$:

$$\int_0^1 v^t \, {}_tp_{50}\mu_{\overline{50}}\,dt = \mu_{\overline{50}}\int_0^1 e^{-(\delta+\mu_{\overline{50}})t}\,dt = \frac{\mu_{\overline{50}}}{\delta+\mu_{\overline{50}}}\left(1-e^{-(\delta+\mu_{\overline{50}})}\right) = 0.00245947 \qquad [1]$$

$$\int_0^1 v^t \, {}_tp_{51}\mu_{\overline{51}}\,dt = \frac{\mu_{\overline{51}}}{\delta+\mu_{\overline{51}}}\left(1-e^{-(\delta+\mu_{\overline{51}})}\right) = 0.00275465 \qquad [1]$$

So:

$$\bar{A}_{50:\overline{2}|} = 0.00245947 + \frac{1}{1.04}\times 0.997492 \times 0.00275465 + \frac{1}{1.04^2}\times 0.997492 \times 0.997191$$

$$= 0.924748 \qquad [1]$$

[Total 6]

18.9 (i) ***Longevity comparison***

The mortality of a life aged $[x]+1$ is equal to that of a life aged $x+1$ because the table has a one-year select period. Thus:

$$E\left[K_{[x]+1}\right] = E\left[K_{x+1}\right] \qquad \text{[Total 1]}$$

(ii) ***Calculation***

Since the select period is one year, we can write:

$$\bar{A}_{[45]:\overline{20}|} = \bar{A}^{\,1}_{[45]:\overline{1}|} + {}_{1|}\bar{A}_{[45]:\overline{19}|} = \bar{A}^{\,1}_{[45]:\overline{1}|} + v\,p_{[45]}\,\bar{A}_{46:\overline{19}|} \qquad [2]$$

Now, assuming deaths occur on average halfway between age 45 and age 46:

$$\bar{A}^{1}_{[45]:\overline{1}|} = v^{0.5} q_{[45]} = v^{0.5} \times 0.9 \times q_{45} = 1.04^{-0.5} \times 0.9 \times 0.00266 = 0.0023475 \qquad [1]$$

Also:

$$v\, p_{[45]} = v\left(1 - q_{[45]}\right) = 1.04^{-1} \times \left(1 - 0.9 \times 0.00266\right) = 0.959236 \qquad [1]$$

and, using a premium conversion relationship with standard ELT15 (Males) mortality (as we are now outside the one-year select period):

$$\bar{A}_{46:\overline{19}|} = 1 - \delta \bar{a}_{46:\overline{19}|} = 1 - \ln(1.04) \times 12.740 = 0.500328 \quad (\textit{Tables}\text{, page 136}) \qquad [1]$$

This gives:

$$\bar{A}_{[45]:\overline{20}|} = 0.0023475 + 0.959236 \times 0.500328 = 0.48228 \qquad [1]$$

[Total 6]

19

Variable benefits and conventional with-profits policies

Syllabus objectives

4.1 Define various assurance and annuity contracts.

4.1.2 Describe the operation of conventional with-profits contracts, in which profits are distributed by the use of regular reversionary bonuses, and by terminal bonuses. Describe the benefits payable under the above assurance-type contracts.

Syllabus objectives continued

4.2 Develop formulae for the means and variances of the payments under various assurance and annuity contracts, assuming constant deterministic interest rates.

4.2.7 Obtain expressions in the form of sums/integrals for the mean of the present value of benefit payments under each contract defined in 4.1.1, in terms of the (curtate) random future lifetime, assuming:

- contingent benefits (increasing or decreasing) are payable at the middle or end of the year of contingent event or continuously
- annuities are paid in advance, in arrears or continuously, and the amount increases or decreases by a constant monetary amount or by a fixed or time-dependent variable rate.

Where appropriate, simplify the above expressions into a form suitable for evaluation by table look-up or other means.

0 Introduction

In this chapter we look at policies for which the benefit amount varies over time, including conventional with-profits contracts. In particular, we examine how to calculate the expected present value of the benefits when they increase by a constant amount or constant percentage each year.

1 Variable payments

The simple contracts discussed so far have had constant payments of the form:

- **a benefit of 1 payable on death; or**
- **a benefit of 1 payable, at some frequency, on survival to the date of payment.**

We consider now the possibility that the payment amount varies.

In the case of a benefit payable on death, let the payment be Y_x if death occurs in the year of age $(x, x+1)$.

Assume first that the benefit is payable at the end of the year of death. The EPV of this death benefit, to a life currently aged x, will be given by, using life table notation:

$$Y_x v \frac{d_x}{l_x} + Y_{x+1} v^2 \frac{d_{x+1}}{l_x} + \dots + Y_{x+t} v^{t+1} \frac{d_{x+t}}{l_x} + \dots$$

This can also be written using summations as:

$$\sum_{j=0}^{\infty} Y_{x+j} v^{j+1} \frac{d_{x+j}}{l_x} = \sum_{j=0}^{\infty} Y_{x+j} v^{j+1} \; {}_j p_x \, q_{x+j}$$

Equivalent integral expressions apply if the benefit is payable immediately on death.

In this case the expected present value of the benefit would be:

$$\int_0^1 Y_x v^t \, {}_t p_x \, \mu_{x+t} \, dt + \int_1^2 Y_{x+1} v^t \, {}_t p_x \, \mu_{x+t} \, dt + \int_2^3 Y_{x+2} v^t \, {}_t p_x \, \mu_{x+t} \, dt + \cdots$$

$$= \sum_{j=0}^{\infty} Y_{x+j} \int_j^{j+1} v^t \, {}_t p_x \, \mu_{x+t} \, dt$$

The above expression may be evaluated directly, and this would be the usual approach when computer power is available. This direct evaluation would be the only approach when no set pattern to the Y_x is imposed.

In this chapter we will describe ways of evaluating the EPV when Y_x is:

- **constant – this evaluation has already been dealt with in an earlier chapter, and the function of interest is A_x**
- **changing by a constant compound rate – we will use A values at an adjusted interest rate**
- **changing by a constant monetary amount – we will use tabulated factors which increase by 1 per annum.**

Corresponding to the assurance evaluation above, we will discuss a similar approach to evaluating annuity benefits where the variation follows one of the patterns just described.

In general, the EPV of an annuity of amount F_{x+t} payable on survival to age $x+t$ to a life currently aged x, assuming, for example, immediate annual payments in arrears, would be evaluated directly from:

$$F_{x+1}\, v \frac{l_{x+1}}{l_x} + F_{x+2}\, v^2 \frac{l_{x+2}}{l_x} + \ldots + F_{x+t}\, v^t \frac{l_{x+t}}{l_x} + \ldots$$

Having evaluated the appropriate assurance and annuity factors, the equivalence principle may then be used to calculate the required premiums and reserves.

Premiums and reserves are covered in a later chapter.

The notation in this chapter for age x can be taken to mean ultimate mortality is being assumed. The algebra and definitions are identical if we assume select mortality, in which case x will be replaced with $[x]$.

2 Payments varying at a constant compound rate

Consider first a whole life assurance issued to a life aged x where the benefit, payable at the end of the year of death, is $(1+b)^k$ if death occurs in the year of age $(x+k, x+k+1)$, $k = 0,1,\ldots$.

The (random) present value of these benefits is:

$$(1+b)^{K_x} v^{K_x+1}$$

This is because K_x is the (integer) policy duration at the start of the year in which death occurs.

Then the EPV of these benefits is:

$$\sum_{k=0}^{\infty} (1+b)^k v^{k+1} {}_{k|}q_x = \frac{1}{1+b} A_x^j$$

where the assurance function is determined on the normal mortality basis but using an interest rate, j, where:

$$j = \frac{(1+i)}{(1+b)} - 1$$

A similar approach may be derived for other types of assurance.

Where b is negative this approach may be used to allow for compound-decreasing benefits.

Question

Calculate the expected present value of a whole life assurance taken out by a life aged 50, where:

- the basic sum assured is £100,000
- the sum assured increases by 1.9231% at the start of each year excluding the first
- the benefits are payable at the end of the year of death.

Assume AM92 Ultimate mortality and 6% *pa* interest.

Solution

The expected present value of the benefit is:

$$100{,}000\left(vq_{50} + 1.019231v^2\,{}_{1|}q_{50} + 1.019231^2v^3\,{}_{2|}q_{50} + \cdots\right)$$

$$= \frac{100{,}000}{1.019231}\left(\frac{1.019231}{1.06}q_{50} + \frac{1.019231^2}{1.06^2}\,{}_{1|}q_{50} + \frac{1.019231^3}{1.06^3}\,{}_{2|}q_{50} + \cdots\right)$$

$$= \frac{100{,}000}{1.019231}A_{50} \text{ @4\%}$$

$$= \frac{100{,}000}{1.019231} \times 0.32907$$

$$= £32{,}286$$

You will often see increases of 1.9231% and $i = 6\%$ in questions because it means we end up evaluating the benefit at 4% interest. However, do take care with questions like these, as you will often have to pull a factor out of the EPV to obtain a standard assurance function. (In the example above, we took out 1.019231^{-1}.)

To consider the evaluation of compound-varying survival benefits, consider, for example, an immediate annuity payable annually in arrears, with the benefit payable on survival to age $x+k$ being $(1+c)^k$, $k = 1,2,\ldots$.

The (random) present value of the annuity is:

$$\sum_{k=1}^{K_x}(1+c)^k v^k$$

This is because the life survives to every (integer) policy duration up to and including duration K_x.

Then the EPV of the annuity is:

$$\sum_{k=1}^{\infty}(1+c)^k v^k\,{}_kp_x = a_x^j$$

where the annuity function a_x^j is determined on the normal mortality basis but using an interest rate j, where $j = \dfrac{1+i}{1+c} - 1$.

Where c is negative this approach may be used to allow for compound-decreasing annuities.

3 Payments varying by a constant monetary amount

3.1 Whole life assurance

Consider, for example, a whole life assurance issued to a life aged x where the benefit, payable at the end of the year of death, is $k+1$ if death occurs in the year of age $(x+k, x+k+1)$, $k=0,1,\ldots$.

The random present value is:

$$(K_x+1)v^{K_x+1}$$

The EPV of this assurance benefit is then:

$$\sum_{k=0}^{\infty}(k+1)v^{k+1}\ {}_{k|}q_x$$

which is given the actuarial symbol $(IA)_x$.

Values of this function are tabulated in the AM92 examination *Tables*.

Where tabulations are available, this provides the quickest way of evaluating such functions. The values are also easy to calculate using a computer.

It is not logical to define a whole life assurance with constant decreases.

If we did, the benefits would eventually become negative if the person lived long enough.

3.2 Term assurance

An increasing temporary assurance, with term n years can now be evaluated using the formula:

$$(IA)^1_{x:\overline{n}|} = (IA)_x - v^n \frac{l_{x+n}}{l_x}\left[(IA)_{x+n} + n\,A_{x+n}\right]$$

$(IA)^1_{x:\overline{n}|}$ is the EPV of payments of $1, 2, \ldots, n$ occurring on death in years $1, 2, \ldots, n$. In this formula, $(IA)_x$ calculates the EPV of death benefits of $1, 2, \ldots, n, n+1, n+2, \ldots$ paid in years $1, 2, \ldots, n, n+1, n+2, \ldots$. So we need to deduct the value of death benefits of $n+1, n+2, \ldots$ paid in years $n+1, n+2, \ldots$.

By deducting $v^n \dfrac{l_{x+n}}{l_x}(IA)_{x+n}$ we deduct the EPV of death benefits of $1, 2, \ldots.$ paid in years $n+1, n+2, \ldots$. The EPV of the remaining death benefits of $n, n, \ldots$ paid in years $n+1, n+2, \ldots$ are deducted using $v^n \dfrac{l_{x+n}}{l_x} n\,A_{x+n}$.

Question

A man aged exactly 40 buys a special 25-year endowment assurance policy that pays £30,000 on maturity. The man pays a premium of £670 at the start of each year throughout the 25 years, or until death if that happens first. Should that happen, all premiums paid so far are returned without interest at the end of the year of death.

Calculate the expected present value of the benefits payable under this policy assuming AM92 Select mortality and 4% *pa* interest.

Solution

The expected present value of the maturity benefit is:

$$\text{EPV} = 30{,}000 \times \frac{D_{65}}{D_{[40]}} = 30{,}000 \times \frac{689.23}{2{,}052.54} = 10{,}073.81$$

The expected present value of the death benefit is:

$$\text{EPV} = 670(IA)^{\;1}_{[40]:\overline{25|}} = 670 \times \left[(IA)_{[40]} - \frac{D_{65}}{D_{[40]}} \left[(IA)_{65} + 25A_{65} \right] \right]$$

$$= 670 \times \left[7.95835 - \frac{689.23}{2{,}052.54} \left(7.89442 + 25 \times 0.52786 \right) \right]$$

$$= 587.02$$

So the total EPV is:

$$10{,}073.81 + 587.02 = 10{,}660.83$$

3.3 Endowment assurance

An increasing endowment assurance, with term *n* years, can be defined.

An example is $(IA)_{x:\overline{n|}}$, which is the EPV of a payment of k paid at the end of the year of death of (x) if the life dies in policy year k $(k = 1, 2, ..., n)$, or a payment of n if (x) is still alive at the end of the term. The EPV of this can be evaluated using:

$$\boldsymbol{(IA)_{x:\overline{n|}} = (IA)^{\;1}_{x:\overline{n|}} + n\,A_{x:\overline{n|}}^{\;\;\;\;1}}$$

3.4 Decreasing term assurance

A decreasing temporary assurance with a term of n years can also be defined. For example, suppose the benefit is n in the first year, and decreases by 1 per subsequent year. Then the EPV can be evaluated using the formula:

$$\boldsymbol{(n+1)A^{\;1}_{x:\overline{n|}} - (IA)^{\;1}_{x:\overline{n|}}}$$

3.5 Increasing assurances payable immediately on death

Increasing assurances payable immediately on death can also be defined. For example, $(I\bar{A})_x$ is the expected present value of a payment of $k+1$ paid immediately on death occurring in the year of age $(x+k, x+k+1), k=0,1,\dots$. It can be calculated using the usual approximations, for example:

$$(I\bar{A})_x \approx (1+i)^{\frac{1}{2}} (IA)_x$$

The formulae in Sections 3.2, 3.3 and 3.4 can be adjusted in a similar way to allow for immediate payment of the benefit on death.

Question

Calculate the value of $(I\bar{A})_{50:\overline{10}|}$ assuming AM92 Ultimate mortality and 4% *pa* interest.

Solution

We want:

$$(I\bar{A})_{50:\overline{10}|} = (I\bar{A})^{\;1}_{50:\overline{10}|} + 10A_{50:\overline{10}|}^{\;\;\;\;1}$$

This is the EPV of:

- k paid immediately on death if the life dies in policy year k $(k \le 10)$
- 10 on surviving to the end of 10 years.

The acceleration of the payment only applies to the death benefit, which is why we need to split the payment into the death benefit and survival benefit components.

So:

$$\begin{aligned}
(I\bar{A})_{50:\overline{10}|} &\approx (1+i)^{\frac{1}{2}} (I\bar{A})^{\;1}_{50:\overline{10}|} + 10A_{50:\overline{10}|}^{\;\;\;\;1} \\
&= (1+i)^{\frac{1}{2}} \left\{ (IA)_{50} - \frac{D_{60}}{D_{50}} \left[(IA)_{60} + 10A_{60} \right] \right\} + 10\frac{D_{60}}{D_{50}} \\
&= 1.04^{\frac{1}{2}} \times \left\{ 8.55929 - \frac{882.85}{1{,}366.61} \times \left[8.36234 + 10 \times 0.45640 \right] \right\} + 10 \times \frac{882.85}{1{,}366.61} \\
&= 6.673
\end{aligned}$$

3.6 Whole life annuity payable annually in arrears

In the case of an annuity that increases by a constant amount each year consider, for example, an immediate annuity payable annually in arrears, with the benefit payable on survival to age $x+k$ being k, $k=1,2,\dots$.

The (random) present value is:

$$\sum_{k=1}^{K_x} k v^k$$

The EPV of this annuity benefit is:

$$\sum_{k=1}^{\infty} k\, v^k\, {}_k p_x$$

which is given the actuarial symbol $(Ia)_x$.

3.7 Whole life annuity payable annually in advance

Similarly we can define the actuarial symbol:

$$(I\ddot{a})_x$$

to represent the EPV of an annuity-due with the first payment being 1 and subsequent payments increasing by 1 per annum.

Values of this function are tabulated in AM92 in the 'Formulae and Tables for Examinations', although again they can be easily calculated using a computer.

It is not logical to define an immediate annuity with constant decreases.

Question

Calculate the value of $(Ia)_{50}$ assuming AM92 mortality and 4% *pa* interest.

Solution

We want:

$$(Ia)_{50} = vp_{50} + 2v^2\,{}_2p_{50} + 3v^3\,{}_3p_{50} + \ldots$$

But we have:

$$(I\ddot{a})_{50} = 1 + 2vp_{50} + 3v^2\,{}_2p_{50} + \ldots$$

So:

$$\begin{aligned}(I\ddot{a})_{50} - (Ia)_{50} &= 1 + vp_{50} + v^2\,{}_2p_{50} + \ldots \\ &= \ddot{a}_{50}\end{aligned}$$

Therefore:

$$(Ia)_{50} = (I\ddot{a})_{50} - \ddot{a}_{50} = 231.007 - 17.444 = 213.563$$

3.8 Temporary annuities

Increasing temporary annuities can now be evaluated. For example, an increasing temporary annuity-due has an EPV given by:

$$(I\ddot{a})_{x:\overline{n}|} = (I\ddot{a})_x - v^n \frac{l_{x+n}}{l_x}\left[(I\ddot{a})_{x+n} + n\ddot{a}_{x+n}\right]$$

A decreasing temporary annuity with a term of n years can also be defined. For example, suppose we have an annuity-due with a payment of n in the first year, and decreasing by 1 per subsequent year. Then the EPV can be evaluated using the formula:

$$(n+1)\ddot{a}_{x:\overline{n}|} - (I\ddot{a})_{x:\overline{n}|}$$

3.9 Annuities payable continuously

Increasing annuities payable continuously can also be defined. For example, $(I\bar{a})_x$ is the EPV of an immediate annuity payable continuously, with the (level) benefit payable over the year of age $(x+k,\ x+k+1)$ being $1+k,\ k=0, 1, \ldots$. The approximate calculation of this is:

$$(I\bar{a})_x \approx (I\ddot{a})_x - \tfrac{1}{2}\ddot{a}_x$$

The formulae in Section 3.8 can be adjusted in a similar way to allow for continuous annuity payments.

Question

Obtain an approximate expression for $(I\bar{a})_{x:\overline{n}|}$ in terms of discrete ($\ddot{a}$) annuity functions.

Solution

First we express the annuity as the difference between whole life annuities:

$$(I\bar{a})_{x:\overline{n}|} = (I\bar{a})_x - v^n \frac{l_{x+n}}{l_x}\left[(I\bar{a})_{x+n} + n\bar{a}_{x+n}\right]$$

Next we put in the relevant approximations for the continuous annuities:

$$(I\bar{a})_{x:\overline{n}|} \approx (I\ddot{a})_x - \tfrac{1}{2}\ddot{a}_x - v^n \frac{l_{x+n}}{l_x}\left[(I\ddot{a})_{x+n} - \tfrac{1}{2}\ddot{a}_{x+n} + n\left(\ddot{a}_{x+n} - \tfrac{1}{2}\right)\right]$$

Bringing together related terms:

$$(I\bar{a})_{x:\overline{n}|} \approx (I\ddot{a})_x - v^n \frac{l_{x+n}}{l_x}\left[(I\ddot{a})_{x+n} + n\ddot{a}_{x+n}\right] - \tfrac{1}{2}\left[\ddot{a}_x - v^n \frac{l_{x+n}}{l_x}\ddot{a}_{x+n}\right] + \tfrac{1}{2}nv^n \frac{l_{x+n}}{l_x}$$

$$= (I\ddot{a})_{x:\overline{n}|} - \tfrac{1}{2}\ddot{a}_{x:\overline{n}|} + \tfrac{1}{2}nv^n \frac{l_{x+n}}{l_x}$$

Question

Calculate the value of $(I\bar{a})_x$ assuming that (x) is subject to a constant force of mortality of 0.02 *pa* and that $\delta = 0.04$ *pa*.

Solution

$(I\bar{a})_x$ is the expected present value of a whole life annuity payable continuously to a life now aged exactly x, where the rate of payment is 1 *pa* in the first year, 2 *pa* in the second year, 3 *pa* in the third year, and so on. So:

$$(I\bar{a})_x = \int_0^1 v^t \; {}_tp_x \, dt + \int_1^2 2v^t \; {}_tp_x \, dt + \int_2^3 3v^t \; {}_tp_x \, dt + \cdots$$

Assuming a constant force of mortality of 0.02 *pa*, we have ${}_tp_x = e^{-0.02t}$.

Also since $v^t = e^{-\delta t} = e^{-0.04t}$, it follows that:

$$\begin{aligned}
(I\bar{a})_x &= \int_0^1 e^{-0.06t}\,dt + \int_1^2 2e^{-0.06t}\,dt + \int_2^3 3e^{-0.06t}\,dt + \cdots \\
&= \left[-\frac{1}{0.06}e^{-0.06t}\right]_0^1 + \left[-\frac{2}{0.06}e^{-0.06t}\right]_1^2 + \left[-\frac{3}{0.06}e^{-0.06t}\right]_2^3 + \cdots \\
&= \frac{1}{0.06}\left[1-e^{-0.06}\right] + \frac{2}{0.06}\left[e^{-0.06} - e^{-0.06\times 2}\right] \\
&\qquad + \frac{3}{0.06}\left[e^{-0.06\times 2} - e^{-0.06\times 3}\right] + \cdots \\
&= \frac{1}{0.06}\left[1 - e^{-0.06} + 2e^{-0.06} - 2e^{-0.06\times 2} + 3e^{-0.06\times 2} - 3e^{-0.06\times 3} + \cdots\right] \\
&= \frac{1}{0.06}\left[1 + e^{-0.06} + e^{-0.06\times 2} + e^{-0.06\times 3} + \cdots\right]
\end{aligned}$$

The expression in brackets in the line above is the sum to infinity of the terms in a geometric progression with first term 1 and common ratio $e^{-0.06}$. So we have:

$$(I\bar{a})_x = \frac{1}{0.06}\left[\frac{1}{1-e^{-0.06}}\right] = 286.19$$

Assurances and annuities that increase (or decrease) continuously at constant monetary rates are not covered by the CM1 syllabus.

4 Conventional with-profits contracts

In this section we see how conventional with-profits business operates. These ideas will be extended in the life insurance course, SP2.

A conventional whole life or endowment policy can be issued on a without-profit or a with-profits basis. On a without-profit basis, both the premiums and benefits under the policy are usually fixed and guaranteed at the date of issue.

When the Core Reading says that the benefits are fixed, it does not mean they are constant. The benefit amounts may be different from one year to the next, but the amount of the benefit in each year is known from the outset. All the contracts with increasing or decreasing benefits described in Sections 2 and 3 , for example, are examples of conventional without-profits contracts.

On a with-profits basis the premiums and/or the benefits can be varied to give an additional benefit to the policyholder in respect of any emerging surplus of assets over liabilities following a valuation. For example, surplus might be used to reduce the premium payable for the same benefit or to increase the sum assured without any additional premium becoming payable.

An alternative way of distributing surplus is to make a cash payment to policyholders.

Where surplus is distributed so as to increase benefits, additions to the sum assured are called bonuses.

At first glance it might seem undesirable from the perspective of both the policyholder and the life insurance company to have contracts whose final benefit is very uncertain, so this begs the question as to why with-profits business has arisen. To understand this, consider a life insurance company deciding on the premium basis for an endowment assurance contract. (The 'premium basis' is just the set of assumptions – such as interest and mortality – that is used to calculate the premium charged.) The future interest rate assumed will be a critical parameter. If the company calculates premiums assuming an interest rate of, say, 2% over the next twenty years, but actual interest rates fall below that level, then the company will almost certainly make a loss on those policies. On the other hand, if interest rates over the term are significantly in excess of 2%, then the policyholders will feel hard done by in comparison with the benefits that they might have received from other mediums such as bank savings.

So it makes sense for both the company and the policyholder to assume a low rate of interest in determining the premiums that are required to meet the initial sum assured, and then to distribute bonuses to policyholders when investment returns exceed those assumed in the premium basis.

So far we have discussed only interest. However, the same principles also apply to mortality and expenses; under with-profits business, the company can make slightly pessimistic assumptions about mortality and future expenses when setting premiums, and then pay bonuses to policyholders when mortality and expense experience prove better than that assumed in the premium basis.

The major types of contract for which a with-profits treatment is suitable are:

- endowment assurance
- whole life assurance
- deferred annuity
- immediate annuity.

With-profits contracts can be regular or single premium.

One type of with-profits contract is an endowment assurance with bonuses added to the sum assured. So, suppose someone takes out a policy with a sum assured of £10,000. At the end of the first year, the life insurance company declares a bonus of 4% of the sum assured. The sum assured is now increased to £10,400. If there were to be no more bonus declarations, the sum assured would remain at £10,400 until the claim was eventually paid out, either on maturity or on earlier death. On the other hand, there may be other bonuses added in future years, and these would increase further the amount that would be eventually paid.

When the company receives a premium from a policyholder with a with-profits contract, that premium (or part of it) will be invested. Over the course of the year the company expects experience to be slightly better than assumed, because its assumptions – for instance, the investment return it assumed when calculating the premiums – were prudent.

Thus we would normally expect at least one of the following events to have occurred:

- investment returns on assets are greater than assumed
- number of death claims is lower than assumed
- the amount of expenses is lower than assumed.

So by the end of the year, the assets in respect of the contract will usually have grown by more than originally assumed. This is what we mean by the creation of surplus.

Once the company has this surplus, it will want to distribute it to the policyholder as a bonus. If the company wants to distribute all of the surplus there and then to the policyholder, it will calculate an increase to the sum assured such that its EPV is exactly equal to the amount of surplus that has been created.

Question

A life company has calculated that the amount of surplus attributable to a with-profits single premium whole life contract issued nine years ago to a 44-year-old male with guaranteed sum assured of \$90,000 is \$2,800. Write down an expression from which the bonus amount of sum assured could be calculated from this surplus.

Solution

The bonus amount of sum assured S satisfies the equation:

$$S\,A_{53} = 2{,}800$$

As we shall see below, the bonus will normally be presented as a percentage increase to the sum assured.

In practice, companies will determine the surplus generated and bonus to be distributed using very broad groupings (*eg* all endowment assurances of term 10 years issued 4 years ago) rather than on a policy-by-policy basis.

Question

Bonuses are usually funded by distributing part of the surplus. Explain why a life insurance company might distribute just part of the surplus rather than all of it.

Solution

If the company is proprietary (*ie* owned by shareholders), then it will want to make a profit on its business. A common procedure both in the UK and in many overseas markets is to give 90% of surplus to policyholders, 10% to shareholders.

In addition, the company may deliberately want to under-distribute surplus now in order:

- to defer eventual surplus distribution (we consider the reasons for this below), and
- to allow smoothing of payouts (the company deliberately pays out less than the surplus generated when investment markets are booming but deliberately pays out more when investment markets are declining). The smoother the bonus payments, the 'safer' the policyholders feel. Ultimately, this can drive up the life insurance company's share price.

4.1 Types of bonus

Various methods of allocating bonuses have been developed, each intended to provide a way of matching the surplus emerging over the duration of the policy. In the past the choice of methods was restricted by the difficulties of completing a valuation quickly and cheaply and by the difficulties of allocating complex bonuses to individual policies. Modern record keeping systems have largely removed these difficulties, and current systems are chosen to match the bonus distribution philosophy conveyed to the policyholders when the policy was issued (in the UK this is called policyholders' reasonable expectations).

By bonus distribution philosophy, we mean matters such as:

- what form bonuses take (of the various forms described below)
- what portion of surplus the company distributes to policyholders
- what degree of smoothing the company operates (*eg* if investment returns in one year are very good, we might not distribute all of the resultant surplus immediately, but instead hold some back to compensate poorer investment returns in future years) and
- to some extent, the broad investment strategy of the company (*eg* the company may aim to pay bonuses that broadly reflect equity market performance).

Clearly what the company says to potential policyholders at the policy sale stage about these aspects will create justifiable expectations, which the company should then try to meet over the lifetime of the policy.

Bonuses are usually allocated annually, which is likely to tie in with the minimum required frequency of valuation for each insurer.

These bonuses are often referred to as *reversionary* bonuses.

When the company declares a reversionary bonus of, say, 3% of sum assured, there is no immediate cash payment to the policyholder. The company is merely promising that the amount of money paid on claim or maturity will be 3% greater than was previously the case. So, cashflows are unchanged until the moment when the policies in question terminate, *ie* when the claims are paid out.

Once added to the sum assured, bonuses become guaranteed benefits, which then need to be reserved for.

Insurance companies are required to hold reserves for each policy sold to ensure that they can meet claim payments when necessary. We will study reserves in more detail later in the course, but for the time being, it is sufficient to know that the higher the benefit promised to the policyholder, the higher the reserve held by the insurance company needs to be.

The implication of this is that the sooner, and greater, the rate at which bonuses are added, the more conservative the insurer is likely to be in choosing its investments.

The last sentence of the above Core Reading is important for insurance companies in practice, as conservative investments generally lead to lower returns in the long run. This will in turn translate into lower payouts for policyholders.

Bonuses can be distributed more slowly, or at a lower rate, which may allow the insurer to choose investments that are more volatile in the short term, but are expected to be more profitable in the long term. A highly effective example of this is choosing to distribute part of the available surplus as a terminal bonus, rather than as an annual bonus. Terminal bonuses are allocated when a policy matures or becomes a claim as a result of the death of the life assured.

This is an example of the insurance company *deferring* the distribution of its surplus to its policyholders, and as the Core Reading states, having a terminal bonus is generally the most effective way of doing this.

Question

Explain why, everything else being equal, a higher amount of profit deferral should lead to a higher expected maturity payment on a with-profits endowment policy.

Solution

A high level of profit deferral (*eg* as a result of paying lower annual bonuses and a higher terminal bonus) will mean that the insurance company has more freedom to invest in volatile investments that have a higher expected return. Should these higher investment returns occur, then this should ultimately translate into higher total benefit payments (which includes the guaranteed sum assured, the annually added reversionary bonuses, and the terminal bonus).

Terminal bonuses are usually allocated as a percentage of the basic sum assured and the bonuses allocated prior to termination. The percentage rate will vary with the term of the policy at the date of payment. Because the policy is being terminated, the terminal bonus rate is usually chosen so as to distribute all the surplus available to the policy, based on asset share.

The asset share of a policy at maturity is essentially the amount of money that the company has accumulated from the policy, net of that policy's share of the company's expenses and of the cost of death benefits that had been covered over the policy term. If the sum assured plus all the declared annual bonuses at the time of maturity is less than this asset share, then the excess can be distributed as terminal bonus.

Typically, bonuses are added by a mixture of annual and terminal components. The annual bonuses will be at variable rates determined from time to time by the insurer based on actual arising surpluses. These bonuses are typically added according to one of the following methods:

- **Simple – the rate of bonus each year is a percentage of the initial (basic) sum assured under the policy. The sum assured will increase linearly over the term of the policy.**

The increase will be linear provided the same (simple) bonus rate is declared each year.

- **Compound – the rate of bonus each year is a percentage of the basic sum assured and the bonuses added in the past. The sum assured increases exponentially over the term of the policy.**

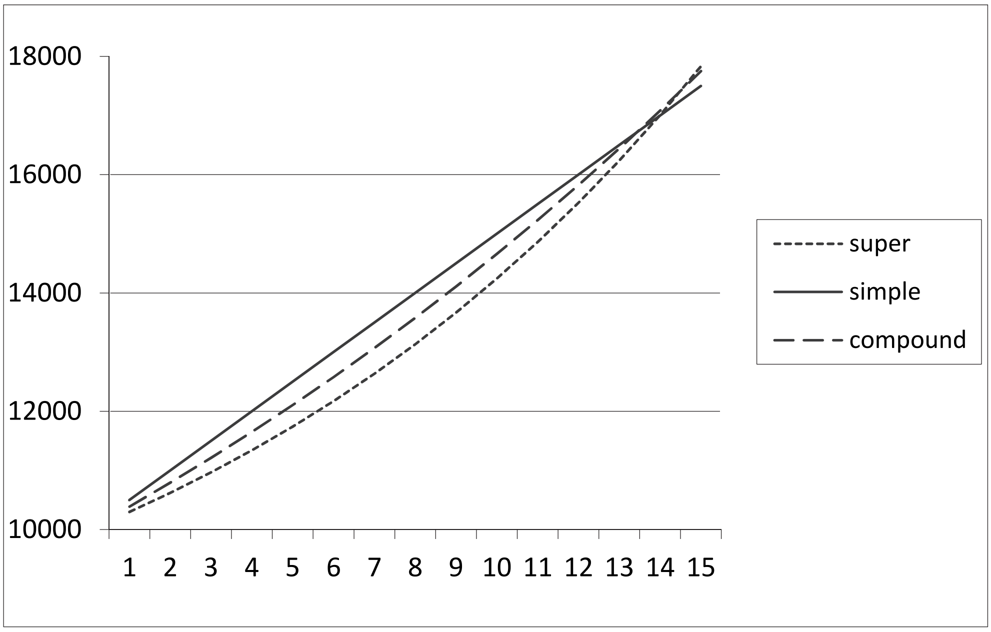
The exponential increase would occur if the same (compound) bonus rate is declared each year.

- **Super compound – two compound bonus rates are declared each year. The first rate (usually the lower) is applied to the basic sum assured. The second rate is applied to the bonuses added to the policy in the past. The sum assured increases exponentially over the term of the policy. The sum assured including bonuses increases more slowly than under a compound allocation in the earlier years, but faster in the later years.**

This pattern of increase would occur if the lower bonus rate was applied to the sum assured.

The graph below shows the build up of the sum assured over the lifetime of a 15-year policy with the following alternative methods, assuming an initial sum assured of £10,000:

- simple bonus at 5% *pa*
- compound bonus at 3.9% *pa*
- super compound bonus at 3% *pa* on basic sum assured, 7.5% *pa* on bonuses.



We see from the graph that:

- the increase in sum assured under a simple bonus arrangement is linear
- under a compound bonus arrangement, the sum assured increases slowly at first and more quickly later in the policy
- under a super compound arrangement, this feature becomes more pronounced, with lower increases earlier on being compensated for by greater increases later on in the policy.

The rates have been chosen so that the final value of the sum assured is approximately the same at the end of 15 years under each of the three different arrangements.

Question

For the three alternative bonus allocation methods specified above, and given an initial guaranteed sum assured of £10,000 in each case, calculate the sum assured (including bonuses) as at the end of Years 1, 2 and 3.

Solution

Simple:	Year 1:	$10,000 \times 1.05 = 10,500$
	Year 2:	$10,000 \times 1.10 = 11,000$
	Year 3:	$10,000 \times 1.15 = 11,500$
Compound:	Year 1:	$10,000 \times 1.039 = 10,390$
	Year 2:	$10,390 \times 1.039 = 10,795$
	Year 3:	$10,795 \times 1.039 = 11,216$
Super-compound:	Year 1:	$10,000 \times 1.03 = 10,300$
	Year 2:	$10,000 \times 1.03 + (10,300 - 10,000) \times 1.075 = 10,623$
	Year 3:	$10,000 \times 1.03 + (10,623 - 10,000) \times 1.075 = 10,969$

Annual bonuses are an allocation in arrears to reflect the growth in the available surplus since the last valuation. Where annual bonuses are given, it is usual at each valuation to declare a rate of interim bonus which will be applied to policies becoming claims before the next valuation. This provides an allocation of bonus for the period from the last valuation to the date of the claim. This rate is applied in the same way as that used for annual bonuses.

An anticipated bonus rate is usually loaded in premium rates by choosing (conservative) rates of bonus allocation and valuing these as benefits in determining the premium to be charged for the policy. The additional premium (above that that would be charged for a without-profit contract with the same (basic) sum assured) is termed the bonus loading.

We saw earlier how with-profits business arose because it allowed companies to price products using conservative interest rate assumptions, which did not hurt the policyholder because as experience proved better than these assumptions, the policyholders would see their benefit increased accordingly. It is now usual to go a step further than this. Companies want to be reasonably sure of being able to pay significant bonuses, which are of similar size to the bonuses which competitors will be paying.

So rather than choose a prudent interest rate (which in reality we expect to exceed by, say 1 to 2% *pa* over the policy's lifetime, giving an equivalent amount of bonus) companies will load explicitly for some specific level of future bonus.

For example, suppose a life insurance company expects an investment return of 4.5% *pa* on the assets backing its with-profits business over the foreseeable future. The company wants to price its contracts to ensure that it can support reversionary bonuses of 2% *pa* (added at the start of each year, using the simple distribution system). For an endowment assurance issued to a life aged x, with a term of n years and a sum assured S payable at the end of the year of death, the premiums will need to meet a benefit with expected present value:

$$S\,A_{x:\overline{n}|} + 0.02\,S\,(IA)_{x:\overline{n}|} \quad \text{when } i = 4.5\%$$

Question

Give an expression for the expected present value of the benefits provided by a with-profits endowment assurance issued to a life aged x with a sum assured of S payable in n years' time or at the end of the year of earlier death. Assume that compound reversionary bonuses of 2% *pa* are declared at the start of each year and interest is 4.5% *pa* effective.

Solution

The EPV of the benefits is:

$$EPV = 1.02 S v q_x + 1.02^2 S v^2 \ {}_{1|}q_x + \cdots + 1.02^n S v^n \ {}_{n-1|}q_x + 1.02^n S v^n \ {}_n p_x$$

$$= S\left[\left(\frac{1.02}{1.045}\right) q_x + \left(\frac{1.02}{1.045}\right)^2 {}_{1|}q_x + \cdots + \left(\frac{1.02}{1.045}\right)^n {}_{n-1|}q_x + \left(\frac{1.02}{1.045}\right)^n {}_n p_x\right]$$

$$= S\left[v' q_x + (v')^2 \ {}_{1|}q_x + \cdots + (v')^n \ {}_{n-1|}q_x + (v')^n \ {}_n p_x\right]$$

where $v' = \dfrac{1.02}{1.045} = 1.0245^{-1}$

So:

$$EPV = S A_{x:\overline{n}|}^{@i'}$$

where $i' = 2.45\%$ *pa*.

In each of these examples, any excess of investment return actually achieved above the 4.5% *pa* assumed would either be paid out in the form of higher annual bonuses (*ie* greater than 2% *pa*), or as a terminal bonus when the contract terminates.

If the actual return achieved is less than the assumed 4.5% *pa*, then the company could declare annual bonuses averaging less than 2% *pa*, and so avoid making losses on the business.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes

Chapter 19 Summary

Variable benefits

In this chapter we have developed ways of calculating the expected present value of benefits that vary from one year to another according to some predetermined pattern.

Payments varying at a constant compound rate

Assurances

The expected present value of a whole life assurance issued to a life aged x where the benefit, payable at the end of the year of death, is $(1+b)^k$ if death occurs in the year of age $(x+k, x+k+1)$, $k=0,1,...$, is:

$$\sum_{k=0}^{\infty}(1+b)^k v^{k+1}{}_{k|}q_x = \frac{1}{1+b}A_x^j$$

where A_x^j is a whole life assurance function determined using the normal mortality basis, but using an interest rate j such that $j=\dfrac{1+i}{1+b}-1$.

A similar approach can be used for other types of assurance.

Annuities

The expected present value of a whole life annuity payable annually in arrears to a life aged x, with the benefit payable on survival to age $x+k$ being $(1+c)^k$, $k=1,2,...$, is:

$$\sum_{k=1}^{\infty}(1+c)^k v^k{}_k p_x = a_x^j$$

where the annuity function a_x^j is evaluated using the normal mortality basis, but using an interest rate j, such that $j=\dfrac{1+i}{1+c}-1$.

Payments changing by a constant monetary amount

Whole life assurance

The expected present value of a whole life assurance issued to a life aged x where the benefit, payable at the end of the year of death, is $k+1$ if death occurs in the year of age $\left(x+k, x+k+1\right)$, $k=0,1,\ldots$, is:

$$\sum_{k=0}^{\infty}\left(k+1\right)v^{k+1}\,{}_{k|}q_x=(IA)_x$$

Term assurance

An increasing term assurance can be evaluated using the formula:

$$(IA)^1_{x:\overline{n}|}=(IA)_x-v^n\frac{l_{x+n}}{l_x}\left[(IA)_{x+n}+nA_{x+n}\right]$$

The expected present value of a decreasing term assurance with a term of n years, where the benefit is n in the first year and decreases by 1 *pa*, can be evaluated using the formula:

$$(n+1)A^1_{x:\overline{n}|}-(IA)^1_{x:\overline{n}|}$$

Endowment assurance

An increasing endowment assurance can be evaluated using the formula:

$$(IA)_{x:\overline{n}|}=(IA)^1_{x:\overline{n}|}+nA_{x:\overline{n}|}^{\;\;1}=(IA)^1_{x:\overline{n}|}+n\frac{D_{x+n}}{D_x}$$

Assurances with death benefits payable immediately

The expected present value for an increasing whole life assurance, which pays immediately on death, is:

$$\left(I\bar{A}\right)_x\approx(1+i)^{1/2}\left(IA\right)_x$$

Similar formulae can be used for the other assurance types.

Whole life annuity

The expected present value of a whole life annuity payable annually in arrears to a life aged x, with the benefit payable on survival to age $x+k$ being k, $k=1,2,\ldots$, is:

$$\sum_{k=1}^{\infty} k\,v^k\,{}_kp_x = (Ia)_x$$

The symbol $(I\ddot{a})_x$ denotes the expected present value of the corresponding annuity-due, where the first payment is 1 and payments increase at the rate of 1 *pa*, and:

$$(I\ddot{a})_x = (Ia)_x + \ddot{a}_x$$

Temporary annuity

An increasing temporary annuity-due can be evaluated using the formula:

$$(I\ddot{a})_{x:\overline{n}|} = (I\ddot{a})_x - v^n \frac{l_{x+n}}{l_x}\left[(I\ddot{a})_{x+n} + n\ddot{a}_{x+n}\right]$$

The expected present value of a decreasing temporary annuity-due with a term of n years, where the benefit is n in the first year and decreases by 1 *pa*, can be evaluated using the formula:

$$(n+1)\ddot{a}_{x:\overline{n}|} - (I\ddot{a})_{x:\overline{n}|}$$

Annuities payable continuously

The expected present value of a whole life annuity payable continuously to a life aged x, which pays at the rate of k in year k, $k=1,2,\ldots$, is:

$$(I\bar{a})_x \approx (I\ddot{a})_x - \tfrac{1}{2}\ddot{a}_x$$

For continuously payable temporary annuities, a good approach is to start from:

$$(I\bar{a})_{x:\overline{n}|} = (I\bar{a})_x - v^n \frac{l_{x+n}}{l_x}\left[(I\bar{a})_{x+n} + n\bar{a}_{x+n}\right]$$

and then replace each of the annuity factors in the formula by their discrete approximations.

Conventional with-profits contracts

Conventional with-profits contracts are priced on a prudent basis. If experience proves better than the prudent assumptions, surplus arises. This surplus is distributed to policyholders as an increase to the benefit.

This increase can be applied as an annual reversionary bonus during the lifetime of the policy, using one of the following methods:

- simple
- compound
- super-compound

and also at claim or maturity as a terminal bonus.

The more a company defers the distribution of surplus as bonus, the more potential the company has to invest in more volatile assets with higher expected investment returns, and thus obtain greater eventual payouts to policyholders. Terminal bonus can be viewed as an extreme example of such deferment.

Reversionary bonuses are often allowed for explicitly when determining the premium basis for a contract.

Chapter 19 Practice Questions

19.1 For each of the following expected present values, describe the payments involved and calculate the value assuming AM92 mortality and 6% *pa* interest.

(i) $(I\ddot{a})_{[45]:\overline{20|}}$

(ii) $(I\bar{A})_{49:\overline{6|}}$

(iii) $(Ia)_{[50]:\overline{10|}}$

19.2 Graham, aged 40, purchases a conventional with-profits whole life assurance with sum assured £2,000 plus attaching bonuses, payable at the end of the year of death. Assuming allowance for simple bonuses of 3% *pa*, which are added at the start of each policy year, calculate the expected present value of Graham's policy benefits.

Basis: AM92 Ultimate mortality, 4% *pa* interest.

19.3 A special term assurance policy is to be issued to a life currently aged 52 exact. The policy term is 8 years, and the sum assured is paid at the end of the year of death. The benefit is 120,000 in the first year, increasing by 10,000 at the end of each year, so that if death occurs in the final policy year 190,000 will be paid.

Calculate the EPV of this policy benefit using the following basis:

Mortality: AM92 Ultimate
Interest: 4% *pa*

19.4 The payments under a special deferred annuity are payable continuously from age 60 and increase continuously at the rate of 5% *pa* compound. The payment stream starts at the rate of £200 *pa*. Assuming AM92 Select mortality before age 60 and PFA92C20 mortality after age 60, calculate the expected present value of the annuity for a female life now aged 40, if interest is 5% *pa* effective.

19.5 A life insurance company has issued a decreasing term assurance with a 20-year term to a person aged exactly 40. The sum assured is 100,000 in the first year, decreasing by 1,000 each year, so that the death benefit in the final policy year is 81,000. The benefit is paid at the end of the policy year of death.

Calculate the EPV of this benefit assuming AM92 Select mortality and 4% *pa* interest.

19.6 A 17-year with-profits endowment assurance is issued to a life aged exactly 48, having a basic guaranteed sum assured of 25,000. The sum assured plus all declared reversionary bonuses to date are paid on survival to the end of the term or immediately on earlier death. Calculate the expected present value of this policy benefit assuming:

- future bonuses are declared at the rate of 2.5% *pa* compound, being added in full at the start of each policy year
- AM92 Select mortality
- 6.6% *pa* interest.

19.7 Exam style A woman aged 67 exact takes out an annuity that makes monthly payments in arrears. The first monthly payment is £1,500, and payments increase by 0.23726% each month.

Calculate the expected present value of the annuity using the following basis:

Mortality: PFA92C20
Interest: 7% per annum [4]

19.8 Exam style A whole life assurance policy pays 20,000 on death in Year 1, 20,100 on death in Year 2, and so on increasing by 100 each year. The payment is made immediately on death of a life currently aged 35 exact.

(i) Write down an expression for the present value random variable of this payment, in terms of the curtate future lifetime K_x, and/or the complete future lifetime T_x. [1]

(ii) Calculate the expected present value of these benefits, assuming:

(a) AM92 Select mortality and 6% *pa* interest

(b) a constant force of mortality of 0.015 *pa* and force of interest 0.03 *pa*. [11]
[Total 12]

19.9 Exam style A whole life annuity with continuous payments is due to commence in 15 years' time. It will be payable to a life that is currently aged exactly 50, provided that person is still alive when the annuity is due to start. Payments commence at the rate of 20,000 *pa*, and increase continuously thereafter at a rate of 2% *pa* compound.

Calculate the expected present value of these payments on the following basis:

Mortality: PMA92C20 prior to age 65
A constant force of 0.038 *pa* at ages over 65
Interest: 3% *pa* effective [6]

19.10 Exam style

A life insurance company is considering selling with-profit endowment policies with a term of twenty years and initial sum assured of £100,000. Death benefits are payable at the end of the policy year of death. Bonuses will be added at the end of each policy year.

The company is considering three different bonus structures:

(1) simple reversionary bonuses of 4.5% per annum

(2) compound reversionary bonuses of 3.84615% per annum

(3) super compound bonuses where the original sum assured receives a bonus of 3% each year and all previous bonuses receive an additional bonus of 6% each year.

(i) Calculate the amount payable at maturity under the three structures. [4]

(ii) Calculate the expected present value of benefits under structure (2) for an individual aged 45 exact at the start, using the following basis:

Interest	8% per annum
Mortality	AM92 Select
Expenses	ignore

[4]

(iii) Calculate the expected present value of benefits, using the same policy and basis as in (ii) but reflecting the following changes:

(a) Bonuses are added at the start of each policy year (the death benefit is payable at the end of the policy year of death).

(b) The death benefit is payable immediately on death (bonuses are added at the end of each policy year).

(c) The death benefit is payable immediately on death, and bonuses are added continuously. [3]

[Total 11]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 19 Solutions

19.1 (i) $(I\ddot{a})_{[45]:\overline{20}|}$

This is the EPV of a 20-year increasing temporary annuity-due. Payments are 1 at the start of Year 1, 2 at the start of Year 2, and so on, so that 20 is paid at the start of Year 20. Payments continue for 20 years or until the earlier death of a select life that is currently aged exactly 45.

We have:

$$(I\ddot{a})_{[45]:\overline{20}|} = (I\ddot{a})_{[45]} - v^{20}\frac{l_{65}}{l_{[45]}}\left[(I\ddot{a})_{65} + 20\ddot{a}_{65}\right]$$

$$= 185.197 - 1.06^{-20} \times \frac{8,821.2612}{9,798.0837} \times \left[89.374 + 20 \times 10.569\right]$$

$$= 100.77$$

(ii) $(I\bar{A})_{49:\overline{6}|}$

This is the EPV of a six-year increasing endowment assurance. On death in the first year the payment would be 1, in the second year the payment would be 2, and so on so that on death in Year 6 the payment would be 6. A payment of 6 would alternatively be paid if the person (currently aged 49) survived to the end of the term. The death benefits are payable immediately on death.

We have:

$$(I\bar{A})_{49:\overline{6}|} \approx (1+i)^{½}\left((IA)_{49} - v^{6}\frac{l_{55}}{l_{49}}\left[(IA)_{55} + 6A_{55}\right]\right) + 6v^{6}\frac{l_{55}}{l_{49}}$$

$$= 1.06^{½} \times \left(4.75618 - 1.06^{-6} \times \frac{9,557.8179}{9,733.8865} \times \left[5.22868 + 6 \times 0.26092\right]\right)$$

$$+ 6 \times 1.06^{-6} \times \frac{9,557.8179}{9,733.8865}$$

$$= 4.20800$$

(iii) $(Ia)_{[50]:\overline{10}|}$

This is the EPV of a 10-year increasing temporary annuity. Payments are 1 at the end of Year 1, 2 at the end of Year 2, and so on, so that 10 is paid at the end of Year 10. Payments continue for 10 years or until the earlier death of a select life that is currently aged exactly 50.

We can use the same formula for the annuity in arrears as we did for the annuity-due, provided we remove the 'double-dots' above all the annuity symbols:

$$(Ia)_{[50]:\overline{10}|} = (Ia)_{[50]} - v^{10}\frac{l_{60}}{l_{[50]}}\left[(Ia)_{60} + 10a_{60}\right]$$

Now:

$$(Ia)_x = (I\ddot{a})_x - \ddot{a}_x$$

and

$$a_x = \ddot{a}_x - 1$$

So:

$$(Ia)_{[50]:\overline{10}|} = (I\ddot{a})_{[50]} - \ddot{a}_{[50]} - v^{10}\frac{l_{60}}{l_{[50]}}\left[(I\ddot{a})_{60} - \ddot{a}_{60} + 10\left(\ddot{a}_{60} - 1\right)\right]$$

$$= 162.597 - 14.051 - 1.06^{-10} \times \frac{9,287.2164}{9,706.0977} \times [113.516 - 11.891 + 10 \times 10.891]$$

$$= 36.058$$

19.2 The expected present value is:

$$2,000A_{40} + 0.03 \times 2,000(IA)_{40} = 2,000 \times 0.23056 + 60 \times 7.95699 = £938.54$$

19.3 The EPV is:

$$110,000A^{\ 1}_{52:\overline{8}|} + 10,000(IA)^{\ 1}_{52:\overline{8}|}$$

where:

$$A^{\ 1}_{52:\overline{8}|} = A_{52:\overline{8}|} - \frac{D_{60}}{D_{52}} = 0.73424 - \frac{882.85}{1,256.80} = 0.031781$$

and:

$$(IA)^{\ 1}_{52:\overline{8}|} = (IA)_{52} - \frac{D_{60}}{D_{52}}[(IA)_{60} + 8A_{60}]$$

$$= 8.59412 - \frac{882.85}{1,256.80} \times [8.36234 + 8 \times 0.45640]$$

$$= 0.155105$$

So:

$$EPV = 110,000 \times 0.031781 + 10,000 \times 0.155105 = 5,047$$

19.4 The expected present value of the deferred annuity is:

$$v^{20}\frac{l_{60}}{l_{[40]}}\int_0^\infty 200 \times 1.05^t v^t \ {}_tp_{60}\,dt = 200v^{20}\frac{l_{60}}{l_{[40]}}\int_0^\infty 1.05^t \times 1.05^{-t} \times {}_tp_{60}\,dt = 200v^{20}\frac{l_{60}}{l_{[40]}}\int_0^\infty {}_tp_{60}\,dt$$

Now, recall from Chapter 15:

$$\int_0^\infty {}_tp_{60}\, dt = \mathring{e}_{60}$$

So the EPV is now:

$$200v^{20}\frac{l_{60}}{l_{[40]}}\mathring{e}_{60} = 1.05^{-20}\times\frac{9,287.2164}{9,854.3036}\times 27.41 = £1,947$$

19.5 The expected present value is:

$$EPV = 101,000A^{\;1}_{[40]:\overline{20}|} - 1,000(IA)^{\;1}_{[40]:\overline{20}|}$$

where:

$$A^{\;1}_{[40]:\overline{20}|} = A_{[40]:\overline{20}|} - \frac{D_{60}}{D_{[40]}} = 0.46423 - \frac{882.85}{2,052.54} = 0.034104$$

and:

$$(IA)^{\;1}_{[40]:\overline{20}|} = (IA)_{[40]} - \frac{D_{60}}{D_{[40]}}\left[(IA)_{60} + 20A_{60}\right]$$

$$= 7.95835 - \frac{882.85}{2,052.54}\times\left[8.36234 + 20\times 0.45640\right]$$

$$= 0.435307$$

So:

$$EPV = 101,000\times 0.034104 - 1,000\times 0.435307 = 3,009$$

19.6 We need the expected present values of the survival benefit and of the death benefit.

Survival benefit

The expected present value of the survival benefit is:

$$EPV = 25,000\times 1.025^{17}v^{17}\;{}_{17}p_{[48]} = 25,000\times 1.04^{-17}\;{}_{17}p_{[48]}$$

as $1.025\times v = \frac{1.025}{1.066} = \frac{1}{1.04}$.

So:

$$EPV = 25,000\times\frac{D_{65@4\%}}{D_{[48]@4\%}} = 25,000\times\frac{689.23}{1,483.73} = 11,613.13$$

Death benefit

The expected present value of the death benefit is:

$$EPV = 25{,}000 \times \left(1.025 v^{½}\ {}_{0|}q_{[48]} + 1.025^2 v^{1½}\ {}_{1|}q_{[48]} + \cdots + 1.025^{17} v^{16½}\ {}_{16|}q_{[48]}\right)$$

It would be helpful if the first term in the bracket began with $1.025v$. We can achieve this by multiplying and dividing by $v^{½}$, so:

$$EPV = \frac{25{,}000}{v^{½}} \times \left(1.025v\ {}_{0|}q_{[48]} + (1.025v)^2\ {}_{1|}q_{[48]} + \cdots + (1.025v)^{17}\ {}_{16|}q_{[48]}\right)$$

$$= 25{,}000 \times 1.066^{½} \times \left(1.04^{-1}\ {}_{0|}q_{[48]} + 1.04^{-2}\ {}_{1|}q_{[48]} + \cdots + 1.04^{-17}\ {}_{16|}q_{[48]}\right)$$

$$= 25{,}000 \times 1.066^{½} \times A^{\ 1}_{[48]:\overline{17|}\,@4\%}$$

Now:

$$A^{\ 1}_{[48]:\overline{17|}} = A_{[48]:\overline{17|}} - \frac{D_{65}}{D_{[48]}} = 0.52596 - \frac{689.23}{1{,}483.73} = 0.061435$$

So:

$$EPV = 25{,}000 \times 1.066^{½} \times 0.061435 = 1{,}585.74$$

Total EPV

Adding the expected present values of the death benefit and survival benefit together we obtain:

$$11{,}613.13 + 1{,}585.74 = 13{,}198.87$$

19.7 First define:

$$f = 0.0023726$$

The EPV of the annuity payments is:

$$EPV = 1{,}500\left[v^{1/12}\left({}_{1/12}p_{67}\right) + (1+f)v^{2/12}\left({}_{2/12}p_{67}\right) + (1+f)^2 v^{3/12}\ {}_{3/12}p_{67} + \cdots\right]$$

$$= \frac{1{,}500}{1+f}\left[(1+f)v^{1/12}\left({}_{1/12}p_{67}\right) + (1+f)^2 v^{2/12}\left({}_{2/12}p_{67}\right) + (1+f)^3 v^{3/12}\ {}_{3/12}p_{67} + \cdots\right] \qquad [1]$$

Now:

$$(1+f)v^{1/12} = \frac{1.0023726}{1.07^{1/12}} \Rightarrow (1+f)^{12} v = \frac{1.0023726^{12}}{1.07} = \frac{1.0288457}{1.07} = \frac{1}{1.04} \qquad [½]$$

So:

$$EPV = \frac{1,500}{1.0023726}\left[1.04^{-1/12}\left({}_{1/12}p_{67}\right) + 1.04^{-2/12}\left({}_{2/12}p_{67}\right) + 1.04^{-3/12}\,{}_{3/12}p_{67} + \cdots\right]$$

$$= \frac{12 \times 1,500}{1.0023726} \times a_{67}^{(12)} \ @4\% \qquad [1]$$

From the *Tables*:

$$a_{67}^{(12)} = \ddot{a}_{67}^{(12)} - \frac{1}{12} \approx \ddot{a}_{67} - \frac{11}{24} - \frac{1}{12} = \ddot{a}_{67} - \frac{13}{24} = 14.111 - \frac{13}{24} = 13.5693 \qquad [1]$$

So, the EPV of the benefit is:

$$\frac{£18,000}{1.0023726} \times 13.5693 = £243,670 \qquad [½]$$

[Total 4]

19.8 (i) ***Present value random variable***

This is:

$$(20,000 + 100K_{35})v^{T_{35}} \qquad [1]$$

(ii)(a) ***Expected present value using AM92 Select and 6% pa interest***

The expected present value of the benefits is:

$$19,900\bar{A}_{[35]} + 100(I\bar{A})_{[35]} \qquad [1]$$

where:

$$\bar{A}_{[35]} \approx 1.06^{½} \times A_{[35]} = 1.06^{½} \times 0.09475 = 0.097551 \qquad [1]$$

$$(I\bar{A})_{[35]} \approx 1.06^{½} \times (IA)_{[35]} = 1.06^{½} \times 3.33735 = 3.43601 \qquad [½]$$

So the EPV is:

$$19,900 \times 0.097551 + 100 \times 3.43601 = 2,284.87 \qquad [½]$$

(ii)(b) ***Expected present value using force of mortality 0.015 pa and force of interest 0.03 pa***

The expected present value of the benefits is:

$$19,900\bar{A}_{35} + 100(I\bar{A})_{35}$$

With constant forces of interest and mortality we can use integrals to calculate the values of the functions.

We have:

$$\bar{A}_{35} = \int_{t=0}^{\infty} v^t \, {}_tp_{35}\mu_{35+t}dt$$

$$= \int_{t=0}^{\infty} e^{-0.03t}e^{-0.015t} \times 0.015\,dt$$

$$= 0.015 \int_{t=0}^{\infty} e^{-0.045t}dt$$

$$= 0.015\left[\frac{e^{-0.045t}}{-0.045}\right]_0^{\infty}$$

$$= \frac{0.015}{0.045} = \frac{1}{3} \qquad [2]$$

Now consider $(I\bar{A})_{35}$. This is:

$$(I\bar{A})_{35} = \int_{t=0}^{1} v^t \, {}_tp_{35}\,\mu_{35+t}\,dt + 2\int_{t=1}^{2} v^t \, {}_tp_{35}\,\mu_{35+t}\,dt + 3\int_{t=2}^{3} v^t \, {}_tp_{35}\,\mu_{35+t}\,dt + \cdots$$

$$= 0.015\left(\int_{t=0}^{1} e^{-0.045t}\,dt + 2\int_{t=1}^{2} e^{-0.045t}\,dt + 3\int_{t=2}^{3} e^{-0.045t}\,dt + \cdots\right)$$

$$= 0.015\left(\left[\frac{e^{-0.045t}}{-0.045}\right]_0^1 + 2\left[\frac{e^{-0.045t}}{-0.045}\right]_1^2 + 3\left[\frac{e^{-0.045t}}{-0.045}\right]_2^3 + \cdots\right)$$

$$= \frac{0.015}{0.045}\left(1 - e^{-0.045} + 2\left(e^{-0.045} - e^{-0.045\times 2}\right) + 3\left(e^{-0.045\times 2} - e^{-0.045\times 3}\right) + \cdots\right)$$

$$= \frac{0.015}{0.045}\left(1 + e^{-0.045} + e^{-0.045\times 2} + e^{-0.045\times 3} + \cdots\right) \qquad [4]$$

The amount in the bracket is the sum of a geometric progression with first term 1 and common ratio $e^{-0.045}$. So we have:

$$(I\bar{A})_{35} = \frac{0.015}{0.045} \times \frac{1}{1 - e^{-0.045}} = 7.57532 \qquad [1]$$

and the expected present value is:

$$19{,}900 \times \frac{1}{3} + 100 \times 7.57532 = 7{,}390.87 \qquad [1]$$

[Total 11]

19.9 The expected present value is:

$$v^{15}\times\frac{l_{65}}{l_{50}}\times 20{,}000\times\int_{t=0}^{\infty}1.02^t\,v^t\,{}_tp_{65}\,dt \qquad [2]$$

Now:

$$1.02^t\,v^t=\left(\frac{1.02}{1.03}\right)^t=0.9902913^t=e^{(\ln 0.9902913)t}=e^{-0.0097562t} \qquad [1]$$

So:

$$1.02^t\,v^t\,{}_tp_{65}=e^{-0.0097562t}e^{-0.038t}=e^{-0.0477562t} \qquad [1]$$

Then:

$$EPV=1.03^{-15}\times\frac{9{,}647.797}{9{,}941.923}\times 20{,}000\times\int_{t=0}^{\infty}e^{-0.0477562t}\,dt$$

where:

$$\int_{t=0}^{\infty}e^{-0.0477562t}dt=\left[\frac{e^{-0.0477562t}}{-0.0477562}\right]_0^{\infty}=-\frac{0-e^0}{0.0477562}=\frac{1}{0.0477562} \qquad [1]$$

So:

$$EPV=1.03^{-15}\times\frac{9{,}647.797}{9{,}941.923}\times 20{,}000\times\left(\frac{1}{0.0477562}\right)$$

$$=260{,}855 \qquad [1]$$

[Total 6]

19.10 *This question is Subject CT5, September 2008, Question 12.*

(i)(1) ***Simple bonuses maturity amount***

After 20 years, the maturity value is:

$$100{,}000(1+0.045\times 20)=£190{,}000 \qquad [1]$$

(i)(2) ***Compound bonuses maturity amount***

After 20 years, the maturity value is:

$$100{,}000(1.0384615)^{20}=£212{,}719.84 \qquad [1]$$

(i)(3) ***Super compound bonuses maturity amount***

Each year, a bonus of 3,000 is added to the benefit. Each of these bonuses then increases by 6% *pa* compound from the date they are declared up to the maturity date.

So the first bonus of 3,000 is added at time 1. This increases by 6% *pa* compound over the next 19 years and so produces an amount at maturity of $3{,}000 \times 1.06^{19}$.

The second bonus of 3,000 is added at time 2. This also increases by 6% *pa* compound over the next 18 years and so produces $3{,}000 \times 1.06^{18}$ by the maturity date.

Each subsequent bonus of 3,000 is treated in a similar way, with the last of these added at time 20, *ie* at the maturity date itself.

The total maturity value is therefore:

$$100{,}000 + 3{,}000\left(1.06^{19} + 1.06^{18} + \cdots + 1\right) \qquad [1]$$

The function in brackets is $s_{\overline{20}|}$ calculated at 6% *pa*, and so the maturity value is:

$$\begin{aligned} 100{,}000 + 3{,}000 s_{\overline{20}|@6\%} &= 100{,}000 + 3{,}000 \times 36.7856 \\ &= £210{,}356.8 \qquad [1] \end{aligned}$$

Alternatively, we could calculate the function in brackets using the formula for the sum of the terms of a geometric progression, ie as:

$$1 + 1.06 + 1.06^2 + \cdots + 1.06^{19} = \frac{1 - 1.06^{20}}{1 - 1.06} = \frac{1.06^{20} - 1}{0.06} = 36.7856$$

[Total 4]

(ii) ***EPV of benefits under structure (2)***

The expected present value of the maturity benefit is:

$$100{,}000 \times 1.0384615^{20} v^{20} \, {}_{20}p_{[45]} = 100{,}000 \times 1.04^{-20} \, {}_{20}p_{[45]}$$

as $1.0384615 \times v = \dfrac{1.0384615}{1.08} = \dfrac{1}{1.04}$.

So:

$$EPV = 100{,}000 \times \frac{D_{65@4\%}}{D_{[45]@4\%}} = 100{,}000 \times \frac{689.23}{1{,}677.42} = 41{,}088.70 \qquad [1]$$

Writing 1.0384615 as $1+b$, the expected present value of the death benefit is:

$$100{,}000 \times \left(v \, {}_{0|}q_{[45]} + (1+b)v^2 \, {}_{1|}q_{[45]} + (1+b)^2 v^3 \, {}_{2|}q_{[45]} + \cdots + (1+b)^{19} v^{20} \, {}_{19|}q_{[45]}\right)$$

It would be helpful if the first term in the bracket began with $(1+b)v$. We can achieve this by multiplying and dividing by $(1+b)$, so:

$$EPV = \frac{100{,}000}{1+b} \times \left((1+b)v \;{}_{0|}q_{[45]} + (1+b)^2 v^2 \;{}_{1|}q_{[45]} + \cdots + (1+b)^{20} v^{20} \;{}_{19|}q_{[45]} \right)$$

$$= \frac{100{,}000}{1.0384615} \times \left(1.04^{-1} \;{}_{0|}q_{[45]} + 1.04^{-2} \;{}_{1|}q_{[45]} + \cdots + 1.04^{-20} \;{}_{19|}q_{[45]} \right)$$

$$= \frac{100{,}000}{1.0384615} \times A^{\;1}_{[45]:\overline{20|}\,@4\%} \qquad [2]$$

Now:

$$A^{\;1}_{[45]:\overline{20|}} = A_{[45]:\overline{20|}} - \frac{D_{65}}{D_{[45]}} = 0.46982 - \frac{689.23}{1{,}677.42} = 0.058933$$

So:

$$EPV = \frac{100{,}000}{1.0384615} \times 0.058933 = 5{,}675.03$$

Adding the EPV of the maturity benefit and death benefit together we find the total expected present value to be:

$$41{,}088.70 + 5{,}675.03 = £46{,}764 \qquad [1]$$

[Total 4]

(iii)(a) ***Bonuses are added at the start of the year***

If we were to write out the EPV of the death benefit as a summation, we would find that the powers of $1+b$ are all increased by 1 compared to part (ii), so the EPV of the death benefit will just be the previous value multiplied by 1.0384615. So the total EPV is now:

$$41{,}088.70 + 1.0384615 \times 5{,}675.03 = £46{,}982 \qquad [1]$$

(iii)(b) ***Death benefit payable immediately on death***

The death benefit can be assumed to be paid on average half a year earlier than in part (ii), so the EPV of the death benefit will just be the previous value multiplied by $1.08^{½}$. So the total EPV is now:

$$41{,}088.70 + 1.08^{½} \times 5{,}675.03 = £46{,}986 \qquad [1]$$

(iii)(c) ***Bonuses are added continuously***

The sum assured payable on death is increasing continuously at a compound rate of 3.84615% *pa* up to the moment the claim is paid, from which point it is discounted over exactly the same period at a compound interest rate of 8% *pa*. So the EPV of the death benefits is equivalent to the EPV of a level term assurance discounted at a net effective interest rate of 4% *pa*.

Hence the EPV of the death benefit is:

$$100,000\bar{A}^{\;1}_{[45]:\overline{20}|@4\%} = 100,000 \times 1.04^{½} \times A^{\;1}_{[45]:\overline{20}|@4\%}$$

Now $A^{\;1}_{[45]:\overline{20}|@4\%} = 0.058933$ from part (ii), so the total EPV becomes:

$$41,088.70 + 100,000 \times 1.04^{½} \times 0.058933 = £47,099$$ [1]

[Total 3]

End of Part 3

What next?

1. Briefly **review** the key areas of Part 3 and/or re-read the **summaries** at the end of Chapters 14 to 19.
2. Ensure you have attempted some of the **Practice Questions** at the end of each chapter in Part 3. If you don't have time to do them all, you could save the remainder for use as part of your revision.
3. Attempt **Assignment X3**.

Time to consider …

… 'revision' products

Flashcards – These are available in both paper and eBook format. One student said:

> *'The Flashcards were useful for highlighting what we needed to memorise and for revision on-the-go.'*

You can find lots more information, including samples, on our website at www.ActEd.co.uk.

Buy online at www.ActEd.co.uk/estore

20

Gross premiums

Syllabus objectives

6.1 Define the gross random future loss under an insurance contract, and state the principle of equivalence.

6.2 Describe and calculate gross premiums and reserves of assurance and annuity contracts.

6.2.1 Define and calculate gross premiums for the insurance contract benefits as defined in objective 4.1 under various scenarios using the equivalence principle or otherwise:

- contracts may accept only single premium
- regular premiums and annuity benefits may be payable annually, more frequently than annually, or continuously
- death benefits (which increase or decrease by a constant compound rate or by a constant monetary amount) may be payable at the end of the year of death, or immediately on death
- survival benefits (other than annuities) may be payable at defined intervals other than at maturity.

0 Introduction

In this chapter we describe some of the methods that can be used to calculate the premiums for conventional life assurance and annuity contracts.

We show how an equation of value can be used to do this. We first met this method in Chapter 10 in relation to certain (as supposed to random) cashflows. This approach will involve calculating expected present values, as described in Chapters 16-19.

We also discuss an alternative method, where we calculate premiums that satisfy specified probabilities.

This chapter includes the concept of the future loss random variable, on which both of these approaches depend.

While studying this chapter, bear in mind that all of the approaches covered here suffer from a common shortcoming in that they do not allow for the cost of the capital that is necessary for insurers to be able to sell such policies in real life. The methods required to take this into account, which involve projecting cashflows and profit testing, are described later, in Chapters 27 and 28.

1 The gross premium

The income to a life insurer comes from the payments made by policyholders, called the *premiums*. The outgo arises from benefits paid to policyholders and the insurer's expenses.

By expenses we mean the costs incurred in administering a life insurance contract and, more broadly, in managing the insurance company's business as a whole. These costs involve such things as:

- staff salaries
- commission payments to sales intermediaries
- costs of infra-structure (*eg* office buildings, computer systems)
- medical underwriting costs
- costs of managing investments
- other running costs (*eg* policy administration, heating, lighting, internet use).

Some of these costs will be *direct* expenses, which are incurred as a result of activities specifically associated with a particular policy (like commission); other expenses will be *overheads*, which are incurred whether or not a particular policy is actually in force (such as the actuary's salary).

Question

Identify the following costs as overheads or direct:

(i) board of directors' remuneration

(ii) £10,000 bonus payable to sales manager on completion of target new business levels

(iii) head office canteen workers' salaries

(iv) new business administration department's salary costs.

Solution

(i) The board of directors' remuneration is an overhead (unless linked in any way to new business volumes).

(ii) The production bonus is a direct expense (it varies with production).

(iii) Head office canteen workers' salaries are an overhead.

(iv) This is a good example of a 'grey' area where there is no clear right or wrong answer. In the very short term, writing one more policy will not change the costs. But a rush of business over a few weeks may require overtime payments, and a rush of business over a few months may require more staff to be taken on.

So some companies will treat this as an overhead, other companies will treat it as a direct expense. Others could do both by treating the basic salaries of normal staff as overheads, and overtime and/or salaries of temporary staff as a direct expense.

The future levels of expenses are also significantly affected by future inflation, so we should allow for this when calculating premiums.

The *gross premium* is the premium required to meet all the costs under an insurance contract, and is the premium that the policyholder pays. When we talk of 'the premium' for a contract, we mean the gross premium. It is also sometimes referred to as the *office premium*.

In practice, the office premium also contains impact for loaded profits, cost of capital, adjustments for competition and other loadings. This is currently beyond the scope of this subject.

Some of these aspects are, however, taken into account when premiums are calculated using cashflow methods, as we describe in Chapters 27 and 28.

2 Gross future loss random variable

Consider the net random future loss (or just 'net loss') from a policy which is in force – where the net loss, L, is defined to be:

$$L = \text{present value of the future outgo} - \text{present value of the future income}$$

Now L is a random variable, since both terms are random variables which depend on the policyholder's future lifetime. (If premiums are not being paid, the second term is zero but the first term is a random variable, so L is still a random variable.) When the outgo includes benefits and expenses, and the income is the gross premiums, then L is referred to as the *gross future loss random variable*.

We will use FLRV for short.

We illustrate the form of the definition of the gross future loss random variable using the example of a whole life assurance. This approach can be extended to any of the standard contract types for a given allocation of expenses and, in the case of with-profits contracts, method of bonus allocation.

Example: Whole life assurance

Suppose we can allocate expenses as:

- I **initial expenses in excess of those occurring regularly each year**
- e **level annual expenses**
- f **additional expenses incurred when the contract terminates**

We are assuming here that level expenses of e are incurred every year *including* the first, *ie* that I represents the amount by which the total initial expense amount *exceeds* the subsequent regular expense amount.

We need to be careful when dealing with examination questions that talk about 'regular expenses' or 'renewal expenses'. We need to be very clear as to whether the expense amount applies at the start of every year, or only from the start of Year 2 onwards. If the renewal expenses start in Year 2, it may be easier to think of them as starting in Year 1 and then change I to $I-e$. The question should always make it clear what is required. If it doesn't, then we need to state the assumption we are making very clearly.

The gross future loss random variable when a policy is issued to a life aged x is:

$$Sv^{T_x} + I + e\bar{a}_{\overline{T_x}|} + fv^{T_x} - G\bar{a}_{\overline{T_x}|}$$

where a gross premium of G secures a sum assured of S, the sum assured is paid immediately on death and the premium is payable continuously.

It also assumes that the renewal expenses are paid continuously at a constant rate of e *pa* until the death of the policyholder (*ie* until the contract terminates).

Example: Endowment assurance

The FLRV for an endowment assurance is similar to the above, except that the term of the contract is limited (say to n years), and the survival benefit needs to be included.

Question

Write down the gross future loss random variable for an n-year endowment assurance. Assume that death benefits are payable immediately on death, level premiums are payable continuously, and the expenses are as defined in the above example. Use the symbols defined in the above example as appropriate.

Solution

In a case like this it is usually easiest to write down the FLRV in two parts: the first being the outcome if the person dies during the policy term (*ie* where $T_x < n$), and the second where the person is still alive at the maturity date (*ie* where $T_x \geq n$). So the FLRV is:

$$\begin{cases} S v^{T_x} + I + e\,\bar{a}_{\overline{T_x}|} + f v^{T_x} - G\bar{a}_{\overline{T_x}|} & T_x < n \\ S v^{n} + I + e\,\bar{a}_{\overline{n}|} + f v^{n} - G\bar{a}_{\overline{n}|} & T_x \geq n \end{cases}$$

where I, e, f, S and G are defined as before.

Sometimes we can combine these into a single expression. Here we can do this using the min{…} function, so the FLRV can alternatively be written:

$$S v^{\min\{T_x,n\}} + I + e\,\bar{a}_{\overline{\min\{T_x,n\}}|} + f v^{\min\{T_x,n\}} - G\bar{a}_{\overline{\min\{T_x,n\}}|}$$

If death benefits were payable at the end of the year of death, then T_x would be replaced by $K_x + 1$ in the multipliers of S and f. If premiums and regular expenses were payable annually in advance, then T_x would be replaced by $K_x + 1$ in the multipliers of e and G, and the annuity functions would be $\ddot{a}$ rather than $\bar{a}$.

Gross future loss random variable for an in-force policy

We can also consider the FLRV for a policy that is currently in force, *ie* as at some policy duration $t > 0$. For example, for the policy in the previous question the FLRV as at the end of the fourth policy year $(n > 4)$ is:

$$\begin{cases} S v^{T_{x+4}} + e\,\bar{a}_{\overline{T_{x+4}}|} + f v^{T_{x+4}} - G\bar{a}_{\overline{T_{x+4}}|} & T_{x+4} < n-4 \\ S v^{n-4} + e\,\bar{a}_{\overline{n-4}|} + f v^{n-4} - G\bar{a}_{\overline{n-4}|} & T_{x+4} \geq n-4 \end{cases}$$

$$= S v^{\min\{T_{x+4},n-4\}} + e\,\bar{a}_{\overline{\min\{T_{x+4},n-4\}}|} + f v^{\min\{T_{x+4},n-4\}} - G\bar{a}_{\overline{\min\{T_{x+4},n-4\}}|}$$

This reflects the fact that the policyholder is now aged $x+4$, and there are only $n-4$ years of the policy left to run.

Question

Explain why the value of I does not appear in the above FLRV.

Solution

At time 4, the initial expense cashflow is in the past, *ie* it has already been incurred (at time 0), and so it is not part of the *future* loss.

Example: with-profits policy

We can also apply these ideas to with-profits policies (or to any other type of policy with a non-level benefit). If we do so, we need to allow correctly for the number of bonuses that have been added by the current date. (Note that we sometimes describe the adding of bonuses as the *vesting* of bonuses so, if a bonus 'vests', it means that the policy benefit level has been increased by the amount of that bonus.) We must also consider carefully the timing of the bonus additions (start of the year, end of the year, *etc*), as always.

For example, suppose a company sells with-profits endowment assurances with an initial sum assured of £20,000 to lives aged 40 exact. Each policy has a term of 20 years, and the death benefit is payable at the end of the year of death. Annual premiums of £2,500 are payable in advance during the term of the policy, or until earlier death of the policyholder. The company assumes that:

(a) bonuses of £2,000 will be added to the sum assured each year, vesting at the start of each year except the first;

(b) expenses of £200 will be incurred initially, together with renewal expenses of 2% of every premium except the first.

We will write down the gross future loss random variable for the policy at the start of the contract, and also just before the tenth premium is paid.

FLRV at the start of the policy

If bonuses were added at the start of every year *including* the first year, and the policyholder dies during the term, *ie* $K_{40} < 20$, then the sum assured will be increased by $K_{40}+1$ bonuses. For example, if the policyholder were to die in policy year 3 (at which point $K_{40}=2$), a bonus would be added at the start of each of the three years, *ie* three bonuses in total. However, in this case bonuses are added at the start of each year *except* the first, and so the number of bonuses added is one fewer than this, *ie* equal to K_{40}. If the policyholder survives to the end of the term, a total of 19 bonuses will be paid altogether (*ie* one for each policy year except the first).

So the present value of the future benefits can be written:

$$\begin{cases} \left[20{,}000+2{,}000K_{40}\right]v^{K_{40}+1} & K_{40}<20 \\ \left[20{,}000+2{,}000\times 19\right]v^{20} & K_{40}\geq 20 \end{cases}$$

$$=\left[20{,}000+2{,}000\min\{K_{40},19\}\right]v^{\min\{K_{40}+1,\,20\}}$$

Premiums of £2,500 are payable annually in advance. So the present value of the future premiums is:

$$\begin{cases} 2{,}500\ddot{a}_{\overline{K_{40}+1}|} & K_{40}<20 \\ 2{,}500\ddot{a}_{\overline{20}|} & K_{40}\geq 20 \end{cases}$$

$$=2{,}500\,\ddot{a}_{\overline{\min\{K_{40}+1,\,20\}}|}$$

The present value of the future expenses is:

$$\begin{cases} 200+0.02\times 2{,}500a_{\overline{K_{40}}|} & K_{40}<20 \\ 200+0.02\times 2{,}500a_{\overline{19}|} & K_{40}\geq 20 \end{cases}$$

$$=200+50a_{\overline{\min\{K_{40},\,19\}}|}$$

$$=200+50\left(\ddot{a}_{\overline{\min\{K_{40}+1,\,20\}}|}-1\right)$$

$$=150+50\ddot{a}_{\overline{\min\{K_{40}+1,\,20\}}|}$$

Putting all these together, we have:

$$\begin{cases} \left[20{,}000+2{,}000K_{40}\right]v^{K_{40}+1}+150-2{,}450\,\ddot{a}_{\overline{K_{40}+1}|} & K_{40}<20 \\ \left[20{,}000+2{,}000\times 19\right]v^{20}+150-2{,}450\,\ddot{a}_{\overline{20}|} & K_{40}\geq 20 \end{cases}$$

$$=\left[20{,}000+2{,}000\min\{K_{40},\,19\}\right]v^{\min\{K_{40}+1,\,20\}}+150-2{,}450\,\ddot{a}_{\overline{\min\{K_{40}+1,\,20\}}|}$$

FLRV just before the tenth premium is paid

Now suppose the policy is still in force at time 9, *ie* just before the 10th premium is paid. The policyholder is 49 at this point and there are 11 years remaining on the policy.

Eight bonuses will have vested so far (the ninth bonus is about to vest). The sum assured before the ninth bonus vests is £36,000, and, since the next bonus is about to vest immediately, the number of future bonuses is $K_{49}+1$ (if the policyholder dies during the term), or 11 (if the policyholder survives to the end of the term).

So the present value of the future benefits is now:

$$\begin{cases} \left[36{,}000+2{,}000\{K_{49}+1\}\right]v^{K_{49}+1} & K_{49}<11 \\ \left[36{,}000+2{,}000\times 11\right]v^{11} & K_{49}\geq 11 \end{cases}$$

$$=\left[36{,}000+2{,}000\min\{K_{49}+1,11\}\right]v^{\min\{K_{49}+1,\,11\}}$$

The present value of the future premiums is:

$$\begin{cases} 2{,}500\ddot{a}_{\overline{K_{49}+1}|} & K_{49}<11 \\ 2{,}500\ddot{a}_{\overline{11}|} & K_{49}\geq 11 \end{cases}$$

$$=2{,}500\,\ddot{a}_{\overline{\min\{K_{49}+1,\,11\}}|}$$

The present value of the future expenses is:

$$\begin{cases} 50\ddot{a}_{\overline{K_{49}+1}|} & K_{49}<11 \\ 50\ddot{a}_{\overline{11}|} & K_{49}\geq 11 \end{cases}$$

$$=50\,\ddot{a}_{\overline{\min\{K_{49}+1,\,11\}}|}$$

(We no longer need to include the initial expenses.)

So the gross future loss random variable is:

$$\begin{cases} \left[36{,}000+2{,}000\{K_{49}+1\}\right]v^{K_{49}+1}-2{,}450\,\ddot{a}_{\overline{K_{49}+1}|} & K_{49}<11 \\ \left[36{,}000+2{,}000\times 11\right]v^{11}-2{,}450\,\ddot{a}_{\overline{11}|} & K_{49}\geq 11 \end{cases}$$

$$=\left[36{,}000+2{,}000\min\{K_{49}+1,\,11\}\right]v^{\min\{K_{49}+1,\,11\}}-2{,}450\,\ddot{a}_{\overline{\min\{K_{49}+1,\,11\}}|}$$

Question

Now assume in the above example that the bonuses vest at the end of each year (*including* the first year). Write down revised expressions for the gross future loss random variables at times 0 and 9.

Solution

At outset, the number of bonuses on death is still K_{40}, but there are now 20 bonuses altogether (the 20th bonus is payable on survival only). So the gross future loss random variable at time zero will now be:

$$\begin{cases} [20{,}000+2{,}000K_{40}]v^{K_{40}+1}+150-2{,}450\,\ddot{a}_{\overline{K_{40}+1}|} & K_{40}<20 \\ [20{,}000+2{,}000\times 20]v^{20}+150-2{,}450\,\ddot{a}_{\overline{20}|} & K_{40}\geq 20 \end{cases}$$

$$=\left[20{,}000+2{,}000\min\{K_{40},20\}\right]v^{\min\{K_{40}+1,\,20\}}+150-2{,}450\,\ddot{a}_{\overline{\min\{K_{40}+1,\,20\}}|}$$

By time 9, exactly 9 policy years have been completed and so 9 bonuses have been added to the sum assured, making the total benefit level at this point equal to $20{,}000+9\times 2{,}000=38{,}000$. So the future loss random variable at time 9 is:

$$\begin{cases} [38{,}000+2{,}000K_{49}]v^{K_{49}+1}-2{,}450\,\ddot{a}_{\overline{K_{49}+1}|} & K_{49}<11 \\ [38{,}000+2{,}000\times 11]v^{11}-2{,}450\,\ddot{a}_{\overline{11}|} & K_{49}\geq 11 \end{cases}$$

$$=\left[38{,}000+2{,}000\min\{K_{49},11\}\right]v^{\min\{K_{49}+1,\,11\}}-2{,}450\,\ddot{a}_{\overline{\min\{K_{49}+1,\,11\}}|}$$

Example: term assurance policy

Let's assume the same details as for the (without-profit) endowment assurance example we described earlier, except that the sum assured is now only payable on death during the n-year term (there is no payment made on survival), and f is the claim expense.

The FLRV is:

$$\begin{cases} S\,v^{T_x}+I+e\,\bar{a}_{\overline{T_x}|}+f\,v^{T_x}-G\,\bar{a}_{\overline{T_x}|} & T_x<n \\ I+e\,\bar{a}_{\overline{n}|}-G\,\bar{a}_{\overline{n}|} & T_x\geq n \end{cases}$$

Note that it is not possible to express a term assurance FLRV as a single-line expression.

Question

Explain why the term involving f does not appear in the FLRV when $T_x \geq n$.

Solution

f is the claim expense, and if $T_x \geq n$ the policyholder has survived the term and so no claim is payable. Hence there is no claim expense to pay.

2.1 Calculating premiums that satisfy probabilities, using the gross future loss random variable

Premiums (and reserves) can be calculated which satisfy probabilities involving the gross future loss random variable.

(The more important method of calculating premiums, however, uses the principle of equivalence – this is covered in Section 3 below.)

Example

A whole life assurance pays a sum assured of 10,000 at the end of the year of death of a life aged 50 exact at entry. Assuming 3% per annum interest, AM92 Ultimate mortality and expenses of 4% of every premium, calculate the smallest level annual premium payable at the start of each year that will ensure the probability of making a loss under this contract is not greater than 5%.

Solution

We will make a loss on this policy if the total future outgo it generates exceeds the total future income received from it. Future investment returns will affect the overall loss, and one way of taking account of these is to measure the future loss in present value terms, *ie* with all cashflows discounted at the effective interest rate. So, allowing for investment returns, we will make a loss under the policy if the present value of the future loss, *ie* L_0, is greater than zero.

If the annual premium is G, the future loss (random variable) of the policy at outset is:

$$L_0 = 10{,}000 v^{K_{50}+1} - 0.96 G \ddot{a}_{\overline{K_{50}+1}|}$$

We need to find the smallest value of G such that:

$$P(L_0 > 0) \le 0.05$$

***ie* such that**

$$P(L_0 \le 0) \ge 0.95$$

Define G_n to be the annual premium that ensures $L_0 = 0$ for $K_{50} = n$.

The value of n has to be a valid value for K_{50}, which means that $n = 0, 1, 2, \ldots$. So, G_n is the premium we need to charge to cover the benefits *exactly*, if the policyholder were to die between (integer) times n and $n+1$.

This means:

$$G_n = \frac{10{,}000 v^{n+1}}{0.96 \ddot{a}_{\overline{n+1}|}}$$

Now:

$$\boldsymbol{P\left(L_0 \le 0 \middle| G = G_n\right) = P\left(K_{50} \ge n\right)}$$

which reads as 'the probability of not making a loss when the premium is G_n, equals the probability of the policyholder surviving for at least n years'.

To understand why this probability is true, suppose we *do* charge a premium of G_n (meaning that the loss will equal zero if $K_{50} = n$). If death occurs later than this (*ie* if $K_{50} > n$), the present value of the benefit will reduce (because it will be more heavily discounted), and the present value of the premium income will increase (because more premiums are received). So, if $K_{50} \ge n$, and G_n is the premium, we cannot make a loss on the contract (*ie* $L_0 \le 0$).

However, this probability needs to be at least 0.95…

and we need to find the *smallest* premium for which this is true. If we look at the formula for G_n, we can see that as n increases, the premium G_n decreases.

We therefore find the largest value of n that satisfies this condition, and the corresponding value of G_n is then the minimum premium required.

The condition is that $P\left(K_{50} \ge n\right) \ge 0.95$.

So:

$$\boldsymbol{P\left(K_{50} \ge n\right) \ge 0.95}$$

$$\Rightarrow \quad {}_np_{50} \ge 0.95 \quad \Rightarrow \quad \frac{l_{50+n}}{l_{50}} \ge 0.95$$

$$\Rightarrow \quad \boldsymbol{l_{50+n} \ge 0.95 \times l_{50} = 0.95 \times 9{,}712.0728 = 9{,}226.4692}$$

From the *Tables*, $l_{60} = 9{,}287.2164$ and $l_{61} = 9{,}212.7143$, and so the largest value of n that satisfies the required probability is $n = 10$. Hence, the smallest premium that satisfies the required probability is:

$$\boldsymbol{G_{10} = \frac{10{,}000\, v^{11}}{0.96\, \ddot{a}_{\overline{11}|}} = \frac{10{,}000 \times 0.72242}{0.96 \times 9.5302} = 790}$$

A premium calculated in this way is sometimes called a *percentile premium*.

A reserve at policy duration t can be calculated in a similar way, to satisfy a probability specified in terms of the gross future loss random variable at time t. Reserves are described in the next chapter.

Question

Hubert, aged 60, is applying to buy a whole life immediate annuity from an insurance company, with his life savings of £200,000. Calculate the largest amount of level annuity, payable annually in arrear, that the insurer could pay if it requires a probability of loss from the contract of no more than 10%.

Assume PMA92C20 mortality, interest of 5% *pa*, and expenses of 1% of each annuity payment.

Solution

The present value of the loss from the policy is:

$$L = 1.01 X a_{\overline{K_{60}}|} - 200{,}000$$

where X is the annual annuity benefit. If we set $X_{(k)}$ such that:

$$L = 1.01 X_{(k)}\, a_{\overline{k}|} - 200{,}000 = 0$$

then:

$$X_{(k)} = \frac{200{,}000}{1.01\, a_{\overline{k}|}}$$

and the insurer will only make a loss if $K_{60} > k$ (for annuities, the loss will increase the longer the person lives).

So, an annuity of $X_{(k)}$ implies $P(L>0) = P(K_{60} > k)$. To find the largest value of $X_{(k)}$ that satisfies:

$$P(L>0) \le 0.1$$

we need the smallest value of k that satisfies:

$$P(K_{60} > k) \le 0.1 \qquad ie \qquad P(K_{60} \ge k+1) \le 0.1 \qquad \text{or} \qquad {}_{k+1}p_{60} \le 0.1$$

that is, the smallest value of k for which:

$$l_{60+k+1} \le 0.1\, l_{60} = 0.1 \times 9{,}826.131 = 982.6131$$

From the *Tables* we find:

$$l_{95} = 1{,}020.409 \qquad (k=34)$$

$$l_{96} = 798.003 \qquad (k=35)$$

So the required (smallest) value of k is 35, and hence the largest amount of level annuity is:

$$X_{(35)} = \frac{200,000}{1.01\, a_{\overline{35}|}^{5\%}} = \frac{200,000}{1.01 \times 16.3742} = £12,093$$

3 Principle of equivalence

3.1 Definition

The equation of value where payments are certain has already been introduced in Chapter 10. In most actuarial contexts some or all of the cashflows in a contract (usually long-term) are uncertain, depending on the death or survival (or possibly the state of health) of a life.

We therefore extend the concept of the equation of value to deal with this uncertainty, by equating expected present values of uncertain cashflows. The equation of expected present values for a contract, usually referred to as the equation of value, is:

The expected present value of the income

= The expected present value of the outgo

This can also be written as:

$$E\left[\text{present value of future outgo}\right]-E\left[\text{present value of future income}\right]=0$$

$$\Rightarrow \qquad E\left[\text{present value of future loss}\right]=0$$

$$ie: \qquad E\left[L_0\right]=0$$

Alternatively, this is referred to as the *principle of equivalence*.

3.2 Determining gross premiums using the equivalence principle

Given a suitable set of assumptions, which we call the basis, we may use the equation of value to calculate the premium or premiums which a policyholder must pay in return for a given benefit.

This is sometimes referred to as calculating a premium using the principle of equivalence.

We may also calculate the amount of benefit payable for a given premium.

A basis is a set of assumptions regarding expected future experience, *eg*:

- mortality experience
- investment returns
- future expenses
- bonus rates.

The set of assumptions used to calculate a premium is called the *pricing basis*.

The gross premium for a contract, given suitable mortality, interest and expense assumptions would be found from the equation of expected present value.

The equivalence principle states that:

E**[gross future loss] = 0**

which implies that:

$$\text{EPV premiums} = \text{EPV benefits} + \text{EPV expenses}$$

so that in an expected present value context the premiums are equal (equivalent) in value to the expenses and the benefits. This relationship is usually called an equation of (gross expected present) value.

3.3 The basis

The basis for applying the equation of value for a life insurance contract will specify the mortality, interest rates and expenses to be assumed.

Usually the assumptions will not be our best estimates of the individual basis elements, but will be more cautious than the best estimates.

For example, if we expect to earn a rate of interest of 8% *pa* on the invested premiums, a more cautious basis might be to calculate the premiums assuming we earn only 6% *pa*. This lower rate of interest will result in a higher premium than that calculated using the expected rate of 8% *pa*.

Some reasons for the element of caution in the basis are:

1. **To allow a contingency margin, to ensure a high probability that the premiums plus interest income meet the cost of benefits, allowing for random variation. In other words, to ensure a high probability of making a profit.**
2. **To allow for uncertainty in the estimates themselves.**

The size of the appropriate margin to incorporate will depend upon a number of factors, such as:

- The level of risk that the insurer is undertaking. The higher the risk the bigger the margin.
- The competitiveness of the market in which the product needs to be sold. The more competitive the market, the lower the margin.

In the case of the mortality rate, the more cautious basis will depend on the nature of the contract. For example, for a contract which pays out the benefit on death, a heavier mortality rate than that expected will give rise to a higher premium and is therefore the more cautious basis. For an annuity contract, a lighter mortality rate is a more cautious basis.

In actuarial jargon, we refer to this as 'being prudent'.

3.4 Premium payment structures

The premium payment structure will commonly be one of the following:

- **A *single premium* contract, under which benefits are paid for by a single lump sum premium paid at the time the contract is effected. This payment is certain, so that the left-hand side of the equation of value is the certain payment, not the expected value of a payment.**

- **An *annual premium* contract, under which benefits are paid for by a regular annual payment, usually of a level amount, the first premium being due at the time the contract is effected. Premiums continue to be paid each year in advance until the end of some agreed maximum premium term, often the same as the contract term, or until the life dies if this is sooner. Therefore, there would not usually be a premium payment at the end of the contract term.**
- **A *true* m*thly premium* contract, under which benefits are paid for by m level payments made every $1/m$ years. As in the annual premium case, premiums continue to be paid in advance until the end of some agreed maximum premium term or until the life dies if this is sooner. Again, there would not usually be a premium paid at the end of the contract term. Often the premium is paid monthly (that is, $m = 12$). For some types of contract, weekly premiums are possible.**

The use of the word 'true' here refers to a technical distinction (between so-called 'true' and 'instalment' premiums), which we needn't worry about here.

***Premiums are always paid in advance*, so the first payment is always due at the time the policy is effected.**

Given a basis specifying mortality, interest and expenses to be assumed, and given details of the benefits to be purchased, we can use the equation of value to calculate the premium payable.

3.5 Annual premium contracts

We will illustrate the equation of value methodology by using the example of a whole life assurance. The same method can be used for other standard contracts.

In particular, the same method can also be used for single premium contracts with the simplification that the expected present value of future premiums will just be equal to the single premium that the company receives on day 1.

Suppose we have premiums of G payable annually in advance, a benefit of S payable at the end of the year of death, initial expenses of I, regular expenses of e payable annually in advance in each year including the first year, and claim expenses of f.

The equation of value is:

$$S A_x + I + e\ddot{a}_x + f A_x = G\ddot{a}_x$$

and, with a basis (to determine the annuity and assurance values) and an expense allocation, the value of G can be determined.

Let's consider Gordon, who is aged 45, and who wishes to buy a whole life assurance policy with a sum assured of £10,000 payable immediately on death. We are going to calculate the gross premium that Gordon needs to pay annually in advance for ten years or until his earlier death, using the equivalence principle according to the following basis:

- Initial expenses of £160 plus 75% of the annual premium
- Renewal expenses of £50 (incurred throughout life from year 2 onwards) plus 4% of the annual premium (incurred at the time of payment of each premium from year 2 onwards)
- Claim expenses of 2.5% of sum assured incurred when the benefit is payable
- AM92 Ultimate mortality
- 4% *pa* interest

The equation of value is:

EPV premiums = EPV benefits + EPV expenses

Now:

$$\text{EPV premiums} = P\ddot{a}_{45:\overline{10|}} = P\left(\ddot{a}_{45} - \frac{D_{55}}{D_{45}}\ddot{a}_{55}\right)$$

$$= \left(18.823 - \frac{1,105.41}{1,677.97} \times 15.873\right)P = 8.366P$$

$$\text{EPV benefits} = 10,000\bar{A}_{45} \approx 10,000 \times (1+i)^{½} A_{45}$$

$$= 10,000 \times 1.04^{½} \times 0.27605 = 2,815.17$$

and:

$$\text{EPV expenses} = 160 + 0.75P + 50\left(\ddot{a}_{45} - 1\right) + 0.04P\left(\ddot{a}_{45:\overline{10|}} - 1\right) + 0.025 \times 10,000\bar{A}_{45}$$

$$= 160 + 0.75P + 50 \times 17.823 + 0.04 \times 7.366P + 0.025 \times 2,815.17$$

$$= 1.045P + 1,121.53$$

So:

$$8.366P = 2,815.17 + 1.045P + 1,121.53$$

$$\Rightarrow P = £537.69$$

(These figures use full accuracy and are sensitive to rounding.)

If the sum assured is paid immediately on death and the premium is paid continuously we can use functions defined in Chapters 16 and 17 to write:

$$S\bar{A}_x + I + e\bar{a}_x + f\bar{A}_x = G\bar{a}_x$$

which can be solved for G as before.

We can also calculate gross premiums using select mortality.

Question

A 25-year endowment assurance policy provides a payment of £75,000 on maturity or at the end of the year of earlier death. Calculate the annual premium payable for a policyholder who effects this insurance at exact age 45.

Expenses are 75% of the first premium and 5% of each subsequent premium, plus an initial expense of £250.

Assume AM92 Select mortality and 4% *pa* interest.

Solution

If the annual premium is P, then:

$$\begin{aligned}\text{EPV premiums} &= P\ddot{a}_{[45]:\overline{25}|}\\ &= P\left(\ddot{a}_{[45]} - \frac{D_{70}}{D_{[45]}}\ddot{a}_{70}\right)\\ &= P\left(18.829 - \frac{517.23}{1,677.42}\times 10.375\right)\\ &= 15.630P\end{aligned}$$

$$\begin{aligned}\text{EPV benefits} &= 75,000A_{[45]:\overline{25}|}\\ &= 75,000\left(A^{1}_{[45]:\overline{25}|} + A_{[45]:\overline{25}|}^{\;\;\;\;\;\;\;\;1}\right)\\ &= 75,000\left(A_{[45]} - \frac{D_{70}}{D_{[45]}}A_{70} + \frac{D_{70}}{D_{[45]}}\right)\\ &= 75,000\left(0.27583 - \frac{517.23}{1,677.42}\times 0.60097 + \frac{517.23}{1,677.42}\right)\\ &= 75,000\times 0.39887 = £29,915\end{aligned}$$

$$\begin{aligned}\text{EPV expenses} &= 0.75P + 0.05P(\ddot{a}_{[45]:\overline{25}|} - 1) + 250\\ &= 0.75P + 0.05\times 14.630P + 250\\ &= 1.4815P + 250\end{aligned}$$

($\ddot{a}_{[45]:\overline{25}|} - 1$ is the same as $a_{[45]:\overline{24}|}$. We have used this approach to save work, since $\ddot{a}_{[45]:\overline{25}|}$ is already calculated.)

So the equation of value is:

$$15.630P = 29,915 + 1.4815P + 250 \Rightarrow P = 30,165 / 14.1484 = £2,132$$

3.6 Conventional with-profits contracts

Gross premiums for with-profits contracts will include not only loadings for expenses, but also for future bonuses.

Extending the preceding example (the whole life assurance described at the start of Section 3.5 above), recall that the equation of value for this contract was:

$$S\,A_x + I + e\,\ddot{a}_x + f\,A_x = G\,\ddot{a}_x$$

Suppose the contract is now with-profits, and a compound bonus of b per annum is assumed. A compound bonus is added to the basic sum assured and existing bonuses. Let us assume that premiums are annual and that the first bonus is awarded at the end of the first policy year (*ie* no bonus is available to deaths in the first year). In this case, the equation of value is:

$$Svq_x + S(1+b)v^2\,{}_{1|}q_x + S(1+b)^2 v^3\,{}_{2|}q_x + \cdots + I + e\ddot{a}_x + fA_x = G\ddot{a}_x$$

or:

$$S\frac{1}{1+b}\left\{(1+b)vq_x + (1+b)^2 v^2\,{}_{1|}q_x + (1+b)^3 v^3\,{}_{2|}q_x + \cdots\right\} + I + e\ddot{a}_x + fA_x = G\ddot{a}_x$$

Here we are defining b as a proportion (of the sum assured plus previously added bonuses).

This can then be simplified to:

$$S\frac{1}{1+b}A_x^j + I + e\,\ddot{a}_x + f\,A_x = G\,\ddot{a}_x$$

where the assurance function is determined using an interest rate j where $j = \dfrac{(1+i)}{(1+b)} - 1$.

We first saw this approach in Chapter 19. What we have done is to take out a factor of $\dfrac{1}{1+b}$ so that the powers of v match the powers of $1+b$ in all the terms in the big bracket. So we can now think of $(1+b)v$ as the 'v' for some different interest rate, and the expression reduces to an A_x term, but calculated at a new rate of interest, in this case at $j = \dfrac{1+i}{1+b} - 1$.

In the case of a simple bonus, the bonus is added to the sum assured only. Simple bonus cases may therefore be valued using the increasing assurance function defined in Chapter 19.

Question

A man aged 45 buys a 15-year with-profits endowment assurance with a basic sum assured of £25,000. Determine the single premium to be paid for this assurance, assuming that simple reversionary bonuses of 6% *pa* vest at the end of each policy year and that death benefits are payable at the end of the year of death. Assume AM92 Ultimate mortality and 4% *pa* interest. Initial expenses are £200 and renewal expenses are £30 at the start of each policy year, excluding the first.

Solution

The single premium is equal to the expected present value of the benefits and the expenses.

On death during the first year, the benefit is just the basic sum assured of £25,000. The benefit on death in subsequent years will increase each year by $0.06 \times 25,000 = £1,500$ because of the bonuses. So:

$$\text{EPV benefits} = 23,500 A_{\overset{1}{45}:\overline{15|}} + 1,500 (IA)_{\overset{1}{45}:\overline{15|}} + 47,500 A_{45:\overset{\;1}{\overline{15|}}}$$

Since the subscripts sum to 60, the endowment assurance factor is given in the *Tables*. We can write the expected present value using the endowment assurance factor as follows:

$$23,500 A_{45:\overline{15|}} + 1,500 (IA)_{\overset{1}{45}:\overline{15|}} + 24,000 A_{45:\overset{\;1}{\overline{15|}}}$$

Now:

$$A_{45:\overline{15|}} = 0.56206$$

$$A_{45:\overset{\;1}{\overline{15|}}} = \frac{D_{60}}{D_{45}} = \frac{882.85}{1,677.97} = 0.52614$$

$$(IA)_{\overset{1}{45}:\overline{15|}} = (IA)_{45} - \frac{D_{60}}{D_{45}} \left[(IA)_{60} + 15 A_{60} \right]$$

$$= 8.33628 - 0.52614 \left[8.36234 + 15 \times 0.45640 \right]$$

$$= 0.33454$$

So:

$$\text{EPV benefits} = 23,500 \times 0.56206 + 1,500 \times 0.33454 + 24,000 \times 0.52614$$

$$= 26,337.62$$

Also:

$$\text{EPV expenses} = 200 + 30 \left(\ddot{a}_{45:\overline{15|}} - 1 \right) = 200 + 30 \times 10.386 = 511.58$$

So the single premium for the policy is:

$$26{,}337.62 + 511.58 = £26{,}849$$

3.7 Premiums payable m times per year

If premiums are payable m times per year, then the expected present values of premiums and level annual expenses must be determined using expressions for m thly annuities as derived in Chapter 18.

Example: Whole life assurance

The equation of value is:

$$S\bar{A}_x + I + e\ddot{a}_x^{(m)} + f\bar{A}_x = G\ddot{a}_x^{(m)}$$

with a given basis and expenses. Using the approximations from Chapter 18 the equation can be solved for G.

Expenses and premiums can be assumed to be payable at different frequencies.

Question

Sam, aged 40, buys a 20-year term assurance with a sum assured of £150,000 payable immediately on death. Calculate the quarterly premium payable by Sam for this policy. Assume that initial expenses are 60% of the total annual premium plus £110, and renewal expenses are £30 *pa* from Year 2 onwards.

Basis: AM92 Select, 4% *pa* interest

Solution

Let P denote Sam's quarterly premium. Then:

$$\text{EPV premiums} = 4P\ddot{a}^{(4)}_{[40]:\overline{20}|} \approx 4P\left[\ddot{a}_{[40]:\overline{20}|} - \frac{3}{8}\left(1 - \frac{D_{60}}{D_{[40]}}\right)\right]$$

using the formula on page 36 of the *Tables*. From the *Tables*:

$$\ddot{a}_{[40]:\overline{20}|} = 13.930$$

and:

$$\frac{D_{60}}{D_{[40]}} = \frac{882.85}{2{,}052.54} = 0.43013$$

So:

$$\text{EPV premiums} = 4P\ddot{a}^{(4)}_{[40]:\overline{20}|} \approx 4P\left[13.930 - \frac{3}{8}\left(1-0.43013\right)\right] = 54.865P$$

Also:

$$\begin{aligned}\text{EPV benefits} &= 150{,}000\bar{A}^{1}_{[40]:\overline{20}|} \\ &\approx 150{,}000 \times 1.04^{½}\left(A_{[40]:\overline{20}|} - \frac{D_{60}}{D_{[40]}}\right) \\ &= 150{,}000 \times 1.04^{½}\left(0.46423 - 0.43013\right) \\ &= 5{,}216.97\end{aligned}$$

and:

$$\begin{aligned}\text{EPV expenses} &= 0.6 \times 4P + 110 + 30\left(\ddot{a}_{[40]:\overline{20}|} - 1\right) \\ &= 2.4P + 110 + 30 \times 12.930 \\ &= 2.4P + 497.90\end{aligned}$$

Alternatively we can think of the expense terms as 60% of the total annual premium plus £80 initially, and then £30 per year every year including the first. This will give us:

$$0.6 \times 4P + 80 + 30\ddot{a}_{[40]:\overline{20}|}$$

for the expenses. Both give the same numerical value. Use whichever method seems easier.

So the equation of value is:

$$54.865P = 5{,}216.97 + 2.4P + 497.90$$

$$\Rightarrow P = £108.93$$

The same approach can be used to determine equations of value for the gross premiums of annuity contracts with annuity benefits payable m thly.

Question

An annuity of £8,500 *pa* is to be paid monthly in advance for the remaining lifetime of Mrs S, who is currently aged exactly 60. Calculate the single premium that should be paid for this annuity, allowing for initial expenses of 1.5% of the premium and administration expenses payable at the start of each year, including the first. Administration expenses are £120 at the start of Year 1 and increase at the rate of 1.9231% *pa*. Assume AM92 Select mortality and 6% *pa* interest.

Solution

The single premium P is equal to the expected present value of benefits and expenses. We have:

$$\begin{aligned}\text{EPV benefits} &= 8{,}500\ddot{a}_{[60]}^{(12)} \text{ @6\%} \approx 8{,}500\left(\ddot{a}_{[60]} - \frac{11}{24}\right)\\ &= 8{,}500\left(11.919 - \frac{11}{24}\right) = 97{,}415.67\end{aligned}$$

and:

$$\begin{aligned}\text{EPV expenses} &= 0.015P + 120\left(1 + 1.019231v\,p_{[60]} + 1.019231^2 v^2\,{}_2p_{[60]} + \cdots\right)\\ &= 0.015P + 120\ddot{a}_{[60]} \text{ @4\%}\\ &= 0.015P + 120 \times 14.167\\ &= 0.015P + 1{,}700.04\end{aligned}$$

So:

$$P = 97{,}415.67 + 0.015P + 1{,}700.04 \Rightarrow P = £100{,}625$$

4 Calculating gross premiums using simple criteria other than the equivalence principle

So far we have seen two ways of calculating a gross premium:

- to satisfy a probability
- using the equivalence principle.

Using the equivalence principle implies that:

$$E[\text{present value of future loss}] = 0$$

so that on average (provided the assumptions used are true) the contract will 'break even'. It is usual to load premiums for profit so that:

$$E[\text{present value of future loss}] < 0$$

If a criterion based on this expected value is chosen to reflect the 'loading for profit' required, then a gross premium including a profit loading can be determined.

Example: Whole life assurance

If the criterion specifies an expected present value of future loss of $-\pi$, then the equation of value becomes:

$$S\bar{A}_x + I + e\ddot{a}_x^{(m)} + f\bar{A}_x + \pi = G\ddot{a}_x^{(m)}$$

Here we are assuming that premiums and renewal expenses are payable m thly in advance, the death benefit is payable immediately on death, and the expense claim is incurred at the time that the death benefit is paid. We are adding a loading π for profit, which means that, in addition to covering benefits and expenses, our premiums must also cover this profit requirement. So the profit element will have the same sign as benefits and expenses, even though we might intuitively think that profit is 'nice' and should therefore have the opposite sign.

Thinking about the impact on premium, this will give us the right result: adding an explicit profit requirement will give larger premiums if we treat the profit requirement in the same way as benefits and expenses, and we would expect to need greater premiums if the shareholders are demanding greater profits.

Question

Calculate the annual premium, payable monthly in advance, for a deferred annuity of £12,400 *pa* to be paid quarterly in advance from age 60 to a male now aged 40. Initial expenses are 80% of the annual premium, renewal expenses are 4% of the annual premium incurred at the start of each year from Year 2 onwards, annuity payment expenses are £15 per payment, and the EPV of profit is 2% of the annual premium. Assume AM92 Ultimate mortality and 4% *pa* interest.

Solution

Making an allowance for profit, the equation of value is:

EPV premiums = EPV benefits + EPV expenses + EPV profit

For an annual premium P :

$$\text{EPV premiums} = P\ddot{a}^{(12)}_{40:\overline{20}|}$$

$$\approx P\left(\ddot{a}_{40:\overline{20}|} - \frac{11}{24}\left(1-\frac{D_{60}}{D_{40}}\right)\right)$$

$$= P\left(13.927 - \frac{11}{24}\left(1-\frac{882.85}{2,052.96}\right)\right)$$

$$= 13.666P$$

$$\text{EPV benefits} = 12,400\ {}_{20|}\ddot{a}^{(4)}_{40}$$

$$= 12,400\frac{D_{60}}{D_{40}}\ddot{a}^{(4)}_{60}$$

$$\approx 12,400 \times \frac{882.85}{2,052.96}\left(14.134 - \frac{3}{8}\right)$$

$$= 12,400 \times 5.91687$$

$$= 73,369.40$$

$$\text{EPV expenses} = 0.8P + 0.04P\left(\ddot{a}_{40:\overline{20}|} - 1\right) + 4 \times 15\ {}_{20|}\ddot{a}^{(4)}_{40}$$

$$= 0.8P + 0.04P \times 12.927 + 60 \times 5.91687$$

$$= 1.31708P + 355.01$$

$$\text{EPV profit} = 0.02P$$

So we have:

$$13.666P = 73,369.40 + 1.31708P + 355.01 + 0.02P$$

giving:

$$P = \frac{73,724.42}{12.3287} = £5,980$$

Chapter 20 Summary

Future loss random variable

This is defined as:

$$L_t = \{\text{present value of future outgo}\} - \{\text{present value of future income}\}$$

as at exact policy duration t.

Principle of equivalence

The principle states that $E[L_0]=0$. It leads to the equation of value:

Expected present value of income = Expected present value of outgo

This equation is solved to calculate the premium.

The premium depends on the set of assumptions regarding the future experience that has been used. Such a set of assumptions is called the pricing (or premium) basis.

Gross premiums

Gross premiums (also called office premiums) are calculated allowing for expenses, and are the actual premiums payable for a contract. Using the equivalence principle, we find the gross premium by solving:

EPV of premiums = EPV of benefits + EPV of expenses

This may be modified if we have an explicit profit criterion to:

EPV of premiums = EPV of benefits + EPV of expenses + EPV profit

Premiums that satisfy probabilities

A premium for a new policy can be found that satisfies a probability defined in terms of the future loss random variable, L_0. For example, we can define the premium as being the smallest premium for which $P(L_0 > 0) \leq \alpha$, where α is some acceptably small probability.

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 20 Practice Questions

20.1 Write down an expression for the gross future loss random variable at issue for a deferred life annuity with a deferred period of n years. Premiums of amount G are paid annually in advance for a maximum of n years, the annuity benefit of B is paid for life annually in advance starting in n years' time, and no benefit is paid if the life does not survive to time n. Assume that there are regular expenses payable annually in advance during the premium payment term of e, additional initial expenses of I at the start of Year 1, and regular benefit payment expenses of e' payable annually in advance while the benefit is being paid.

20.2 Show how you would modify the premium equation below to allow for the expenses indicated:

$$P\ddot{a}_{x:\overline{n|}} = SA_{x:\overline{n|}}$$

(i) initial expenses of 2% of the sum assured

(ii) renewal expenses of 2% of each premium, including the first

(iii) claim expenses of 2% of the sum assured

(iv) initial expenses of 50% of first premium plus renewal expenses of 3% of each premium excluding the first.

20.3 Exam style

A life office sells 5-year term assurance policies to lives aged 60. Each policy has a sum assured of £10,000 payable at the end of the year of death. Premiums of £200 are payable annually in advance throughout the 5-year term or until earlier death.

Let L denote the present value of the insurer's loss on one of these policies, at policy outset, ignoring expenses.

(i) Write down an expression for L. [2]

(ii) Assuming AM92 Ultimate mortality and 5½% *pa* interest, calculate the expected value and standard deviation of L. [9]

[Total 11]

20.4 Calculate the annual premium payable in advance by a life now aged exactly 32, in respect of a deferred annuity payable from age 60 for 5 years certain and for life thereafter. The amount of the annuity is 400 *pa*, payable annually in arrear, and the insurer incurs an additional administration cost of 2 when each annuity payment is made. The premium is paid throughout the deferred period or until the earlier death of the policyholder.

Basis: AM92 Ultimate mortality, 6% *pa* interest

20.5 Exam style

An insurer issues a combined term assurance and annuity contract to a life aged 35. Level premiums are payable monthly in advance for a maximum of 30 years.

On death before age 65 a benefit is paid immediately. The benefit is £200,000 on death in the first year of the contract, £195,000 on death in the second year, £190,000 on death in the third year, *etc*, with the benefit decreasing by £5,000 each year until age 65. No benefit is payable on death after age 65.

On attaining age 65 the life receives a whole life annuity of £10,000 *pa* payable monthly in arrears.

Calculate the monthly premium using the following basis:

Mortality:	up to age 65:	AM92 Select
	over age 65:	PFA92C20
Interest:	4% *pa*	
Expenses:	Initial:	£350
	Regular:	Assumed incurred annually at the start of each year during the deferred period, equal to 45 in the first year, inflating at the rate of 4% *pa*
	Claim:	0.5% of each annuity payment

[12]

20.6 A life insurance company sells term assurance policies with a term of 2 years, with level premiums paid annually in advance, to male lives aged 60 exact at policy commencement. Each policy has a sum assured of £50,000, which is payable at the end of the year of death. The company prices the product assuming AM92 Ultimate mortality.

The premium is calculated from an equation of value, in which the expected present value of the premiums is set to equal the expected present value of the benefit payments plus 10% of the standard deviation of the present value of the benefit payments. Calculate the premium assuming 4% *pa* interest. Ignore expenses.

20.7 Exam style

An annual premium conventional with-profits 20-year endowment assurance policy, issued to a life aged exactly 40 has a basic sum assured of £10,000 payable at the end of the year of death. Premiums are calculated assuming AM92 Select mortality, 4% *pa* interest, initial expenses of £150 and claim-related expenses of 3% of the basic sum assured (payable at the end of the year of death or on maturity).

(i) Calculate the premium if the policy is assumed to provide simple bonuses of 2% of the sum assured vesting at the end of each policy year (*ie* the basic benefit amount will be increased by £200 at the end of each policy year for future claims). [6]

(ii) Calculate the premium if the policy is assumed to provide compound bonuses of 4% *pa* of the sum assured vesting at the end of each policy year (*ie* the basic benefit amount will be increased by a factor of 1.04 at the end of each policy year for future claims). [5]

[Total 11]

20.8 Exam style

A life insurance company issues whole life assurance policies to lives aged 50 exact for a sum assured of £75,000 payable at the end of the year of death. Premiums are payable annually in advance.

(i) Calculate the annual gross premium for each policy using the basis below. [4]

(ii) Calculate the minimum annual gross premium that the company should charge in order that the probability of making a loss on any one policy would be 10% or less. [6]

Basis:

Mortality:	AM92 Select
Interest:	6% per annum
Initial commission:	100% of the annual gross premium
Initial expenses:	£325
Renewal commission:	2.5% of each annual gross premium excluding the first
Renewal expenses:	£75 per annum at the start of the second and subsequent policy years

[Total 10]

20.9 Exam style

A life aged 60 exact purchases a special deferred term assurance policy for an overall term of 20 years.

Under this policy a sum assured of £100,000 is paid on death but only on death from age 65 exact up to the end of the term. On death between age 60 and 65 the benefit is equal to the total premiums paid without interest.

All payments on death are made at the end of the year of death. An annual premium paid in advance is payable for the full 20 year term.

Calculate the annual premium payable. [7]

Basis:	Mortality	AM92 Ultimate
	Interest	4% per annum
	Expenses	Ignore

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 20 Solutions

20.1 Because premiums, benefits and expenses are payable annually, we can use K_x, the curtate future lifetime, rather than T_x.

It is easiest to split the gross future loss random variable (FLRV) by whether the person dies before or after time n. So:

$$FLRV = I + e\ddot{a}_{\overline{K_x+1}|} - G\,\ddot{a}_{\overline{K_x+1}|} \qquad \text{if } K_x < n$$

(*ie* if the life does not survive to the end of the deferred period), and:

$$I + e\ddot{a}_{\overline{n}|} - G\,\ddot{a}_{\overline{n}|} + B\ddot{a}_{\overline{K_x+1-n}|}v^n + e'\ddot{a}_{\overline{K_x+1-n}|}v^n \qquad \text{if } K_x \geq n$$

(*ie* if the life survives to receive payments under the annuity).

Alternatively we can write:

$$\ddot{a}_{\overline{K_x+1-n}|}v^n = \ddot{a}_{\overline{K_x+1}|} - \ddot{a}_{\overline{n}|}$$

and collecting together terms we obtain:

$$FLRV = \begin{cases} I - (G-e)\,\ddot{a}_{\overline{K_x+1}|} & \text{if } K_x < n \\ I - (G-e)\,\ddot{a}_{\overline{n}|} + (B+e')\left(\ddot{a}_{\overline{K_x+1}|} - \ddot{a}_{\overline{n}|}\right) & \text{if } K_x \geq n \end{cases}$$

20.2 (i) $P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|} + 0.02S$

(ii) $P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|} + 0.02P\ddot{a}_{x:\overline{n}|}$ which simplifies to $0.98P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|}$

(iii) $P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|} + 0.02SA_{x:\overline{n}|}$ which simplifies to $P\ddot{a}_{x:\overline{n}|} = 1.02SA_{x:\overline{n}|}$

(iv) $P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|} + 0.5P + 0.03P(\ddot{a}_{x:\overline{n}|} - 1)$

This simplifies to:

$$0.97P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|} + 0.47P$$

20.3 (i) ***Future loss random variable***

The future loss random variable is:

$$L=\begin{cases}10{,}000v^{K_{60}+1}-200\ddot{a}_{\overline{K_{60}+1|}} & \text{if } K_{60}=0,1,2,3,4\\ -200\ddot{a}_{\overline{5|}} & \text{if } K_{60}\geq 5\end{cases}$$

where K_{60} denotes the curtate future lifetime of a life currently aged 60. [2]

(ii) ***Expectation and variance of future loss***

Since we don't have any tabulated functions for 5½% interest, we will need to do the calculations for the expected value numerically (by considering the possible cases for the time of death), rather than attempting an algebraic method using A functions etc.

The table below shows the possible values of L and their associated probabilities:

Curtate future lifetime, K_{60}	Loss, L	Probability
0	9,278.67	$q_{60}=0.0080220$
1	8,594.95	$p_{60}\,q_{61}=0.0089367$
2	7,946.87	${}_2p_{60}\,q_{62}=0.0099405$
3	7,332.58	${}_3p_{60}\,q_{63}=0.011039$
4	6,750.31	${}_4p_{60}\,q_{64}=0.012234$
≥ 5	–901.03	${}_5p_{60}=0.94983$

[5]

The probabilities can alternatively be calculated as $\dfrac{d_{60}}{l_{60}},\dfrac{d_{61}}{l_{60}},\dfrac{d_{62}}{l_{60}},\dfrac{d_{63}}{l_{60}},\dfrac{d_{64}}{l_{60}}$ *and* $\dfrac{l_{65}}{l_{60}}$.

The expected present value of the future loss random variable is:

$$E(L)=(9{,}278.67\times 0.0080220)+(8{,}594.95\times 0.0089367)+\cdots$$

$$+(-901.03\times 0.94983)$$

$$=-462.06$$ [1]

The variance of L is given by:

$$E\left(L^2\right)-\left[E(L)\right]^2$$

Now:

$$E\left(L^2\right)=\left(9{,}278.67^2\times 0.0080220\right)+\cdots+\left(901.03^2\times 0.94983\right)$$
$$=3{,}900{,}700\ (5\ sf)$$

So:

$$\text{var}(L)=3{,}900{,}700-(-462.06)^2=\left(£1{,}920\right)^2$$

ie the standard deviation of the present value of the loss is £1,920. [3]

[Total 9]

20.4 The premium equation is:

$$P\ddot{a}_{32:\overline{28|}}=(400+2)v^{28}\frac{l_{60}}{l_{32}}\left(a_{\overline{5|}}+v^5\frac{l_{65}}{l_{60}}a_{65}\right)$$

The factors are:

$$\ddot{a}_{32:\overline{28|}}=14.053\ \text{(tabulated on page 106 of the \textit{Tables})}$$

$$v^{28}\frac{l_{60}}{l_{32}}=\frac{1}{1.06^{28}}\times\frac{9{,}287.2164}{9{,}913.3821}=0.18327$$

$$a_{\overline{5|}}=4.2124$$

$$v^5\frac{l_{65}}{l_{60}}=\frac{1}{1.06^5}\times\frac{8{,}821.2612}{9{,}287.2164}=0.70977$$

$$a_{65}=\ddot{a}_{65}-1=9.569$$

So the premium equation becomes:

$$14.053P=402\times 0.18327\times 11.004=810.74$$

$$\Rightarrow P=810.74/14.053=57.69$$

20.5 If the monthly premium is P, the premium equation is:

$$12P\ddot{a}^{(12)}_{[35]:\overline{30|}}=205{,}000\bar{A}^{\ 1}_{[35]:\overline{30|}}-5{,}000(I\bar{A})^{\ 1}_{[35]:\overline{30|}}+1.005\times 10{,}000\frac{D_{65}}{D_{[35]}}a^{(12)}_{65}$$
$$+350+45\ddot{a}_{[35]:\overline{30|}@0\%}$$ [3]

The regular expenses inflate at the same compound rate at which they are discounted, so we can value them using an annuity factor calculated at a zero interest rate.

The factors (calculated using the appropriate mortality and interest rates) are:

$$\ddot{a}^{(12)}_{[35]:\overline{30|}} \approx \ddot{a}_{[35]:\overline{30|}} - \frac{11}{24}\left(1-\frac{D_{65}}{D_{[35]}}\right) = 17.631 - \frac{11}{24}\times\left(1-\frac{689.23}{2,507.02}\right) = 17.299 \qquad [1]$$

using the formula on page 36 of the *Tables.*

$$\bar{A}^{\;1}_{[35]:\overline{30|}} \approx 1.04^{½}\left(A_{[35]:\overline{30|}} - \frac{D_{65}}{D_{[35]}}\right) = 1.04^{½}\left(0.32187 - \frac{689.23}{2,507.02}\right) = 0.04788 \qquad [1]$$

$$(I\bar{A})^{\;1}_{[35]:\overline{30|}} \approx 1.04^{½}\left((IA)_{[35]} - \frac{D_{65}}{D_{[35]}}\left((IA)_{65} + 30A_{65}\right)\right)$$

$$= 1.04^{½}\left(7.47005 - \frac{689.23}{2,507.02}\left(7.89442 + 30\times 0.52786\right)\right)$$

$$= 0.96487 \qquad [2]$$

$$a^{(12)}_{65} = \ddot{a}^{(12)}_{65} - \frac{1}{12} \approx \ddot{a}_{65} - \frac{11}{24} - \frac{1}{12} = 14.871 - \frac{13}{24} = 14.329 \qquad [1]$$

$$\ddot{a}_{[35]:\overline{30|}@0\%} = \ddot{a}_{[35]@0\%} - (v@0\%)^{30}\frac{l_{65}}{l_{[35]}}\ddot{a}_{65@0\%} \qquad [1]$$

But:

$$v@0\% = 1$$

$$\ddot{a}_{x@0\%} = 1 + e_x$$

So, using AM92 Select mortality:

$$\ddot{a}_{[35]:\overline{30|}@0\%} = 1 + e_{[35]} - \frac{l_{65}}{l_{[35]}}\left(1+e_{65}\right)$$

$$= 1 + 43.909 - \frac{8,821.2612}{9,892.9151}\times\left(1+16.645\right)$$

$$= 29.175 \qquad [2]$$

So the premium equation becomes:

$$12P\times 17.299 = 205,000\times 0.04788 - 5,000\times 0.96487 + 10,050\times\frac{689.23}{2,507.02}\times 14.329$$

$$+350 + 45\times 29.175$$

So: $P = \dfrac{46,245.05}{12\times 17.299} = £222.78$ per month [1]

[Total 12]

20.6 The expected present value of the benefit outgo is:

$$50{,}000A^{1}_{60:\overline{2|}} = 50{,}000\left(A_{60} - \frac{D_{62}}{D_{60}}A_{62}\right)$$

$$= 50{,}000\left(0.45640 - \frac{802.40}{882.85}\times 0.48458\right)$$

$$= 50{,}000 \times 0.0159775$$

$$= £798.88$$ [1]

The variance per unit sum assured of the present value is equal to ${}^{2}A^{1}_{60:\overline{2|}} - (A^{1}_{60:\overline{2|}})^2$.

From first principles, the variance is:

$$v'q_{60} + v'^2 p_{60}q_{61} - (0.0159775)^2$$ [1]

where v' is calculated at $1.04^2 - 1 = 8.16\%$. So the variance is:

$$\left(1.0816^{-1}\times 0.008022 + 1.0816^{-2}\times 0.991978\times 0.009009\right) - (0.0159775)^2$$

$$= 0.014801$$ [2]

Therefore, the variance for a sum assured of £50,000 is:

$$50{,}000^2 \times 0.014801 = 6{,}082.9^2$$

and so the standard deviation is $6{,}082.9$. [1]

The expected present value of the premiums is:

$$P\ddot{a}_{60:\overline{2|}} = P\left(1 + v\,p_{60}\right) = P\left(1 + \frac{0.991978}{1.04}\right) = 1.953825P$$ [1]

Hence:

$$P = \frac{798.88 + 608.29}{\ddot{a}_{60:\overline{2|}}} = \frac{1{,}407.17}{1.953825} = £720.21$$ [1]

[Total 7]

20.7 (i) ***Simple bonuses***

The benefit amount on death will be £10,000 for the first year, £10,200 for the second year, £10,400 for the third year, *etc*, and the benefit amount on maturity will be £14,000.

So the premium equation can be expressed as:

$$P\ddot{a}_{[40]:\overline{20|}} = 9{,}800A_{[40]:\overline{20|}} + 200(IA)^{\;1}_{[40]:\overline{20|}} + 4{,}200\frac{D_{60}}{D_{[40]}} + 150 + 300A_{[40]:\overline{20|}}$$ [2]

The increasing assurance function can be calculated as:

$$(IA)^{\;1}_{[40]:\overline{20}|} = (IA)_{[40]} - \frac{D_{60}}{D_{[40]}}\left[(IA)_{60} + 20A_{60}\right]$$

$$= 7.95835 - \frac{882.85}{2{,}052.54}\left[8.36234 + 20 \times 0.45640\right]$$

$$= 0.43531 \qquad [2]$$

So the premium equation becomes:

$$13.930P = 9{,}800 \times 0.46423 + 200 \times 0.43531 + 4{,}200 \times \frac{882.85}{2{,}052.54}$$

$$+150 + 300 \times 0.46423$$

$$= 6{,}732.31 \qquad [1]$$

Hence:

$$P = \frac{6{,}732.31}{13.930} = £483.30 \qquad [1]$$

[Total 6]

(ii) ***Compound bonuses***

The benefit amount on death will be £10,000 for the first year, $£10{,}000 \times 1.04$ for the second year, $£10{,}000 \times 1.04^2$ for the third year, *etc*, and the benefit amount on maturity will be $£10{,}000 \times 1.04^{20}$.

The expected present value of the death benefit is:

$$10{,}000 \times \left(v\,{}_{0|}q_{[40]} + 1.04v^2\,{}_{1|}q_{[40]} + 1.04^2 v^3\,{}_{2|}q_{[40]} + \cdots + 1.04^{19} v^{20}\,{}_{19|}q_{[40]}\right)$$

$$= \frac{10{,}000}{1.04} \times \left(1.04v\,{}_{0|}q_{[40]} + 1.04^2 v^2\,{}_{1|}q_{[40]} + 1.04^3 v^3\,{}_{2|}q_{[40]} + \cdots + 1.04^{20} v^{20}\,{}_{19|}q_{[40]}\right)$$

But:

$$1.04v = \frac{1.04}{1.04} = 1$$

So the EPV of the death benefit becomes:

$$\frac{10{,}000}{1.04} \times \left({}_{0|}q_{[40]} + {}_{1|}q_{[40]} + {}_{2|}q_{[40]} + \cdots + {}_{19|}q_{[40]}\right) = \frac{10{,}000}{1.04} \times {}_{20}q_{[40]}$$

The EPV of the maturity benefit is:

$$10{,}000 \times 1.04^{20} v^{20}\,{}_{20}p_{[40]} = 10{,}000\,{}_{20}p_{[40]}$$

So the premium equation can be expressed as:

$$P\ddot{a}_{[40]:\overline{20}|} = \frac{10{,}000}{1.04} \times {}_{20}q_{[40]} + 10{,}000\,{}_{20}p_{[40]} + 150 + 300A^{@4\%}_{[40]:\overline{20}|} \qquad [2]$$

The probabilities are:

$${}_{20}p_{[40]} = \frac{l_{60}}{l_{[40]}} = \frac{9{,}287.2164}{9{,}854.3036} = 0.94245 \qquad [½]$$

and:

$${}_{20}q_{[40]} = 1 - {}_{20}p_{[40]} = 1 - 0.94245 = 0.05755 \qquad [½]$$

So the premium equation becomes:

$$13.930P = \frac{10{,}000}{1.04} \times 0.05755 + 10{,}000 \times 0.94245 + 150 + 300 \times 0.46423$$

$$= 10{,}267.14 \qquad [1]$$

and hence:

$$P = \frac{10{,}267.14}{13.930} = £737.05 \qquad [1]$$

[Total 5]

20.8 *This question is Subject CT5, April 2013, Question 12.*

(i) ***Gross premium***

Letting G denote the gross premium, the expected present value of the gross premiums is:

$$G\ddot{a}_{[50]} = 14.051G \qquad [1]$$

The expected present value of the benefits is:

$$75{,}000A_{[50]} = 75{,}000 \times 0.20463 = 15{,}347.25 \qquad [1]$$

The expected present value of the commission and expenses is:

$$G + 325 + 0.025G(\ddot{a}_{[50]} - 1) + 75(\ddot{a}_{[50]} - 1)$$

Using $\ddot{a}_{[50]} = 14.051$, this simplifies to:

$$1.326275G + 1{,}303.825 \qquad [1]$$

The premium equation is therefore:

$$14.051G = 15,347.25 + 1.326275G + 1,303.825$$

$$\Rightarrow \quad G = \frac{16,651.075}{12.724725} = £1,308.56$$

The gross premium is £1,308.56. [1]

[Total 4]

(ii) ***Minimum gross premium so that the probability of loss is 10% or less***

The insurer's gross future loss random variable for the policy is:

$$L = 75,000v^{K_{[50]}+1} + G + 325 + (0.025G + 75)a_{\overline{K_{[50]}|}} - G\ddot{a}_{\overline{K_{[50]}+1|}}$$

$$= 75,000v^{K_{[50]}+1} + 325 + 75a_{\overline{K_{[50]}|}} - 0.975Ga_{\overline{K_{[50]}|}} \qquad [2]$$

where G is the gross annual premium. Let G_n denote the premium such that $L=0$ when $K_{[50]}=n$, *ie* to make the policy breakeven if the person dies in year $[n,\ n+1]$. Then:

$$0 = 75,000v^{n+1} + 325 + 75a_{\overline{n|}} - 0.975G_n a_{\overline{n|}}$$

$$\Rightarrow G_n = \frac{75,000v^{n+1} + 325 + 75a_{\overline{n|}}}{0.975a_{\overline{n|}}} \qquad [1]$$

Now we make a loss on the policy if the policyholder dies before time n. So:

$$P\left(L > 0 \middle| G = G_n\right) = P\left(K_{[50]} < n\right)$$

and this probability needs to be at most 10%. G_n reduces as n increases, so to find the minimum premium we need to find the largest value of n such that:

$$P\left(K_{[50]} < n\right) \le 0.1$$

This is equivalent to finding n such that:

$$P\left(K_{[50]} \ge n\right) = {}_n p_{[50]} = \frac{l_{[50]+n}}{l_{[50]}} \ge 0.9$$

Using the AM92 table, $l_{[50]} = 9,706.0977$, so we have:

$$l_{[50]+n} \ge 0.9 \times 9,706.0977 = 8,735.488$$

From the *Tables*: $l_{65} = 8,821.2612$ and $l_{66} = 8,695.6199$.

So, the largest value of n to satisfy this inequality is 15. [2]

The minimum premium required is then:

$$G_{15} = \frac{75{,}000v^{16} + 325 + 75a_{\overline{15}|}}{0.975a_{\overline{15}|}}$$

Using page 58 of the *Tables*:

$$a_{\overline{15}|} = 9.7122$$

Substituting this into our equation gives:

$$G_{15} = \frac{75{,}000v^{16} + 325 + 75 \times 9.7122}{0.975 \times 9.7122} = \frac{30{,}576.89}{9.4694} = £3{,}229.02$$

So, the minimum premium to ensure that the probability of making a loss is at most 10% is £3,229.02. [1]

[Total 6]

20.9 *This question is Subject CT5, September 2014, Question 9.*

The premium equation is:

$$P\ddot{a}_{60:\overline{20}|} = 100{,}000 \times \frac{D_{65}}{D_{60}} \times A^{\ \ 1}_{65:\overline{15}|} + P(IA)^{\ \ 1}_{60:\overline{5}|}$$ [2]

Evaluating the various terms:

$$\ddot{a}_{60:\overline{20}|} = \ddot{a}_{60} - \frac{D_{80}}{D_{60}}\ddot{a}_{80} = 14.134 - \frac{228.48}{882.85} \times 6.818 = 12.370$$ [1]

$$A^{\ \ 1}_{65:\overline{15}|} = A_{65} - \frac{D_{80}}{D_{65}}A_{80} = 0.52786 - \frac{228.48}{689.23} \times 0.73775 = 0.28330$$ [1]

and:

$$(IA)^{\ \ 1}_{60:\overline{5}|} = (IA)_{60} - \frac{D_{65}}{D_{60}}\left[(IA)_{65} + 5A_{65}\right]$$

$$= 8.36234 - \frac{689.23}{882.85}\left[7.89442 + 5 \times 0.52786\right] = 0.13880$$ [2]

Putting this all together, we find that:

$$12.370P = 100{,}000 \times \frac{689.23}{882.85} \times 0.28330 + 0.13880P$$

Solving this, we find that $P = 1{,}808.28$, or about £1,808. [1]

[Total 7]

21

Gross premium reserves

Syllabus objectives

4.2 Develop formulae for the means and variances of the payments under various assurance and annuity contracts, assuming constant deterministic interest rates.

4.2.8 Define and evaluate the expected accumulations in terms of expected values for the contracts described in 4.1.1 and contract structures described in 4.2.7.

6.2 Describe and calculate gross premiums and reserves of assurance and annuity contracts.

6.2.2 State why an insurance company will set up reserves.

6.2.3 Define and calculate gross prospective and retrospective reserves.

6.2.4 State the conditions under which, in general, the prospective reserve is equal to the retrospective reserve allowing for expenses.

6.2.5 Prove that, under the appropriate conditions, the prospective reserve is equal to the retrospective reserve, with or without allowance for expenses, for all fixed benefit and increasing / decreasing benefit contracts.

6.2.6 Obtain recursive relationships between successive periodic gross premium reserves, and use this relationship to calculate the profit earned from a contract during the period.

6.2.7 Outline the concepts of net premiums and net premium valuation and how they relate to gross premiums and gross premium valuation respectively.

0 Introduction

In this chapter we introduce the concept of a reserve, and how it can be calculated using either prospective or retrospective approaches. The prospective formula is based on the gross future loss random variable, which we introduced in Chapter 20, while the retrospective formula requires the use of accumulations, which are described in this chapter.

A reserve represents the amount of money that an insurer sets aside in respect of a policy that is currently in force, in order to meet future payments on that policy. It is important in real life, because an insurer that holds insufficient reserves will ultimately run out of money before all of its policyholders have made their claims, meaning that the company becomes insolvent and policyholders lose money, which can potentially be very large amounts.

Reserve calculations are required for all the kinds of contracts covered in the course so far. The same approaches will be used to calculate reserves for contracts on multiple lives, which we describe in Chapters 22 and 23. The reserve is an essential part of calculating mortality profit, an important topic which is covered in Chapter 24. In the profit testing chapters (Chapters 27 and 28), we will also see an important alternative methodology for calculating reserves that uses cashflow projections.

We also show in this chapter how thinking about the change in the reserve from one year to the next leads to a recursive relationship between successive years' reserves, and which also leads to an important definition of the profit earned by a contract during any given year. The whole concept of profit testing is based on this result.

We generally talk throughout about *gross premium* reserves. These, like gross premiums themselves (Chapter 20), include assumptions about future expenses in their calculation. At the end of this chapter we briefly describe the idea of *net premium* reserves, which make no explicit allowance for expenses. These are included in the course mainly for historical interest, as they are no longer much used in practice; however, a basic knowledge of what they are, and how they should be calculated for conventional without-profit contracts, is required for this subject.

1 Why hold reserves?

In many life insurance contracts, the expected cost of paying benefits increases over the term of the contract. Let us consider an endowment assurance as an example. The probability that the benefit will be paid in the first few years is relatively low because the life is young and expected to be in good health. Later, the expected cost increases as the life ages and the probability of a claim by death increases. In the final year the payment is certain, either on death during the year or on maturity at the end of that year, therefore the expected cost of the benefit will be close to the actual sum assured.

If we ignore discounting, the expected cost in the final year is *equal* to the sum assured.

Although on average the cost of the benefit increases over the term, the premiums which pay for these benefits are usually level. This means that premiums received in the early years of a contract are more than enough to pay the expected benefits that fall due in those early years, but in the later years the premiums are too small to pay for the expected benefits.

It is, therefore, prudent for the premiums which are not required in the early years of a contract to be set aside, or reserved, to fund the shortfall in the later years. While funds are reserved, they are invested so that interest also contributes to the cost of benefits.

If the life insurance company was to spend all the premiums received, perhaps by distributing the surplus to shareholders, in later years it may not have sufficient funds to pay for the excess of the cost of benefits over the premiums received. The company sets up reserves to ensure (as far as possible) that this does not happen, and to remain solvent.

Consider the case of a term assurance sold to a 30-year old British male, with a 10-year term and a sum assured of £500,000. The premium is £48 per month, paid over the 10-year term or until the earlier death of the policyholder. The premium is level (£48 per month) but the expected cost of the benefits is increasing, since the policyholder is getting older and UK mortality rates increase after age 30.

For example, at age 30 the expected cost is $500{,}000q_{30}$, but at age 39 it is $500{,}000q_{39}$, which is bigger than $500{,}000q_{30}$.

When designing the products sold by the insurer, the actuary will always try to ensure that in the later years the benefit payments are greater than the premiums payable.

To understand why this is the case, suppose now that the product is redesigned so that the sum assured is falling over the term of the contract, say by £50,000 per year. The premium is recalculated to be £23 per month.

In that case, the average cost of benefits may well be falling over time (not rising as in the previous contract) since the sum assured is falling. For example, at age 30 the expected cost is $500{,}000q_{30}$, but at age 39 it is $50{,}000q_{39}$.

This can lead to an issue for the insurer if the policyholder decides to stop paying premiums before the full term of the policy has expired. (This event is referred to as a surrender, or lapse, of the policy. Generally we use the term 'surrender' to refer to the case where the policyholder is given some cash payment (the 'surrender value') at the point where premiums cease, and we use the term 'lapse' where no cash payment is made. For term assurances it is normal practice for such policies to lapse without any surrender value being payable.)

Returning to our example, if the premiums are greater than the expected value of the benefits, say in the final few years of the contract, then this may encourage the policyholder to stop paying premiums, and hence cause the policy to lapse. The result is that the policyholder will have held the contract for the time period for which the benefit is greater than the premium but not for the subsequent period when premiums exceed benefits.

Question

Consider the situation described in the last paragraph above. Explain the effect this would have on the insurer.

Solution

We can explain this by thinking of the future period for which premiums are not going to be paid. During the final few years of this contract the expected present value of the premiums payable exceed the expected present value of the benefits that would be paid out – in other words, this period of the policy term, considered on its own, is expected to produce a profit for the insurer. So if we remove this profitable period, then overall the profitability of the contract must fall.

So, whenever any policy is lapsed in this situation, the insurer will expect to lose money.

2 Prospective reserves

The prospective reserve for a life insurance contract which is in force (that is, has been written but has not yet expired through claim or reaching the end of the term) is defined to be, for a given basis:

The expected present value of the future outgo

less

the expected present value of the future income

This is the prospective reserve because it looks forward to the future cashflows of the contract. The prospective reserve is important because if the company holds funds equal to the reserve, and the future experience follows the reserve basis, then, averaging over many policies, the combination of reserve and future income will be sufficient to pay the future liabilities.

This last sentence is very important and forms the foundation of life insurance reserving and hence the rest of this chapter. That is to say that the money an insurer needs to set aside to meet its future payments is the expected present value of future outgo (benefits plus expenses) less the expected present value of future income (premiums).

The reserve, therefore, gives the office a measure of the minimum funds it needs to hold at any point *during* the term of a contract. The process of calculating a reserve is called the *valuation* of the policy.

Reserves are also calculated for other reasons, such as the calculation of surrender values. The insurer may set the surrender value (*ie* the amount paid to the policyholder) by reference to the reserve.

Question

An annual premium endowment policy is surrendered one year before the maturity date. Explain why the reserve is a sensible surrender value for the insurer to pay.

Solution

If the policy remains in force, the insurer has to make a benefit payment in the next year (either on maturity or on the death of the policyholder). However, if the policy is surrendered, no benefit payment will be made and also the final renewal expense will not be incurred. It is therefore reasonable to set the surrender benefit equal to the sum assured plus final year's renewal expenses, less the final premium and the cost of administrative expenses associated with processing the surrender.

2.1 Calculating gross premium prospective reserves

The expression for the gross future loss at policy duration t can be used to determine the gross premium prospective reserve at policy duration t. This is done by determining the expected value of the gross future loss random variable.

In general we use the notation ${}_tV$ to represent the reserve held at policy duration t.

It should be noted that the calculation of gross premium prospective reserves will use mortality, interest, expense, and (if applicable) future bonus assumptions specifically chosen for this purpose. This set of assumptions, called the gross premium valuation basis, may differ from the underlying basis used to calculate the actual gross premiums which the company charges to policyholders.

The premium used in the calculation of the gross premium reserve is always the actual gross premium being charged.

Example: Whole life assurance

Consider a whole life assurance contract that has:

- a sum assured of S payable immediately on death
- annual premiums of G payable for the duration of the contract
- regular expenses of e *pa* incurred at the start of each year while the policy is in force
- claim expenses of f
- age at entry equal to x.

The gross future loss random variable after exactly t years ($t = 1, 2, \ldots$) is:

$$L_t = Sv^{T_{x+t}} + e\ddot{a}_{\overline{K_{x+t}+1}|} + fv^{T_{x+t}} - G\ddot{a}_{\overline{K_{x+t}+1}|}$$

The prospective reserve after exactly t years is therefore:

$$\begin{aligned} {}_tV^{pro} &= E\left[L_t\right] \\ &= S \times E\left[v^{T_{x+t}}\right] + e \times E\left[\ddot{a}_{\overline{K_{x+t}+1}|}\right] + f \times E\left[v^{T_{x+t}}\right] - G \times E\left[\ddot{a}_{\overline{K_{x+t}+1}|}\right] \\ &= S\bar{A}_{x+t} + e\ddot{a}_{x+t} + f\bar{A}_{x+t} - G\ddot{a}_{x+t} \end{aligned}$$

where 'pro' stands for 'prospective'.

So we can think of the gross premium prospective reserve as:

EPV of future benefits + EPV of future expenses – EPV of future premiums

Now consider the more general case for this contract, where premiums and renewal expenses are payable $m$thly in advance.

The expected value of the gross future loss random variable at policy duration t is:

$$S\,\bar{A}_{x+t} + e\,\ddot{a}^{(m)}_{x+t} + f\,\bar{A}_{x+t} - G\,\ddot{a}^{(m)}_{x+t}$$

This can be evaluated using an assumed valuation basis to determine the values of the annuity and assurance functions, including an assumption for future expenses. G is the gross (*ie* actual) premium payable under the contract.

In order to calculate a gross premium reserve for regular premium products, we therefore first need to know the amount of the gross premium. (In the exam this may mean the first thing we might have to do is to *calculate* the gross premium.) This will not be necessary for single premium policies.

Question

Calculate the gross premium prospective reserve that should be held at the end of the tenth policy year of a 25-year regular premium endowment assurance policy with a sum assured of £75,000 payable on maturity or at the end of the year of earlier death. The policy was taken out by a person aged exactly 45 at entry.

The gross premium, which is being paid annually in advance for the duration of the policy, is £2,132 *pa*.

Expenses are 75% of the first premium and 5% of each subsequent premium, plus an initial expense of £250.

Assume AM92 Ultimate mortality and 4% *pa* interest.

Solution

The prospective reserve at the end of Year 10 is:

$${}_{10}V^{pro} = \text{EPV future benefits and expenses} - \text{EPV future premiums}$$

The components are:

$$\text{EPV future premiums} = P\ddot{a}_{55:\overline{15|}}$$

where:

$$\ddot{a}_{55:\overline{15|}} = \ddot{a}_{55} - \frac{D_{70}}{D_{55}}\ddot{a}_{70} = 15.873 - \frac{517.23}{1{,}105.41} \times 10.375 = 11.0185$$

So:

$$\text{EPV future premiums} = 2{,}132 \times 11.0185 = 23{,}491$$

Now:

$$\text{EPV future benefits} = 75{,}000 A_{55:\overline{15}|}$$

Using premium conversion:

$$A_{55:\overline{15}|} = 1 - d\ddot{a}_{55:\overline{15}|} = 1 - \left(\frac{0.04}{1.04}\right) \times 11.0185 = 0.57621$$

So:

$$\text{EPV future benefits} = 75{,}000 \times 0.57621 = 43{,}216$$

Also:

$$\text{EPV future expenses} = 0.05P\ddot{a}_{55:\overline{15}|} = 0.05 \times 23{,}491 = 1{,}175$$

So we have:

$${}_{10}V^{pro} = 43{,}216 + 1{,}175 - 23{,}491 = £20{,}899$$

We now consider a prospective reserve for an annuity contract.

Question

Andy, aged 40, purchases a single premium whole life annuity of £8,400 *pa* payable monthly in advance from age 60. Initial expenses are 2% of the premium and renewal expenses are £60 *pa* from Year 2 onwards, including during payment of the annuity, assumed incurred annually in advance throughout.

Calculate the reserve for Andy's policy at the end of the tenth policy year. Assume interest of 4% *pa*, mortality AM92 Ultimate in deferment and PMA92C20 from age 60.

Solution

It is a single premium contract so we do not need to calculate the premium in order to calculate the prospective reserve. The gross premium prospective reserve is:

EPV of future benefits + EPV of future expenses

The expected present value of the future benefits is:

$$8{,}400\ {}_{10|}\ddot{a}_{50}^{(12)} = 8{,}400 \frac{D_{60}}{D_{50}} \ddot{a}_{60}^{(12)}$$

where D_{50} and D_{60} are taken from the AM92 Table, and $\ddot{a}_{60}^{(12)}$ is taken from the PMA92C20 Table.

Therefore the EPV of the future benefits is:

$$8{,}400\times\frac{882.85}{1{,}366.61}\times\left(15.632-\tfrac{11}{24}\right)=82{,}340$$

The expected present value of the future expenses is:

$$60\ddot{a}_{50}=60\left(\ddot{a}_{50:\overline{10|}}+\frac{D_{60}}{D_{50}}\ddot{a}_{60}\right)$$

with $\ddot{a}_{50:\overline{10|}}$ taken from the AM92 Table.

So the expected value of the future expenses is:

$$60\left(8.314+\frac{882.85}{1{,}366.61}\times15.632\right)=1{,}105$$

We need to be careful about valuing the pre-age 60 and the post-age 60 elements separately due to the different mortality assumptions before and after age 60.

Thus the reserve is:

$$82{,}340+1{,}105=£83{,}445$$

2.2 Calculating prospective reserves that satisfy probabilities

In the previous chapter, we explained how gross premiums can be calculated that satisfy probabilities. A similar approach can be used to calculate prospective reserves.

Suppose an insurance company is carrying out a valuation of its in-force business. The following details relate to one particular policy that is in force on the valuation date:

Policy type:	Whole life assurance
Benefit:	£75,000 payable at the end of the year of death
Entry age:	50 exact
Current duration in force:	8 years exact
Annual premium:	£1,500 (paid at the start of each year)

Let us now calculate the smallest reserve that could be held at the valuation date, which will ensure that the insurance company can cover the liability under this contract with a probability of at least 97.5%, assuming interest of 3% *pa* and that mortality follows the AM92 Ultimate table. We shall ignore expenses for simplicity.

The future loss that needs to be covered by the reserve at the present time is:

$$L=75{,}000v^{K_{58}+1}-1{,}500\ddot{a}_{\overline{K_{58}+1|}}$$

If V is the reserve, then we need the smallest value of V such that:

$$P(L \le V) \ge 0.975$$

Let V_n be the reserve needed to cover the loss *exactly* when $K_{58} = n$ $(n = 0, 1, 2, \ldots)$, that is, assuming the policyholder dies in year $n+1$, counting from now. So:

$$V_n = 75{,}000v^{n+1} - 1{,}500\ddot{a}_{\overline{n+1|}} \qquad (*)$$

Suppose we were now to *hold* a reserve of amount V_n. The probability that this reserve is large enough to cover the loss is:

$$P\left(L \le V \middle| V = V_n\right) = P(K_{58} \ge n)$$

because the larger the value of K_{58} (*ie* the later that death occurs) the smaller will be the value of L.

However, we also need:

$$P(K_{58} \ge n) \ge 0.975$$

Now, as:

$$P(K_{58} \ge n) = {}_n p_{58} = \frac{l_{58+n}}{l_{58}}$$

we require:

$$\frac{l_{58+n}}{l_{58}} \ge 0.975 \quad \Rightarrow \quad l_{58+n} \ge 0.975 \times l_{58} = 0.975 \times 9{,}413.8004 = 9{,}178.4554 \qquad (**)$$

We now need to choose the particular value of n that produces the smallest reserve that satisfies this condition.

Question

Without performing any calculations, by considering expression (*) above explain whether a larger or a smaller value of n would cause the reserve amount V_n to reduce.

Solution

A larger value of n will reduce the present value of the benefits, by discounting the payment over a longer period; and it will increase the number (and hence present value) of the premiums deducted. Both of these effects will cause the value of V_n to reduce.

So, to find the smallest reserve to cover the loss, we need the largest value of n that satisfies (**). From the *Tables*, we find:

$$l_{61} > 9{,}178.4554 > l_{62}$$

and so the largest value of n that satisfies (**) is 3. Therefore the smallest reserve we can hold to cover the loss with a probability of at least 97.5% is:

$$V_3 = 75{,}000v^4 - 1{,}500\ddot{a}_{\overline{4}|} = 75{,}000 \times 1.03^{-4} - 1{,}500 \times \left[\frac{1-1.03^{-4}}{0.03/1.03}\right] = £60{,}894$$

Question

Suppose that the insurer in the above example has two of these policies in force, on independent lives.

The insurer now wishes to calculate the smallest total reserve that would ensure it could meet its liabilities under both policies with a probability of at least 97.5%.

Without performing any more calculations, explain whether the total reserve would be:

I less than double

II exactly double, or

III more than double

the reserve that is required for one policy when considered on its own.

Solution

Option I is correct.

The two losses are independent and identically distributed. If L_1 and L_2 are the present values of the losses on the two policies, then:

$$E[L_1 + L_2] = 2E[L]$$

and:

$$\text{var}[L_1 + L_2] = 2\,\text{var}[L]$$

This means that the standard deviation of the sum of the losses is:

$$SD[L_1 + L_2] = \sqrt{2\,\text{var}[L]} = 1.414\,SD[L]$$

As the standard deviation has increased by (considerably) less than double, then $P(L_1 + L_2 > 2 \times 60{,}894)$ will be much smaller than 2.5%. That is, we can hold a total reserve that is considerably smaller than $2 \times 60{,}894$ and still be able to meet the required probability of loss.

In the above question, we see the beginnings of the effect of the 'law of large numbers', by which all insurance companies are able to pool their risks by insuring a large number of independent lives or policies (according to the Central Limit Theorem of statistics). The greater the number of independent risks the insurer has on its books, the smaller the reserve required *per policy* that will leave the insurer with an acceptable risk of loss. With enough policies in force, the reserve required per policy will be close in size to the expectation of the present value of the loss – that is, a prospective reserve calculation.

2.3 Gross premium prospective reserves for conventional with-profits policies

In the case of conventional with-profits policies, we must define carefully how bonuses are to be allowed for, both in the future benefits to be valued and the gross premium.

If asked to calculate a gross premium reserve for a with-profits policy in the CM1 exam, just follow the instructions given in the question.

Normally, the future benefits to be valued will include at least the level of bonuses added to the point of calculation of the reserve. However, normally also the full value of the future gross premium payments will be deducted, and this will effectively discount all the loadings for bonuses contained within the gross premiums that are still to be received. In the context of a reserve calculated at some point in a policy's life, the historical premiums paid to date plus the discounted value of the future premiums will effectively capitalise all the premium loadings for bonuses. This would include both those already added as at the date of the reserve calculation and any which could be added thereafter. It can then be seen that calculating benefits allowing just for bonuses to date, but then deducting the full value of the gross premiums, may produce a rather weak, *ie* low, reserve.

For this reason, some level of future bonus is normally also valued as a prospective future benefit. This may or may not include an allowance for terminal bonus, depending mainly on the purpose of the calculation.

Question

A conventional with-profits whole life assurance policy with an initial sum assured of £100,000 was taken out 5 years ago by a man who was then aged exactly 45, for an annual premium of £2,500. The insurance company has declared compound reversionary bonuses of 3% *pa* each year during this period. All benefits are paid at the end of the year of death.

Calculate the prospective gross premium reserve at the present time using the following assumptions:

Mortality:	AM92 Ultimate
Interest:	6% *pa*
Future bonuses:	1.9231% *pa* compound, vesting at the end of each year
Future expenses:	£40 *pa* paid at the start of each year, plus £300 payable on a claim

Solution

The prospective gross premium reserve is given by:

EPV of future benefits + EPV of future expenses – EPV of future premiums

The current sum assured, allowing for five years of past bonuses of 3% *pa* compound, is:

$$100{,}000 \times 1.03^5 = 115{,}927.41$$

Defining $b = 0.019231$ the EPV of the future benefit payments is:

$$115{,}927.41 \times \left(v\ {}_{0|}q_{50} + (1+b)v^2\ {}_{1|}q_{50} + (1+b)^2 v^3\ {}_{2|}q_{50} + \cdots \right)$$

$$= \frac{115{,}927.41}{1+b} \times \left((1+b)v\ {}_{0|}q_{50} + (1+b)^2 v^2\ {}_{1|}q_{50} + (1+b)^3 v^3\ {}_{2|}q_{50} + \cdots \right)$$

$$= 115{,}927.41 \times \frac{A_{50}^{4\%}}{1.019231}$$

because $(1+b)v = \dfrac{1.019231}{1.06} = \dfrac{1}{1.04} = v$ calculated at 4% interest.

So the expected present value of the benefits is:

$$115{,}927.41 \times \frac{0.32907}{1.019231} = 37{,}428.45$$

The EPV of the future expenses is:

$$40\ddot{a}_{50} + 300A_{50} \quad @6\%$$

$$= 40 \times 14.044 + 300 \times 0.20508 = 623.28$$

The EPV of the future premium payments is:

$$2{,}500\ddot{a}_{50} = 2{,}500 \times 14.044 = 35{,}110$$

So the gross premium prospective reserve is:

$${}_5V^{pro} = 37{,}428.45 + 623.28 - 35{,}110 = 2{,}941.73$$

2.4 Reserve conventions

We often calculate reserves at integer durations, *ie* whole years. In this case, we calculate the reserve just before any payment of premium due on that date, and just after any payment of benefit payable in arrears due on that date.

The benefit being referred to here is any payment that's made at the end of the year if the policyholder survives to the end of the year, such as an annuity payable in arrears, for example. We don't have to worry about the timing of *death* benefits because reserves are only required for policies that are still in force.

The general rule is, for valuation on the t th policy anniversary, payments in respect of the year $t-1$ to t payable in arrears (*ie* on the t th anniversary) are assumed to have been paid, payments in respect of the year t to $t+1$ payable in advance (and so are also due on the t th anniversary) are assumed not yet to have been paid.

3 Retrospective reserves

The retrospective reserve for a life insurance contract that is in force is defined to be, for a given basis:

The accumulated value allowing for interest and survivorship of the premiums received to date

less

the accumulated value allowing for interest and survivorship of the benefits and expenses paid to date

By 'accumulated value' we mean the *average* amount of money that would be accumulated per policy by a group of identical (but independent) policies over a period of time. In essence this reserve is calculated by 'looking backwards' to the payments that were expected to have occurred under the policy up to now.

The retrospective reserve on a given basis tells us how much the premiums less expenses and claims have accumulated to, averaging over a large number of policies.

To calculate this, we need to understand retrospective accumulations.

3.1 Retrospective accumulations

In mathematics of finance there are two common viewpoints from which a stream of cashflows may be considered.

(1) Prospectively, leading to the calculation of present values

(2) Retrospectively, leading to the calculation of accumulations.

In this section we discuss the latter approach allowing for the presence of mortality.

The 'retrospective accumulation' can be thought of as the 'pot of money' accumulated in respect of a policy, *ie* premiums paid, plus interest, less expenses, less the cost of life cover. The amount is often referred to as the 'asset share' of a policy.

The basic idea is that we consider a *group* of lives, who are regarded as identical and stochastically independent as far as mortality is concerned. At age x, each life transacts an identical life insurance contract. Under these contracts, payments will be made (the direction of the payments is immaterial), depending on the experience of the members of the group. We imagine these payments being accumulated in a fund at rate of interest i. After n years, we divide this fund equally among the surviving members of the group. (If the fund is negative we imagine charging the survivors in equal shares.) The *retrospective accumulation* is defined as the amount that each survivor would receive, as the group size tends towards infinity.

Question

A fund of £1,000,000 has 10,000 members aged 40. The fund accumulates at an interest rate of 4% per annum effective and will be divided between all members who survive to age 60.

Based on AM92 Ultimate mortality, calculate the expected payout for each survivor.

Solution

The accumulated fund will be:

$$1,000,000 \times 1.04^{20} = 2,191,123$$

The expected number of survivors will be:

$$10,000 \times \frac{l_{60}}{l_{40}} = 10,000 \times \frac{9,287.2164}{9,856.2863} = 9,422.63$$

So the expected payout per survivor is:

$$\frac{2,191,123}{9,422.63} = £232.54$$

In the solution to this question we divided the projected fund (£2,191,123) by the *expected* number of survivors (9,422.63). However, the number of survivors, N say, is actually a random variable. So logically, we should be looking at the expected value of $\frac{2,191,123}{N}$, not $\frac{2,191,123}{E(N)}$.

From Subject CS1, we know that, in general, $E\left(\frac{2,191,123}{N}\right) \neq \frac{2,191,123}{E(N)}$. So we need to justify the approach we have just used, which we can do as follows.

Suppose that there are L_n survivors at age $x+n$ out of L 'starters' at age x, and that the accumulated fund at age $x+n$ is $F_n(L)$. The retrospective accumulation of the benefit under consideration is defined to be:

$$\lim_{L \to \infty} \frac{F_n(L)}{L_n}$$

Clearly L_n and $F_n(L)$ are random variables. The process of taking the limit eliminates the awkward possibility that $L_n = 0$, but also since:

$$\lim_{L \to \infty} \frac{F_n(L)}{L} = E[F_n(1)]$$

and:

$$\lim_{L \to \infty} \frac{L_n}{L} = {}_n p_x$$

by the law of large numbers:

$$\lim_{L \to \infty} \frac{F_n(L)}{L_n} = \frac{E[F_n(1)]}{{}_n p_x}$$

This proof justifies the method we used in the last question.

Example 1: pure endowment

Consider the case for a pure endowment, that pays 1 on survival to time n. We need only consider a single life and calculate $E[F_n(1)]$. We can see that $F_n(1)$ has the following distribution:

$$F_n(1) = 0 \qquad \text{if } K_x < n$$

$$F_n(1) = 1 \qquad \text{if } K_x \geq n$$

So

$$E[F_n(1)] = 0 \times {}_nq_x + 1 \times {}_np_x = {}_np_x$$

Hence the accumulation of the pure endowment benefit is:

$$\frac{E[F_n(1)]}{{}_np_x} = \frac{{}_np_x}{{}_np_x} = 1$$

Question

Write down the retrospective accumulation after 10 years of the benefit payable under a pure endowment contract, which has a benefit amount of 50,000 and a term of 10 years.

Solution

50,000

We can explain this result intuitively as follows:

- the payment (of 50,000) is made at the accumulation date, so no interest will be added to accumulate it
- the payment is only paid to those who survive to the accumulation date, and so we are directly given the amount 'per survivor' without having to make any further adjustments for survival.

Example 2: term assurance

For an example less trivial than the pure endowment, consider a term assurance that pays 1 at the end of the year of death occurring within n years. Now:

$$F_n(1) = (1+i)^{n-(K_x+1)} \qquad \text{if } K_x < n$$

$$F_n(1) = 0 \qquad \text{if } K_x \geq n$$

That is, if (x) dies during the n years ($K_x < n$), the payment (of 1) is made at time $K_x + 1$ (which is the end of the year of death). We then accumulate this for the remaining period, *ie* over $n-(K_x+1)$ years.

If (x) survives the n years ($K_x \geq n$), no payment is made as there is no survival benefit for a term assurance contract.

So:

$$E[F_n(1)] = \sum_{k=0}^{n-1} (1+i)^{n-(k+1)} {}_{k|}q_x = (1+i)^n A^{\,1}_{x:\overline{n}|}$$

Hence the accumulation of the term assurance benefit is:

$$\frac{(1+i)^n A^{\,1}_{x:\overline{n}|}}{{}_np_x}$$

This can also be written as $A^{\,1}_{x:\overline{n}|} \dfrac{D_x}{D_{x+n}}$.

We can calculate accumulated values at any point in time, not just at the end of the term.

Question

Write down an expression for the retrospective accumulation at the end of t years (where $t < n$) of the payments made under the term assurance contract by that time.

Solution

The retrospective accumulation after t years is $A^{\,1}_{x:\overline{t}|} \dfrac{D_x}{D_{x+t}}$, *ie* we just change the n's to t's.

Now we will do an example calculation of a retrospective accumulation.

Question

John, aged exactly 35, buys a term assurance policy that pays a benefit of £100,000 at the end of the year of his death if he dies before age 65. Calculate the retrospective accumulation of the benefits to time 10.

Basis: AM92 Ultimate, 6% *pa* interest

Solution

The retrospective accumulation to the end of 10 years is:

$$100{,}000A^{\;1}_{35:\overline{10|}}\times\frac{(1+i)^{10}}{{}_{10}p_{35}}=100{,}000\left(A_{35}-v^{10}\,{}_{10}p_{35}A_{45}\right)\times\frac{(1+i)^{10}}{{}_{10}p_{35}}$$

$$=100{,}000\left(A_{35}\times(1+i)^{10}\times\frac{l_{35}}{l_{45}}-A_{45}\right)$$

$$=100{,}000\left(0.09488\times1.06^{10}\times\frac{9{,}894.4299}{9{,}801.3123}-0.15943\right)$$

$$=£1{,}210$$

We now look at one more example of a retrospective accumulation, this time for an annuity.

Example 3: temporary annuity-due

Now consider a temporary annuity-due paying 1 per annum with a term of n years. $F_n(1)$ has the following distribution:

$$F_n(1)=(1+i)^{n-(K_x+1)}\ddot{s}_{\overline{K_x+1|}} \qquad \text{if } K_x<n$$

$$F_n(1)=\ddot{s}_{\overline{n|}} \qquad \text{if } K_x\geq n$$

Recall that $\ddot{s}_{\overline{n|}}$ is the accumulated value of certain payments of 1 *pa*, payable annually in advance for n years, accumulated to the end of the n years.

Hence:

$$E[F_n(1)]=\sum_{k=0}^{n-1}(1+i)^{n-(k+1)}\ddot{s}_{\overline{k+1|}}\;{}_{k|}q_x \;+\; \ddot{s}_{\overline{n|}}\;{}_np_x$$

$$=(1+i)^n\left(\sum_{k=0}^{n-1}\ddot{a}_{\overline{k+1|}}\;{}_{k|}q_x \;+\; \ddot{a}_{\overline{n|}}\;{}_np_x\right)$$

$$=(1+i)^n\,\ddot{a}_{x:\overline{n|}}$$

Therefore the accumulation of the temporary annuity-due is:

$$\frac{(1+i)^n\,\ddot{a}_{x:\overline{n|}}}{{}_np_x}$$

Notation

To each annuity EPV there corresponds an accumulation denoted by an 's' symbol instead of an 'a' symbol. Thus in Example 3 above we define:

$$\ddot{s}_{x:\overline{n}|} = \frac{(1+i)^n \ddot{a}_{x:\overline{n}|}}{{}_n p_x}$$

and we define symbols for the accumulation of other annuities similarly. There is no actuarial notation for the accumulation of assurance benefits.

Question

Consider an annuity of 1 *pa* that has expected present value, at the start of the contract, of $\ddot{a}_{x:\overline{n}|}$.

Write down an expression for the retrospective accumulation to integer time t ($t < n$), of the payments made by that date.

Solution

The retrospective accumulation after t years is $\dfrac{(1+i)^t}{{}_t p_x}\ddot{a}_{x:\overline{t}|}$, *ie* we just change the n's to t's.

It is easy to see that we can write down the retrospective accumulation of any benefit after n years in the same way – we multiply its EPV by $\dfrac{(1+i)^n}{{}_n p_x}$. If $i = 4\%$ and we are assuming AM92 mortality, the accumulation can be calculated more efficiently from the *Tables* by multiplying the EPV by the commutation function expression $\dfrac{D_x}{D_{x+n}}$.

3.2 Gross premium retrospective reserve

Gross premium retrospective reserves at policy duration t take account of expected expenses incurred between policy duration 0 and policy duration t, as well as the expected premiums and benefits paid. All the retrospective accumulations are determined at policy duration t.

With a retrospective reserve we are determining the 'pot of money' represented by the retrospective accumulation of premiums and interest, less the average cost of cover provided and of expenses incurred to date.

So a generic definition of a gross premium retrospective reserve is:

Retrospective accumulation of past premiums – *Retrospective accumulation of past benefits and expenses*

Example: Whole life assurance

The gross premium retrospective reserve at policy duration t is:

$$\frac{l_x}{l_{x+t}}(1+i)^t\left\{G\,\ddot{a}^{(m)}_{x:\overline{t|}} - S\,\bar{A}^{1}_{x:\overline{t|}} - I - e\,\ddot{a}^{(m)}_{x:\overline{t|}} - f\,\bar{A}^{1}_{x:\overline{t|}}\right\}$$

Again, we are assuming that the premiums and renewal expenses are payable $m$thly in advance, and that claims and claim expenses are payable immediately on death.

The expression in brackets gives us 'the expected present value at time 0 of the premiums less benefits and expenses payable in the first t policy years'. The factors outside the brackets then translate that into a value at time t.

To evaluate this expression we need assumptions to determine the assurance and annuity functions and information or assumptions about expenses incurred.

Similar expressions can be determined using the same approach for other standard contracts.

Question (*)

Calculate the gross premium retrospective reserve at the end of the second policy year for a 5-year single premium endowment assurance with sum assured £30,000 payable on maturity or at the end of the year of earlier death, issued to a 48-year old. Assume AM92 Select mortality, interest of 4% *pa* effective, initial expenses of £360 and renewal expenses of £45 at the start of each year excluding the first.

Solution

We first need to calculate the premium paid at inception. This is given by:

$$P = 30{,}000A_{[48]:\overline{5|}} + 360 + 45\left(\ddot{a}_{[48]:\overline{5|}} - 1\right)$$

Now:

$$A_{[48]:\overline{5|}} = A_{[48]} - v^5\,{}_5p_{[48]}\,A_{53} + v^5\,{}_5p_{[48]}$$

where:

$$v^5\,{}_5p_{[48]} = (1+i)^{-5}\times\frac{l_{53}}{l_{[48]}} = 1.04^{-5}\times\frac{9{,}630.0522}{9{,}748.8603} = 0.81191$$

So:

$$\begin{aligned}A_{[48]:\overline{5|}} &= A_{[48]} - v^5\,{}_5p_{[48]}\,A_{53} + v^5\,{}_5p_{[48]}\\ &= 0.30664 - 0.81191\times0.36448 + 0.81191\\ &= 0.82263\end{aligned}$$

and:

$$\ddot{a}_{[48]:\overline{5}|} = \ddot{a}_{[48]} - v^5 {}_5p_{[48]} \ddot{a}_{53} = 18.027 - 0.81191 \times 16.524 = 4.611$$

So:

$$P = 30{,}000 \times 0.82263 + 360 + 45(4.611 - 1) = £25{,}201.25$$

The retrospective reserve at the end of the second policy year is:

$$\left(25{,}201.25 - 30{,}000 A^{\;1}_{[48]:\overline{2}|} - 360 - 45 a_{[48]:\overline{1}|}\right)(1+i)^2 \frac{l_{[48]}}{l_{50}}$$

where:

$$A^{\;1}_{[48]:\overline{2}|} = A_{[48]} - v^2 \frac{l_{50}}{l_{[48]}} A_{50} = 0.30664 - 1.04^{-2} \times \frac{9{,}712.0728}{9{,}748.8603} \times 0.32907 = 0.0035444$$

and:

$$a_{[48]:\overline{1}|} = v \frac{l_{[48]+1}}{l_{[48]}} = 1.04^{-1} \times \frac{9{,}733.1938}{9{,}748.8603} = 0.95999$$

The required retrospective reserve is then:

$$= (25{,}201.25 - 30{,}000 \times 0.0035444 - 360 - 45 \times 0.95999) \times 1.04^2 \times \frac{9{,}748.8603}{9{,}712.0728}$$

$$= £26{,}808$$

We could have alternatively calculated the $v^n\left(l_{x+n} / l_x\right)$ factors using D_{x+n} / D_x throughout, which could have saved a bit of time. The numerical answer will differ slightly due to rounding in the Tables.

The assurance function we use to value the past claim payments in this reserve is a term assurance function. This is because the only benefit that could have been paid out in the first two years is a death benefit.

We can also see how this formula develops recursively from policy inception. The retrospective reserve at the end of Year 1 is the premium less initial expenses, plus interest and less the expected cost of cover, divided by the probability of surviving the year:

$$\frac{(25{,}201.25 - 360) \times 1.04 - 30{,}000\, q_{[48]}}{p_{[48]}} = \frac{25{,}834.90 - 30{,}000 \times 0.001607}{1 - 0.001607} = £25{,}828.20$$

The retrospective reserve at the end of Year 2 is then the end-of-Year-1 reserve less renewal expenses, plus interest and less the expected cost of cover, divided by the probability of surviving that year:

$$\frac{(25{,}828.20-45)\times 1.04-30{,}000\times q_{[48]+1}}{p_{[48]+1}}=\frac{26{,}814.53-30{,}000\times 0.002170}{1-0.002170}=£26{,}807.60$$

which rounds to the same answer of £26,808 that we obtained before.

Later in the chapter we shall return to this idea of calculating the reserve at the end of a given policy year from the reserve at the end of the previous year.

Question

Calculate the gross premium retrospective reserve at the end of the third policy year for the policy described in the previous question. Justify the answer by reconciling it with the reserve at the end of the second policy year.

Solution

The retrospective reserve at the end of the third policy year is:

$${}_3V^{retro}=\left(25{,}201.25-30{,}000A^{\,1}_{[48]:\overline{3}|}-360-45a_{[48]:\overline{2}|}\right)(1+i)^3\,\frac{l_{[48]}}{l_{51}}$$

where 'retro' stands for 'retrospective'.

The required factors are:

$$(1+i)^3\,\frac{l_{[48]}}{l_{51}}=1.04^3\times\frac{9{,}748.8603}{9{,}687.7149}=1.131964$$

$$A^{1}_{[48]:\overline{3}|}=A_{[48]}-v^3\,{}_3p_{[48]}\,A_{51}=0.30664-\frac{1}{1.131964}\times 0.34058=0.00576$$

$$\begin{aligned}
a_{[48]:\overline{2}|}&=\ddot{a}_{[48]:\overline{3}|}-1\\
&=\ddot{a}_{[48]}-v^3\,{}_3p_{[48]}\,\ddot{a}_{51}-1\\
&=18.027-\frac{1}{1.131964}\times 17.145-1\\
&=1.881
\end{aligned}$$

So:

$$\begin{aligned}
{}_3V^{retro}&=(25{,}201.25-30{,}000\times 0.00576-360-45\times 1.881)\times 1.131964\\
&=£27{,}828
\end{aligned}$$

If instead we start from the end-of-Year-2 reserve, we obtain:

$$\frac{(26,807.60-45)\times 1.04-30,000q_{50}}{p_{50}}=\frac{27,833.10-30,000\times 0.002508}{1-0.002508}=£27,827.66$$

which again rounds to the same answer as before.

4 Equality of prospective and retrospective reserves

4.1 Conditions for equality

If:

1. **the retrospective and prospective reserves are calculated on the same basis, and**

2. **this basis is the same as the basis used to calculate the premiums used in the reserve calculation,**

then the retrospective reserve will be equal to the prospective reserve.

Question

Calculate the prospective reserve at the end of the second policy year for the policy considered in the question marked (*) in Section 3.2 above.

Solution

The prospective reserve at the end of the second policy year is:

$$_2V^{pro} = 30{,}000A_{50:\overline{3}|} + 45\ddot{a}_{50:\overline{3}|}$$

Now:

$$A_{50:\overline{3}|} = A_{50} - v^3\,{}_3p_{50}\,A_{53} + v^3\,{}_3p_{50}$$

where:

$$v^3\,{}_3p_{50} = (1+i)^{-3} \times \frac{l_{53}}{l_{50}} = 1.04^{-3} \times \frac{9{,}630.0522}{9{,}712.0728} = 0.88149$$

So:

$$A_{50:\overline{3}|} = 0.32907 - 0.88149 \times 0.36448 + 0.88149 = 0.88927$$

and:

$$\ddot{a}_{50:\overline{3}|} = \ddot{a}_{50} - v^3\,{}_3p_{50}\,\ddot{a}_{53} = 17.444 - 0.88149 \times 16.524 = 2.878$$

So:

$$_2V^{pro} = 30{,}000 \times 0.88927 + 45 \times 2.878 = £26{,}808$$

This is equal to the retrospective reserve calculated in the earlier question, because both reserves have been calculated using the assumptions of the gross premium basis.

In practice these conditions rarely hold, since the assumptions that are appropriate for the retrospective calculation are not generally appropriate for the prospective calculation. The conditions experienced over the duration of the contract up to the valuation date may no longer be suitable assumptions for the remainder of the policy term. Furthermore, the assumptions considered appropriate at the time the premium was calculated may not be appropriate for either the retrospective or prospective reserve some years later.

This last point is important and is worth reiterating. The retrospective reserve is based on actual experience, in terms of interest, expenses, mortality and so on. This will *not* usually be the same as that assumed when the premiums were set. For example:

- The number of people who actually die will differ from the number expected.
- Investment returns will not exactly match what was assumed.
- We may have had margins in our assumptions (as discussed in the Core Reading in Chapter 20). For example, we may have been expecting 8% interest but assumed only 6%. If we actually did earn 8% then the prospective and retrospective reserves would not be the same.
- The actuary's expectation of future interest rates may have changed, so that now the expected return on money held is 5%.

Question

Explain how the situation in the last bullet point above would affect each of the retrospective and prospective reserves.

Solution

The retrospective reserve would be unchanged (because it is based on the past experience). The prospective reserve would increase. (If we expect to earn the lower rate of 5% interest on future premiums, we will need to set aside more money now in order to meet the cost of the benefits.)

4.2 Demonstrating the equality of prospective and retrospective reserves

We will demonstrate that under the conditions given above, the retrospective and prospective gross premium reserves for a whole life assurance are equal. This method can be extended to all standard contracts, including those with increasing or decreasing benefits.

Example: Whole life assurance

The three key expressions are those for prospective reserves, retrospective reserves and the original equation of value used to determine the gross premium.

These are:

(i) $S\,\bar{A}_{x+t} + e\,\ddot{a}_{x+t}^{(m)} + f\,\bar{A}_{x+t} - G\,\ddot{a}_{x+t}^{(m)}$

(ii) $\dfrac{l_x}{l_{x+t}}(1+i)^t \left\{ G\,\ddot{a}_{x:\overline{t}|}^{(m)} - S\,\bar{A}_{x:\overline{t}|}^{1} - I - e\,\ddot{a}_{x:\overline{t}|}^{(m)} - f\,\bar{A}_{x:\overline{t}|}^{1} \right\}$

(iii) $G\,\ddot{a}_{x}^{(m)} - S\,\bar{A}_{x} - I - e\,\ddot{a}_{x}^{(m)} - f\,\bar{A}_{x} = 0$

Then, if we add:

$$\frac{l_x}{l_{x+t}}(1+i)^t \left\{ G\,\ddot{a}_{x}^{(m)} - S\,\bar{A}_{x} - I - e\,\ddot{a}_{x}^{(m)} - f\,\bar{A}_{x} \right\}$$

(*ie* zero) to the expression for the prospective reserve, and rearrange the terms, we will obtain the expression for the retrospective reserve.

Alternatively, we could split the premium equation up at time t and rearrange as follows:

$$G\ddot{a}_x^{(m)} = (S+f)\bar{A}_x + I + e\ddot{a}_x^{(m)}$$

$$\Rightarrow G\left(\ddot{a}_{x:\overline{t}|}^{(m)} + v^t\,{}_tp_x\,\ddot{a}_{x+t}^{(m)}\right) = (S+f)\left(\bar{A}_{x:\overline{t}|}^{1} + v^t\,{}_tp_x\bar{A}_{x+t}\right) + I + e\left(\ddot{a}_{x:\overline{t}|}^{(m)} + v^t\,{}_tp_x\,\ddot{a}_{x+t}^{(m)}\right)$$

$$\Rightarrow G\ddot{a}_{x:\overline{t}|}^{(m)} - (S+f)\bar{A}_{x:\overline{t}|}^{1} - I - e\ddot{a}_{x:\overline{t}|}^{(m)} = v^t\,{}_tp_x\left((S+f)\bar{A}_{x+t} + e\ddot{a}_{x+t}^{(m)} - G\ddot{a}_{x+t}^{(m)}\right)$$

Dividing both sides by $v^t\,{}_tp_x$ then gives:

$$\left(G\ddot{a}_{x:\overline{t}|}^{(m)} - (S+f)\bar{A}_{x:\overline{t}|}^{1} - I - e\ddot{a}_{x:\overline{t}|}^{(m)}\right)\frac{(1+i)^t}{{}_tp_x} = (S+f)\bar{A}_{x+t} + e\ddot{a}_{x+t}^{(m)} - G\ddot{a}_{x+t}^{(m)}$$

The expression on the left-hand side of this equation is the gross premium retrospective reserve at time t and the expression on the right is the gross premium prospective reserve at time t.

A similar result can be shown when all expenses are assumed to be zero ($I = e = f = 0$).

Question

Prove that the retrospective and prospective reserves are equal at time t for an immediate annuity (payable annually in arrears) of amount B with initial expenses I and renewal expenses of R (incurred at the start of every year except the first year).

Solution

The premium equation is:

$$P = Ba_x + I + R(\ddot{a}_x - 1)$$

Now split the premium equation at time t, taking care to include t past annuity payments but only $t-1$ past renewal expense payments. We obtain:

$$P = B\left(a_{x:\overline{t}|} + v^t\ {}_tp_x\, a_{x+t}\right) + I + R\left(\ddot{a}_{x:\overline{t}|} - 1 + v^t\ {}_tp_x\, \ddot{a}_{x+t}\right)$$

Rearranging:

$$P - Ba_{x:\overline{t}|} - I - R\left(\ddot{a}_{x:\overline{t}|} - 1\right) = v^t\ {}_tp_x\left(Ba_{x+t} + R\ddot{a}_{x+t}\right)$$

Finally dividing through by $v^t\ {}_tp_x$:

$$\left[P - Ba_{x:\overline{t}|} - I - R\left(\ddot{a}_{x:\overline{t}|} - 1\right)\right]\frac{(1+i)^t}{{}_tp_x} = Ba_{x+t} + R\ddot{a}_{x+t}$$

ie ${}_tV^{retro} = {}_tV^{pro}$.

The fact that prospective and retrospective reserves are equal for equality of bases has two immediate uses:

- some policies with complicated and varying future benefit levels may require complex calculations to arrive at the reserve prospectively, but a retrospective calculation may be much easier, and
- you may find that retrospective reserves are a more tangible concept than prospective reserves, in which case thinking about aspects of life insurance involving reserves may be easier if you think in retrospective terms.

On the other hand, under some circumstances the prospective calculation will be easier. For example for policies where there are no further premiums to be paid, the prospective calculation is often simpler, because the term 'expected present value of future premiums' disappears.

For equality of prospective and retrospective reserves we require equality of bases. In practice, we shall often find that the bases are not equal. The retrospective basis may be just the past experience – for instance, the mortality experienced by our policyholders was 86% of AM92 – while the prospective basis may be our estimate of future experience, and it might be deliberately prudent (especially in the context of calculating reserves to demonstrate the solvency of the company).

However, we might want to ignore recent experience and 'arbitrarily' set the past basis to be equal to the future basis, in order to use the retrospective method to calculate reserves rather than the prospective method.

However, this will only give a valid answer if the reserving basis is identical to the pricing basis of the gross premium involved. While in most of our examples so far the two bases have been assumed to be identical, in real life they are usually different. On the other hand, *in the exam* it is quite common to have the same basis. So, unless a question specifically states the approach to be adopted, you would then be free to choose retrospective or prospective calculations. In such cases, however, *the prospective method is usually the easier method to do*.

Very often, companies will price products using 'best estimates' of future experience for interest, mortality and expenses, or best estimate with a small margin for prudence. However, the supervisory authority may insist that the reserving basis is chosen to be much more prudent, in order to be more certain that life companies will be capable of honouring their financial commitments in the event of deteriorating future conditions. For instance, if our best estimate of future interest rates was 6½% we might calculate premiums using 5%, but then might need to calculate statutory reserves using 3% interest.

In this case we would *have* to calculate the reserve prospectively, as the reserving basis is different from the pricing basis.

The effect of these bases being different is studied in Chapter 28.

5 Recursive relationship between reserves for annual premium contracts

If the expected cashflows (*ie* premiums, benefits and expenses) during the policy year $(t, t+1)$ are evaluated and allowance is made for the time value of money, we can develop a recursive relationship linking gross premium policy values in successive years.

(Policy value is just another name for reserve.)

We illustrate this using a whole life assurance secured by level annual premiums, but the method extends to all standard contracts.

Example: Whole life assurance

Gross premium policy value at duration t	${}_tV'$
Premium less expenses paid at t	$G-e$
Expected claims plus expenses paid at $t+1$	$q_{x+t}(S+f)$
Gross premium policy value at duration $t+1$	${}_{t+1}V'$

Then the equation of value at time $t+1$ for these cashflows is:

$$\left({}_tV' + G - e\right)(1+i) - q_{x+t}(S+f) = (1-q_{x+t})\,{}_{t+1}V' \qquad (*)$$

Question

(i) Summarise the above relationship in words.

(ii) Write down the relationship for a single premium whole life assurance policy.

Solution

(i) The reserves at the end of the year for those policies still in force, are equal to the reserves at the beginning of the year plus premiums and investment income, less expenses and the expected cost of cover.

(ii) $\left({}_tV' - e\right)(1+i) - q_{x+t}(S+f) = (1-q_{x+t})\,{}_{t+1}V'$

Equation (*) above will only be satisfied if all quantities are calculated on mutually consistent bases: *ie* using the same interest, mortality and expense assumptions for the reserves, premium and experience over the year.

Dividing through by $p_{x+t} = 1 - q_{x+t}$, we obtain a formula for calculating the reserve at the end of the year from the reserve at the beginning of the year:

$$\frac{\left({}_tV' + G - e\right)(1+i) - q_{x+t}\left(S + f\right)}{p_{x+t}} = {}_{t+1}V'$$

In this case, the equation then gives a recursive relationship between policy values in successive years.

Question

Consider a 25-year regular premium endowment assurance policy with a sum assured of £75,000 payable on maturity (or at the end of the year of earlier death), taken out by a 45-year old. Expenses are 75% of the first premium and 5% of each subsequent premium, plus an initial expense of £250.

Given a gross annual premium of £2,132 and a gross premium reserve at the end of Year 10 of £20,898, calculate the gross premium reserve at the end of Year 11.

Assume AM92 Select mortality and 4% *pa* interest.

Solution

The recursive relationship is:

$${}_{11}V' = \frac{\left({}_{10}V' + G - e\right)(1+i) - Sq_{[45]+10}}{1 - q_{[45]+10}} = \frac{\left({}_{10}V' + G - e\right)(1+i) - Sq_{55}}{1 - q_{55}}$$

So:

$${}_{11}V' = \frac{\left(20{,}898 + 2{,}132 \times 0.95\right) \times 1.04 - 75{,}000 \times 0.004469}{1 - 0.004469} = £23{,}611$$

The recursive relationship can also be used in reverse, *ie* to calculate the reserve at the start of a policy year given the reserve at the end of that year.

Question

For the policy described in the previous question, calculate ${}_9V'$ using the recursive relationship and the value ${}_{10}V' = £20{,}898$.

Solution

We have:

$$\left({}_9V' + G - e\right)(1+i) - Sq_{54} = p_{54}\ {}_{10}V'$$

This gives:

$$\left({}_9V'+2{,}132\times 0.95\right)\times 1.04-75{,}000\times 0.003976=(1-0.003976)\times 20{,}898$$

So:

$${}_9V'=£18{,}276$$

We can also rearrange equation (*) to give:

$$\left({}_tV'+G-e\right)\left(1+i\right)-q_{x+t}\left(S+f-{}_{t+1}V'\right)={}_{t+1}V'$$

So the end-year reserve is the start-year reserve, plus premium and less expenses, accumulated with one year's interest, less the expected cost of claims *in excess of the end-year reserve*. This excess claim cost is known as the *sum at risk*, or the *death strain at risk*, and is important in the calculation of mortality profit, which we cover in Chapter 24.

Where any of these bases differ, the equation (*) can be reformulated to represent the profit over the year, *ie*:

$$\boldsymbol{PRO_t=\left({}_tV'+G-e\right)(1+i)-q_{x+t}(S+f)-(1-q_{x+t})\,{}_{t+1}V'}$$

This profit relates to the year *starting* at policy duration t, that is for policy *year* $t+1$.

We can consider the elements of this formula in different ways, depending on what kind of 'profit' we wish to calculate.

(1) Actual profit over the year

In this case, the elements will represent the actual experience *that has happened* over the year, rather than being the same as the premium basis assumptions. The reserves may also be calculated using a different basis from the gross premium basis, which is something that happens very often in practice.

Question

For the same 25-year endowment assurance described in the previous two questions, calculate the profit earned during Year 11 if:

- the start and end of year reserves were as previously given/calculated
- the insurer earned 3.8% on its investments during the year
- renewal expenses of £78 were incurred at the start of the year
- claim expenses of £150 per death claim were incurred
- 73% of expected mortality occurred during the year.

Solution

The actual profit earned in Year 11 is:

$$PRO_{10} = (20{,}898 + 2{,}132 - 78) \times 1.038 - (75{,}000 + 150) \times 0.73 \times q_{55}$$
$$- (1 - 0.73 \times q_{55}) \times 23{,}611$$

From the *Tables*, $q_{55} = 0.004469$, which gives:

$$PRO_{10} = £45.04$$

(2) Expected profit over the year

Alternatively, we may want to calculate the profit we would *expect* to earn during the year if the actual experience turns out in a particular way, *ie* according to our 'expected future experience' assumptions. The approach is identical to the calculation in (1), but it is based on *assumptions* of what might happen during the year, rather than on what has *actually* happened.

6 Net premium reserves for conventional without profit contracts

6.1 Difference from gross premium reserve

The differences between the net premium reserve and the gross premium reserve for any conventional without profit contract are:

1. **all expenses are ignored; and**

2. **the premium used in the reserve calculation is the *net premium*, as defined below.**

The net premium reserve is the prospective reserve, where we make no allowance for future expenses, and where the premium used in the calculation is a notional net premium. This net premium is calculated using the equivalence principle and using the same assumptions as the reserve basis, and again making no allowance for future expenses.

So, with a net premium reserve, we have just one basis, which we first use to calculate a premium using the equivalence principle in the normal way, but with no expenses in the equation of value (this gives us the so-called 'net premium'). Then we calculate the reserve prospectively as the expected present value of the future benefits less the expected present value of these future net premiums. No explicit allowance is made anywhere either for expenses or for the actual amount of premium received from the policyholder. The result is the 'net premium reserve'.

Though this may appear artificial, the net premium valuation has been an important feature in life insurance for many years.

One reason why this is the case is that the reserve is simple to calculate. The net premium method was used before computers, spreadsheets or calculators were available.

The notional net premium calculated and valued as the *future income* element of the reserve is generally considerably smaller than the actual premium being paid. It is considered that the excess of the actual premium over the notional premium will be sufficient to cover the expenses that are not specifically valued.

Example: whole life assurance

The net premium reserve at policy duration t is:

$$S\bar{A}_{x+t} - P\ddot{a}_{x+t}^{(m)}$$

where

$$P = \frac{S\bar{A}_x}{\ddot{a}_x^{(m)}}$$

and the notional net premium P is calculated on the same assumptions as the reserve basis.

In this example the sum assured is payable immediately on death and net premiums of P/m are payable m times a year.

Here P is found by solving the equation of value for this policy at duration 0, ignoring expenses. That is, we find P from:

$$P\ddot{a}_x^{(m)} = S\bar{A}_x$$

Net premium reserves for with-profits contracts are not covered in the CM1 syllabus.

6.2 A special result for the net premium reserve for some endowment and whole life assurance contracts

A special result for the net premium reserve for some endowment and whole life assurance contracts follows from the fact that the net premium is calculated on the same basis as the reserve basis.

Example: Whole life assurance

${}_tV_x$ is the net premium reserve at duration t for a whole life assurance policy, with sum assured of 1 payable at the end of the year of death, and with level annual premiums payable during the duration of the policy. The net premium P_x for this contract is:

$$\boldsymbol{P_x = \frac{A_x}{\ddot{a}_x}}$$

and so:

$$\begin{aligned} {}_tV_x &= A_{x+t} - P_x\,\ddot{a}_{x+t} \\ &= A_{x+t} - \frac{A_x}{\ddot{a}_x}\ddot{a}_{x+t} \\ &= (1 - d\,\ddot{a}_{x+t}) \; - \; (1 - d\,\ddot{a}_x)\frac{\ddot{a}_{x+t}}{\ddot{a}_x} \end{aligned}$$

This uses the premium conversion formula for A_x and A_{x+t} as given on page 37 of the *Tables*.

Since the net premium and the net premium reserve are both calculated using the same basis, the formula for the net premium reserve simplifies to:

$${}_tV_x = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}$$

The net premium reserve for the above contract with a sum assured of S would then be:

$$S\,{}_tV_x = S\left(1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}\right) \qquad (**)$$

${}_tV_x$ is international actuarial notation for the net premium reserve for this particular type of whole life assurance, as defined at the start of this example. Net premium reserves for other contract variations have different notation.

Similar formulae to (**) can be obtained for endowment assurances with level annual premiums and death benefits paid at the end of the year of death, also for whole life and endowment assurances with continuously payable premiums and death benefits payable immediately on death. These results are shown in the *Formulae and Tables for Examinations*.

(They are to be found on page 37.)

It is important to understand the particular symbols listed below, and to use the special formulae to calculate their values, where appropriate, in the CM1 exam. In all cases the symbol represents the amount of reserve held at time t :

${}_tV_x$ whole life assurance, pays at end of year of death, premium payable annually in advance

${}_tV_{x:\overline{n|}}$ endowment assurance with a term of n years $(t < n)$, pays at end of year of death or at end of term, premium payable annually in advance

${}_t\bar{V}_x$ whole life assurance, pays immediately on death, premium payable continuously

${}_t\bar{V}_{x:\overline{n|}}$ endowment assurance with a term of n years $(t < n)$, pays immediately on death or at end of term, premium payable continuously.

In each case listed above, the sum assured is assumed to be 1.

In all other cases, including m -thly premium cases, these special formulae do not hold and the net premium reserves would need to be calculated from the standard formulae such as shown in Section 6.1 above.

Question

Calculate the net premium reserve after exactly 10 years for a 15-year endowment assurance issued to a life aged exactly 50 at entry, with a sum assured of 10,000 paid at the end of the term or at the end of the year of death if earlier:

(i) where premiums are paid annually

(ii) where premiums are paid monthly.

Basis: AM92 Ultimate mortality and 4% *pa* interest.

Solution

(i) ***Premiums paid annually***

We can use the special result for this contract, as it is one of the examples cited in the above Core Reading (and included on page 37 of the *Tables*). The required net premium reserve is:

$$S\,{}_tV_{x:\overline{n|}} = S\left(1 - \frac{\ddot{a}_{x+t:\overline{n-t|}}}{\ddot{a}_{x:\overline{n|}}}\right)$$

where $S = 10{,}000$, $x = 50$, $n = 15$ and $t = 10$.

So, we need:

$$10{,}000\,{}_{10}V_{50:\overline{15}|} = 10{,}000 \times \left(1 - \frac{\ddot{a}_{60:\overline{5}|}}{\ddot{a}_{50:\overline{15}|}}\right) = 10{,}000 \times \left(1 - \frac{4.550}{11.253}\right) = 5{,}957$$

(ii) ***Premiums paid monthly***

This is not one of the specified contract types, so we cannot use the special result for this.

First we need to calculate the net annual premium from the equation of value. We have:

$$P\ddot{a}^{(12)}_{50:\overline{15}|} = 10{,}000 A_{50:\overline{15}|}$$

where:

$$A_{50:\overline{15}|} = 0.56719$$

$$\ddot{a}^{(12)}_{50:\overline{15}|} \approx \ddot{a}_{50:\overline{15}|} - \frac{11}{24}\left(1 - \frac{D_{65}}{D_{50}}\right)$$ (using the formula on page 36 of the *Tables*)

Putting in the numbers:

$$\ddot{a}^{(12)}_{50:\overline{15}|} \approx 11.253 - \frac{11}{24} \times \left(1 - \frac{689.23}{1{,}366.61}\right) = 11.026$$

The annual amount of net premium is then:

$$P = 10{,}000 \frac{A_{50:\overline{15}|}}{\ddot{a}^{(12)}_{50:\overline{15}|}} = 10{,}000 \times \frac{0.56719}{11.026} = 514.42$$

The net premium reserve at time 10 is then:

$$10{,}000 \times A_{60:\overline{5}|} - 514.42 \times \ddot{a}^{(12)}_{60:\overline{5}|}$$

where:

$$A_{60:\overline{5}|} = 0.82499$$

$$\ddot{a}^{(12)}_{60:\overline{5}|} \approx \ddot{a}_{60:\overline{5}|} - \frac{11}{24}\left(1 - \frac{D_{65}}{D_{60}}\right) = 4.550 - \frac{11}{24} \times \left(1 - \frac{689.23}{882.85}\right) = 4.449$$

Hence the required reserve is:

$$10{,}000 \times 0.82499 - 514.42 \times 4.449 = 5{,}961$$

Chapter 21 Summary

Reserves

A reserve is money set aside by the insurer, to pay policyholders' benefits and, where appropriate, future expenses.

Gross premium prospective reserves

Gross premium prospective reserves can be calculated as:

EPV of future benefits + EPV of future expenses – EPV of future premiums

Retrospective accumulations

The retrospective accumulation of a t-year payment stream can be calculated as:

$$(AV)_t = \{EPV \text{ of the payments as at the start of the } t \text{ years}\} \times \frac{(1+i)^t}{{}_t p_x}$$

for a life that is initially aged x. Retrospective accumulations are defined as the amount per survivor to time t.

Gross premium retrospective reserves

Gross premium retrospective reserves can be calculated as:

Retrospective accumulation of (past premiums – past benefits and expenses)

Calculating reserves that satisfy probabilities

A reserve at time t can be found that satisfies a probability defined in terms of the future loss random variable, L_t. For example, we can define the reserve as being the smallest reserve ${}_tV$ for which $P\left(L_t \le {}_tV\right) \ge \alpha$, where α is some acceptably large probability.

Equality of reserves

Given equality of bases used to calculate the premium and the reserves, the prospective and retrospective reserves of any policy at any given time t will be equal.

Recursive formula for reserves

Reserves at successive values of time t are related by the equation:

$$\left({}_tV' + G - e\right)(1+i) - q_{x+t}\left(S + f\right) = p_{x+t}\ {}_{t+1}V'$$

Profit

The profit for the year between policy durations t and $t+1$ (*ie* for policy year $t+1$) can be calculated using:

$$PRO_t = \left({}_tV' + G - e\right)\left(1+i\right) - q_{x+t}\left(S+f\right) - p_{x+t}\ {}_{t+1}V'$$

Net premium reserves for without-profit contracts

These are identical to gross premium reserves except that expenses are not included in any of the calculations.

The net premiums used in the formulae are calculated from the premium equation of value, with the same interest and mortality assumptions as used for the reserves, and no expenses.

Calculations can sometimes be speeded up using special formulae, which apply only to certain endowment and whole life assurances. The net premium reserve at time t for an annual premium n-year endowment assurance policy, with discrete-time payments and a sum assured of 1, is given by:

$${}_tV_{x:\overline{n}|} = A_{x+t:\overline{n-t}|} - P_{x:\overline{n}|} \times \ddot{a}_{x+t:\overline{n-t}|} = 1 - \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}$$

Similar formulae apply (a) for whole life assurances (by removing the policy term status) and (b) for the equivalent contracts with continuous-time payments (by putting bars over all the actuarial symbols). These are shown on page 37 of the *Tables*.

Chapter 21 Practice Questions

21.1 A 10-year term assurance with a sum assured of £500,000 payable at the end of the year of death, is issued to a male aged 30 for a level annual premium of £330.05. Calculate the prospective and retrospective reserves at the end of the fifth policy year, *ie* just before the sixth premium has been paid, assuming AM92 Ultimate mortality and 4% *pa* interest. Ignore expenses.

21.2 An annual premium conventional with-profits endowment assurance policy is issued to a life aged 35. The initial sum assured is £50,000, the gross premium is £1,500 and the term of the policy is 25 years. The death benefit of the sum assured and attaching bonuses is payable at the end of the year of death. The office declares compound reversionary bonuses, vesting at the end of each policy year. Given that bonuses of 3% *pa* have been declared for each year of the contract so far, calculate the prospective gross premium reserve at the end of the fifth policy year.

Basis: Future bonuses of 1.92308% *pa* compound
AM92 Ultimate mortality
6% *pa* interest
Renewal expenses of 5% of each premium
Claim expenses of £350

21.3 A 10-year regular-premium term assurance policy is issued to a life aged 40. The sum assured is £20,000 and is payable at the end of the year of death. Expenses of £72 are assumed to be incurred at the start of each year in which the policy is in force, except at the start of the first year when the expense is £425. The gross premium is £1,700 *pa*.

Write down an expression for the gross premium retrospective reserve immediately before the 6th premium is due.

21.4 A temporary annuity of £3,000 *pa* payable annually in arrears for a term of 10 years was purchased one year ago by a life then aged exactly 60 by the payment of a single premium. Show algebraically that the current retrospective and prospective gross premium reserves are equal, assuming that the pricing and reserving bases are the same. The company assumes that each policy incurs initial expenses of £200 and annual expenses of 1.5% of each annuity payment.

21.5 A special ten-year increasing endowment assurance policy payable by level annual premiums provides a sum assured of £10,000 during the first year, which increases by £1,000 in each subsequent year. The sum payable on maturity at age 60 is £25,000. Write down an expression for the net premium reserve immediately before the 5th premium is paid.

21.6 An annual premium whole life assurance policy provides a sum assured of £30,000 payable immediately on death. Write down an expression for the gross premium retrospective reserve after 20 years in respect of a life aged 30 at entry, who is paying a gross annual premium of £250. Expenses are £100 payable initially, with renewal expenses of 5% of each premium except the first.

21.7 The premiums under a whole life assurance with sum assured S issued to a life aged x are payable annually in advance throughout life. The annual premium P is calculated assuming that the following expenses will be incurred:

Initial expenses:	I
Renewal expenses:	$100k$ % of each premium after the first
Claim related expenses:	$100c$ % of the sum assured

Write down equations linking the gross premium reserves at the end of successive policy years.

21.8 Calculate the retrospective accumulation of £200 paid by a person aged exactly 25, assuming ELT15 (Females) mortality and 7.5% *pa* interest, at the end of:

(i) 5 years

(ii) 20 years.

21.9 Exam style

A whole life assurance policy pays a benefit of £50,000 at the end of the year of death. The policyholder is currently aged 30 and is paying an annual premium of £700 at the start of each year. A premium has just been paid.

Use the following basis to calculate the reserve the company needs to hold at the present time so that the probability of covering the liability in full is at least 99%.

Mortality:	AM92 Select
Interest:	3% *pa*
Expenses:	5% of each future premium

[5]

21.10 Exam style

A life office sells decreasing term assurance policies with an initial sum assured of £150,000 to lives aged 50 exact. The term of the policies is 10 years, and the sum assured decreases by £10,000 at the start of each year from Year 2 onwards. The benefit is payable immediately on death. Premiums are payable annually in advance throughout the term of the policy. The office calculates premiums using AM92 Ultimate mortality and 4% *pa* interest, initial expenses of £300, renewal expenses of £43 at the start of each year except the first, and claim expenses of £400.

(i) Using P for the annual premium, write down the future loss random variable for the policy at the start of the term, and also just before the payment of the fifth premium, assuming that the policy is still in force at that time. [4]

(ii) Show that the premium for the policy is £491.31. [4]

(iii) Calculate the gross premium prospective reserve for the policy just before the payment of the fifth premium. Assume that the reserving basis is the same as that used to calculate the premium. [4]

(iv) Comment on your answer to part (iii). [2]

[Total 14]

21.11 Exam style

A life insurance company issues a 30-year with profits endowment assurance policy to a life aged 35 exact. The sum assured of £100,000 plus declared reversionary bonuses are payable on survival to the end of the term or immediately on death if earlier. The quarterly premium payable in advance throughout the term of the policy or until earlier death is £615.61.

At the end of the 25th policy year, the actual past bonus additions to the policy have been £145,000.

Calculate the gross prospective policy reserve at the end of that policy year immediately before the premium then due.

Policy reserving basis:

Mortality:	AM92 Ultimate
Interest:	4% per annum
Bonus loading:	4% of the sum assured and attaching bonuses, compounded and vesting at the end of each policy year
Renewal commission:	2.5% of each quarterly premium
Renewal expenses:	£90 at the start of each policy year
Claim expense:	£1,000 on death; £500 on maturity

[6]

21.12 Exam style

A life insurance company issues a with profit whole life assurance policy to a life aged 55 exact. The sum assured is £75,000 together with any attaching bonuses and is payable immediately on death. Level premiums are payable monthly in advance ceasing on the policyholder's death or on reaching age 85 if earlier.

Simple annual bonuses are added at the end of each policy year (*ie* the death benefit does not include any bonus relating to the policy year of death).

The company calculates the premium on the following basis:

Mortality	AM92 Select
Interest	4% per annum
Expenses	
Initial	£275
Renewal	£65 at the start of the second and subsequent policy years and payable until death
Claim	£200 on death
Commission	
Initial	75% of the total premium payable in the first policy year
Renewal	2.5% of the second and subsequent monthly premiums
Bonuses	Simple bonus of 2.0% of basic sum assured per annum

(i) Calculate the monthly premium for this policy. [6]

(ii) Calculate the gross prospective policy value at the end of the 30th policy year given that the total actual past bonus additions to the policy have followed the assumptions stated in the premium basis above (including the bonus just vested).

Policy value basis:

Mortality	AM92 Ultimate
Interest	4% per annum
Expenses	
Renewal	£80 at the start of each policy year and payable until death
Claim	£250 on death
Commission	
Renewal	2.5% of the monthly premiums
Bonuses	Simple bonus of 2.5% of basic sum assured per annum

[4]
[Total 10]

21.13 Exam style

On 1 January 2008, a life insurance company issued a number of without profit endowment policies maturing at age 60 to lives then aged 40 exact. The sum assured is payable at the end of year of death or on survival to the end of the term and level premiums are payable annually in advance throughout the term of the contract.

Premiums and reserves on each policy are both calculated on the following basis:

Mortality:	AM92 Select
Interest:	4% per annum
Initial commission:	60% of the first premium
Renewal commission:	6% of each annual premium excluding the first

(i) Calculate the annual office premium per £1,000 sum assured for each policy. [2]

(ii) Calculate the gross premium prospective reserve per £1,000 sum assured for each policy in force at 31 December 2012. [2]

(iii) Calculate the profit or loss to the company in 2013 in respect of these policies given the following information:

- The total sums assured in force on 1 January 2013 were £15,500,000.
- The company incurred expenses relating to these policies of £76,500 on 1 January 2013 (including renewal commission).
- The total sums assured paid on 31 December 2013 in respect of deaths during 2013 were £295,000.
- The total sums assured surrendered during 2013 were £625,000. The surrender value on each policy (which was paid on 31 December 2013) was calculated as 85% of the gross premium prospective reserve applicable at the date of payment of the surrender value.
- The company earned interest of 3.5% per annum on its assets during 2013. [10]

[Total 14]

21.14 Exam style

Under a policy issued by a life insurance company, the death benefit payable at the end of year of death is a return of premiums paid without interest. A level premium of £3,000 is payable annually in advance throughout the term of the policy.

For a policy in force at the start of the 12th policy year, you are given the following information:

Reserve at the start of the policy year	£25,130
Reserve at the end of the policy year per survivor	£28,950
Probability of death during the policy year	0.03
Expenses incurred at the start of the policy year	£90
Rate of interest earned	4% per annum

Reserves given above are immediately before payment of the premium due.

Calculate the profit/loss expected to emerge at the end of the 12th policy year per policy in force at the start of that year. [3]

21.15 Exam style

Calculate $s_{55:\overline{10}|}$.

Basis:

Mortality: PFA92C20

Interest: 4% per annum [1]

Chapter 21 Solutions

21.1 ***Prospective calculation***

The expected present value of the future premiums is:

$$330.05\ddot{a}_{35:\overline{5}|} = 330.05\left(\ddot{a}_{35} - v^5\,{}_5p_{35}\,\ddot{a}_{40}\right)$$

$$= 330.05\left(21.003 - 1.04^{-5} \times \frac{9,856.2863}{9,894.4299} \times 20.005\right)$$

$$= 330.05 \times 4.6237$$

$$= 1,526.06$$

The expected present value of the future benefits is:

$$500,000\,A^{1}_{35:\overline{5}|} = 500,000\left(A_{35} - v^5\,{}_5p_{35}\,A_{40}\right)$$

$$= 500,000\left(0.19219 - 1.04^{-5} \times \frac{9,856.2863}{9,894.4299} \times 0.23056\right)$$

$$= 500,000 \times 0.00341703$$

$$= 1,708.52$$

Hence the prospective reserve is:

$$1,708.52 - 1,526.06 = £182 \text{ to the nearest } £1$$

Retrospective calculation

The accumulated value of the past premiums is:

$$330.05\ddot{a}_{30:\overline{5}|} \times \frac{(1+i)^5}{{}_5p_{30}} = 330.05\left(\ddot{a}_{30} - v^5\,{}_5p_{30}\,\ddot{a}_{35}\right) \times \frac{(1+i)^5}{{}_5p_{30}}$$

$$= 330.05\left(21.834 \times 1.04^5 \times \frac{9,925.2094}{9,894.4299} - 21.003\right)$$

$$= 330.05 \times 5.64404$$

$$= 1,862.81$$

The accumulated value of the past benefits is:

$$500{,}000\,A^{1}_{30:\overline{5}|}\times\frac{(1+i)^5}{{}_5p_{30}}=500{,}000\left(A_{30}-v^5\,{}_5p_{30}\,A_{35}\right)\times\frac{(1+i)^5}{{}_5p_{30}}$$

$$=500{,}000\left(0.16023\times1.04^5\times\frac{9{,}925.2094}{9{,}894.4299}-0.19219\right)$$

$$=500{,}000\times0.0033607$$

$$=1{,}680.36$$

Hence the retrospective reserve is $1{,}862.81-1{,}680.36=£182$, as before.

21.2 The reserve at the end of the fifth policy year is:

$${}_5V^{pro}=EPV\text{ future benefits}+EPV\text{ future expenses}-EPV\text{ future premiums}$$

Now, writing $b=0.0192308$:

EPV future benefits $=$

$$50{,}000\times1.03^5\times\left[\frac{1}{1.06}\times{}_{0|}q_{40}+\frac{1+b}{1.06^2}\times{}_{1|}q_{40}+\cdots+\frac{(1+b)^{19}}{1.06^{20}}\times{}_{19|}q_{40}\right]$$

$$+50{,}000\times1.03^5\times\frac{(1+b)^{20}}{1.06^{20}}\times{}_{20}p_{40}$$

$$=\frac{50{,}000\times1.03^5}{1+b}\times\left[\frac{1+b}{1.06}\times{}_{0|}q_{40}+\left(\frac{1+b}{1.06}\right)^2\times{}_{1|}q_{40}+\cdots+\left(\frac{1+b}{1.06}\right)^{20}\times{}_{19|}q_{40}\right]$$

$$+50{,}000\times1.03^5\times1.04^{-20}\times{}_{20}p_{40}$$

$$=50{,}000\times1.03^5\times\left[\frac{A^{1\ @4\%}_{40:\overline{20}|}}{1+b}+\frac{D_{60}^{@4\%}}{D_{40}^{@4\%}}\right]$$

where:

$$\frac{D_{60}^{@4\%}}{D_{40}^{@4\%}}=\frac{882.85}{2{,}052.96}=0.43004$$

$$A^{1\ @4\%}_{40:\overline{20}|}=A^{@4\%}_{40:\overline{20}|}-\frac{D_{60}^{@4\%}}{D_{40}^{@4\%}}=0.46433-0.43004=0.03429$$

So:

$$EPV\text{ future benefits}=50{,}000\times1.03^5\times\left[\frac{0.03429}{1.0192308}+0.43004\right]=26{,}876.78$$

Hence:

$$
\begin{aligned}
{}_5V^{pro} &= 26{,}876.78 + 350 A^{@6\%}_{40:\overline{20|}} - (1-0.05)\times 1{,}500 \times \ddot{a}^{@6\%}_{40:\overline{20|}} \\
&= 26{,}876.78 + 350\times 0.32088 - 0.95\times 1{,}500\times 11.998 \\
&= £9{,}892
\end{aligned}
$$

21.3 The retrospective gross premium reserve can be calculated as the expected present value at the outset of (premiums – benefits – expenses), accumulated with interest and allowing for survival to age 45. This gives:

$$\frac{(1+i)^5}{{}_5p_{40}} \times \left[1{,}700\ddot{a}_{40:\overline{5|}} - 20{,}000 A^{\ 1}_{40:\overline{5|}} - 425 - 72\times\left(\ddot{a}_{40:\overline{5|}} - 1\right)\right]$$

21.4 The prospective reserve is:

$${}_1V^{pro} = 1.015\times 3{,}000\times a_{61:\overline{9|}}$$

The retrospective reserve is the retrospective accumulation of the premium less benefits and expenses:

$${}_1V^{retro} = \frac{D_{60}}{D_{61}} \times \left(P - 1.015\times 3{,}000 a_{60:\overline{1|}} - 200\right)$$

where P is the single premium.

Now:

$$P = 1.015\times 3{,}000 a_{60:\overline{10|}} + 200$$

so:

$$
\begin{aligned}
{}_1V^{retro} &= \frac{D_{60}}{D_{61}} \times \left(1.015\times 3{,}000 a_{60:\overline{10|}} + 200 - 1.015\times 3{,}000 a_{60:\overline{1|}} - 200\right) \\
&= 1.015\times 3{,}000 \frac{D_{60}}{D_{61}} \left(a_{60:\overline{10|}} - a_{60:\overline{1|}}\right)
\end{aligned}
$$

Since the difference between the annuities in the brackets represents the value at age 60 of annual payments made in arrears for the 9 years between ages 61 and 70, this can be written as:

$${}_1V^{retro} = 1.015\times 3{,}000 \frac{D_{60}}{D_{61}} \times \left(\frac{D_{61}}{D_{60}} \times a_{61:\overline{9|}}\right) = 1.015\times 3{,}000 a_{61:\overline{9|}} = {}_1V^{pro}$$

21.5 The policy starts when the policyholder is age 50. The 5th premium is paid on the policyholder's 54th birthday, when the remaining term will be 6 years.

If death occurs at age 54 last birthday, the benefit amount will be £14,000, which will increase by £1,000 each year. The maturity value is £25,000. So the net premium reserve is:

$$ {}_4V^{pro} = 13{,}000 A^{\,1}_{54:\overline{6|}} + 1{,}000 (IA)^{\,1}_{54:\overline{6|}} + 25{,}000 \frac{D_{60}}{D_{54}} - P\ddot{a}_{54:\overline{6|}} $$

where the net premium is given by:

$$ P\ddot{a}_{50:\overline{10|}} = 9{,}000 A^{\,1}_{50:\overline{10|}} + 1{,}000 (IA)^{\,1}_{50:\overline{10|}} + 25{,}000 \frac{D_{60}}{D_{50}} $$

In order to specify the calculation of the reserve precisely, it is necessary to state how the premium is calculated. This is because, for a net premium reserve, the net premium is always calculated on the same basis as the reserve.

21.6 The gross premium retrospective reserve at the end of year 20 is:

$$ {}_{20}V^{retro} = \frac{D_{30}}{D_{50}} \left[250\ddot{a}_{30:\overline{20|}} - 30{,}000 \bar{A}^{\,1}_{30:\overline{20|}} - 100 - 0.05 \times 250 \times \left(\ddot{a}_{30:\overline{20|}} - 1 \right) \right] $$

21.7 The reserve at time 0 is zero, and so in the first year the equation of equilibrium is:

$$ (P - I)(1 + i) = (1 + c) S q_x + {}_1V\, p_x $$

For subsequent years, *ie* $t = 1, 2, \ldots$:

$$ ({}_tV + P - kP)(1 + i) = (1 + c) S q_{x+t} + {}_{t+1}V\, p_{x+t} $$

21.8 (i) The retrospective accumulation at the end of 5 years is:

$$ 200(1.075)^5 \frac{l_{25}}{l_{30}} = 200(1.075)^5 \times \frac{98{,}797}{98{,}617} = £287.65 $$

(ii) The retrospective accumulation at the end of 20 years is:

$$ 200(1.075)^{20} \frac{l_{25}}{l_{45}} = 200(1.075)^{20} \times \frac{98{,}797}{97{,}315} = £862.51 $$

21.9 The reserve V should be such that the probability of making a positive future loss is less than 1%, *ie* such that:

$$ P\left(50{,}000 v^{K_{[30]}+1} - 0.95 \times 700 a_{\overline{K_{[30]}|}} - V > 0 \right) < 0.01 \qquad [1] $$

noting that the next premium is due in one year's time, hence we use the annuity function for payments in arrears.

We need to calculate the reserve as:

$$V(r)=50{,}000v^{r+1}-0.95\times 700a_{\overline{r}|} \qquad [½]$$

for a value of r such that:

$$P\left(K_{[30]}<r\right)<0.01 \quad \text{and} \quad P\left(K_{[30]}<r+1\right)\geq 0.01 \qquad [1½]$$

Rearranging the above we require:

$$P\left(K_{[30]}\geq r\right)>0.99 \quad \text{and} \quad P\left(K_{[30]}\geq r+1\right)\leq 0.99$$

or $\quad {}_rp_{[30]}>0.99 \quad$ and $\quad {}_{r+1}p_{[30]}\leq 0.99$

or $\quad l_{[30]+r}>0.99l_{[30]} \quad$ and $\quad l_{[30]+r+1}\leq 0.99l_{[30]}$

From the *Tables* we find $l_{[30]}=9{,}923.7497$ which makes $0.99l_{[30]}=9{,}824.5122$. From the *Tables* we also find that $l_{[30]+13}=l_{43}=9{,}826.2060$ and $l_{[30]+14}=l_{44}=9{,}814.3359$, which means that we take $r=13$. [1½]

So the required reserve is:

$$V(13)=50{,}000v^{14}-0.95\times 700\left(\frac{1-v^{13}}{0.03}\right)=£25{,}984 \qquad [½]$$

[Total 5]

21.10 (i) ***Future loss random variables***

The future loss random variable at time zero is the present value of the future benefits and expenses paid out, less the present value of the future premiums received.

If the life dies, we need to use K_{50} to determine the value of the sum assured (since this decreases by discrete amounts), but T_{50} for the discount factor (since the benefit is paid immediately on death). We add 400 to the total paid on death for the claim expenses.

The future loss random variable at the outset is:

$$\begin{cases}\left(150{,}400-10{,}000K_{50}\right)v^{T_{50}}+300+43a_{\overline{K_{50}}|}-P\ddot{a}_{\overline{K_{50}+1}|} & K_{50}<10 \\ 300+43a_{\overline{9}|}-P\ddot{a}_{\overline{10}|} & K_{50}\geq 10\end{cases} \qquad [2]$$

Just before the payment of the fifth premium, the life is aged 54, and the balance of the term is 6 years. So the future loss random variable is now:

$$\begin{cases} \left(110{,}400-10{,}000K_{54}\right)v^{T_{54}}+43\ddot{a}_{\overline{K_{54}+1}|}-P\ddot{a}_{\overline{K_{54}+1}|} & K_{54}<6 \\ 43\ddot{a}_{\overline{6}|}-P\ddot{a}_{\overline{6}|} & K_{54}\geq 6 \end{cases}$$ [2]

[Total 4]

(ii) ***Premium***

The premium equation using the equivalence principle is:

$$P\ddot{a}_{50:\overline{10}|}=160{,}400\bar{A}^{\,1}_{50:\overline{10}|}-10{,}000\left(I\bar{A}\right)^{\,1}_{50:\overline{10}|}+300+43\times\left(\ddot{a}_{50:\overline{10}|}-1\right)$$ [1]

The level annuity factor is tabulated. The level term assurance factor is:

$$\bar{A}^{\,1}_{50:\overline{10}|}\approx(1+i)^{½}A^{\,1}_{50:\overline{10}|}=\sqrt{1.04}\left[A_{50:\overline{10}|}-\frac{D_{60}}{D_{50}}\right]$$

$$=\sqrt{1.04}\left[0.68024-\frac{882.85}{1{,}366.61}\right]=0.03490$$ [1]

The increasing term assurance factor is:

$$\left(I\bar{A}\right)^{\,1}_{50:\overline{10}|}\approx(1+i)^{½}(IA)^{\,1}_{50:\overline{10}|}=\sqrt{1.04}\left[(IA)_{50}-\frac{D_{60}}{D_{50}}\left((IA)_{60}+10A_{60}\right)\right]$$

$$=\sqrt{1.04}\left[8.55929-\frac{882.85}{1{,}366.61}\left(8.36234+10\times 0.45640\right)\right]=0.21282$$ [1½]

So we have:

$$8.314P=160{,}400\times 0.03490-10{,}000\times 0.21282+300+43\times 7.314$$

This gives us a premium of £491.31. [½]

[Total 4]

(iii) ***Gross premium prospective reserve***

The gross premium prospective reserve for the policy at time 4 is the expected present value of the future benefits and expenses less the expected present value of the future premiums:

$${}_4V=120{,}400\bar{A}^{\,1}_{54:\overline{6}|}-10{,}000\left(I\bar{A}\right)^{\,1}_{54:\overline{6}|}-\left(491.31-43\right)\ddot{a}_{54:\overline{6}|}$$ [1]

We calculate the term assurance factors as before:

$$\bar{A}^{\,1}_{54:\overline{6}|} \approx (1+i)^{½} A^{\,1}_{54:\overline{6}|} = \sqrt{1.04}\left[A_{54:\overline{6}|} - \frac{D_{60}}{D_{54}}\right]$$

$$= \sqrt{1.04}\left[0.79264 - \frac{882.85}{1,154.22}\right] = 0.02830 \qquad [1]$$

and:

$$(I\bar{A})^{\,1}_{54:\overline{6}|} \approx (1+i)^{½} (IA)^{\,1}_{54:\overline{6}|} = \sqrt{1.04}\left[(IA)_{54} - \frac{D_{60}}{D_{54}}\left((IA)_{60} + 6A_{60}\right)\right]$$

$$= \sqrt{1.04}\left[8.59381 - \frac{882.85}{1,154.22}(8.36234 + 6 \times 0.45640)\right] = 0.10502 \qquad [1½]$$

The annuity function is again tabulated.

So the reserve is:

$${}_4V = 120,400 \times 0.02830 - 10,000 \times 0.10502 - 448.31 \times 5.391 = -59.60 \qquad [½]$$

[Total 4]

(iv) ***Comment***

The gross premium reserve at time 4 is negative. This is typical of a decreasing term assurance where the cost of claims and expenses are relatively high in the early years of the policy and low in the later years. [½]

As the premiums are level, the future premiums have greater expected present value than the future claims and expenses, producing the negative reserve value. [1]

The insurer would incur an overall loss from a group of policies of this type, should some of them lapse when their reserves are negative. [½]

The insurer can avoid this problem if the premium paying term is reduced, *eg* to 6 years for a 10-year policy, so for reserves at later policy durations the EPV of the future premiums is reduced, causing the reserve value to rise. [½]

[Maximum 2]

21.11 *This question is Subject CT5, April 2010, Question 14, part (ii).*

We want the expected present value of the future benefits, expenses and commission, less the expected present value of the future premiums.

The basic sum assured plus past declared bonuses is £245,000. Allowing for a compound future bonus rate of 4% (and remembering that the bonuses do not vest until the end of the year), the EPV of the death benefits is approximately:

$$245,000\left[q_{60}v^{½} + 1.04\,{}_{1|}q_{60}v^{1½} + 1.04^2\,{}_{2|}q_{60}v^{2½} + 1.04^3\,{}_{3|}q_{60}v^{3½} + 1.04^4\,{}_{4|}q_{60}v^{4½}\right] \qquad [½]$$

Taking out a factor of $v^{½}$, this is:

$$245{,}000v^{½}\left[q_{60}+1.04\,{}_{1|}q_{60}v^{1}+1.04^{2}\,{}_{2|}q_{60}v^{2}+1.04^{3}\,{}_{3|}q_{60}v^{3}+1.04^{4}\,{}_{4|}q_{60}v^{4}\right] \qquad [½]$$

But since $v=\dfrac{1}{1.04}$, this simplifies to:

$$245{,}000v^{½}\left[q_{60}+{}_{1|}q_{60}+{}_{2|}q_{60}+{}_{3|}q_{60}+{}_{4|}q_{60}\right]$$

$$=245{,}000v^{½}\times{}_{5}q_{60}=245{,}000\times\frac{1}{1.04^{½}}\times\left(1-\frac{8{,}821.2612}{9{,}287.2164}\right)$$

$$=12{,}053.357 \qquad [1]$$

Similarly, the EPV of the survival benefit is:

$$245{,}000\times{}_{5}p_{60}\times v^{5}\times1.04^{5}=245{,}000\times{}_{5}p_{60}=245{,}000\times\frac{8{,}821.2612}{9{,}287.2164}=232{,}707.940 \qquad [1]$$

The EPV of the claim expense on death is:

$$1{,}000A^{1\,@4\%}_{60:\overline{5|}}=1{,}000\left[A_{60:\overline{5|}}-\frac{D_{65}}{D_{60}}\right]$$

$$=1{,}000\left[0.82499-\frac{689.23}{882.85}\right]=44.302 \qquad [1]$$

The EPV of the claim expense on maturity is:

$$500\times{}_{5}p_{60}\times v^{5}_{4\%}=390.345 \qquad [½]$$

So the gross prospective reserve at time 25 can be written as follows:

$${}_{25}V^{pro}=12{,}053.357+232{,}707.94+90\ddot{a}^{@4\%}_{60:\overline{5|}}+44.302+390.345$$

$$+0.025\times4\times615.61\ddot{a}^{(4)@4\%}_{60:\overline{5|}}-4\times615.61\ddot{a}^{(4)@4\%}_{60:\overline{5|}} \qquad [½]$$

Calculating the values of the annuity factors:

$$\ddot{a}_{60:\overline{5|}}=4.550$$

$$\ddot{a}^{(4)}_{60:\overline{5|}}\approx4.550-\frac{3}{8}\left(1-\frac{D_{65}}{D_{60}}\right)=4.468 \qquad [½]$$

So:

$$
\begin{aligned}
{}_{25}V^{pro} &= 12{,}053.357 + 232{,}707.940 + 90 \times 4.550 + 44.302 + 390.345 \\
&\qquad + 0.025 \times 2{,}462.44 \times 4.468 - 2{,}462.44 \times 4.468 \\
&= 234{,}878.32
\end{aligned}
$$

or about £234,900. [½]

[Total 6]

21.12 *This question is Subject CT5, October 2012, Question 13.*

(i) ***Monthly premium***

Let G be the total amount of premium paid in a year. The expected present value of the premiums is:

$$G\ddot{a}^{(12)}_{[55]:\overline{30|}}$$ [½]

The expected present value of the benefits is:

$$73{,}500\bar{A}_{[55]} + 1{,}500(I\bar{A})_{[55]}$$ [1]

The expected present value of the expenses and commission is:

$$275 + 65\left[\ddot{a}_{[55]} - 1\right] + 200\bar{A}_{[55]} + 0.75G + 0.025G\left[\ddot{a}^{(12)}_{[55]:\overline{30|}} - \frac{1}{12}\right]$$ [1½]

Now:

$$\ddot{a}_{[55]:\overline{30|}} = \ddot{a}_{[55]} - \frac{D_{85}}{D_{[55]}}\ddot{a}_{85} = 15.891 - \frac{120.71}{1{,}104.05} \times 5.333 = 15.308$$ [½]

So:

$$\ddot{a}^{(12)}_{[55]:\overline{30|}} \approx \ddot{a}_{[55]:\overline{30|}} - \frac{11}{24}\left(1 - \frac{D_{85}}{D_{[55]}}\right) = 15.308 - \frac{11}{24}\left(1 - \frac{120.71}{1{,}104.05}\right) = 14.900$$ [½]

We also need:

$$\bar{A}_{[55]} \approx 1.04^{½} \times A_{[55]} = 1.04^{½} \times 0.38879 = 0.39649$$

$$(I\bar{A})_{[55]} \approx 1.04^{½} \times (IA)_{[55]} = 1.04^{½} \times 8.58908 = 8.75918$$ [½]

So, using the equivalence principle, we have:

$$G\ddot{a}^{(12)}_{[55]:\overline{30}|} = 73,500\bar{A}_{[55]} + 1,500\left(I\bar{A}\right)_{[55]} + 275 + 65\left[\ddot{a}_{[55]} - 1\right]$$

$$+200\bar{A}_{[55]} + 0.75G + 0.025G\left[\ddot{a}^{(12)}_{[55]:\overline{30}|} - \frac{1}{12}\right] \qquad [½]$$

Substituting in the values:

$$14.900G = 73,500 \times 0.39649 + 1,500 \times 8.75918 + 275 + 65 \times 14.891$$

$$+200 \times 0.39649 + 0.75G + 0.025G \times 14.816 \qquad [½]$$

So:

$$G = \frac{43,602.96}{13.7793} = 3,164.38$$

So the monthly premium is $\dfrac{3,164.38}{12} = £263.70$. [½]

[Total 6]

(ii) ***Gross prospective policy value***

At the end of the 30th policy year, the 30th past bonus has just been added, bringing the current sum assured to £120,000. [½]

So, the gross premium prospective reserve at time 30 is:

$$_{30}V = 118,125\bar{A}_{85} + 1,875(I\bar{A})_{85} + 80\ddot{a}_{85} + 250\bar{A}_{85} \qquad [2]$$

Evaluating:

$$\bar{A}_{85} \approx 1.04^{½} \times A_{85} = 1.04^{½} \times 0.79490 = 0.81064$$

$$(I\bar{A})_{85} \approx 1.04^{½} \times (IA)_{85} = 1.04^{½} \times 4.40856 = 4.49587$$

$$\ddot{a}_{85} = 5.333 \qquad [1]$$

So:

$$_{30}V = 118,125 \times 0.81064 + 1,875 \times 4.49587 + 80 \times 5.333 + 250 \times 0.81064$$

$$= 104,816 \qquad [½]$$

or about £104,800.

[Total 4]

21.13 *This question is Subject CT5, April 2014, Question 11.*

(i) ***Annual office premium***

Let P be the annual office premium for a sum assured of £1,000.

The equation of value, for a sum assured of £1,000, is:

$$P\ddot{a}_{[40]:\overline{20|}} = 1{,}000A_{[40]:\overline{20|}} + 0.6P + 0.06P\left(\ddot{a}_{[40]:\overline{20|}} - 1\right)$$ [1]

Looking up the values in the AM92 tables, using an interest rate of 4%:

$$13.930P = 1{,}000 \times 0.46423 + 0.6P + 0.06P(13.930 - 1)$$ [½]

Rearranging:

$$P(13.930 - 0.6 - 0.06 \times 12.930) = 464.23$$

$$\Rightarrow \quad P = \frac{464.23}{12.5542} = 36.98$$ [½]

So, the annual office premium for a sum assured of £1,000 is £36.98.

[Total 2]

(ii) ***Gross premium prospective reserve at time 5***

The policies were issued on 1 January 2008, so 31 December 2012 is 5 years later. This means we need the gross premium prospective reserve at time 5, when the life is aged 45 and the remaining term is 15 years.

The gross premium prospective reserve at time 5 is:

$$1{,}000A_{45:\overline{15|}} + 0.06 \times 36.98\ddot{a}_{45:\overline{15|}} - 36.98\ddot{a}_{45:\overline{15|}}$$

$$= 1{,}000A_{45:\overline{15|}} - 0.94 \times 36.98\ddot{a}_{45:\overline{15|}}$$ [1½]

Looking up values in the AM92 tables using an interest rate of 4% gives:

$$1{,}000 \times 0.56206 - 0.94 \times 36.98 \times 11.386 = 166.27$$ [½]

So, the gross premium prospective reserve at time 5 for a sum assured of £1,000 is £166.27.

[Total 2]

(iii) ***Overall profit or loss in 2013***

The total reserve held at the start of 2013 for policies with total sum assured £15,500,000 is:

$$166.27 \times \frac{15{,}500{,}000}{1{,}000} = £2{,}577{,}185$$ [1]

The total premiums received at the start of 2013 are:

$$36.98 \times \frac{15,500,000}{1,000} = £573,190 \qquad [½]$$

Interest is earned over 2013 at a rate of 3.5%, so by the end of the year, the accumulated value of the opening reserve plus the premiums less the expenses will be:

$$(2,577,185 + 573,190 - 76,500) \times 1.035 = £3,181,460.63 \qquad [1½]$$

This represents the total funds available to the company at the end of 2013.

The total amount paid out at the end of 2013 in respect of surrenders is:

$$0.85 \times {}_6V \times \frac{625,000}{1,000} \qquad [1]$$

where ${}_6V$ denotes the gross premium prospective reserve at time 6 (*ie* 31 December 2013) for a sum assured of £1,000. We can calculate ${}_6V$ as:

$$1,000 A_{46:\overline{14|}} - 0.94 \times 36.98 \ddot{a}_{46:\overline{14|}} \qquad [1]$$

Using AM92 mortality and an interest rate of 4%:

$${}_6V = 1,000 \times 0.58393 - 0.94 \times 36.98 \times 10.818 = 207.88 \qquad [½]$$

So, the total amount paid out at the end of 2013 in respect of surrenders is:

$$0.85 \times 207.88 \times \frac{625,000}{1,000} = £110,436.25 \qquad [½]$$

The total sum assured in force at the end of 2013 (after death claims and surrenders) is:

$$15,500,000 - 295,000 - 625,000 = £14,580,000 \qquad [1]$$

So, the total reserve required at the end of 2013 for the policies remaining in force is:

$$207.88 \times \frac{14,580,000}{1,000} = £3,030,890.40 \qquad [1]$$

This means that the total funds needed at the end of 2013 to pay death and surrender claims, and to set up the reserves needed for surviving policies is:

$$295,000 + 110,436.25 + 3,030,890.40 = £3,436,326.65 \qquad [1]$$

The overall profit in 2013 will be the funds available at the end of the year minus the funds needed at the end of the year:

$$3,181,460.63 - 3,436,326.65 = -£254,866.02 \qquad [1]$$

ie there is a loss of £254,866.

[Total 10]

21.14 *This question is Subject CT5, April 2012, Question 2.*

The left-hand side of the equation of equilibrium takes the form:

Reserve + premium – expenses + interest [½]

Here, this gives us:

$$(25{,}130 + 3{,}000 - 90) \times 1.04 = 29{,}161.60$$ [1]

The required cashflow is:

Expected reserve at the end of the year + expected death benefit [½]

$$= 0.97 \times 28{,}950 + 0.03 \times 36{,}000 = 29{,}161.50$$ [½]

The expected cashflow is greater than the required cashflow by £0.10. So this is our expected profit emerging at the end of the 12th policy year (and is approximately zero). [½]

[Total 3]

21.15 *This question is Subject CT5, September 2013, Question 11, part (c).*

This is evaluated as follows:

$$s_{55:\overline{10}|} = (1+i)^{10} \frac{l_{55}}{l_{65}} a_{55:\overline{10}|} = (1+i)^{10} \frac{l_{55}}{l_{65}} \left[a_{55} - v^{10} \frac{l_{65}}{l_{55}} a_{65} \right]$$

$$= (1+i)^{10} \frac{l_{55}}{l_{65}} a_{55} \; - \; a_{65}$$

$$= 1.04^{10} \times \frac{9{,}917.623}{9{,}703.708} (18.210 - 1) - (14.871 - 1)$$

$$= 26.03659 - 13.871 = 12.166$$ [1]

22

Joint life and last survivor functions

Syllabus objectives

5.1 Define and use assurance and annuity functions involving two lives.

5.1.1 Extend the techniques of objectives 4.2 to deal with cashflows dependent upon the death or survival of either or both of two lives.

0 Introduction

In this chapter, we extend the concepts developed earlier to deal with situations involving two lives; for instance, 'what is the value of an annuity payable until the last of two lives dies?'

This chapter covers the following topics:

- random variables describing basic joint life and last survivor functions
- determining simple joint life and last survivor probabilities
- present values of simple joint life and last survivor policies
- calculating premiums and reserves for policies based on two lives.

1 Random variables to describe joint life functions

1.1 Single life functions

So far we have described annuity and assurance functions that depend upon the death or survival of a single life aged x. Central to the development of these functions is the random variable measuring the future lifetime of a life now aged x, T_x, or its curtate counterpart K_x.

Recall that K_x (the curtate future lifetime of x) is the integer part of T_x (the complete future lifetime of x).

1.2 Joint life functions

We now consider annuity and assurance functions that depend upon the death or survival of two lives. The random variables of interest are T_x and T_y, the future lifetimes of two lives, one aged x and the other aged y. Throughout the analysis of these problems, we assume that T_x and T_y are independent random variables.

So we assume that the mortality of life x is independent of the mortality of life y. In practice, this is seldom the case since many joint life policies involve husband/wife, or similar, partnerships. The mortality of these partners is not independent because of the significant possibility of both lives dying as a result of the same accident or illness. However, the assumption of independence makes the theory far more manageable.

The random variable T_{xy} measures the joint lifetime of (x) and (y), *ie* the time while both (x) and (y) remain alive, which is the time until the first death of (x) and (y). We can write:

$$T_{xy} = \min\{T_x, T_y\}$$

The cumulative distribution function of this random variable can be written:

$$F_{T_{xy}}(t) = P\left[T_{xy} \le t\right]$$

which we write as:

$$\begin{aligned}
F_{T_{xy}}(t) &= 1 - {}_tp_{xy} \\
&= P\left[\min\left\{T_x, T_y\right\} \le t\right] \\
&= 1 - P\left[T_x > t \text{ and } T_y > t\right] \\
&= 1 - P\left[T_x > t\right] P\left[T_y > t\right]
\end{aligned}$$

since the random variables T_x and T_y are independent.

So:

$$F_{T_{xy}}(t) = 1 - {}_tp_x\,{}_tp_y$$

using the life table notation.

It is helpful at this point to recall the idea of a *status*, which we met earlier when considering single life assurances and annuities.

The random variable T_{xy} is characterised by the *joint life* status xy, which is the status of both lives x and y being alive. If either one of x or y dies, the joint life status is said to *fail*. The random variable T_{xy} represents the future time until the failure of the status, which in this case is the joint life status xy.

With this in mind, the probability notation ${}_tp_{xy}$ introduced above represents the probability that the joint life status of x and y survives for t years, *ie* both lives survive for at least t years.

In the same way, when we consider the single life temporary annuity $\ddot{a}_{x:\overline{n}|}$, we are looking at an annuity payable while the joint status $x:\overline{n}|$ is still active, that is, the life x is still alive and the n-year period has not yet expired. In this case the joint status is made up of an active life and a time period, but the underlying logic is still the same.

The density function of T_{xy} can be obtained by differentiating the cumulative distribution function:

$$f_{T_{xy}}(t) = \frac{d}{dt}\left[1 - {}_tp_x\,{}_tp_y\right]$$

$$= -\frac{d}{dt}\,{}_tp_x\,{}_tp_y$$

$$= -{}_tp_x\frac{d}{dt}\,{}_tp_y - {}_tp_y\frac{d}{dt}\,{}_tp_x \qquad \text{(using the product rule)}$$

$$= -{}_tp_x\left(-{}_tp_y\mu_{y+t}\right) - {}_tp_y\left(-{}_tp_x\mu_{x+t}\right)$$

The last line above follows from the fact that the PDF of the random variable T_x is ${}_tp_x\,\mu_{x+t}$, so this must be equal to the derivative of the cumulative distribution function:

$$f_{T_x}(t) = {}_tp_x\,\mu_{x+t} = \frac{d}{dt}F_{T_x}(t) = \frac{d}{dt}P(T_x \le t) = \frac{d}{dt}\,{}_tq_x = \frac{d}{dt}\left(1 - {}_tp_x\right) = -\frac{d}{dt}\,{}_tp_x$$

Tidying up the expression above, the PDF of T_{xy} is:

$$f_{T_{xy}}(t) = {}_tp_x\,{}_tp_y\left(\mu_{x+t} + \mu_{y+t}\right)$$

By considering the infinitesimally small time interval $(t, t+\delta t)$, we can interpret:

$$ {}_tp_x\, {}_tp_y \left(\mu_{x+t} + \mu_{y+t}\right)\delta t $$

as the approximate probability that the joint life status xy fails over the interval of time δt. This is the product of:

- the probability of both x and y surviving to time t, ${}_tp_x\, {}_tp_y \left(= {}_tp_{xy}\right)$, and
- the probability of x or y dying in the interval from time t to time $t+\delta t$, $\left(\mu_{x+t} + \mu_{y+t}\right)\delta t$.

1.3 Joint lifetime random variables and joint life table functions

When functions of a single life *eg* T_x are considered, it is helpful to introduce the life table functions l_x, d_x and q_x as an aid to the calculation of the numerical values of expressions that are the solution of actuarial problems.

In an exactly similar way it is helpful to develop the joint life functions l_{xy}, d_{xy} and q_{xy} to help in the numerical evaluation of expressions that are the solution to problems involving more than one life. We define these functions in terms of the single life functions. Recall that:

$$ {}_tp_{xy} = {}_tp_x\, {}_tp_y $$

Using the independence assumption:

$$ {}_tp_{xy} = \frac{l_{x+t}}{l_x} \cdot \frac{l_{y+t}}{l_y} $$

So we write:

$$ l_{xy} = l_x l_y $$

and:

$$ {}_tp_{xy} = \frac{l_{x+t:y+t}}{l_{xy}} $$

only separating the subscripts with colons when the exact meaning of the function would be unclear if the colons were omitted.

Then:

$$ d_{xy} = l_{xy} - l_{x+1:y+1} $$

$$ q_{xy} = \frac{d_{xy}}{l_{xy}} $$

q_{xy} is the probability that the joint life status fails by the end of the year, *ie* it is the probability that *at least one* of x and y dies within the year.

Similarly, ${}_tq_{xy}$ represents the probability that the joint life status fails within the next t years, *ie* by the end of t years there is at least one death.

The force of failure of the joint life status can be derived in the usual way:

$$\mu_{x+t:y+t} = -\frac{1}{l_{x+t:y+t}}\frac{d}{dt}l_{x+t:y+t}$$

This can then be related to the forces of mortality in the life tables for the single lives *x* and *y:*

$$\mu_{x+t:y+t} = -\frac{d}{dt}\log_e l_{x+t:y+t}$$

$$= -\frac{d}{dt}\log_e l_{x+t}\, l_{y+t}$$

$$= -\frac{d}{dt}\left\{\log_e l_{x+t} + \log_e l_{y+t}\right\}$$

$$= \mu_{x+t} + \mu_{y+t}$$

Notice that:

- **this relationship is additive, which is in contrast to the previous relationships, which were multiplicative, and**
- **that there is no 'simple' relationship for d_{xy}.**

So to get the joint *force of mortality* we *add* the forces for the individual lives. For the joint *probability of survival* we *multiply* the individual probabilities of survival.

So we can write:

$$f_{T_{xy}}(t) = {}_tp_x\, {}_tp_y\left(\mu_{x+t} + \mu_{y+t}\right)$$

$$= {}_tp_{xy}\ \mu_{x+t:y+t}$$

We can define a discrete random variable that measures the curtate joint future lifetime of x and y :

$$K_{xy} = \text{integer part of } T_{xy}$$

and develop the probability function of K_{xy} :

$$P\left[K_{xy} = k\right] = P\left[k \le T_{xy} < k+1\right]$$

$$= F_{T_{xy}}(k+1) - F_{T_{xy}}(k)$$

$$= \left(1 - {}_{k+1}p_{xy}\right) - \left(1 - {}_k p_{xy}\right)$$

$$= {}_k p_{xy} - {}_{k+1}p_{xy}$$

$$= {}_k p_{xy} - {}_k p_{xy}\, p_{x+k:y+k}$$

$$= {}_k p_{xy}\, q_{x+k:y+k}$$

$$= {}_{k|}q_{xy}$$

This is a deferred probability that the joint life status fails. Specifically, it is the probability that the joint life status fails within a one-year period, starting in k years' time, *ie* it is the probability that both lives survive for k years, and then at least one of the lives dies in the following year.

The joint life table functions l_{xy} , d_{xy} , q_{xy} and μ_{xy} are not tabulated in the Formulae and Tables for Examinations. However, these functions can be evaluated using the tabulated single life functions l_x , q_x and μ_x .

There is limited information in the *Tables* for joint life functions. The only values provided are those of joint life annuities-due for males and females of different ages based on PMA92C20 mortality for the male life, PFA92C20 for the female life, and 4% *pa* interest. These appear on page 115 of the *Tables*, and we will use these later on in this chapter.

Question

Assuming that both lives are independently subject to AM92 mortality, calculate the following:

(i) ${}_3p_{45:41}$

(ii) $q_{66:65}$

(iii) $\mu_{38:30}$

Solution

(i) $${}_3p_{45:41} = \frac{l_{48:44}}{l_{45:41}} = \frac{l_{48}}{l_{45}} \times \frac{l_{44}}{l_{41}} = \frac{9{,}753.4714}{9{,}801.3123} \times \frac{9{,}814.3359}{9{,}847.0510} = 0.991813$$

(ii) $$q_{66:65} = 1 - p_{66:65} = 1 - \frac{l_{67:66}}{l_{66:65}} = 1 - \frac{l_{67}}{l_{66}} \times \frac{l_{66}}{l_{65}} = 1 - \frac{l_{67}}{l_{65}} = 1 - \frac{8{,}557.0118}{8{,}821.2612} = 0.029956$$

(iii) $$\mu_{38:30} = \mu_{38} + \mu_{30} = 0.000788 + 0.000585 = 0.001373$$

1.4 Last survivor lifetime random variables

Two common types of policy are:

- an annuity payable to a couple while at least one of them is alive, and
- an assurance payable on the second death of a couple.

These are both examples of last survivor policies, where the payment is contingent on what happens to the second life to die, rather than the first.

The random variable $T_{\overline{xy}}$ measures the time until the last death of (x) and (y), *ie* the time while at least one of (x) and (y) remains alive. We can write:

$$T_{\overline{xy}} = \max\{T_x, T_y\}$$

So the last survivor status fails on the second death. The last survivor status is indicated by $\overline{xy}$.

The cumulative distribution function of this random variable can be written:

$$F_{T_{\overline{xy}}}(t) = P\left[T_{\overline{xy}} \le t\right] = {}_tq_{\overline{xy}}$$

This is the probability that the last survivor status fails within t years, *ie* the probability that both lives are dead by the end of t years. Now:

$$\begin{aligned} F_{T_{\overline{xy}}}(t) &= P\left[\max\{T_x, T_y\} \le t\right] \\ &= P\left[T_x \le t \text{ and } T_y \le t\right] \\ &= P\left[T_x \le t\right] P\left[T_y \le t\right] \\ &= {}_tq_x \times {}_tq_y \end{aligned}$$

since the random variables T_x and T_y are independent.

So:

$$F_{T_{\overline{xy}}}(t) = \left(1 - {}_tp_x\right)\left(1 - {}_tp_y\right)$$

$$= 1 - {}_tp_x - {}_tp_y + {}_tp_x\,{}_tp_y$$

$$= \left(1 - {}_tp_x\right) + \left(1 - {}_tp_y\right) - \left(1 - {}_tp_x\,{}_tp_y\right)$$

$$= F_{T_x}(t) + F_{T_y}(t) - F_{T_{xy}}(t)$$

using the life table notation.

This last result is a particular example of a general relationship that will be key to enabling us to calculate multiple life functions easily.

Important result

$$\begin{pmatrix}\text{function of last}\\ \text{survivor status } \overline{xy}\end{pmatrix} = \begin{pmatrix}\text{function of single}\\ \text{life status } x\end{pmatrix} + \begin{pmatrix}\text{function of single}\\ \text{life status } y\end{pmatrix} - \begin{pmatrix}\text{function of joint}\\ \text{life status } xy\end{pmatrix}$$

The function involved must be the same for all statuses. We will subsequently refer to this relationship as just:

$$\text{last survivor (L)} = \text{single (S)} + \text{single (S)} - \text{joint (J)}$$

In the above Core Reading, the function is the cumulative distribution function of the lifetime of the status at time t, *ie* the probability of the status failing by time t. So we have:

$${}_tq_{\overline{xy}} = {}_tq_x + {}_tq_y - {}_tq_{xy}$$

The density function of $T_{\overline{xy}}$ can be obtained by differentiating the cumulative distribution function:

$$f_{T_{\overline{xy}}}(t) = \frac{d}{dt}\left[1 - {}_tp_x - {}_tp_y + {}_tp_x\,{}_tp_y\right]$$

$$= {}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_x\,{}_tp_y\left(\mu_{x+t} + \mu_{y+t}\right)$$

Using the results from Section 1.3 above, we can write:

$$f_{T_{\overline{xy}}}(t) = {}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t}$$

$$= f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t)$$

In other words, we have the $L = S + S - J$ relationship again, but this time for the probability density function at time t.

We could have alternatively found the above result from:

$$f_{T_{\overline{xy}}}(t) = \frac{d}{dt} F_{T_{\overline{xy}}}(t) = \frac{d}{dt}\left[F_{T_x}(t) + F_{T_y}(t) - F_{T_{xy}}(t)\right] = f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t)$$

Tip

For joint life statuses, it is often easier to work with p -type (survival) functions, since ${}_t p_{xy} = {}_t p_x \times {}_t p_y$.

For last survivor statuses, it is often easier to work with q -type (mortality) functions, since ${}_t q_{\overline{xy}} = {}_t q_x \times {}_t q_y$.

2 Simple probabilities involving two lives

We now apply the random variable theory developed above to see how joint life and last survivor probabilities of death and survival can be evaluated.

2.1 Evaluating probabilities of death or survival of either or both of two lives

We can define a discrete random variable that measures the curtate last survivor lifetime of x and y:

$$K_{\overline{xy}} = \text{integer part of } T_{\overline{xy}}$$

and develop the probability function of $K_{\overline{xy}}$:

$$P\left[K_{\overline{xy}} = k\right] = P\left[k \le T_{\overline{xy}} < k+1\right]$$

$$= {}_{k|}q_{\overline{xy}}$$

$$= F_{T_{\overline{xy}}}(k+1) - F_{T_{\overline{xy}}}(k)$$

$$= F_{T_x}(k+1) + F_{T_y}(k+1) - F_{T_{xy}}(k+1) - \left\{F_{T_x}(k) + F_{T_y}(k) - F_{T_{xy}}(k)\right\}$$

$$= P\left[K_x = k\right] + P\left[K_y = k\right] - P\left[K_{xy} = k\right]$$

$$= {}_{k|}q_x + {}_{k|}q_y - {}_{k|}q_{xy}$$

Note that this is again the $L = S + S - J$ relationship:

$${}_{k|}q_{\overline{xy}} = {}_{k|}q_x + {}_{k|}q_y - {}_{k|}q_{xy}$$

As we saw in the last section, **the cumulative distribution function of $T_{\overline{xy}}$ is given by:**

$${}_tq_{\overline{xy}} = 1 - {}_tp_x - {}_tp_y + {}_tp_x\,{}_tp_y$$

So the survival function of $T_{\overline{xy}}$ is:

$${}_tp_{\overline{xy}} = P\left[T_{\overline{xy}} > t\right] = 1 - P\left[T_{\overline{xy}} \le t\right] = {}_tp_x + {}_tp_y - {}_tp_x\,{}_tp_y$$

(*ie* $L = S + S - J$).

This can be factorised into:

$$_tp_x\,_tp_y + \left(1 - _tp_x\right)_tp_y + \left(1 - _tp_y\right)_tp_x$$

where each of the three terms corresponds to one of the mutually exclusive and exhaustive events which result in the last survivor of (x) and (y) living for at least t years, *ie*:

- **both (x) and (y) alive after t years,**
- **(x) dead, but (y) alive after t years, and**
- **(x) alive, but (y) dead after t years.**

These probabilities can be evaluated directly. The probability of the complementary event that both lives die within t years is:

$$_tq_{\overline{xy}} = (1 - _tp_x)\,(1 - _tp_y) = _tq_x\,_tq_y$$

as we saw earlier.

In summary, there are three alternatives for calculating $_tp_{\overline{xy}}$:

- $1 - _tq_{\overline{xy}}$ (1)
- $_tp_x + _tp_y - _tp_{xy}$ (2)
- $_tp_x\,_tp_y + \left(1 - _tp_x\right)_tp_y + \left(1 - _tp_y\right)_tp_x$ (3)

All three of these can be useful, but it's generally best to try using (1) first and (3) last, as more calculations are involved as we move down the list.

We also have $p_{\overline{xy}} = p_x + p_y - p_{xy}$ (for example), where $p_{\overline{xy}}$ is the probability that the last survivor status remains active for at least one year.

Question

Calculate $P\left(5 < K_{\overline{50:60}} < 10\right)$ assuming that the two lives are both independently subject to AM92 mortality.

Solution

Here we want the curtate joint future lifetime to be 6, 7, 8 or 9 years. This means that the first death of the two lives must occur between time 6 years and time 10 years. So both lives must survive for 6 years, and then at least one life must die in the next 4 years.

We can evaluate this as follows:

$$P(5 < K_{50:60} < 10) = {}_{6|4}q_{50:60} = \frac{l_{56}}{l_{50}} \times \frac{l_{66}}{l_{60}} \times \left[1 - \left(\frac{l_{60}}{l_{56}} \times \frac{l_{70}}{l_{66}}\right)\right]$$

$$= \frac{9{,}515.1040}{9{,}712.0728} \times \frac{8{,}695.6199}{9{,}287.2164} \times \left[1 - \left(\frac{9{,}287.2164}{9{,}515.1040} \times \frac{8{,}054.0544}{8{,}695.6199}\right)\right]$$

$$= 0.088028$$

2.2 Evaluating last survivor functions

We have already derived:

$$F_{T_{\overline{xy}}}(t) = F_{T_x}(t) + F_{T_y}(t) - F_{T_{xy}}(t) = 1 - {}_tp_x - {}_tp_y + {}_tp_{xy}$$

$$f_{T_{\overline{xy}}}(t) = f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t) = {}_tp_x\,\mu_{x+t} + {}_tp_y\,\mu_{y+t} - {}_tp_{xy}\,\mu_{x+t:y+t}$$

and so it seems that all last survivor functions can be expressed in terms of single life and joint life functions. This is true and provides a method of evaluating such functions without the need to develop any additional functions to help in computation.

This is the result of the relationship between the joint lifetime and last survivor lifetime random variables:

$$T_{xy} + T_{\overline{xy}} = \min\{T_x, T_y\} + \max\{T_x, T_y\} = T_x + T_y$$

Similarly:

$$K_{xy} + K_{\overline{xy}} = \min\{K_x, K_y\} + \max\{K_x, K_y\} = K_x + K_y$$

which gives the result:

$$P\left[K_{\overline{xy}} = k\right] = P\left[K_x = k\right] + P\left[K_y = k\right] - P\left[K_{xy} = k\right]$$

This is the rationale underlying the important relationship that we stated in Section 1.4 above.

So curtate last survivor functions can be evaluated from the corresponding joint life and single life functions.

Question

Calculate:

(i) $p_{\overline{62:65}}$

(ii) ${}_3q_{\overline{50:50}}$

assuming that the two lives are both independently subject to AM92 Ultimate mortality.

Solution

(i) We can calculate this as:

$$p_{\overline{62:65}} = 1 - q_{\overline{62:65}} = 1 - q_{62}\,q_{65} = 1 - 0.010112 \times 0.014243 = 0.999856$$

or, alternatively:

$$\begin{aligned} p_{\overline{62:65}} &= p_{62} + p_{65} - p_{62:65} \\ &= \frac{l_{63}}{l_{62}} + \frac{l_{66}}{l_{65}} - \frac{l_{63:66}}{l_{62:65}} \\ &= \frac{9,037.3973}{9,129.7170} + \frac{8,695.6199}{8,821.2612} - \frac{9,037.3973}{9,129.7170} \times \frac{8,695.6199}{8,821.2612} \\ &= 0.999856 \end{aligned}$$

(ii) We can calculate this as:

$${}_3q_{\overline{50:50}} = \left({}_3q_{50}\right)^2 = \left(1 - \frac{l_{53}}{l_{50}}\right)^2 = \left(1 - \frac{9,630.0522}{9,712.0728}\right)^2 = 0.000071$$

3 Present values involving two lives

There is no new theory in this section, as we re-use the same assurance and annuity functions that we met when considering single lives, but using multiple life statuses instead of the single life status.

3.1 Present values of joint life and last survivor assurances

Consider an assurance under which the benefit (of 1) is paid immediately on the ending (failure) of a status u. This status u could be any joint lifetime or last survivor status, eg $xy, \overline{xy}$. Let T_u be a continuous random variable representing the future lifetime of the status u and let $f_{T_u}(t)$ be the probability density function of T_u.

The present value of the assurance can be represented by the random variable:

$$\bar{Z}_u = v_i^{T_u}$$

where i is the valuation rate of interest. The expected value of $\bar{Z}_u$ is denoted by $\bar{A}_u$ where:

$$E\left[\bar{Z}_u\right] = \bar{A}_u = \int_{t=0}^{t=\infty} v^t f_{T_u}(t)dt$$

and the variance can be written as:

$$\begin{aligned}\text{var}\left(\bar{Z}_u\right) &= E\left[\bar{Z}_u^2\right] - \left(E\left[\bar{Z}_u\right]\right)^2 \\ &= \int_{t=0}^{t=\infty} v^{2t} f_{T_u}(t)dt - \bar{A}_u^2 \\ &= {}^2\bar{A}_u - \left(\bar{A}_u\right)^2\end{aligned}$$

where ${}^2\bar{A}_u$ is evaluated at a valuation rate of interest of $i^* = 2i + i^2$.

The final expression above follows because:

$$\int_{t=0}^{\infty} v^{2t} f_{T_u}(t)dt = \int_{t=0}^{\infty} \left(v^2\right)^t f_{T_u}(t)dt = \int_{t=0}^{\infty} \left(v^*\right)^t f_{T_u}(t)dt$$

which is equal to ${}^2\bar{A}_u$ where the pre-superscript of 2 denotes a rate of interest i^* such that:

$$i^* = \frac{1}{v^*} - 1 = \frac{1}{v^2} - 1 = (1+i)^2 - 1 = 2i + i^2$$

Equivalently, the pre-superscript of 2 can be thought of as denoting a force of interest δ^* such that:

$$\delta^* = -\ln v^* = -\ln v^2 = -2\ln v = 2\delta$$

For example, using the results from Sections 1.2 and 1.3, if $u = xy$ we can write the mean (*ie* the expected value) and variance of the present value of an assurance payable immediately on the ending of the joint lifetime of (x) and (y) as:

Mean: $$\bar{A}_{xy} = \int_{t=0}^{t=\infty} v^t\, {}_tp_{xy}\mu_{x+t:y+t}dt = \int_{t=0}^{t=\infty} v^t\, {}_tp_{xy}\left(\mu_{x+t} + \mu_{y+t}\right)dt$$

Variance: $${}^2\bar{A}_{xy} - \left(\bar{A}_{xy}\right)^2$$

Again, using the results of Section 1.4, if $u = \overline{xy}$ we can write the mean and variance of the present value of an assurance payable immediately on the death of the last survivor of (x) or (y) as:

Mean: $$\bar{A}_{\overline{xy}} = \int_{t=0}^{t=\infty} v^t\left({}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t}\right)dt = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$$

Variance: $${}^2\bar{A}_{\overline{xy}} - \left(\bar{A}_{\overline{xy}}\right)^2 = \left({}^2\bar{A}_x + {}^2\bar{A}_y - {}^2\bar{A}_{xy}\right) - \left(\bar{A}_x + \bar{A}_y - \bar{A}_{xy}\right)^2$$

So last survivor functions can be evaluated in terms of single and joint life functions.

The integral expression given above for the mean can also be written as:

$$\begin{aligned}
\bar{A}_{\overline{xy}} &= \int_{t=0}^{t=\infty} v^t\left({}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t}\right)dt \\
&= \int_{t=0}^{t=\infty} v^t\left({}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_x\,{}_tp_y\left(\mu_{x+t} + \mu_{y+t}\right)\right)dt \\
&= \int_{t=0}^{t=\infty} v^t\left({}_tp_x\mu_{x+t}\left(1 - {}_tp_y\right) + {}_tp_y\mu_{y+t}\left(1 - {}_tp_x\right)\right)dt \\
&= \int_{t=0}^{t=\infty} v^t\left({}_tp_x\mu_{x+t}\,{}_tq_y + {}_tp_y\mu_{y+t}\,{}_tq_x\right)dt
\end{aligned}$$

The integrand in this expression reflects the payment of the benefit at time t, requiring a discount factor of v^t, and the two events that would cause the benefit to be paid at this time, namely:

- x dies at time t (with probability density ${}_tp_x\mu_{x+t}$), after y has already died (with probability ${}_tq_y$)
- y dies at time t (with probability density ${}_tp_y\mu_{y+t}$), after x has already died (with probability ${}_tq_x$).

Integrating over all possible values of t gives the EPV of the benefit.

If the assurance benefit is paid at the end of the year in which the status ends, then we can use a discrete random variable K_u with a present value function:

$$Z_u = v_i^{K_u+1}$$

where i is the valuation rate of interest.

In the above expression we need the extra '1' to take us to the end of the year in which the status fails, which is when the payment is made, because K_u will take us only to the start of that year. For example, if the status fails in the first year, the present value of the benefit is v and not $v^K = v^0 = 1$.

A similar analysis for K_{xy} and $K_{\overline{xy}}$ gives the means and variances of the present values of the joint life and last survivor assurances with sums assured payable at the end of the year of death.

Joint life

Mean: $$A_{xy} = \sum_{t=0}^{t=\infty} v^{t+1} {}_{t|}q_{xy}$$

Variance: $${}^2A_{xy} - \left(A_{xy}\right)^2$$

Question

Prove the above results for the mean and variance.

Solution

Using the result for the probability function of K_{xy} developed in Section 1.3, the mean is:

$$E\left[v^{K_{xy}+1}\right] = \sum_{t=0}^{\infty} v^{t+1} P(K_{xy}=t) = \sum_{t=0}^{\infty} v^{t+1} {}_tp_{xy}\, q_{x+t:y+t} = \sum_{t=0}^{\infty} v^{t+1} {}_{t|}q_{xy}$$

The variance is:

$$\begin{aligned}\text{var}\left[v^{K_{xy}+1}\right] &= E\left[\left(v^{K_{xy}+1}\right)^2\right] - \left(E\left[v^{K_{xy}+1}\right]\right)^2 \\ &= E\left[\left(v^2\right)^{K_{xy}+1}\right] - \left(A_{xy}\right)^2 \\ &= {}^2A_{xy} - \left(A_{xy}\right)^2\end{aligned}$$

Last survivor

Mean: $A_{\overline{xy}} = A_x + A_y - A_{xy}$

Variance: ${}^2A_{\overline{xy}} - \left(A_{\overline{xy}}\right)^2 = \left({}^2A_x + {}^2A_y - {}^2A_{xy}\right) - \left(A_x + A_y - A_{xy}\right)^2$

3.2 Present values of joint life and last survivor annuities

Consider an annuity under which a benefit of 1 *pa* is paid continuously so long as a status u continues. The present value of these annuity payments can be represented by the random variable:

$$\bar{a}_{\overline{T_u}|}$$

The expected present value of this benefit is denoted by $\bar{a}_u$ where:

$$E\left[\bar{a}_{\overline{T_u}|}\right] = \bar{a}_u = \int_{t=0}^{t=\infty} \bar{a}_{\overline{t}|}\, f_{T_u}(t)\,dt$$

This is most simply expressed by using assurance functions:

$$E\left[\bar{a}_{\overline{T_u}|}\right] = E\left[\frac{1-v^{T_u}}{\delta}\right] = \frac{1-E\left[v^{T_u}\right]}{\delta} = \frac{1-\bar{A}_u}{\delta}$$

The variance can be expressed in a similar way:

$$\begin{aligned}
\text{var}\left(\bar{a}_{\overline{T_u}|}\right) &= \text{var}\left(\frac{1-v^{T_u}}{\delta}\right) \\
&= \text{var}\left(\frac{1}{\delta} - \frac{1}{\delta}v^{T_u}\right) \\
&= \frac{1}{\delta^2}\text{var}\left(v^{T_u}\right) \\
&= \frac{1}{\delta^2}\left\{{}^2\bar{A}_u - \left(\bar{A}_u\right)^2\right\}
\end{aligned}$$

The results from Section 3.1 can be used to determine the means and variances for $u = xy$ (the joint life annuity) and $u = \overline{xy}$ (the last survivor annuity).

The means and variances of the present values of annuities payable in advance and in arrears can be evaluated using the (discrete) random variables:

- **in advance:** $\ddot{a}_{\overline{K_u+1}|}$
- **in arrears:** $a_{\overline{K_u}|}$

giving the results:

	In advance	In arrears
Mean	$\ddot{a}_u = \dfrac{1-A_u}{d}$	$a_u = \ddot{a}_u - 1 = \dfrac{(1-d)-A_u}{d}$
Variance	$\dfrac{1}{d^2}\left\{{}^2A_u-(A_u)^2\right\}$	$\dfrac{1}{d^2}\left\{{}^2A_u-(A_u)^2\right\}$

which can be applied to joint lifetime and last survivor statuses.

The relationship $\ddot{a}_u = \dfrac{1-A_u}{d}$ is a rearrangement of the premium conversion formula we met earlier in the course.

Question

Prove the above results for:

(i) an annuity payable annually in advance

(ii) an annuity payable annually in arrears.

Solution

(i) ***Annuity in advance***

The mean of the present value of an annuity payable in advance is:

$$E\left[\ddot{a}_{\overline{K_u+1}|}\right]=E\left[\frac{1-v^{K_u+1}}{d}\right]=\frac{1-E\left[v^{K_u+1}\right]}{d}=\frac{1-A_u}{d}$$

The variance of the present value of an annuity payable in advance is:

$$\text{var}\left[\ddot{a}_{\overline{K_u+1}|}\right]=\text{var}\left[\frac{1-v^{K_u+1}}{d}\right]=\frac{1}{d^2}\text{var}\left[v^{K_u+1}\right]=\frac{1}{d^2}\left\{{}^2A_u-\left(A_u\right)^2\right\}$$

(ii) ***Annuity in arrears***

The expected present value of an annuity payable in arrears is:

$$E\left[a_{\overline{K_u}|}\right]=E\left[\frac{1-v^{K_u}}{i}\right]=\frac{1-E\left[v^{K_u}\right]}{i}=\frac{1-\frac{1}{v}E\left[v^{K_u+1}\right]}{i}=\frac{v-A_u}{iv}=\frac{(1-d)-A_u}{d}$$

The variance of the present value of an annuity payable in arrears is:

$$\text{var}\left[a_{\overline{K_u}|}\right]=\text{var}\left[\frac{1-v^{K_u}}{i}\right]=\frac{1}{i^2}\text{var}\left[v^{K_u}\right]=\frac{1}{i^2v^2}\text{var}\left\{v^{K_u+1}\right\}=\frac{1}{d^2}\left\{{}^2A_u-\left(A_u\right)^2\right\}$$

4 Calculations, premiums, reserves

We are now in a position to calculate numerical values for some joint life functions. Once we have done this, we can apply the equivalence principle to calculate premiums for policies involving more than one life. In addition, we will adapt the techniques we have covered in earlier chapters for calculating reserves. This will enable us to calculate reserves for these types of policy.

In many of our examples, we will use the PA92C20 mortality tables, where the life represented by the first subscript is subject to PMA92C20 mortality, and the life represented by the second subscript is subject to PFA92C20 mortality. If the interest rate is 4% *pa*, we will be able to use the functions tabulated on pages 114 and 115 of the *Tables.*

Question

Calculate $\ddot{a}_{\overline{55:51}}$, assuming that the 55-year-old's mortality follows PMA92C20, the 51-year-old's mortality follows PFA92C20, and the annual effective interest rate is 4%.

Solution

Using a superscript of m to denote an annuity based on male mortality and a superscript of f to denote an annuity based on female mortality, we have:

$$\ddot{a}_{\overline{55:51}} = \ddot{a}^m_{55} + \ddot{a}^f_{51} - \ddot{a}_{55:51}$$

The single life annuity factors appear on page 114 of the *Tables*, and the value of the joint life annuity factor appears on page 115 of the *Tables* (with $x=55$ and $y=51$, so the age difference $d=y-x=-4$). We have:

$$\ddot{a}_{\overline{55:51}} = 17.364 + 19.291 - 16.506 = 20.149$$

We can also calculate actuarial functions using a constant force of mortality (possibly different) for each of the two lives.

Question

Calculate $\bar{A}_{60:50}$.

Basis: a constant force of mortality for (60) of 0.005 *pa*,

a constant force of mortality for (50) of 0.002 *pa*,

a force of interest of 5% *pa.*

Solution

This is a joint life assurance, where the benefit is paid immediately on the first death of (60) and (50) whenever that occurs in the future.

We can calculate this using the integral expression:

$$\bar{A}_{60:50} = \int_0^\infty v^t \, {}_tp_{60:50} \, \mu_{60+t:50+t} \, dt = \int_0^\infty e^{-\delta t} \, {}_tp_{60} \, {}_tp_{50} \left(\mu_{60+t} + \mu_{50+t} \right) dt$$

Now, $\delta = 0.05$, $\mu_{60+t} = 0.005$ for all t and $\mu_{50+t} = 0.002$ for all t, so:

$$\bar{A}_{60:50} = \int_0^\infty e^{-0.05t} \times e^{-0.005t} \times e^{-0.002t} \times \left(0.005 + 0.002\right) dt = \int_0^\infty 0.007 e^{-0.057t} \, dt$$

$$= \left[\frac{0.007}{-0.057} e^{-0.057t} \right]_0^\infty = \frac{0.007}{0.057} = 0.122807$$

An alternative, and possibly quicker, approach here is to find the value of the corresponding continuous annuity, and then use a premium conversion formula.

By analogy with the single life expression:

$$\bar{a}_x = \int_0^\infty v^t \, {}_tp_x \, dt$$

for a continuous joint life annuity we have:

$$\bar{a}_{xy} = \int_0^\infty v^t \, {}_tp_{xy} \, dt = \int_0^\infty e^{-\delta t} \, {}_tp_x \, {}_tp_y \, dt$$

So, here:

$$\bar{a}_{60:50} = \int_0^\infty e^{-0.05t} \times e^{-0.005t} \times e^{-0.002t} \, dt = \int_0^\infty e^{-0.057t} \, dt = \left[\frac{e^{-0.057t}}{-0.057} \right]_0^\infty = \frac{1}{0.057}$$

Using a premium conversion relationship, we have:

$$\bar{A}_{60:50} = 1 - \delta \bar{a}_{60:50} = 1 - \frac{0.05}{0.057} = 0.122807$$

as before.

To calculate the value of $\bar{A}_{60:50}$ in the case that the two lives are subject to PMA92C20 and PFA92C20 mortality, we would have to use the premium conversion approach as there are no joint life assurance functions for this mortality basis given in the *Tables*.

So, we would look up $\ddot{a}_{60:50}$ in the *Tables*, calculate the approximate value of $\bar{a}_{60:50}$ as $\ddot{a}_{60:50} - 0.5$, and then use the premium conversion formula $\bar{A}_{60:50} = 1 - \delta \bar{a}_{60:50}$.

Alternatively, we could use premium conversion to calculate $A_{60:50} = 1 - d\ddot{a}_{60:50}$, and then use $\bar{A}_{60:50} \approx (1+i)^{0.5} A_{60:50}$.

4.1 Evaluating premiums

We are now able to calculate premiums for policies involving two lives. As for single life policies, the usual method is to use the equivalence principle.

Question

A life office sells joint whole life assurances to male lives aged 60 and female lives aged 55 exact. The benefits, payable at the end of the year of death in each case, are £100,000 on the first death and £50,000 on the second death. Level premiums are paid annually in advance while the policy is in force.

Calculate the annual premium payable.

Basis: PMA92C20/PFA92C20 mortality, 4% *pa* interest. Ignore expenses.

Solution

The policy will remain in force until the second death. So the annuity we use to value the premiums is a last survivor annuity.

Using the equivalence principle:

$$P\ddot{a}_{\overline{60:55}} = 100{,}000 A_{60:55} + 50{,}000 A_{\overline{60:55}}$$

The relevant annuity from page 115 of the *Tables* is $\ddot{a}_{60:55} = 14.756$. So, using the relevant single life annuities (tabulated on page 114):

$$\ddot{a}_{\overline{60:55}} = \ddot{a}^{m}_{60} + \ddot{a}^{f}_{55} - \ddot{a}_{60:55} = 15.632 + 18.210 - 14.756 = 19.086$$

We can calculate the assurance factors using a premium conversion formula:

$$A_{60:55} = 1 - d\,\ddot{a}_{60:55} = 1 - \frac{0.04}{1.04} \times 14.756 = 0.43246$$

$$A_{\overline{60:55}} = 1 - d\,\ddot{a}_{\overline{60:55}} = 1 - \frac{0.04}{1.04} \times 19.086 = 0.26592$$

So the premium equation becomes:

$$19.086P = 100{,}000 \times 0.43246 + 50{,}000 \times 0.26592$$

This gives us an annual premium of £2,962.50.

4.2 Calculating reserves

We can calculate reserves for policies based on two lives, again adapting the methods we have met earlier for reserve calculations for single life policies. The prospective calculation involves calculating the EPV of the future benefits and expenses (if appropriate), and subtracting the EPV of any future premiums.

Question

For the policy in the previous question, calculate the prospective reserve just before the payment of the 10th annual premium, assuming both lives are still alive at that point, and using the same basis as was used to calculate the premium.

Solution

Just before the 10th premium is paid (*ie* at time 9) we have:

$${}_9V = 100{,}000 A_{69:64} + 50{,}000 A_{\overline{69:64}} - 2{,}962.50\,\ddot{a}_{\overline{69:64}}$$

The last survivor annuity is calculated in the usual way:

$$\ddot{a}_{\overline{69:64}} = \ddot{a}^m_{69} + \ddot{a}^f_{64} - \ddot{a}_{69:64} = 11.988 + 15.242 - 10.933 = 16.297$$

The assurance factors are calculated using premium conversion:

$$A_{69:64} = 1 - d\,\ddot{a}_{69:64} = 1 - \frac{0.04}{1.04} \times 10.933 = 0.57950$$

$$A_{\overline{69:64}} = 1 - d\,\ddot{a}_{\overline{69:64}} = 1 - \frac{0.04}{1.04} \times 16.297 = 0.37319$$

So the reserve is:

$${}_9V = 100{,}000\times 0.57950 + 50{,}000\times 0.37319 - 2{,}962.50\times 16.297 = £28{,}329.75$$

In this question, we were told to assume that both lives are still alive at the time of the reserve calculation.

It is important when calculating a reserve for a last survivor assurance to remember that it would be necessary to establish whether both lives remain alive or one has already died. This is because, of course, the contract still remains in force whether one or two lives remain alive. The premium being paid will still be the original premium calculated on a last survivor basis.

Question

For a policy where the male and female are both aged 63, list the three possible states that a last survivor assurance may be in after a few years.

Identify which of these three states requires the largest reserve.

Solution

The three possible states are:

- both lives are still alive
- only the male life is still alive
- only the female life is still alive.

We would expect the sum assured to be paid out earliest on the policy where only the male is still alive (as male life expectancy is shorter than female life expectancy for lives of the same age). This policy is therefore expected to have the largest reserve.

Thus for example if both *x* and *y* are alive, a net premium reserve for a whole life last survivor contract would be:

$${}_tV_{\overline{x:y}} = A_{\overline{x+t:y+t}} - P_{\overline{x:y}}\,\ddot{a}_{\overline{x+t:y+t}}$$

whereas if say (*y*) had previously died the reserve would be:

$${}_tV_{\overline{x:y}} = A_{x+t} - P_{\overline{x:y}}\,\ddot{a}_{x+t}$$

Thus on first death a significant increase in reserve will take place.

Here $P_{\overline{x:y}}$ is the net premium (*ie* ignoring expenses) payable annually in advance for the last survivor assurance contract, and ${}_tV_{\overline{x:y}}$ is the net premium reserve for this contract at time t.

Above, we have used a prospective approach to calculating reserves. We can also calculate reserves retrospectively, determining the expected accumulated value of the premiums, less the expected accumulated value of the benefits and expenses (if appropriate). However, other than for simple joint life assurances and annuities, the retrospective calculation is very complex, so the prospective method is generally used.

4.3 Future loss random variable

As we have seen in previous chapters, the prospective reserve is the expected value of an underlying random variable, known as the future loss random variable. We can apply the same techniques as before to write down future loss random variables for policies based on two lives.

Question

A life office sells joint whole life assurances to male lives aged 60 and female lives aged 55 exact. The benefits, payable at the end of the year of death in each case, are £100,000 on the first death and £50,000 on the second death. Level premiums of £2,962.50 are paid annually in advance while the policy is in force.

Write down the future loss random variable at the end of the 9th policy year for this policy, in the case where both lives are still alive, ignoring expenses.

Solution

We use a 'min' function for the benefit payable on first death, and a 'max' function for the benefit payable on second death. Since the premiums are payable until the second death, a 'max' function will also be needed for these.

So the future loss random variable at the end of the 9th policy year is:

$$100{,}000v^{\min\{K_{69}+1,K_{64}+1\}}+50{,}000v^{\max\{K_{69}+1,K_{64}+1\}}-2{,}962.50\ddot{a}_{\overline{\max\{K_{69}+1,K_{64}+1\}}|}$$

We could alternatively write this using the joint curtate future lifetime random variable and the corresponding last survivor future lifetime random variable, *ie*:

$$100{,}000v^{K_{69:64}+1}+50{,}000v^{K_{\overline{69:64}}+1}-2{,}962.50\ddot{a}_{\overline{K_{\overline{69:64}}+1}|}$$

Chapter 22 Summary

We have developed the following random variables:

Random variable	***Modelling***
$T_{xy} = \min\{T_x, T_y\}$	Time to failure of the joint life status xy, *ie* the time until the first death of x and y
$T_{\overline{xy}} = \max\{T_x, T_y\}$	Time to failure of the last survivor status $\overline{xy}$, *ie* the time until the second death of x and y
$K_{xy} = \min\{K_x, K_y\}$	Curtate time to failure of the joint life status xy, *ie* the curtate time until the first death of x and y
$K_{\overline{xy}} = \max\{K_x, K_y\}$	Curtate time to failure of the last survivor status $\overline{xy}$, *ie* the curtate time until the second death of x and y

We have also developed the following notation, with associated formulae:

Symbol	***Description***	***Formula***
l_{xy}	Life table survival function for two independent lives x and y	$l_x \, l_y$
μ_{xy}	Force of failure of the joint life status xy	$\mu_x + \mu_y$
${}_tp_{xy}$	Probability that the joint life status xy is still active in t years' time, *ie* the probability that both x and y survive for at least t years	${}_tp_x \; {}_tp_y$
${}_tp_{\overline{xy}}$	Probability that the last survivor status $\overline{xy}$ is still active in t years' time, *ie* the probability that at least one of x and y survive for at least t years	$1 - {}_tq_x \; {}_tq_y$ $= {}_tp_x + {}_tp_y - {}_tp_x \; {}_tp_y$
${}_tq_{xy}$	Probability that the joint life status xy fails within t years, *ie* the probability that at least one of x and y dies in next t years	$1 - {}_tp_x \; {}_tp_y$ $= {}_tq_x + {}_tq_y - {}_tq_x \; {}_tq_y$
${}_tq_{\overline{xy}}$	Probability that the last survivor status $\overline{xy}$ fails within t years, *ie* the probability that both x and y die in next t years	${}_tq_x \; {}_tq_y$

The most common benefits, and their values, are:

Function	***Value (algebraic)***	***Value (stochastic)***
$\bar{a}_{xy}$	$\int_0^\infty v^t \, {}_tp_{xy} \, dt$	$E\left[\bar{a}_{\overline{\min\{T_x,T_y\}}\rvert}\right]$
$\bar{a}_{\overline{xy}}$	$\int_0^\infty v^t \, {}_tp_{\overline{xy}} \, dt$	$E\left[\bar{a}_{\overline{\max\{T_x,T_y\}}\rvert}\right]$
$\bar{A}_{xy}$	$\int_0^\infty v^t \, {}_tp_{xy}(\mu_{x+t} + \mu_{y+t}) \, dt$	$E\left[v^{\min\{T_x,T_y\}}\right]$
$\bar{A}_{\overline{xy}}$	$\int_0^\infty v^t ({}_tp_y \, \mu_{y+t} \, {}_tq_x + {}_tp_x \, \mu_{x+t} \, {}_tq_y) \, dt$	$E\left[v^{\max\{T_x,T_y\}}\right]$

Equivalent results hold for discrete functions, *eg*:

$$a_{xy} = \sum_{t=1}^{\infty} v^t \, {}_tp_{xy} = E\left[a_{\overline{\min\{K_x,K_y\}}\rvert}\right]$$

We can adapt the premium conversion relationships for single life functions to help us to calculate the corresponding joint life and last survivor values. For example:

$$A_{xy} = 1 - d\ddot{a}_{xy} \qquad \bar{A}_{xy} = 1 - \delta\bar{a}_{xy}$$

and: $$A_{\overline{xy}} = 1 - d\ddot{a}_{\overline{xy}} \qquad \bar{A}_{\overline{xy}} = 1 - \delta\bar{a}_{\overline{xy}}$$

Chapter 22 Practice Questions

22.1 (i) Explain what it means for the last survivor status $\overline{50:60}$ to remain active for at least 10 years.

(ii) Calculate the probability that the event described in part (i) occurs, assuming the two lives are independent with respect to mortality and:

(a) the mortality of each life follows the ELT15 (Males) table

(b) each life is subject to a constant force of mortality of 0.025 *pa*.

22.2 (i) Explain what it means for the joint life status $50:60$ to fail within the next 10 years.

(ii) Calculate the probability that the event described in part (i) occurs, assuming the two lives are independent with respect to mortality and:

(a) the mortality of each life follows the ELT15 (Females) table

(b) each life is subject to a constant force of mortality of 0.025 *pa*.

22.3 Given that ${}_nq_x = 0.3$ and ${}_nq_y = 0.5$, calculate ${}_nq_{xy}$ and ${}_nq_{\overline{xy}}$.

22.4 Consider each of the symbols listed below:

(a) $p_{\overline{xy}}$

(b) $\bar{A}_{xy}$

(c) $\bar{A}_{\overline{xy}}$

Explain carefully the meaning of each of these symbols and calculate the value of each, assuming that:

- (x) is subject to a constant force of mortality of 0.01 *pa*
- (y) is subject to a constant force of mortality of 0.02 *pa*
- the force of interest is 0.04 *pa*.

22.5 A life insurance company issues 1,000 last survivor annuities to pairs of lives aged 60. Each pair comprises one male and one female, and the annuity pays £5,000 *pa* continuously until the second of the two lives dies. The single premium charged is £90,000.

Calculate the expected present value of the profit to the life office and the standard deviation of this profit in respect of this group of policies.

Basis: Mortality: PMA92C20 for the male life, PFA92C20 for the female life

Interest: 4% *pa* effective

Using this mortality assumption, $\bar{A}_{\overline{60:60}} = 0.20021$ at an interest rate of 8.16% *pa*.

22.6 Exam style

William, aged 75, and Laura, aged 80, are the guardians of a child. They take out a life assurance policy that provides a payment of £25,000 immediately when the second of them dies. Level annual premiums are payable in advance whilst the policy is in force.

(i) Calculate the annual gross premium, using the basis given below. [4]

(ii) Calculate the gross premium prospective reserve just before the sixth premium is paid, using the basis given below, assuming that both William and Laura are still alive at that time. [3]

[Total 7]

Basis: Mortality: PMA92C20 for William, PFA92C20 for Laura

Interest: 4% *pa* effective

Expenses: Initial: £250

Renewal: 5% of each premium, excluding the first

22.7 Exam style

A life insurance company issues an annuity to a male, aged 68, and a female, aged 65. The annuity of £10,000 *pa* is payable annually in arrears and continues until both lives have died.

The insurance company values this benefit using PMA92C20 mortality for the male life, PFA92C20 mortality for the female life and 4% *pa* interest.

(i) Calculate the expected present value of this annuity. [2]

(ii) Derive an expression for the variance of the present value of this annuity in terms of appropriate single life and joint life assurance functions. [4]

Let X be the present value of the insurer's profit from this policy.

(iii) If the insurance company charges a premium of £150,000 for this policy, calculate $P(X>0)$. [4]

[Total 10]

22.8 Exam style

A man aged 60 exact and a woman aged 65 exact wish to purchase an annuity that provides:

- £25,000 *pa* payable while they are both alive,
- £20,000 *pa* payable for the remainder of the woman's life, if the man dies first,
- £15,000 *pa* payable for the remainder of the man's life, if the woman dies first.

All the annuity payments are made annually in arrears.

(i) Write down an expression for the present value random variable of this benefit. [2]

(ii) Calculate the expected present value of this annuity benefit, using the following basis:

Mortality: PMA92C20 for the male life, PFA92C20 for the female life

Interest: 4% *pa* effective [2]

[Total 4]

Chapter 22 Solutions

22.1 (i) For the last survivor status $\overline{50:60}$ to remain active for at least 10 years, at least one of the two lives, currently aged 50 and 60, must survive for at least 10 years. So, the time until the second death must be greater than or equal to 10 years.

(ii)(a) The probability that the last survivor status $\overline{50:60}$ remains active for at least 10 years can be written as:

$$_{10}p_{\overline{50:60}} = 1 - {_{10}q_{\overline{50:60}}} = 1 - {_{10}q_{50}}\ {_{10}q_{60}}$$

Using ELT15 (Males) mortality:

$$_{10}q_{50} = 1 - \frac{l_{60}}{l_{50}} = 1 - \frac{86,714}{93,925} = 0.076774$$

and: $$_{10}q_{60} = 1 - \frac{l_{70}}{l_{60}} = 1 - \frac{68,055}{86,714} = 0.215179$$

So:

$$_{10}p_{\overline{50:60}} = 1 - 0.076774 \times 0.215179 = 0.983480$$

(ii)(b) Assuming a constant force of mortality of 0.025 *pa*, we have:

$$_{10}q_{50} = {_{10}q_{60}} = 1 - e^{-0.025\times 10} = 1 - e^{-0.25}$$

So:

$$_{10}p_{\overline{50:60}} = 1 - \left(1 - e^{-0.25}\right)^2 = 0.951071$$

22.2 (i) For the joint life status $50:60$ to fail within the next 10 years, at least one of the two lives, currently aged 50 and 60, must die within 10 years. So, the time until the first death must be less than 10 years.

This does not exclude the possibility that the second death might also occur within 10 years.

(ii)(a) The probability that the joint life status $50:60$ fails within the next 10 years can be written as:

$$_{10}q_{50:60} = 1 - {_{10}p_{50:60}} = 1 - {_{10}p_{50}}\ {_{10}p_{60}}$$

Using ELT15 (Females) mortality:

$$_{10}p_{50} = \frac{l_{60}}{l_{50}} = \frac{91,732}{96,247} = 0.953089$$

and: $$_{10}p_{60} = \frac{l_{70}}{l_{60}} = \frac{79,970}{91,732} = 0.871779$$

So:

$$_{10}q_{50:60} = 1 - 0.953089 \times 0.871779 = 0.169117$$

(ii)(b) Assuming a constant force of mortality of 0.025 *pa*, we have:

$$_{10}p_{50} = {}_{10}p_{60} = e^{-0.025\times 10} = e^{-0.25}$$

So:

$$_{10}q_{50:60} = 1 - \left(e^{-0.25}\right)^2 = 0.393469$$

22.3 We have:

$$_nq_{xy} = 1 - {}_np_{xy} = 1 - {}_np_x\,{}_np_y = 1 - (1 - {}_nq_x)\left(1 - {}_nq_y\right) = 1 - 0.7 \times 0.5 = 0.65$$

and: $${}_nq_{\overline{xy}} = {}_nq_x\;{}_nq_y = 0.15$$

22.4 (a) $p_{\overline{xy}}$ is the probability that the last survivor status based on lives (x) and (y) is still active in one year's time, *ie* it is the probability that at least one of (x) and (y) is alive in one year's time.

We can calculate this as follows:

$$p_{\overline{xy}} = 1 - q_{\overline{xy}} = 1 - q_x\,q_y = 1 - \left(1 - e^{-0.01}\right)\left(1 - e^{-0.02}\right) = 0.999803$$

(b) $\bar{A}_{xy}$ is the expected present value of a benefit of 1 payable immediately on the failure of the joint life status xy. So the benefit is paid immediately upon the first death of the two lives.

We can calculate this using the integral expression:

$$\bar{A}_{xy} = \int_0^\infty v^t\,{}_tp_{xy}\,\mu_{x+t:y+t}\,dt = \int_0^\infty v^t\,{}_tp_x\,{}_tp_y\left(\mu_{x+t} + \mu_{y+t}\right)dt$$

So this gives:

$$\bar{A}_{xy} = \int_0^\infty e^{-0.04t} e^{-0.01t} e^{-0.02t} \left(0.01+0.02\right) dt$$

$$= 0.03 \int_0^\infty e^{-0.07t}\, dt$$

$$= 0.03 \left[\frac{-e^{-0.07t}}{0.07} \right]_0^\infty = \frac{3}{7}$$

(c) $\bar{A}_{\overline{xy}}$ is the expected present value of a benefit of 1 payable immediately on the failure of the last survivor status $\overline{xy}$. So the benefit is paid immediately upon the second death.

We can write:

$$\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$$

where:

$$\bar{A}_x = \int_0^\infty v^t \,{}_tp_x\, \mu_{x+t}\, dt = 0.01 \int_0^\infty e^{-0.04t} e^{-0.01t}\, dt = 0.01 \left[\frac{-e^{-0.05t}}{0.05} \right]_0^\infty = \frac{1}{5}$$

$$\bar{A}_y = \int_0^\infty v^t \,{}_tp_y\, \mu_{y+t}\, dt = 0.02 \int_0^\infty e^{-0.04t} e^{-0.02t}\, dt = 0.02 \left[\frac{-e^{-0.06t}}{0.06} \right]_0^\infty = \frac{1}{3}$$

So:

$$\bar{A}_{\overline{xy}} = \frac{1}{5} + \frac{1}{3} - \frac{3}{7} = \frac{11}{105} = 0.104762$$

22.5 The present value of the profit, X, on a single policy is:

$$X = 90{,}000 - 5{,}000\, \bar{a}_{\overline{T_{\overline{xy}}}|}$$

The expected value of X is:

$$E\left[90{,}000 - 5{,}000\, \bar{a}_{\overline{T_{\overline{xy}}}|} \right] = 90{,}000 - 5{,}000\, E\left[\bar{a}_{\overline{T_{\overline{xy}}}|} \right] = 90{,}000 - 5{,}000\, \bar{a}_{\overline{xy}}$$

Here x and y are both equal to 60, and we can calculate the last survivor annuity from:

$$\begin{aligned}\bar{a}_{\overline{60:60}} &= \bar{a}^m_{60} + \bar{a}^f_{60} - \bar{a}_{60:60} \\ &\approx \left(\ddot{a}^m_{60} - 0.5\right) + \left(\ddot{a}^f_{60} - 0.5\right) - \left(\ddot{a}_{60:60} - 0.5\right) \\ &= (15.632 - 0.5) + (16.652 - 0.5) - (14.090 - 0.5) \\ &= 17.694\end{aligned}$$

So, the expected present value of the profit on a single policy is:

$$90{,}000 - 5{,}000 \times 17.694 = £1{,}530$$

The expected present value of the profit on the block of 1,000 policies is £1,530,000.

The variance of X is:

$$\text{var}\left[90{,}000 - 5{,}000\,\bar{a}_{\overline{T_{\overline{xy}}}|}\right] = 5{,}000^2\,\text{var}\left[\frac{1 - v^{T_{\overline{xy}}}}{\delta}\right] = \left(\frac{5{,}000}{\delta}\right)^2 \text{var}\left[v^{T_{\overline{xy}}}\right]$$

where:

$$\text{var}\left[v^{T_{\overline{xy}}}\right] = {}^2\bar{A}_{\overline{xy}} - (\bar{A}_{\overline{xy}})^2$$

Using the premium conversion relationship $\bar{A}_{\overline{xy}} = 1 - \delta\bar{a}_{\overline{xy}}$, we obtain:

$$\bar{A}_{\overline{60:60}} = 1 - \ln 1.04 \times 17.694 = 0.30603$$

and we are given ${}^2\bar{A}_{\overline{60:60}} = 0.20021$ in the question.

So the variance of the present value of the profit for a single policy is:

$$\frac{5{,}000^2}{\delta^2}(0.20021 - 0.30603^2) = 1{,}731{,}764{,}026$$

The variance of the present value of the profit on the block of 1,000 policies is then:

$$1{,}000 \times 1{,}731{,}764{,}026$$

assuming independent policies. So the standard deviation of the total profit is:

$$\sqrt{1{,}000 \times 1{,}731{,}764{,}026} = £1{,}315{,}965$$

22.6 (i) ***Gross annual premium***

The premium equation is:

$$P\ddot{a}_{\overline{75:80}} = 25{,}000\bar{A}_{\overline{75:80}} + 250 + 0.05P\left(\ddot{a}_{\overline{75:80}} - 1\right) \quad [1]$$

We can calculate $\ddot{a}_{\overline{75:80}}$ using:

$$\ddot{a}_{\overline{75:80}} = \ddot{a}^m_{75} + \ddot{a}^f_{80} - \ddot{a}_{75:80} = 9.456 + 8.989 - 6.822 = 11.623 \quad [1]$$

The assurance factor can be calculated by premium conversion:

$$\bar{A}_{\overline{75:80}} \approx 1.04^{0.5} A_{\overline{75:80}} = 1.04^{0.5}\left(1 - d\,\ddot{a}_{\overline{75:80}}\right) = 1.04^{0.5}\left(1 - \frac{0.04}{1.04} \times 11.623\right) = 0.56391 \quad [1]$$

The premium equation becomes:

$$11.623P = 25{,}000 \times 0.56391 + 250 + 0.05P(11.623 - 1)$$

So the premium is:

$$P = \frac{14{,}347.81}{11.09185} = £1{,}293.55 \quad [1]$$

[Total 4]

(ii) ***Gross premium prospective reserve at time 5***

Just before the sixth premium payment (*ie* at time 5), William is aged 80 and Laura is aged 85. The gross premium prospective reserve is given by:

$$_5V = 25{,}000\bar{A}_{\overline{80:85}} + 0.05P\ddot{a}_{\overline{80:85}} - P\ddot{a}_{\overline{80:85}} = 25{,}000\bar{A}_{\overline{80:85}} - 0.95P\ddot{a}_{\overline{80:85}} \quad [1]$$

where $P = £1{,}293.55$ from part (i).

Using the same approach as in part (i):

$$\ddot{a}_{\overline{80:85}} = \ddot{a}^m_{80} + \ddot{a}^f_{85} - \ddot{a}_{80:85} = 7.506 + 7.220 - 5.161 = 9.565 \quad [½]$$

$$\bar{A}_{\overline{80:85}} \approx 1.04^{0.5} A_{\overline{80:85}} = 1.04^{0.5}\left(1 - d\,\ddot{a}_{\overline{80:85}}\right) = 1.04^{0.5}\left(1 - \frac{0.04}{1.04} \times 9.565\right) = 0.64463 \quad [½]$$

So the gross premium prospective reserve at time 5 is:

$$_5V = 25{,}000 \times 0.64463 - 0.95 \times 1{,}293.55 \times 9.565 = £4{,}361.68 \quad [1]$$

[Total 3]

22.7 (i) ***Expected present value***

The expected present value of the annuity is:

$$10{,}000\,a_{\overline{68:65}} = 10{,}000\left(\ddot{a}_{\overline{68:65}} - 1\right) = 10{,}000\left(\ddot{a}^{m}_{68} + \ddot{a}^{f}_{65} - \ddot{a}_{68:65} - 1\right) \qquad [1]$$

Looking up the values, the EPV of the annuity is:

$$10{,}000\left(12.412 + 14.871 - 11.112 - 1\right) = £151{,}710 \qquad [1]$$

[Total 2]

(ii) ***Variance of the present value***

The present value random variable for the benefits from this contract is:

$$10{,}000\,a_{\left.\overline{K_{\overline{68:65}}}\right|} \qquad [½]$$

The variance of this random variable is given by:

$$\begin{aligned}
\text{var}\left(10{,}000\,a_{\left.\overline{K_{\overline{68:65}}}\right|}\right) &= 10{,}000^2\,\text{var}\left(\frac{1 - v^{K_{\overline{68:65}}}}{i}\right)\\
&= \frac{10{,}000^2}{i^2}\,\text{var}\left(v^{K_{\overline{68:65}}}\right)\\
&= \frac{10{,}000^2}{i^2v^2}\,\text{var}\left(v^{K_{\overline{68:65}}+1}\right)\\
&= \frac{10{,}000^2}{d^2}\,\text{var}\left(v^{K_{\overline{68:65}}+1}\right) \qquad [1½]
\end{aligned}$$

Now:

$$\text{var}\left(v^{K_{\overline{68:65}}+1}\right) = E\left[\left(v^{K_{\overline{68:65}}+1}\right)^2\right] - \left[E\left(v^{K_{\overline{68:65}}+1}\right)\right]^2 = {}^2A_{\overline{68:65}} - \left(A_{\overline{68:65}}\right)^2$$

where the pre-superscript of 2 indicates that the function is evaluated at an interest rate of $1.04^2 - 1 = 8.16\%$. [1]

Also:

$$A_{\overline{68:65}} = A_{68} + A_{65} - A_{68:65}$$

and:

$${}^2A_{\overline{68:65}} = {}^2A_{68} + {}^2A_{65} - {}^2A_{68:65}$$

So:

$$\text{var}\left(10{,}000\,a_{\overline{K_{\overline{68:65}}}|}\right)=\frac{10{,}000^2}{d^2}\left[{}^2A_{\overline{68:65}}-\left(A_{\overline{68:65}}\right)^2\right]$$

$$=\frac{10{,}000^2}{d^2}\left[{}^2A_{68}+{}^2A_{65}-{}^2A_{68:65}-\left(A_{68}+A_{65}-A_{68:65}\right)^2\right] \qquad [1]$$

[Total 4]

(iii) ***Probability that the insurance company makes a profit***

The life insurance company charges a premium of £150,000. The present value of the profit is:

$$X=150{,}000-10{,}000\,a_{\overline{K_{\overline{68:65}}}|}$$

So:

$$P(X>0)=P\left(10{,}000\,a_{\overline{K_{\overline{68:65}}}|}<150{,}000\right) \qquad [½]$$

Now:

$$10{,}000\,a_{\overline{K_{\overline{68:65}}}|}<150{,}000 \iff a_{\overline{K_{\overline{68:65}}}|}<15$$

$$\iff \frac{1-v^{K_{\overline{68:65}}}}{i}<15$$

$$\iff v^{K_{\overline{68:65}}}>1-15i$$

$$\iff K_{\overline{68:65}}<\frac{\ln\left(1-15\times 0.04\right)}{-\ln 1.04}=23.4 \qquad [1½]$$

Since $K_{\overline{68:65}}$ can take only integer values:

$$P\left(K_{\overline{68:65}}<23.4\right)=P\left(K_{\overline{68:65}}\le 23\right)=P\left(K_{\overline{68:65}}<24\right) \qquad [½]$$

Alternatively, we could obtain this result by inspection. Looking up annuity values at 4% in the Tables, we see that $a_{\overline{23}|}=14.8568$ *but* $a_{\overline{24}|}=15.2470$*, so* $P\left(a_{\overline{K_{\overline{68:65}}}|}<15\right)=P\left(K_{\overline{68:65}}<24\right)$.

We want the probability that the last survivor status fails within 24 years. This will happen if both lives die within 24 years. The required probability, $P(X>0)$, is therefore:

$${}_{24}q_{68}\times{}_{24}q_{65}=\left(1-\frac{l_{92}}{l_{68}}\right)\left(1-\frac{l_{89}}{l_{65}}\right)$$

$$=\left(1-\frac{1{,}911.771}{9{,}440.717}\right)\left(1-\frac{4{,}533.230}{9{,}703.708}\right)=0.424935 \qquad [1½]$$

[Total 4]

22.8 (i) ***Present value random variable***

This is:

$$20{,}000a_{\overline{K_{65}}|} + 15{,}000a_{\overline{K_{60}}|} - 10{,}000a_{\overline{K_{60:65}}|} \qquad \text{[Total 2]}$$

We can consider this situation as a single life annuity of £20,000 pa payable to the woman for the whole of her life, plus a single life annuity of £15,000 pa payable to the man for the whole of his life, minus a joint life annuity of £10,000 pa payable while both lives are alive. Then:

- *when only the woman is alive, she receives £20,000 pa*
- *when only the man is alive, he received £15,000 pa, and*
- *when both lives are alive, the annual payment is* $20{,}000+15{,}000-10{,}000=£25{,}000$

as required.

(ii) ***Expected present value***

The expected present value is:

$$20{,}000a_{65}^{f} + 15{,}000a_{60}^{m} - 10{,}000a_{60(m):65(f)}$$

$$= 20{,}000\left(\ddot{a}_{65}^{f} - 1\right) + 15{,}000\left(\ddot{a}_{60}^{m} - 1\right) - 10{,}000\left(\ddot{a}_{60(m):65(f)} - 1\right) \qquad [1]$$

Looking up the values:

$$EPV = 20{,}000(14.871-1) + 15{,}000(15.632-1) - 10{,}000(13.101-1) = £375{,}890 \qquad [1]$$

[Total 2]

23

Contingent and reversionary benefits

Syllabus objectives

5.1 Define and use assurance and annuity functions involving two lives.

5.1.1 Extend the techniques of objectives 4.2 to deal with cashflows dependent upon the death or survival of either or both of two lives.

5.1.2 Extend the technique of 5.1.1 to deal with functions dependent upon a fixed term as well as age.

0 Introduction

In this chapter we will consider the implication of specifying the order in which the two lives die, which leads to the two types of benefit:

- *contingent assurances* – these are payable on the death of one life, contingent upon another life being in a specified state (alive or dead); and
- *reversionary annuities* – these are payable to one life following the death of another life.

We will also look at multiple life functions that depend on duration in a variety of ways, and annuities based on two lives that are payable more frequently than annually.

1 Contingent probabilities of death

So far we have examined events which depend on the joint lifetime xy or last survivor lifetime $\overline{xy}$ of two lives. We now look at events that depend on the order in which the deaths occur.

We will study two events:

- **the event that (x) is the first to die of two lives (x) and (y):** $\overset{1}{x}y$
- **the event that (x) is the second to die of two lives (x) and (y):** $\overset{2}{x}y$

Events that depend upon the order in which the lives die are called *contingent* events.

Then we use ${}_nq^{1}_{xy}$ and ${}_nq^{2}_{xy}$ to denote the probabilities that each of these two events occurs in the next n years.

Let's consider the meaning of ${}_nq^{1}_{xy}$. The way to read this is by first considering *only* the probability as it would relate to the life *with the number above it* ((x) in this case), *ie* initially ignoring the other life (y). So we first read this as we would ${}_nq_x$:

- the probability that (x) dies within n years.

We then bring in the second life, and notice that the number superscript over the x is a '1', so this tells us that (x) has to die first. However, we have already said something about (x), so we need to express this in terms of what happens to (y); that is, in addition to the above, we have:

- (y) must die after (x).

So the whole thing reads:

$${}_nq^{1}_{xy} = \text{ probability that } (x) \text{ dies within } n \text{ years, and } (y) \text{ dies (any time) after } (x).$$

(In this probability, y may or may not die within the n-year period – y can die as long as x is already dead.)

Similarly:

$${}_nq^{2}_{xy} = \text{ probability that } (x) \text{ dies within } n \text{ years, and } (y) \text{ dies before } (x)$$

This is because the superscript over the x is a '2', so (x) has to die second.

These probabilities can be evaluated by an appropriate integration of the density functions of the random variables T_x and T_y.

For example, we can write:

$$ {}_nq^{\;1}_{xy} = \int_{t=0}^{t=n} \int_{s=t}^{s=\infty} {}_tp_x\,\mu_{x+t}\;{}_sp_y\mu_{y+s}\,ds\;dt $$

which corresponds to the event $T_x \le T_y$, $T_x \le n$.

If we rewrite this as:

$$ {}_nq^{\;1}_{xy} = \int_{t=0}^{t=n} {}_tp_x\,\mu_{x+t}\left\{\int_{s=t}^{s=\infty} {}_sp_y\mu_{y+s}\,ds\right\}dt $$

$$ = \int_{t=0}^{t=n} {}_tp_x\,\mu_{x+t}\;{}_tp_y\;dt $$

then we are calculating the product of:

- the probability that (x) dies at exact moment t ($\approx {}_tp_x\,\mu_{x+t}\,dt$)
- the probability that (y) is still living at exact moment t ($= {}_tp_y$), *ie* that (y) dies after time t and therefore after (x) dies

and then summing (*ie* integrating) over all possible times at which (x) could die over the next n years.

Finally we can neaten this up slightly by writing:

$$ {}_nq^{\;1}_{xy} = \int_{t=0}^{t=n} {}_tp_{xy}\,\mu_{x+t}\;dt $$

The ability to express probabilities of life and death as integrals for single and joint lives is important.

The key considerations are:

- what event should be modelled as occurring at time t (*eg* should it be a death or the survival of a particular life)?
- who is dying at time t (if anyone)?
- who has to be alive/dead at time t?
- over what range of times t should the integral be evaluated?

So, in expressing ${}_nq^{\ 1}_{xy}$ as an integral, we could proceed as follows:

- we want to model (x)'s death as occurring at time t (we choose (x) rather than (y) on this occasion because we know that (x) has to die within n years, so when we integrate over 0 to n we will have covered all the possible times that this can happen);
- at time t, (x) must have survived to that point in order then to die, and then die instantly (so we want the factor ${}_tp_x\,\mu_{x+t}\,dt$);
- at time t, (y) needs to be alive (so we want the factor ${}_tp_y$).

So the integral must be: $\displaystyle\int_{t=0}^{n} {}_tp_x\,\mu_{x+t}\,{}_tp_y\,dt$.

Question

Express the probability ${}_nq^{\ 2}_{xy}$ as an integral, using the above approach.

Solution

In the probability ${}_nq^{\ 2}_{xy}$, (x) has to die within the n-year period. So we model (x) as dying at time t, meaning that we need a factor of:

$$ {}_tp_x\,\mu_{x+t}\,dt $$

This can be thought of as the probability that (x) dies at the exact future time t.

However, since (x) has to be the second life to die, we also need to include the probability that (y) has already died by this time, *ie* the probability that (y) dies before time t. This is:

$$ {}_tq_y $$

Since the time of (x)'s death must be between time 0 and time n, the integral is:

$$ {}_nq^{\ 2}_{xy} = \int_0^n {}_tp_x\,\mu_{x+t}\,{}_tq_y\,dt $$

Using the solution above and writing ${}_tq_x = \displaystyle\int_{s=0}^{s=t} {}_sp_y\,\mu_{y+s}\,ds$, we see that:

$$ {}_nq^{\ 2}_{xy} = \int_0^n {}_tp_x\,\mu_{x+t}\left(\int_{s=0}^{s=t} {}_sp_y\,\mu_{y+s}\,ds\right)dt \qquad (1) $$

This can be rewritten as:

$$ {}_nq^{2}_{xy} = \int_{t=0}^{t=n} \int_{s=0}^{s=t} {}_tp_x\, \mu_{x+t}\, {}_sp_y\, \mu_{y+s}\, ds\, dt $$

This corresponds to the event $T_y < T_x \le n$.

If we substitute $\left(1-{}_tp_y\right)$ for ${}_tq_y$ in the solution to the previous question, then we have:

$$ {}_nq^{2}_{xy} = \int_{t=0}^{t=n} (1-{}_tp_y)\, {}_tp_x\, \mu_{x+t}\, dt $$

$$ = \int_0^n {}_tp_x\, \mu_{x+t}\, dt - \int_0^n {}_tp_{xy}\, \mu_{x+t}\, dt $$

$$ = {}_nq_x - {}_nq^{1}_{xy} $$

This result implies that 'second death' probabilities can always be expressed in terms of 'single death' and 'first death' probabilities. This provides a method of evaluating 'second death' probabilities.

The truth of this expression can be argued by general reasoning if it is rewritten as:

$$ {}_nq_x = {}_nq^{1}_{xy} + {}_nq^{2}_{xy} $$

The right-hand side is the probability that a life aged x dies in an n-year period either before or after a life aged y. As these are the only two possibilities for (x)'s death in relation to (y), this is equal to the probability that (x) dies at some point in the n-year period.

By changing the order of integration in expression (1) for ${}_nq^{2}_{xy}$ we can show that:

$$ {}_nq^{2}_{xy} = {}_nq^{1}_{xy} - {}_np_x\, {}_nq_y $$

Question

Prove this result.

Solution

From expression (1), we have:

$$ {}_nq^{2}_{xy} = \int_0^n {}_tp_x\, \mu_{x+t} \left(\int_0^t {}_sp_y\, \mu_{y+s}\, ds \right) dt $$

Changing the order of integration, we have:

$$ {}_nq^{2}_{xy} = \int_0^n {}_sp_y\, \mu_{y+s} \left(\int_s^n {}_tp_x\, \mu_{x+t}\, dt \right) ds $$

In the original double integral we have $0 \le s \le t \le n$. So, when we change the order, we have s going from 0 to n and t going from s to n.

Now $\int_s^n {}_tp_x\, \mu_{x+t}\, dt$ is the probability that (*x*) dies between time s and time n, and so:

$$ \int_s^n {}_tp_x\, \mu_{x+t}\, dt = {}_sp_x - {}_np_x $$

Substituting this expression into the equation above gives:

$$ \begin{aligned} {}_nq^{2}_{xy} &= \int_0^n {}_sp_y\, \mu_{y+s} \left({}_sp_x - {}_np_x \right) ds \\ &= \int_0^n {}_sp_y\, \mu_{y+s}\, {}_sp_x\, ds - {}_np_x \int_0^n {}_sp_y\, \mu_{y+s}\, ds \\ &= {}_nq^{\;1}_{xy} - {}_np_x\, {}_nq_y \end{aligned} $$

as required.

This relationship can also be argued by considering the events defined by each of the functions.

So, the events defined by ${}_nq^{\;1}_{xy}$ are:

(A) (*y*) dies followed by (*x*) dying, all within the *n*-year period, or

(B) (*y*) dies within the *n*-year period and (*x*) dies after the *n*-year period has elapsed.

On the other hand, only event (A) is covered by ${}_nq^{2}_{xy}$. So ${}_nq^{\;1}_{xy} - {}_nq^{2}_{xy}$ must be the probability of event (B) only, which is ${}_nq_y\, {}_np_x$, as above.

The expressions become simple and easy to evaluate when $x = y$. We can write:

$$ \boldsymbol{{}_nq_x = {}_nq^{1}_{xx} + {}_nq^{2}_{xx}} $$

$$ \boldsymbol{{}_nq^{2}_{xx} = {}_nq^{1}_{xx} - {}_np_x\, {}_nq_x} $$

by substituting x in place of y in the previous two results. When we use these formulae, we are assuming not only that the two lives are the same age, but that they are subject to identical mortality, and that mortality operates independently between the two lives.

Recall that:

$$ \boldsymbol{{}_nq_{xx} = 1 - {}_np_{xx} = 1 - {}_np_x\, {}_np_x} $$

So we have:

$$\begin{aligned}{}_nq_{\overset{2}{x}x} &= {}_nq_{\overset{1}{x}x} - {}_np_x(1-{}_np_x) \\ &= {}_nq_{\overset{1}{x}x} - {}_np_x + {}_np_{xx} \\ &= {}_nq_{\overset{1}{x}x} - (1-{}_nq_x) + (1-{}_nq_{xx}) \\ &= {}_nq_{\overset{1}{x}x} + {}_nq_x - {}_nq_{xx} \\ &= {}_nq_{\overset{1}{x}x} + \left({}_nq_{\overset{1}{x}x} + {}_nq_{\overset{2}{x}x}\right) - {}_nq_{xx}\end{aligned}$$

Cancelling the ${}_nq_{\overset{2}{x}x}$ terms, we obtain:

$$0 = 2\,{}_nq_{\overset{1}{x}x} - {}_nq_{xx}$$

to give:

$$\boldsymbol{{}_nq_{\overset{1}{x}x} = \tfrac{1}{2}\,{}_nq_{xx}}$$

If $n=\infty$ then ${}_\infty q_{xx} = 1$, leading to:

$$\boldsymbol{{}_\infty q_{\overset{1}{x}x} = {}_\infty q_{\overset{2}{x}x} = \tfrac{1}{2}}$$

Note also that:

$${}_nq_{\overset{2}{x}x} = \tfrac{1}{2}\,{}_nq_{\overline{xx}} = \tfrac{1}{2}\left({}_nq_x\right)^2$$

Arguments of symmetry can often lead to simplifications when we have joint life problems involving lives of equal ages. However, it is often useful to start off by considering the more general case with unequal ages.

Question

Simplify the sum ${}_nq_{\overset{2}{x}y} + {}_nq_{x\overset{2}{y}}$, and use the result to prove that ${}_nq_{\overset{2}{x}x} = \frac{1}{2}\,{}_nq_{\overline{xx}}$.

Solution

${}_nq_{\overset{2}{x}y} + {}_nq_{x\overset{2}{y}}$ is the probability that either (x) is the second of the two lives to die within the n-year period, or (y) is the second of the two lives to die within the n-year period. So, overall, this is the probability that both lives die within the n-year period, *ie* ${}_nq_{\overline{xy}}$.

So:

$${}_nq_{\overset{2}{x}y} + {}_nq_{x\overset{2}{y}} = {}_nq_{\overline{xy}}$$

Now if we have two lives of equal age who both experience the same mortality, we have:

$$ {}_nq^{\,2}_{xx} = {}_nq_{x\,x}^{\;\;2} $$

so the expression above becomes:

$$ {}_nq^{\,2}_{xx} + {}_nq_{x\,x}^{\;\;2} = {}_nq_{\overline{xx}} \;\;\Rightarrow\;\; 2\,{}_nq^{\,2}_{xx} = {}_nq_{\overline{xx}} \;\;\Rightarrow\;\; {}_nq^{\,2}_{xx} = \tfrac{1}{2}\,{}_nq_{\overline{xx}} $$

The formulae:

$$ {}_nq^{\,1}_{xx} = \tfrac{1}{2}\,{}_nq_{xx} \quad \text{and} \quad {}_nq^{\,2}_{xx} = \tfrac{1}{2}\,{}_nq_{\overline{xx}} $$

enable us to calculate order of death probabilities in terms of single life probabilities.

Question

Calculate:

(i) ${}_5q^{\,1}_{40:40}$

(ii) ${}_5q^{\,2}_{40:40}$

using AM92 mortality.

Solution

(i) We have:

$$ {}_5q^{\,1}_{40:40} = \frac{1}{2}\,{}_5q_{40:40} = \frac{1}{2}\left(1 - {}_5p_{40:40}\right) = \frac{1}{2}\left(1 - \left({}_5p_{40}\right)^2\right) = \frac{1}{2}\left(1 - \left(\frac{l_{45}}{l_{40}}\right)^2\right) $$

$$ = \frac{1}{2}\left(1 - \left(\frac{9{,}801.3123}{9{,}856.2863}\right)^2\right) = 0.005562 $$

(ii) We have:

$$ {}_5q^{\,2}_{40:40} = \frac{1}{2}\,{}_5q_{\overline{40:40}} = \frac{1}{2}\left({}_5q_{40}\right)^2 = \frac{1}{2}\left(1 - \frac{l_{45}}{l_{40}}\right)^2 = \frac{1}{2}\left(1 - \frac{9{,}801.3123}{9{,}856.2863}\right)^2 = 0.000016 $$

2 Contingent assurances

In Section 1 we saw that contingent events depending on the future lifetime of two lives (x) and (y) can be written in terms of the random variables T_x and T_y. The random variables representing the present value of contingent assurances can be expressed as functions of these two random variables.

For example, the present value of a sum assured of 1 paid immediately on the death of (x) provided that (y) is still alive can be expressed as:

$$\bar{Z} = \begin{cases} v_i^{T_x} & \text{if } T_x \leq T_y \\ 0 & \text{if } T_x > T_y \end{cases}$$

where i is the valuation rate of interest.

It is important to realise that the sum assured is not necessarily paid *to* (y) when (x) dies.

Using similar methods to those used in Section 1 the mean of $\bar{Z}$ is:

$$E\left[\bar{Z}\right] = \bar{A}^{1}_{xy} = \int_{t=0}^{t=\infty} v^t \, {}_tp_{xy} \, \mu_{x+t} \, dt$$

The actuarial notation is consistent with the term assurance notation $\bar{A}^{1}_{x:\overline{n}|}$ that we defined earlier in the course, but here it involves a second life status *y* instead of the duration status.

So, $\bar{A}^{1}_{xy}$ represents the EPV of a sum assured of 1 paid immediately on the death of life (*x*), provided that life (*x*) dies before life (*y*).

A point to note is that the *positioning* of the number (over the *x*) tells us *on whose death* the benefit will be paid. Then the *value* of that number (1 in this case) tells us the required *order* in which the deaths must occur.

Using a stochastic approach, we obtain the expectation as shown above. We can alternatively derive this integral using general reasoning. As we saw when expressing probabilities as integrals, we consider an integral based on some event happening at time t and what conditions have to hold at time t. In addition, we need to discount the benefit payment back to time 0. It is therefore normally necessary for the event that we are modelling at time t to be the event that triggers payment.

So for this assurance we have:

- a benefit payable on (x)'s death provided (y) is still alive, so we model (x)'s death at time t
- at time t, we require (x) to have survived to t and then to die, giving ${}_tp_x\mu_{x+t}$
- at time t, we require (y) to be alive, giving ${}_tp_y$
- we discount to time 0 from the moment of benefit payment at time t, giving v^t
- and we integrate over all possible values of t (from 0 to ∞) as the term is not limited.

So the integral is:

$$\int_{t=0}^{\infty} v^t\, {}_tp_x\, {}_tp_y\, \mu_{x+t}\, dt = \int_{t=0}^{\infty} v^t\, {}_tp_{xy}\mu_{x+t}\, dt$$

The variance of $\bar{Z}$ is:

$$\mathbf{var}(\bar{Z}) = {}^2\bar{A}^1_{xy} - \left(\bar{A}^1_{xy}\right)^2$$

where ${}^2\bar{A}$ is evaluated at a valuation rate of interest $i^2 + 2i$.

To obtain this formula we start by noting that if:

$$\bar{Z} = \begin{cases} v_i^{T_x} & \text{if } T_x \le T_y \\ 0 & \text{if } T_x > T_y \end{cases}$$

then:

$$\bar{Z}^2 = \begin{cases} \left(v_i^{T_x}\right)^2 = \left(v_i^2\right)^{T_x} & \text{if } T_x \le T_y \\ 0 & \text{if } T_x > T_y \end{cases}$$

We see that $\bar{Z}^2$ has the same form as $\bar{Z}$ with v_i replaced by v_i^2, and so we can replace v^t by $\left(v^2\right)^t$ in the expression for $E\left[\bar{Z}\right]$ to obtain:

$$E\left[\bar{Z}^2\right] = \int_{t=0}^{\infty} \left(v^2\right)^t\, {}_tp_{xy}\mu_{x+t}\, dt$$

This is equal to ${}^2\bar{A}^1_{xy}$ at an interest rate of $i^2 + 2i$ (or force of interest 2δ). Hence:

$$\text{var}\left(\bar{Z}\right) = E\left[\bar{Z}^2\right] - \left(E\left[\bar{Z}\right]\right)^2 = {}^2\bar{A}^1_{xy} - \left(\bar{A}^1_{xy}\right)^2$$

These functions are usually evaluated by using numerical methods, such as Simpson's rule to determine the values of the integrals.

Simpson's rule is similar to the trapezium rule except that, instead of joining pairs of consecutive points with straight-line segments, we take groups of three consecutive points and fit quadratics to them.

In some cases, joint and single life values obtained from tables can be useful in conjunction with the following and similar relationships:

(i) $\bar{A}_{xy} = \bar{A}^{1}_{x\,y} + \bar{A}_{x\,y}^{\;\,1}$

We can prove this result as follows:

$$\bar{A}^{1}_{x\,y} + \bar{A}_{x\,y}^{\;\,1} = \int_{t=0}^{\infty} v^t \, {}_tp_{xy} \mu_{x+t} \, dt + \int_{t=0}^{\infty} v^t \, {}_tp_{xy} \mu_{y+t} \, dt$$

$$= \int_{t=0}^{\infty} v^t \, {}_tp_{xy} \left(\mu_{x+t} + \mu_{y+t} \right) dt$$

$$= \int_{t=0}^{\infty} v^t \, {}_tp_{xy} \, \mu_{x+t:y+t} \, dt$$

$$= \bar{A}_{xy}$$

$\bar{A}_{xy}$ is the expected present value of 1, paid immediately when the first of the two lives dies. The first life to die can either be (x) or (y), so $\bar{A}_{xy}$ is equal to the sum of an assurance that makes a payment immediately if (x) is the first life to die ($\bar{A}^{1}_{x\,y}$) and an assurance that makes a payment immediately if (y) is the first life to die ($\bar{A}_{x\,y}^{\;\,1}$).

(ii) $\bar{A}_{x} = \bar{A}^{1}_{x\,y} + \bar{A}^{2}_{x\,y}$

We can see that this is true by general reasoning.

$\bar{A}_x$ is the expected present value of 1 paid immediately on the death of (x). In relation to (y), (x) must either die first or second. Since these two possibilities are mutually exclusive and exhaustive, $\bar{A}_x$ is equal to the sum of an assurance that makes a payment immediately if (x) is the first life to die ($\bar{A}^{1}_{x\,y}$) and an assurance that makes a payment immediately if (x) is the second life to die ($\bar{A}^{2}_{x\,y}$).

We can also prove this algebraically as follows:

$$\begin{aligned}
\bar{A}_{\overset{1}{x}y} + \bar{A}_{\overset{2}{x}y} &= \int_0^\infty v^t \, {}_tp_x \mu_{x+t} \, {}_tp_y \, dt + \int_0^\infty v^t \, {}_tp_x \mu_{x+t} \, {}_tq_y \, dt \\
&= \int_0^\infty v^t \, {}_tp_x \mu_{x+t} \left({}_tp_y + {}_tq_y \right) dt \\
&= \int_0^\infty v^t \, {}_tp_x \mu_{x+t} \, dt \\
&= \bar{A}_x
\end{aligned}$$

(iii) $\boldsymbol{\bar{A}_{\overset{1}{x}x} = \tfrac{1}{2}\bar{A}_{xx}}$

We can prove this result by making use of the formula:

$$\bar{A}_{xy} = \bar{A}_{\overset{1}{x}y} + \bar{A}_{x\overset{1}{y}}$$

Making both lives the same age gives:

$$\bar{A}_{xx} = \bar{A}_{\overset{1}{x}x} + \bar{A}_{x\overset{1}{x}}$$

By symmetry:

$$\bar{A}_{\overset{1}{x}x} = \bar{A}_{x\overset{1}{x}}$$

So:

$$\bar{A}_{\overset{1}{x}x} = \tfrac{1}{2}\bar{A}_{xx}$$

(iv) $\boldsymbol{\bar{A}_{\overset{2}{x}x} = \tfrac{1}{2}\bar{A}_{\overline{xx}}}$

To derive this result, we start by obtaining another relationship.

Question

Prove that $\bar{A}_{\overset{2}{x}y} + \bar{A}_{x\overset{2}{y}} = \bar{A}_{\overline{xy}}$.

Solution

By rearranging relationship (ii) above we have:

$$\bar{A}_{\overset{2}{x}y} = \bar{A}_x - \bar{A}_{\overset{1}{x}y}$$

Similarly:

$$\bar{A}_{x\overset{2}{y}} = \bar{A}_y - \bar{A}_{x\overset{1}{y}}$$

So:

$$\begin{aligned}
\bar{A}_{\overset{2}{x}y} + \bar{A}_{x\overset{2}{y}} &= \bar{A}_x - \bar{A}_{\overset{1}{x}y} + \bar{A}_y - \bar{A}_{x\overset{1}{y}} \\
&= \bar{A}_x + \bar{A}_y - \left(\bar{A}_{\overset{1}{x}y} + \bar{A}_{x\overset{1}{y}}\right) \\
&= \bar{A}_x + \bar{A}_y - \bar{A}_{xy} \\
&= \bar{A}_{\overline{xy}}
\end{aligned}$$

where the last line above uses the 'last survivor $(\text{L}) = \text{single }(\text{S}) + \text{single }(\text{S}) - \text{joint }(\text{J})$' result that we met in the previous chapter.

Now, using $\bar{A}_{\overset{2}{x}y} + \bar{A}_{x\overset{2}{y}} = \bar{A}_{\overline{xy}}$ with $y = x$:

$$\bar{A}_{\overset{2}{x}x} + \bar{A}_{x\overset{2}{x}} = \bar{A}_{\overline{xx}}$$

By symmetry:

$$\bar{A}_{\overset{2}{x}x} = \bar{A}_{x\overset{2}{x}}$$

So:

$$\bar{A}_{\overset{2}{x}x} = \tfrac{1}{2}\bar{A}_{\overline{xx}}$$

Question

Two lives are aged 50 and 60. The 50-year-old is subject to a constant force of mortality of 0.02 *pa*, and the 60-year-old is subject to a constant force of mortality of 0.025 *pa*.

Assuming that the constant force of interest is 6% *pa*, calculate:

(i) $\bar{A}^{\;1}_{50:60}$

(ii) $\bar{A}^{\;2}_{50:60}$

Solution

Since both lives experience a constant force of mortality, we have:

$$\mu_{50+t} = 0.02 \quad \text{and} \quad {}_tp_{50} = e^{-0.02t}$$

and: $$\mu_{60+t} = 0.025 \quad \text{and} \quad {}_tp_{60} = e^{-0.025t}$$

for all t. In addition, $v^t = e^{-0.06t}$.

(i) We have:

$$\bar{A}^{\,1}_{50:60} = \int_0^\infty v^t \, {}_tp_{50:60}\mu_{50+t}\, dt = \int_0^\infty v^t \, {}_tp_{50} \, {}_tp_{60} \, \mu_{50+t} \, dt$$

$$= \int_0^\infty e^{-0.06t} \times e^{-0.02t} \times e^{-0.025t} \times 0.02 \, dt$$

$$= 0.02 \int_0^\infty e^{-0.105t} \, dt = 0.02 \left[\frac{e^{-0.105t}}{-0.105} \right]_0^\infty = \frac{0.02}{0.105} = 0.19048$$

(ii) We have:

$$\bar{A}_{50} = \int_0^\infty v^t \, {}_tp_{50} \, \mu_{50+t} \, dt$$

$$= \int_0^\infty e^{-0.06t} \times e^{-0.02t} \times 0.02 \, dt$$

$$= 0.02 \int_0^\infty e^{-0.08t} \, dt = 0.02 \left[\frac{e^{-0.08t}}{-0.08} \right]_0^\infty = \frac{0.02}{0.08} = 0.25$$

So, using relationship (ii):

$$\bar{A}^{\,2}_{50:60} = \bar{A}_{50} - \bar{A}^{\,1}_{50:60} = 0.25 - \frac{0.02}{0.105} = 0.05952$$

Alternatively, we could calculate this as:

$$\bar{A}^{\,2}_{50:60} = \int_0^\infty v^t \, {}_tp_{50} \, \mu_{50+t} \, {}_tq_{60} \, dt$$

The four relationships we have met above all relate to payments made immediately on death.

If the benefit is payable at the end of the policy year in which the contingent event occurs, then we can show that:

$$A^{1}_{xy} = \sum_{t=0}^{t=\infty} v^{t+1}\,{}_tp_{xy}\,q^{\;1}_{x+t:y+t}$$

with analogous expressions for the variance to those derived for assurances payable immediately on the occurrence of the contingent event.

Such expressions are usually evaluated by using the approximate relationship:

$$A^{1}_{xy} \approx (1+i)^{-\frac{1}{2}}\,\bar{A}^{1}_{xy}$$

and similar expressions.

3 Reversionary annuities

The simplest form of a reversionary annuity is one that begins on the death of (x), if (y) is then alive, and continues during the lifetime of (y). The life (x) is called the counter or failing life, and the life (y) is called the annuitant. The random variable $\bar{Z}$ representing the present value of this reversionary annuity if it is payable continuously can be written as a function of the random variables T_x and T_y, where:

$$\bar{Z} = \begin{cases} \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_x}|} & \text{if } T_y > T_x \\ 0 & \text{if } T_y \le T_x \end{cases}$$

So, if $T_y > T_x$, $\bar{Z}$ is the present value of a continuously payable annuity of 1 *pa* beginning in exactly T_x years' time (when (*x*) dies) and ending exactly T_y years from now (when (*y*) dies). So we can write, for $T_y > T_x$:

$$\bar{Z} = \int_{T_x}^{T_y} v^t\,dt = \int_0^{T_y} v^t\,dt - \int_0^{T_x} v^t\,dt = \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_x}|}$$

as before.

Also, we can see that, by substituting $r = t - T_x$:

$$\bar{Z} = \int_0^{T_y - T_x} v^{T_x + r}\,dr = v^{T_x} \int_0^{T_y - T_x} v^r\,dr = v^{T_x}\,\bar{a}_{\overline{T_y - T_x}|}$$

(remembering that this is only for $T_y > T_x$).

So the reversionary annuity is paid for a total of $T_y - T_x$ years, beginning (and therefore discounted from) T_x years from now.

As an alternative to the above expressions for the present value of the reversionary annuity, we can write:

$$\bar{Z} = \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{\min\{T_x, T_y\}}|}$$

so that when $T_x < T_y$ the value is $\bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_x}|}$, and when $T_x > T_y$ the value is $\bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_y}|} = 0$. Then, because $\min\{T_x, T_y\} = T_{xy}$, we can conveniently write:

$$\bar{Z} = \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_{xy}}|}$$

This is a very useful form for the present value of a reversionary annuity.

Using similar methods to those used for contingent assurances, we can show that:

$$E\left[\bar{Z}\right] = \bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy} = \frac{\bar{A}_{xy} - \bar{A}_y}{\delta}$$

$$= \int_{t=0}^{t=\infty} v^t \, \bar{a}_{y+t} \; {}_tp_{xy} \, \mu_{x+t} \, dt$$

The variance can also be expressed as an integral in this way.

The actuarial notation for the EPV of this reversionary annuity is $\bar{a}_{x|y}$. The vertical bar in the subscript represents a period of deferment, as before. Here, it specifically means that the annuity is paid to (y) but is deferred for the lifetime of (x).

$\bar{a}_y - \bar{a}_{xy}$ is generally the most useful result for calculating expected present values of reversionary annuities. This follows easily from our final expression for $\bar{Z}$ above:

$$E\left[\bar{Z}\right] = E\left[\bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_{xy}}|}\right] = E\left[\bar{a}_{\overline{T_y}|}\right] - E\left[\bar{a}_{\overline{T_{xy}}|}\right] = \bar{a}_y - \bar{a}_{xy}$$

Also, since:

$$\bar{a}_y = \frac{1-\bar{A}_y}{\delta} \quad \text{and} \quad \bar{a}_{xy} = \frac{1-\bar{A}_{xy}}{\delta}$$

by premium conversion, we have:

$$\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy} = \frac{1-\bar{A}_y}{\delta} - \frac{1-\bar{A}_{xy}}{\delta} = \frac{\bar{A}_{xy} - \bar{A}_y}{\delta}$$

Question

Explain the integral expression:

$$\bar{a}_{x|y} = \int_{t=0}^{t=\infty} v^t \, \bar{a}_{y+t} \; {}_tp_{xy} \, \mu_{x+t} \, dt$$

by general reasoning.

Solution

Under this reversionary annuity, payments will begin at exact time t if (x) dies at that moment (with probability density ${}_tp_x \, \mu_{x+t}$) and (y) is still alive (with probability ${}_tp_y$). This gives a factor of ${}_tp_x \, \mu_{x+t} \times {}_tp_y = {}_tp_{xy} \, \mu_{x+t}$.

From time t onwards, an annuity of 1 *pa* is payable continuously until the subsequent death of (y). As (y) is aged $y+t$ at the time the annuity starts, the expected present value of the annuity as at time t is $\bar{a}_{y+t}$. Discounting to time 0 requires further multiplication by v^t.

Finally, integrating over all possible values of t covers all possible start times for the annuity payments (*ie* all the moments at which (x) could die with (y) still living).

A simpler alternative integral expression is:

$$\bar{a}_{x|y} = \int_0^\infty v^t \, {}_tp_y \, {}_tq_x \, dt$$

The logic here is that payments are made at (and hence discounted from) each time t provided (y) is alive at that moment and (x) is dead by that time.

Using this integral expression, we can again obtain the result $\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy}$:

$$\begin{aligned}\bar{a}_{x|y} &= \int_0^\infty v^t \, {}_tp_y \, {}_tq_x \, dt = \int_0^\infty v^t \, {}_tp_y \left(1 - {}_tp_x\right) dt \\ &= \int_0^\infty v^t \, {}_tp_y \, dt - \int_0^\infty v^t \, {}_tp_{xy} \, dt \\ &= \bar{a}_y - \bar{a}_{xy}\end{aligned}$$

If the annuity begins at the end of the year of death of (x) and is then paid annually in arrears during the lifetime of (y), the random variable Z representing the present value of the payments can be written as a function of K_x and K_y, where:

$$\boldsymbol{Z} = \begin{cases} \boldsymbol{a_{\overline{K_y}|} - a_{\overline{K_x}|}} & \textbf{if } \boldsymbol{K_y > K_x} \\ \boldsymbol{0} & \textbf{if } \boldsymbol{K_y \le K_x} \end{cases}$$

$$= a_{\overline{K_y}|} - a_{\overline{K_{xy}}|}$$

We can show that:

$$\boldsymbol{E[Z] = a_{x|y} = a_y - a_{xy} = \frac{A_{xy} - A_y}{d}}$$

Question

Calculate $a_{65|60}$, assuming:

(i) (65) is male and (60) is female

(ii) (65) is female and (60) is male.

Basis: Mortality: PMA92C20 for the male life, PFA92C20 for the female life

Interest: 4% *pa* effective

Solution

(i) $a^{m\ f}_{65|60} = a^{f}_{60} - a^{m\ f}_{65:60} = 15.652 - 11.682 = 3.970$

(ii) $a^{f\ m}_{65|60} = a^{m}_{60} - a^{m\ f}_{60:65} = 14.632 - 12.101 = 2.531$

To find the values of reversionary annuities in arrears, we take the tabulated values of the annuities-due in the *Tables*, and subtract one from both. The two –1 terms cancel out, so that we can say:

$$a_{x|y} = a_y - a_{xy} = (\ddot{a}_y - 1) - (\ddot{a}_{xy} - 1) = \ddot{a}_y - \ddot{a}_{xy} = \ddot{a}_{x|y}$$

4 Joint life functions dependent on term

All of the theory involving assurance and annuity benefits considered above can be easily modified to allow for policies with a specified term.

4.1 Expected present values of joint life assurances and annuities which also depend upon term

Joint life assurances that are dependent on a fixed term of n years can be term assurances or endowment assurances. Their expected present values, if they are paid immediately on death, can be expressed as:

$$\bar{A}^{\;1}_{\overset{\frown}{xy}:\overline{n}|} = \int_{t=0}^{t=n} v^t \, {}_tp_{xy} \, \mu_{x+t:y+t} \, dt$$

$$\bar{A}_{xy:\overline{n}|} = \bar{A}^{\;1}_{\overset{\frown}{xy}:\overline{n}|} + \bar{A}_{xy:\overset{\;\;1}{\overline{n}|}}$$

where:

$$\bar{A}_{xy:\overset{\;\;1}{\overline{n}|}} = {}_np_{xy} v^n$$

The bracket used in the notation for the term assurance $\bar{A}^{\;1}_{\overset{\frown}{xy}:\overline{n}|}$ indicates that the joint life status must end within the fixed term of n years.

We want to indicate that the joint life status has to fail before the n-year term runs out, but we don't want to specify that a particular life has to die. So we draw a bracket over the joint life status to denote that we are treating the joint lives as a single entity.

Question

Describe precisely the meaning of each of the three assurance functions used above.

Solution

$\bar{A}^{\;1}_{\overset{\frown}{xy}:\overline{n}|}$ = EPV of 1 paid immediately on the first death out of (x) and (y), provided that death occurs within n years.

$\bar{A}_{xy:\overline{n}|}$ = EPV of 1 paid in n years' time, or immediately on the first death of (x) and (y) if this occurs sooner.

$\bar{A}_{xy:\overset{\;\;1}{\overline{n}|}}$ = EPV of 1 paid in n years' time, provided neither (x) nor (y) has died by then.

The expected present value of the temporary joint life annuity payable continuously can be written as:

$$\bar{a}_{xy:\overline{n}|} = \int_{t=0}^{t=n} v^t \, {}_tp_{xy} \, dt$$

Similar expressions involving summation operators can be developed if assurances are paid at the end of the year of death or if annuities are payable annually in advance or in arrears.

4.2 Expected present values of last survivor assurances and annuities that also depend upon term

Last survivor assurances that are dependent on a fixed term of n years can be term assurances or endowment assurances. Their expected present values can be expressed in terms of single and joint life functions by making use of the results set out in Chapter 22, which can be generalised to any two statuses u and v. So we write:

$$T_{\overline{uv}} = T_u + T_v - T_{uv}$$

$$K_{\overline{uv}} = K_u + K_v - K_{uv}$$

The resulting expressions for assurances payable immediately on death are:

$$\bar{A}_{\overline{xy}:\overline{n}|} = \bar{A}_{x:\overline{n}|} + \bar{A}_{y:\overline{n}|} - \bar{A}_{xy:\overline{n}|}$$

$$\bar{A}^{1}_{\overline{xy}:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} + \bar{A}^{1}_{y:\overline{n}|} - \bar{A}^{1}_{xy:\overline{n}|}$$

The expected present value of the temporary last survivor annuity payable continuously can be written as:

$$\bar{a}_{\overline{xy}:\overline{n}|} = \bar{a}_{x:\overline{n}|} + \bar{a}_{y:\overline{n}|} - \bar{a}_{xy:\overline{n}|}$$

Similar expressions involving summation operators can be developed if assurances are paid out at the end of the year of death or if annuities are payable annually in advance or arrears.

Question

Express $A_{\overline{60:60}:\overline{5}|}$ in terms of single and joint life functions (assuming the two lives have identical, and independent, mortality).

Solution

$$A_{\overline{60:60}:\overline{5}|} = 2A_{60:\overline{5}|} - A_{60:60:\overline{5}|}$$

With last survivor assurances and annuities we must be very careful when allowing for duration. For example:

$$\ddot{a}_{\overline{xy}:\overline{n}|} \neq \ddot{a}_{\overline{xy}} - v^n \, {}_np_{\overline{xy}} \, \ddot{a}_{\overline{x+n:y+n}}$$

Question

Explain why.

Solution

In this kind of formula, we are considering the temporary annuity as being equal to a whole life annuity minus another whole life annuity from higher ages, reflecting the payments that will not be received after the *n*-year term has elapsed.

In this case, after *n* years, the status $\overline{xy}$ is still active if:

- (x) only is alive
- (y) only is alive
- both (x) and (y) are alive.

However, the annuity value $\ddot{a}_{\overline{x+n:y+n}}$ is conditional on *both* lives being alive at time *n*.

To correct the formula, we have to allow for all three possibilities separately. That is:

$$\ddot{a}_{\overline{xy}:\overline{n}|} = \ddot{a}_{\overline{xy}} - v^n \left[{}_np_x \, {}_nq_y \, \ddot{a}_{x+n} + {}_nq_x \, {}_np_y \, \ddot{a}_{y+n} + {}_np_{xy} \, \ddot{a}_{\overline{x+n:y+n}} \right]$$

The formulae shown in the Core Reading are therefore simpler.

For similar reasons:

$$A^{\;1}_{\overline{xy}:\overline{n}|} \neq A_{\overline{xy}} - v^n \, {}_np_{\overline{xy}} \, A_{\overline{x+n:y+n}}$$

Question

Using PA92C20 mortality and 4% *pa* interest, calculate $\ddot{a}_{\overline{50:50}:\overline{20}|}$, assuming that one life is male and the other is female.

Solution

The last survivor temporary annuity can be thought of as the sum of the two single life temporary annuities less the temporary joint life annuity:

$$\ddot{a}_{\overline{50:50}:\overline{20}|} = \ddot{a}_{50:\overline{20}|} + \ddot{a}_{50:\overline{20}|} - \ddot{a}_{50:50:\overline{20}|}$$

Now:

$$\ddot{a}_{50:\overline{20|}} = \ddot{a}_{50} - v^{20}\frac{l_{70}}{l_{50}}\ddot{a}_{70}$$

which for males gives:

$$\ddot{a}_{50:\overline{20|}} = 18.843 - 1.04^{-20} \times \frac{9,238.134}{9,941.923} \times 11.562 = 13.940$$

and for females gives:

$$\ddot{a}_{50:\overline{20|}} = 19.539 - 1.04^{-20} \times \frac{9,392.621}{9,952.697} \times 12.934 = 13.968$$

Also:

$$\ddot{a}_{50:50:\overline{20|}} = \ddot{a}_{50:50} - 1.04^{-20} \times \frac{9,238.134}{9,941.923} \times \frac{9,392.621}{9,952.697} \times \ddot{a}_{70:70}$$

$$= 17.688 - 1.04^{-20} \times \frac{9,238.134}{9,941.923} \times \frac{9,392.621}{9,952.697} \times 9.766$$

$$= 13.780$$

So:

$$\ddot{a}_{\overline{50:50}:\overline{20|}} = 13.940 + 13.968 - 13.780 = 14.128$$

4.3 More complex conditions

We frequently employ a technique of describing probabilities and expected present values with appropriately constructed integrals. We can generally use this technique to deal with more complex conditions, and this is often the most efficient way to solve such problems. The technique can be summarised as follows:

(1) identify the critical dates / times in the problem,

(2) express as an integral,

(3) simplify (often by making some appropriate substitution), and

(4) re-express in terms of simpler (non-integral) functions.

For discrete benefits we would use a summation expression rather than an integral.

Question

Two sisters, Xanthe and Yolanda, are aged x and y, respectively.

(i) Derive an expression for the probability that Xanthe will die more than 5 years after the death of Yolanda, giving the answer in terms of single life probabilities and probabilities based on the first death.

Xanthe wishes to take out an assurance with a sum assured of £50,000 payable immediately on her death under the conditions described in part (i).

(ii) Derive an expression for the expected present value of this benefit, giving the answer in terms of single life assurances and assurances payable on the first death.

Solution

(i) ***Probability***

When deriving probabilities for two lives, we can usually condition on either death. In this case, if Yolanda dies at time t, we require Xanthe to be alive 5 years later. This can be expressed in integral form as:

$$\int_0^\infty {}_{t+5}p_x \; {}_tp_y \; \mu_{y+t} \, dt$$

Since ${}_{t+5}p_x = {}_5p_x \; {}_tp_{x+5}$, the integral can be written:

$${}_5p_x \int_0^\infty {}_tp_{x+5} \; {}_tp_y \; \mu_{y+t} \, dt = {}_5p_x \; {}_\infty q^{\;\;\;\;1}_{x+5:y}$$

(ii) ***Expected present value***

When deriving the expected present value of assurances, it is easier to condition on the assured life. If Xanthe dies at time t, the sum assured will be paid provided Yolanda died before time $t-5$, for $t \geq 5$. The expected present value of the assurance per £1 sum assured is therefore:

$$\int_5^\infty v^t \, {}_tp_x \, \mu_{x+t} \; {}_{t-5}q_y \, dt = \int_5^\infty v^t \, {}_tp_x \, \mu_{x+t} (1 - {}_{t-5}p_y) \, dt$$

Substituting $u = t-5$, the integral becomes:

$$\int_0^\infty v^{u+5} \; {}_{u+5}p_x \, \mu_{x+u+5} \, (1 - {}_up_y) \, du = v^5 \, {}_5p_x \int_0^\infty v^u \; {}_up_{x+5} \, \mu_{x+u+5} (1 - {}_up_y) \, du$$

since ${}_{u+5}p_x = {}_5p_x \; {}_up_{x+5}$.

Multiplying out the bracket, we have:

$$v^5 \, {}_5p_x \left[\int_0^\infty v^u \, {}_up_{x+5} \, \mu_{x+u+5} \, du - \int_0^\infty v^u \, {}_up_{x+5} \, \mu_{x+u+5} \, {}_up_y \, du \right]$$

Expressing this in terms of assurance functions and multiplying by the sum assured of £50,000 gives an expected present value of:

$$50{,}000v^5 \, {}_5p_x (\bar{A}_{x+5} - \bar{A}^{\,1}_{x+5:y})$$

We will use this kind of approach when developing formulae in the next section.

4.4 Expected present values of reversionary annuities that depend upon term

There are several different types of reversionary annuities that depend on term, and some of the possibilities are listed below.

Type 1 – an annuity payable after a fixed term has elapsed

A reversionary annuity in which the counter or failing status is a fixed term of n years is exactly equivalent to a deferred life annuity. The expected present value of an annuity that is paid continuously can be written:

$$\bar{a}_{\overline{n}|\,|y} = {}_{n|}\bar{a}_y = \bar{a}_y - \bar{a}_{y:\overline{n}|}$$

However, for calculation purposes it is much quicker to use:

$$_{n|}\bar{a}_y = v^n \, {}_np_y \, \bar{a}_{y+n}$$

as we have done before.

Type 2 – an annuity payable to (*y*) on the death of (*x*), but ceasing at time *n*

If a reversionary annuity ceases in any event after n years, *ie* is payable to (y) after the death of (x) with no payment being made after n years, the expected present value can be expressed as:

$$\bar{a}_{y:\overline{n}|} - \bar{a}_{xy:\overline{n}|}$$

We can obtain this expression using integrals, by considering the payment made at time t. A payment will be made at time t if:

- (y) is alive at time t
- (x) is dead by time t
- $t < n$.

So the expected present value is:

$$\int_0^n v^t\,{}_tp_y\,{}_tq_x\,dt = \int_0^n v^t\,{}_tp_y\left(1-{}_tp_x\right)dt = \int_0^n v^t\,{}_tp_y\,dt - \int_0^n v^t\,{}_tp_{xy}\,dt = \bar{a}_{y:\overline{n}|} - \bar{a}_{xy:\overline{n}|}$$

Question

Ralph and Betty are both aged 65 exact. Upon Betty's death, Ralph will receive £20,000 *pa* payable annually in advance starting from the end of the year of Betty's death. There will be no payments on or beyond Ralph's 80th birthday in any circumstances.

Ralph's mortality follows PMA92C20, Betty's mortality follows PFA92C20 and the interest rate for all future years is 4% *pa*.

Calculate the EPV of this benefit to Ralph.

Solution

This is a reversionary annuity payable to Ralph, after Betty's death, but ceasing after 15 years.

We have:

$$\ddot{a}^{m}_{65:\overline{15}|} - \ddot{a}^{m\ \ f}_{65:65:\overline{15}|} = \ddot{a}_{65} - v^{15}\,{}_{15}p_{65}\,\ddot{a}_{80} - \left[\ddot{a}_{65:65} - v^{15}\,{}_{15}p_{65:65}\,\ddot{a}_{80:80}\right]$$

$$= 13.666 - 1.04^{-15} \times \frac{6,953.536}{9,647.797} \times 7.506$$

$$- \left[11.958 - 1.04^{-15} \times \frac{6,953.536}{9,647.797} \times \frac{7,724.737}{9,703.708} \times 5.857\right]$$

$$= 0.5700$$

So the EPV to Ralph is:

$$£20,000 \times 0.5700 = £11,401$$

Type 3 – an annuity payable to (*y*) on the death of (*x*) provided that (*x*) dies within *n* years

If the payment commences on the death of (x) within n years and then continues until the death of (y), the expected present value can be expressed as:

$$\int_{t=0}^{t=n} v^t\,{}_tp_{xy}\,\mu_{x+t}\,\bar{a}_{y+t}\,dt = \bar{a}_y - \bar{a}_{xy} - v^n\,{}_np_{xy}\left(\bar{a}_{y+n} - \bar{a}_{x+n:y+n}\right)$$

$$= \bar{a}_{x|y} - v^n\,{}_np_{xy}\,\bar{a}_{x+n|y+n}$$

This is what we might most accurately describe as a 'temporary reversionary annuity'.

The rationale for the integral expression:

$$\int_{t=0}^{n} v^t \, {}_t p_{xy} \, \mu_{x+t} \, \bar{a}_{y+t} \, dt$$

is that the only restriction compared to the normal whole life version is that payments made as a result of the death of (x) after n years will not be made. We can therefore calculate the expected present value by integrating from $t = 0$ to n rather than to infinity.

Question

Prove that $\int_{t=0}^{n} v^t \, {}_t p_{xy} \, \mu_{x+t} \, \bar{a}_{y+t} \, dt = \bar{a}_{x|y} - v^n \, {}_n p_{xy} \, \bar{a}_{x+n|y+n}$.

Solution

We already have, from Section 3, the result:

$$\int_{t=0}^{\infty} v^t \, \bar{a}_{y+t} \, {}_t p_{xy} \, \mu_{x+t} \, dt = a_{x|y} = \bar{a}_y - \bar{a}_{xy}$$

So we can write:

$$\begin{aligned}\int_{t=0}^{n} v^t \, \bar{a}_{y+t} \, {}_t p_{xy} \, \mu_{x+t} \, dt &= \int_{t=0}^{\infty} v^t \, \bar{a}_{y+t} \, {}_t p_{xy} \, \mu_{x+t} \, dt - \int_{t=n}^{\infty} v^t \, \bar{a}_{y+t} \, {}_t p_{xy} \, \mu_{x+t} \, dt \\ &= \bar{a}_y - \bar{a}_{xy} - \int_{s=0}^{\infty} v^{s+n} \, \bar{a}_{y+s+n} \, {}_{s+n} p_{xy} \, \mu_{x+s+n} \, ds\end{aligned}$$

using the substitution $s = t - n$.

Pulling a factor of $v^n \, {}_n p_{xy}$ outside the integral gives:

$$\begin{aligned}\int_{t=0}^{n} v^t \, \bar{a}_{y+t} \, {}_t p_{xy} \, \mu_{x+t} \, dt &= \bar{a}_y - \bar{a}_{xy} - v^n \, {}_n p_{xy} \int_{s=0}^{\infty} v^s \, \bar{a}_{y+s+n} \, {}_s p_{x+n:y+n} \, \mu_{x+s+n} \, ds \\ &= \bar{a}_y - \bar{a}_{xy} - v^n \, {}_n p_{xy} \left(\bar{a}_{y+n} - \bar{a}_{x+n:y+n} \right) \\ &= \bar{a}_{x|y} - v^n \, {}_n p_{xy} \, \bar{a}_{x+n|y+n}\end{aligned}$$

The expression:

$$\bar{a}_{x|y} - v^n\, {}_np_{xy}\, \bar{a}_{x+n|y+n}$$

can be justified by noting that, compared to the normal whole life reversionary annuity $\bar{a}_{x|y}$, we need to deduct all annuity payments that relate to the death of (x) after n years with (y) alive at the time of (x)'s death. (Note that the death of (x) after n years, with (y) dead at the time of (x)'s death, is already excluded from $\bar{a}_{x|y}$). The required deduction therefore equates to a reversionary annuity from time n, with both (x) and (y) alive at that point, appropriately discounted to the start of the contract.

Question

Ralph and Betty are both aged 65 exact. Upon Betty's death, Ralph will receive £20,000 *pa* payable annually in advance for the rest of his life starting from the end of the year of Betty's death, provided that Betty dies in the next 10 years.

Ralph's mortality follows PMA92C20, Betty's mortality follows PFA92C20 and the interest rate for all future years is 4% *pa*.

Calculate the EPV of this benefit to Ralph.

Solution

We have:

$$\ddot{a}_{65}^{m} - \ddot{a}_{65:65} - v^{10}\, {}_{10}p_{65:65}(\ddot{a}_{75}^{m} - \ddot{a}_{75:75})$$

$$= 13.666 - 11.958 - 1.04^{-10} \times \frac{8,405.160}{9,647.797} \times \frac{8,784.955}{9,703.708} \times (9.456 - 7.679)$$

$$= 0.76117$$

So, the EPV to Ralph is:

$$£20,000 \times 0.76117 = £15,223$$

Type 4 – an annuity payable to (*y*) on the death of (*x*) for a maximum of *n* years

If the conditions of payment say that the payment will:

- **begin on the death of (x) and**
- **cease on the death of (y) or n years after the death of (x) (whichever event occurs first),**

then the expected present value can be expressed as:

$$\int_{t=0}^{t=\infty} v^t \, {}_tp_{xy} \, \mu_{x+t} \, \bar{a}_{y+t:\overline{n}|} \, dt = \bar{a}_{y:\overline{n}|} + v^n \, {}_np_y \, \bar{a}_{x:y+n} - \bar{a}_{xy}$$

Similar expressions involving summation operators can be developed if the annuity payments are made at annual intervals from the date on which the counter status fails.

Question

Explain in words the integral formula on the left-hand side of the equation above.

Solution

The annuity begins at time *t*, which is the instant of (*x*)'s death. This gives a factor of ${}_tp_x \, \mu_{x+t} \, dt$.

The annuity is payable to (*y*) for life thereafter, but for a maximum of *n* years. Hence:

- (*y*) has to be alive at time *t* (with probability ${}_tp_y$), and
- the expected present value of the temporary annuity from that time is $\bar{a}_{y+t:\overline{n}|}$.

We then need to discount this value to time zero, using v^t , and integrate over all times t at which (*x*) can die.

To obtain the expression on the right-hand side above:

$$\bar{a}_{y:\overline{n}|} + v^n \, {}_np_y \, \bar{a}_{x:y+n} - \bar{a}_{xy}$$

it is easiest to consider an alternative integral expression for this annuity.

First, consider an annuity that starts *n* years after the death of (*x*) and is payable to (*y*). This annuity will be in payment at time $t + n$ if:

- (*x*) has died before time *t* and
- (*y*) is alive at time $t + n$.

So, the expected present value of this annuity is:

$$\int_0^\infty v^{t+n}\ {}_tq_x\ {}_{t+n}p_y\, dt$$

Now, the Type 4 annuity is equal to a reversionary annuity less this one above. So, the expected present value of the Type 4 annuity is:

$$\bar{a}_{x|y} - \int_0^\infty v^{t+n}\ {}_tq_x\ {}_{t+n}p_y\, dt = \bar{a}_{x|y} - v^n\ {}_np_y \int_0^\infty v^t\ {}_tq_x\ {}_tp_{y+n}\, dt$$

$$= \bar{a}_{x|y} - v^n\ {}_np_y\ \bar{a}_{x|y+n}$$

Finally, using the formulae for a reversionary annuity, we can write:

$$\bar{a}_{x|y} - v^n\ {}_np_y\ \bar{a}_{x|y+n} = \bar{a}_y - \bar{a}_{xy} - v^n\ {}_np_y\left(\bar{a}_{y+n} - \bar{a}_{x:y+n}\right)$$

$$= \bar{a}_y - v^n\ {}_np_y\ \bar{a}_{y+n} + v^n\ {}_np_y\ \bar{a}_{x:y+n} - \bar{a}_{xy}$$

$$= \bar{a}_{y:\overline{n}|} + v^n\ {}_np_y\ \bar{a}_{x:y+n} - \bar{a}_{xy}$$

Question

Ralph and Betty are both aged 70 exact. Upon Betty's death, Ralph will receive £20,000 *pa* payable annually in advance starting from the end of the year of Betty's death and ceasing on Ralph's death. Ralph will receive a maximum of 20 payments.

Ralph's mortality follows PMA92C20, Betty's mortality follows PFA92C20 and the interest rate for all future years is 4% *pa*.

Calculate the EPV of this benefit to Ralph.

Solution

We have:

$$\ddot{a}^{m}_{70:\overline{20}|} + v^{20}\ {}_{20}p^{m}_{70}\ \ddot{a}^{f\ m}_{70:90} - \ddot{a}_{70:70}$$

$$= 11.562 - 1.04^{-20} \times \frac{2{,}675.203}{9{,}238.134} \times 4.527 + 1.04^{-20} \times \frac{2{,}675.203}{9{,}238.134} \times 4.339 - 9.766$$

$$= 1.771154$$

So, the EPV to Ralph is:

$$£20{,}000 \times 1.77115 = £35{,}423$$

Type 5 – an annuity payable to (*y*) on the death of (*x*) and guaranteed for *n* years

The expected present value of this benefit is:

$$\bar{A}^{1}_{x:y}\,\bar{a}_{\overline{n}|}+v^{n}\,{}_{n}p_{y}\,\bar{a}_{x|y+n}$$

Question

Prove this result.

Solution

We can write the expected present value of this benefit as an integral by considering time t to be the instant of (*x*)'s death, which is when the annuity begins. This gives a factor of ${}_{t}p_{x}\,\mu_{x+t}\,dt$.

The annuity is payable to (*y*) for life thereafter, but for a minimum of *n* years. Hence:

- (*y*) has to be alive at time *t* (with probability ${}_{t}p_{y}$), and
- the expected present value of the guaranteed annuity from that time is $\bar{a}_{\overline{y+t:n}|}$.

We then need to discount the payments to time zero, using v^t , and integrate over all times t at which (*x*) can die, to give:

$$\int_0^\infty v^t\,{}_{t}p_{x}\,\mu_{x+t}\,{}_{t}p_{y}\,\bar{a}_{\overline{y+t:n}|}\,dt=\int_0^\infty v^t\,{}_{t}p_{x}\,\mu_{x+t}\,{}_{t}p_{y}\left(\bar{a}_{\overline{n}|}+v^{n}\,{}_{n}p_{y+t}\,\bar{a}_{y+t+n}\right)dt$$

If we multiply out the brackets, then the first term is:

$$\bar{a}_{\overline{n}|}\int_0^\infty v^t\,{}_{t}p_{x}\,\mu_{x+t}\,{}_{t}p_{y}\,dt=\bar{a}_{\overline{n}|}\,\bar{A}^{1}_{x:y}$$

The second term can be written as:

$$\int_0^\infty v^{t+n}\,{}_{t}p_{x}\,\mu_{x+t}\,{}_{t+n}p_{y}\,\bar{a}_{y+t+n}\,dt=v^{n}\,{}_{n}p_{y}\int_0^\infty v^t\,{}_{t}p_{x}\,\mu_{x+t}\,{}_{t}p_{y+n}\,\bar{a}_{y+t+n}\,dt$$

$$=v^{n}\,{}_{n}p_{y}\,\bar{a}_{x|y+n}$$

So the overall EPV is:

$$\bar{A}^{1}_{x:y}\,\bar{a}_{\overline{n}|}+v^{n}\,{}_{n}p_{y}\,\bar{a}_{x|y+n}$$

The first term in the above expression is the expected present value of the guaranteed benefit, which is paid to (y) following the death of (x). The second term is the expected present value of the benefit paid to (y) once the n-year guarantee period has elapsed.

Question

Ralph and Ted are both aged 60 exact. Upon Ted's death, Ralph will receive £20,000 *pa* for the rest of his life payable annually in advance starting from the end of the year of Ted's death. The payments to Ralph are guaranteed for 5 years.

Ralph's mortality and Ted's mortality both follow PMA92C20 and the interest rate for all future years is 4% *pa*.

Given that $A_{60:60} = 0.47585$ and $A_{60:65} = 0.51084$, where both lives follow PMA92C20, calculate the EPV of this benefit to Ralph.

Solution

We need to calculate:

$$A^{1}_{60:60}\,\ddot{a}_{\overline{5}|} + v^5\,{}_5p_{60}\,\ddot{a}_{60|65}$$

The first term in the expression above relates to the 5 guaranteed annuity payments, the first of which is made at the end of the year of Ted's death. We can think of the value of these as an assurance that pays a benefit of $\ddot{a}_{\overline{5}|}$ at the end of the year of Ted's death, provided Ted dies first. The second term relates to the annual annuity payments made after the 5-year guarantee period if Ralph is alive.

Since, by symmetry:

$$A^{1}_{60:60} = 0.5A_{60:60}$$

we have, using premium conversion:

$$0.5A_{60:60}\,\ddot{a}_{\overline{5}|} + v^5\,{}_5p_{60}\left(\ddot{a}_{65} - \ddot{a}_{60:65}\right)$$

$$= 0.5A_{60:60}\,\ddot{a}_{\overline{5}|} + v^5\,{}_5p_{60}\left(\ddot{a}_{65} - \frac{1 - A_{60:65}}{d}\right)$$

$$= 0.5 \times 0.47585 \times \frac{1 - 1.04^{-5}}{0.04/1.04} + 1.04^{-5} \times \frac{9{,}647.797}{9{,}826.131} \times \left(13.666 - \frac{1 - 0.51084}{0.04/1.04}\right)$$

$$= 1.86648$$

So, the EPV to Ralph is:

$$£20{,}000 \times 1.86648 = £37{,}330$$

Type 6 – an annuity payable to (*y*) on the death of (*x*) and continuing for *n* years after (*y*)'s death

The expected present value of this benefit is:

$$\bar{a}_{x|y} + \bar{A}_{x:\overset{2}{y}}\,\bar{a}_{\overline{n}|}$$

The first term is the expected present value of the benefit payable after the death of (x) while (y) is still alive.

The second term is the expected present value of the annuity paid for n years following the death of (y), provided that (y) dies after x. We can think of this as an assurance that provides a lump sum payment of $\bar{a}_{\overline{n}|}$ immediately on the death of (*y*), provided (*y*) dies second.

Question

Ralph and Ted are both aged 60 exact. Upon Ted's death, Ralph will receive £20,000 *pa* payable annually in advance for the rest of his life, starting from the end of the year of Ted's death. The payments to Ralph will continue for 12 years after Ralph has died. No payments are made if Ralph dies first.

Ralph's mortality and Ted's mortality both follow PMA92C20 and the interest rate for all future years is 4% *pa*.

Given that $A_{60:60} = 0.47585$ where both lives follow PMA92C20, calculate the EPV of this benefit to Ralph.

Solution

We need to calculate:

$$\ddot{a}_{60|60} + A_{60:\overset{2}{60}}\,\ddot{a}_{\overline{12}|}$$

Here we use a factor of $A_{60:\overset{2}{60}}$ *as the first annuity payment is made at the end of the year of Ted's death.*

Since, by symmetry:

$$A_{60:\overset{2}{60}} = 0.5A_{\overline{60:60}}$$

we have, using premium conversion:

$$\ddot{a}_{60} - \ddot{a}_{60:60} + 0.5A_{\overline{60:60}}\,\ddot{a}_{\overline{12}|} = \ddot{a}_{60} - \frac{1 - A_{60:60}}{d} + 0.5A_{\overline{60:60}}\,\ddot{a}_{\overline{12}|}$$

Now:

$$\begin{aligned} A_{\overline{60:60}} &= A_{60} + A_{60} - A_{60:60} \\ &= 2\left(1 - \frac{0.04}{1.04} \times 15.632\right) - 0.47585 \\ &= 0.32169 \end{aligned}$$

So:

$$\begin{aligned} \ddot{a}_{60} - \frac{1 - A_{60:60}}{d} + 0.5 A_{\overline{60:60}}\, \ddot{a}_{\overline{12}|} &= 15.632 - \frac{1 - 0.47585}{0.04/1.04} + 0.5 \times 0.32169 \times \frac{1 - 1.04^{-12}}{0.04/1.04} \\ &= 3.57402 \end{aligned}$$

The EPV to Ralph is therefore:

$$£20,000 \times 3.57402 = £71,480$$

The different types of reversionary annuity above are not exhaustive. In unusual cases, where it is not obvious what formula to use, we have to think carefully to develop the correct expressions.

Question

Jack, aged 60, wants to buy a reversionary annuity. If he dies before age 65 and before his wife Vera, who is also now aged 60, she will receive an income of £10,000 *pa*. The income will be paid annually in arrears (from the end of the year of Jack's death) until Vera's 75th birthday or until her earlier death.

Calculate the single premium payable assuming PMA92C20 mortality for Jack, PFA92C20 mortality for Vera and 4% *pa* interest.

Solution

Here, we should consider the period before Jack and Vera are aged 65, and the period after they are aged 65, separately. This is because before age 65 the annuity payments can commence at any time (due to Jack's death), but after age 65, if the annuity payments have not already started, they never will.

The payments made to Vera before she is aged 65 are a Type 2 annuity, *ie* they are payments to Vera, after the death of Jack, but ceasing after 5 years. The expected present value of these payments is:

$$10,000\left(a^{f}_{60:\overline{5}|} - a^{m\ f}_{60:60:\overline{5}|}\right)$$

Payments will only be made in the period from age 65 to age 75 if Jack dies before age 65 (with probability ${}_5q_{60}^{m}$) and Vera is still alive at age 65 (with probability ${}_5p_{60}^{f}$). The EPV at age 65 of the payments made to Vera in the period from age 65 to age 75 is $10{,}000a_{65:\overline{10}|}^{f}$ and this will need to be discounted back to age 60. So the EPV of this part of the benefit is:

$$10{,}000v^5\,{}_5q_{60}^{m}\,{}_5p_{60}^{f}\,a_{65:\overline{10}|}^{f}$$

Overall, the single premium is given by:

$$P = 10{,}000\left(a_{60:\overline{5}|}^{f} - a_{60:60:\overline{5}|}^{m\ f} + v^5\,{}_5q_{60}^{m}\,{}_5p_{60}^{f}\,a_{65:\overline{10}|}^{f}\right)$$

Now:

$$a_{60:\overline{5}|}^{f} = a_{60}^{f} - v^5\,{}_5p_{60}^{f}\,a_{65}^{f}$$

$$= 15.652 - 1.04^{-5} \times \frac{9{,}703.708}{9{,}848.431} \times 13.871$$

$$= 4.419$$

$$a_{60:60:\overline{5}|}^{m\ f} = a_{60:60}^{m\ f} - v^5\,{}_5p_{60}^{m}\,{}_5p_{60}^{f}\,a_{65:65}^{m\ f}$$

$$= 13.090 - 1.04^{-5} \times \frac{9{,}647.797}{9{,}826.131} \times \frac{9{,}703.708}{9{,}848.431} \times 10.958$$

$$= 4.377$$

$$a_{65:\overline{10}|}^{f} = a_{65}^{f} - v^{10}\,{}_{10}p_{65}^{f}\,a_{75}^{f}$$

$$= 13.871 - 1.04^{-10} \times \frac{8{,}784.955}{9{,}703.708} \times 9.933$$

$$= 7.796$$

So:

$$P = 10{,}000\left[4.419 - 4.377 + 1.04^{-5} \times \left(1 - \frac{9{,}647.797}{9{,}826.131}\right) \times \frac{9{,}703.708}{9{,}848.431} \times 7.796\right]$$

$$= £1{,}566$$

4.5 Expected present values of contingent assurances that depend upon term

Only term assurances are meaningful in this context. The expected present value of an assurance payable immediately on the death of (x) within n years provided (y) is then alive can be written:

$$\bar{A}^{\,1}_{xy:\overline{n|}} = \int_{t=0}^{t=n} v^t \, {}_tp_{xy} \mu_{x+t} dt$$

with a similar expression involving summation operators if the sum assured is payable at the end of the year of death.

If the sum assured is payable at the end of the year of (x)'s death, then the expected present value is:

$$A^{1}_{xy:\overline{n|}} = \sum_{k=0}^{n-1} v^{k+1} \, {}_kp_{xy} \, q^{\;1}_{x+k:y+k}$$

5 Expected present value of annuities payable *m* times a year

In Chapter 18 we determined (for a single life status x) the approximations:

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{(m-1)}{2m}$$

and:

$$a_x^{(m)} \approx a_x + \frac{m-1}{2m}$$

The first of these formulae appears on page 36 of the *Tables*.

It is important to note that the nature of the above approximation means that the single life status x can equally be replaced by any life status, 'u' say.

In particular, with $u = xy$, we obtain the joint life annuity approximation:

$$a_{xy}^{(m)} \approx a_{xy} + \frac{m-1}{2m}$$

For a last survivor annuity we write:

$$a_{\overline{xy}}^{(m)} = a_x^{(m)} + a_y^{(m)} - a_{xy}^{(m)}$$

and use the result with statuses x, y and xy to obtain:

$$a_{\overline{xy}}^{(m)} \approx a_x + a_y - a_{xy} + \frac{m-1}{2m}$$

in its simplest form.

For a reversionary annuity we write:

$$a_{x|y}^{(m)} = a_y^{(m)} - a_{xy}^{(m)}$$

and use the result with statuses y and xy to obtain:

$$a_{x|y}^{(m)} \approx a_y - a_{xy}$$

in its simplest form. Notice that there is no 'correction term' in this case because they are cancelled out.

Similar expressions may be developed if the $m$thly annuities are temporary.

We can write, for the generalised status u:

$$a_{u:\overline{n}|}^{(m)} = a_u^{(m)} - {}_{n|}a_u^{(m)}$$

We know from above that $a_u^{(m)} \approx a_u + \dfrac{m-1}{2m}$.

Also:

$$
\begin{aligned}
{}_{n|}a_u^{(m)} &= v^n \frac{l_{u+n}}{l_u} a_{u+n}^{(m)} \\
&\approx v^n \frac{l_{u+n}}{l_u}\left(a_{u+n} + \frac{m-1}{2m}\right) \\
&= {}_{n|}a_u + \frac{m-1}{2m} v^n \frac{l_{u+n}}{l_u}
\end{aligned}
$$

Hence:

$$a_{u:\overline{n}|}^{(m)} \approx a_u + \frac{m-1}{2m} - \left({}_{n|}a_u + \frac{m-1}{2m} v^n \frac{l_{u+n}}{l_u}\right)$$

or:

$$a_{u:\overline{n}|}^{(m)} \approx a_{u:\overline{n}|} + \frac{m-1}{2m}\left(1 - v^n \frac{l_{u+n}}{l_u}\right)$$

Hence we obtain the following expressions:

$$a_{xy:\overline{n}|}^{(m)} \approx a_{xy:\overline{n}|} + \frac{m-1}{2m}\left(1 - v^n \frac{l_{x+n}\, l_{y+n}}{l_x\, l_y}\right)$$

and:

$$
\begin{aligned}
a_{\overline{xy}:\overline{n}|}^{(m)} &= a_{x:\overline{n}|}^{(m)} + a_{y:\overline{n}|}^{(m)} - a_{xy:\overline{n}|}^{(m)} \\
&\approx a_{x:\overline{n}|} + a_{y:\overline{n}|} - a_{xy:\overline{n}|} + \frac{m-1}{2m}\left(1 - v^n \frac{l_{x+n}}{l_x} - v^n \frac{l_{y+n}}{l_y} + v^n \frac{l_{x+n}\, l_{y+n}}{l_x\, l_y}\right)
\end{aligned}
$$

For a reversionary annuity which ceases in any event after n years we can write:

$$a_{y:\overline{n}|}^{(m)} - a_{xy:\overline{n}|}^{(m)} \approx a_{y:\overline{n}|} - a_{xy:\overline{n}|} + \frac{m-1}{2m}\left(v^n \frac{l_{x+n}\, l_{y+n}}{l_x\, l_y} - v^n \frac{l_{y+n}}{l_y}\right)$$

Similar expressions can be developed for annuities payable in advance and, letting $m \to \infty$, continuous annuities.

Question

Jim and Dot, both aged 60, buy an annuity payable monthly in advance for at most 20 years, where payments are made while at least one of them is alive.

Calculate the expected present value of the annuity assuming PMA92C20 mortality for Jim, PFA92C20 mortality for Dot, and interest of 4% *pa*.

Solution

The expected present value of this temporary last survivor annuity is:

$$\ddot{a}^{(12)}_{\overline{60:60}:\overline{20}|} = \ddot{a}^{(12)m}_{60:\overline{20}|} + \ddot{a}^{(12)f}_{60:\overline{20}|} - \ddot{a}^{(12)}_{60:60:\overline{20}|}$$

Now, using the approximation on page 36 of the *Tables*:

$$\ddot{a}^{(12)m}_{60:\overline{20}|} = \ddot{a}^{(12)m}_{60} - v^{20}\,{}_{20}p^m_{60}\,\ddot{a}^{(12)m}_{80}$$

$$\approx \ddot{a}^m_{60} - \frac{11}{24} - v^{20}\,{}_{20}p^m_{60}\left(\ddot{a}^m_{80} - \frac{11}{24}\right)$$

$$= 15.632 - \frac{11}{24} - 1.04^{-20} \times \frac{6,953.536}{9,826.131} \times \left(7.506 - \frac{11}{24}\right)$$

$$= 12.898$$

Similarly:

$$\ddot{a}^{(12)f}_{60:\overline{20}|} \approx \ddot{a}^f_{60} - \frac{11}{24} - v^{20}\,{}_{20}p^f_{60}\left(\ddot{a}^f_{80} - \frac{11}{24}\right)$$

$$= 16.652 - \frac{11}{24} - 1.04^{-20} \times \frac{7,724.737}{9,848.431} \times \left(8.989 - \frac{11}{24}\right)$$

$$= 13.140$$

Finally, the joint life annuity is given by:

$$\ddot{a}^{(12)}_{60:60:\overline{20}|} = \ddot{a}^{(12)}_{60:60} - v^{20}\,{}_{20}p_{60:60}\,\ddot{a}^{(12)}_{80:80}$$

$$\approx \ddot{a}_{60:60} - \frac{11}{24} - v^{20}\,{}_{20}p_{60:60}\left(\ddot{a}_{80:80} - \frac{11}{24}\right)$$

$$= 14.090 - \frac{11}{24} - 1.04^{-20} \times \frac{6,953.536}{9,826.131} \times \frac{7,724.737}{9,848.431} \times \left(5.857 - \frac{11}{24}\right)$$

$$= 12.264$$

So:

$$\ddot{a}^{(12)}_{\overline{60:60}:\overline{20}|} \approx 12.898 + 13.140 - 12.264 = 13.77$$

6 Further aspects

We round off this chapter with consideration of:

- premium conversion relationships,
- the premium payment term.

6.1 Premium conversion relationships

Earlier in the course we met the premium conversion relationships for single life policies:

$$A_x = 1 - d\ddot{a}_x$$

and: $$A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}$$

As we have seen, these have analogous equivalents for joint life and last survivor statuses. These relationships are particularly useful for calculating the value of joint life and last survivor assurances, as the *Tables* provide annuity values only.

Question

Marge and Homer, both aged exactly 55, take out a policy that provides a lump sum of £50,000 payable immediately when the second of them dies. Premiums are payable annually in advance while at least one of Marge and Homer is alive.

Calculate the annual premium for the policy assuming PMA92C20 mortality for Homer, PFA92C20 mortality for Marge, 4% *pa* interest, and no expenses.

Solution

The payment of the premiums and the payment of the benefits both depend on the last survivor status. So, the equation of value is:

$$P\ddot{a}_{\overline{55:55}} = 50{,}000\bar{A}_{\overline{55:55}}$$

The annuity function is:

$$\ddot{a}_{\overline{55:55}} = \ddot{a}^m_{55} + \ddot{a}^f_{55} - \ddot{a}_{55:55} = 17.364 + 18.210 - 16.016 = 19.558$$

By premium conversion, we have:

$$A_{\overline{55:55}} = 1 - d\,\ddot{a}_{\overline{55:55}} = 1 - \frac{0.04}{1.04} \times 19.558 = 0.24777$$

So:

$$\bar{A}_{\overline{55:55}} \approx 1.04^{0.5} \times 0.24777 = 0.25268$$

Therefore:

$$P = \frac{50{,}000 \times 0.25268}{19.558} = £645.97$$

6.2 Premium payment term

One of the principal uses of the theory developed in this chapter is to determine the premium suitable for any given assurance or annuity benefit involving two lives. However, we may then find a complication in that it is possible for the joint life function defining the premium payment annuity to be different from that defining the desired benefit.

For example, if a man took out a reversionary annuity contract to provide a pension for his wife (specified by name) after his death, it would be unreasonable to expect him to continue paying premiums if his wife died before him. So premium payment would continue only up to the first death of the pair, *ie* failure of their joint life status.

Normally for contracts involving two lives, premiums will be payable until one of the following events occurs:

- the benefit is paid,
- the term of the contract expires,
- the person paying the premium dies, or
- it becomes impossible for the benefit to be paid at any time in the future.

Question

The following assurance functions represent the factors used for valuing the benefits from annual premium insurance contracts:

(a) A_{xy}

(b) $A_{\overline{xy}}$

(c) $A_{\overset{1}{x}y}$

(d) $A_{x\overset{2}{y}}$

Specify the appropriate annuity factor to be used when valuing premiums.

Solution

(a) A_{xy} represents a benefit payable when the first death of the two lives occurs. No premiums will be paid after the first death has occurred (as the benefit will already have been paid).

So premiums will be payable until the first death occurs, *ie* while both lives are alive, and the appropriate annuity factor is $\ddot{a}_{xy}$.

(b) $A_{\overline{xy}}$ represents a benefit payable when the second death of the two lives occurs. No premiums will be paid after the second death has occurred (as the benefit has already been paid, and both policyholders are dead).

So premiums will be payable until the second death occurs, and the appropriate annuity factor is $\ddot{a}_{\overline{xy}}$.

(c) A^{1}_{xy} represents a benefit payable if (x) dies first. If (x) dies first, the benefit is paid, so no premiums will be paid after (x)'s death. If (y) dies first, the benefit can never be paid, so no premiums will be paid after (y)'s death.

Therefore premiums will be payable until the first death occurs, and the appropriate annuity factor is $\ddot{a}_{xy}$.

(d) $A^{\;2}_{xy}$ represents a benefit payable if (y) dies second. When (y) dies, either the benefit will be paid (if (x) is already dead), or the benefit will never be paid (if (x) is still alive).

So (y)'s death is the deciding factor and premiums should stop when (y) dies. The appropriate annuity factor is therefore $\ddot{a}_y$.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 23 Summary

Contingent probabilities

${}_tq^1_{xy}$ represents the probability that (x) dies within the next t years, with (y) still alive at the time of (x)'s death. It can be expressed in terms of an integral as follows:

$${}_tq^1_{xy} = \int_0^t {}_sp_x\,\mu_{x+s}\,{}_sp_y\,ds$$

${}_tq^2_{xy}$ represents the probability that (x) dies within the next t years, with (y) already dead at the time of (x)'s death. It can be expressed in terms of an integral as follows:

$${}_tq^2_{xy} = \int_0^t {}_sp_x\,\mu_{x+s}\,{}_sq_y\,ds$$

By writing ${}_sq_y = 1 - {}_sp_y$ in the integral above, we see that:

$${}_tq^2_{xy} = \int_0^t {}_sp_x\,\mu_{x+s}\left(1 - {}_sp_y\right)ds = \int_0^t {}_sp_x\,\mu_{x+s}\,ds - \int_0^t {}_sp_x\,\mu_{x+s}\,{}_sp_y\,ds = {}_tq_x - {}_tq^1_{xy}$$

When we have two lives of the same age (and we assume that their mortality is identical), we can use a symmetry argument to write:

$${}_tq^1_{xx} = ½\,{}_tq_{xx} \quad \text{and} \quad {}_tq^2_{xx} = ½\,{}_tq_{\overline{xx}}$$

Contingent assurances

The present value of a benefit of 1 payable immediately on the death of (x) provided that (y) is still alive is:

$$\bar{Z} = \begin{cases} v^{T_x} & \text{if } T_x \le T_y \\ 0 & \text{if } T_x > T_y \end{cases}$$

The expected present value of this benefit is denoted by $\bar{A}^1_{xy}$ and can be expressed in integral form as follows:

$$E\left(\bar{Z}\right) = \bar{A}^1_{xy} = \int_0^\infty v^t\,{}_tp_x\,\mu_{x+t}\,{}_tp_y\,dt$$

The variance of the present value random variable is:

$$\text{var}\left(\bar{Z}\right) = {}^2\bar{A}^1_{xy} - \left(\bar{A}^1_{xy}\right)^2$$

Similar expressions involving summation operators can be developed if the assurance is payable at the end of the year of death.

Reversionary annuities

The present value of an annuity of 1 *pa* payable continuously throughout life to (y) following the death of (x) is:

$$\bar{Y} = \begin{cases} \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_x}|} & \text{if } T_y > T_x \\ 0 & \text{if } T_y \le T_x \end{cases}$$

The expected present value of this benefit is denoted by $\bar{a}_{x|y}$ and can be expressed in integral form as:

$$E\left(\bar{Y}\right) = \bar{a}_{x|y} = \int_0^\infty v^t \ {}_tp_x \ \mu_{x+t} \ {}_tp_y \ \bar{a}_{y+t} \, dt$$

We can also write:

$$\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy}$$

Similar expressions involving summation operators can be developed if the reversionary annuity is payable at discrete intervals.

Functions with specified terms

We can adapt the relationships we have observed for single life functions to help us to calculate the corresponding joint life values. For example:

$$\ddot{a}_{xy:\overline{n}|} = \ddot{a}_{xy} - {}_np_x \times {}_np_y \times v^n \times \ddot{a}_{x+n:y+n}$$

$$A_{xy:\overline{n}|} = A^{\ \ 1}_{\overset{\frown}{xy}:\overline{n}|} + A_{xy:\overset{1}{\overline{n}|}}$$

Premium conversion formulae

$$A_{xy} = 1 - d\ddot{a}_{xy} \qquad A_{xy:\overline{n}|} = 1 - d\ddot{a}_{xy:\overline{n}|}$$

$$\bar{A}_{xy} = 1 - \delta\bar{a}_{xy} \qquad \bar{A}_{xy:\overline{n}|} = 1 - \delta\bar{a}_{xy:\overline{n}|}$$

Similar results hold for last survivor annuities and assurances.

Chapter 23 Practice Questions

23.1 Express each of the following symbols:

- in terms of the random variables T_x and T_y, and
- as an integral.

(i) $_{\infty}q_{xy}^{1}$

(ii) $\bar{A}_{xy}^{1}$

(iii) $\bar{A}_{xy}^{\;2}$

23.2 Two lives, each aged x, are subject to the same mortality table. According to this mortality table, and at a certain rate of interest, $A_x = 0.4$ and $A_{xx} = 0.6$.

Calculate the value of $A_{xx}^{\;2}$ based on this mortality table and interest rate.

23.3 Given that:

$$\mu_x = \frac{1}{100 - x} \quad \text{for} \quad 0 \le x < 100$$

calculate the value of $_{30}q_{50:60}^{\;2}$.

23.4 Calculate:

(i) $_{10}q_{xy}^{1}$

(ii) $\bar{a}_{y|x}$

assuming that:

- (x) is subject to a constant force of mortality of 0.01 *pa*
- (y) is subject to a constant force of mortality of 0.02 *pa*
- the force of interest is 0.04 *pa*.

23.5 Calculate $a_{68:58}^{(4)}$, assuming PMA92C20 mortality for the life aged 68, PFA92C20 mortality for the life aged 58, and 4% *pa* interest.

23.6 Calculate $a^{(12)}_{70|60}$ on the following basis:

Mortality:	70-year-old:	PMA92C20
	60-year-old:	PFA92C20
Interest:	4% *pa*	

23.7 Two lives, aged 40 and 44, purchase a policy from an insurance company that pays 50,000 in 20 years' time, if at least one of them is still alive at that time.

Assuming that the mortality of each life follows the AM92 Select table, and the annual effective interest rate is 6%, calculate the expected present value of the benefits from this policy.

23.8 Exam style

Two lives aged x and y take out a policy that will pay £15,000 immediately on the death of (x) provided that (y) has died at least 5 years earlier and no more than 15 years earlier.

(i) Express the present value of this benefit in terms of the random variables denoting the future lifetimes of (x) and (y). [2]

(ii) Write down an integral expression (in terms of single integrals only) for the expected present value of the benefit. [3]

(iii) Prove that the expected present value is equal to:

$$15{,}000\left[v^5 \, {}_5p_x \, \bar{A}^{\,2}_{x+5:y} - v^{15} \, {}_{15}p_x \, \bar{A}^{\,2}_{x+15:y}\right] \quad [3]$$

(iv) Describe the appropriate premium payment term for this policy, assuming premiums are to be paid annually in advance. [2]

[Total 10]

23.9 Exam style

A 65-year-old male and a 62-year-old female take out a joint whole life policy with a sum assured of £10,000 that is payable immediately on the first death. Premiums are payable monthly in advance while the policy is in force for at most 5 years.

(i) Show that the monthly premium is £100, using the basis given below. [6]

(ii) Calculate the prospective reserve at the end of the third policy year, using the basis given below. [4]

Basis:	Mortality:	PMA92C20 for the male life, PFA92C20 for the female life
	Interest:	4% *pa* effective
	Expenses:	None

[Total 10]

23.10 A male and a female, aged 60 and 64 respectively, take out a policy under which the benefits are:

Exam style

- A lump sum of £50,000 payable at the end of the year of the first death provided this occurs within 10 years.
- An annuity payable annually in advance with the first payment due to be made 10 years from the date of issue. The annuity payments will be £10,000 *pa* for as long as both lives are still alive or £5,000 while only one of them is alive.

Level premiums are payable annually in advance for at most 10 years and will cease on the first death if this occurs earlier.

Calculate the amount of the annual premium on the following basis:

Interest:	4% *pa*	
Mortality:	PMA92C20 for the male life and PFA92C20 for the female life	
Expenses:	Initial:	£750
	Renewal:	3% of each premium excluding the first

[8]

23.11 A pension scheme provides the following benefit to the spouse of a member, following the death of the member in retirement:

Exam style

A pension of £25,000 *pa* payable during the lifetime of the spouse, but ceasing 20 years after the death of the member if that is earlier. All payments are made on the anniversary of the member's retirement.

Calculate the expected present value of the spouse's benefit in the case of a female member retiring now on her 65th birthday, who has a husband aged exactly 55.

Basis:	Mortality:	PMA92C20 for the male life, PFA92C20 for the female life
	Interest:	4% *pa* effective
	Expenses:	None

[7]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 23 Solutions

23.1 (i) $_{\infty}q^{1}_{xy}$ represents the probability that (x) dies before (y).

In terms of T_x and T_y:

$$_{\infty}q^{1}_{xy} = P(T_x < T_y)$$

As an integral:

$$_{\infty}q^{1}_{xy} = \int_0^{\infty} {}_tp_x\, \mu_{x+t}\, {}_tp_y\, dt = \int_0^{\infty} {}_tp_{xy}\, \mu_{x+t}\, dt$$

(ii) $\bar{A}^{1}_{xy}$ represents the expected present value of a payment of 1 unit made immediately on the death of (x), provided (x) dies before (y).

In terms of T_x and T_y:

$$\bar{A}^{1}_{xy} = E\left[g(T_x, T_y)\right] \quad \text{where} \quad g(T_x, T_y) = \begin{cases} v^{T_x} & \text{if } T_x < T_y \\ 0 & \text{if } T_x \geq T_y \end{cases}$$

As an integral:

$$\bar{A}^{1}_{xy} = \int_0^{\infty} v^t\, {}_tp_x\, \mu_{x+t}\, {}_tp_y\, dt = \int_0^{\infty} v^t\, {}_tp_{xy}\, \mu_{x+t}\, dt$$

(iii) $\bar{A}_{x\overset{2}{y}}$ represents the expected present value of a payment of 1 unit made immediately on the death of (y), provided (y) dies after (x).

In terms of T_x and T_y:

$$\bar{A}_{x\overset{2}{y}} = E\left[g(T_x, T_y)\right] \quad \text{where} \quad g(T_x, T_y) = \begin{cases} v^{T_y} & \text{if } T_y > T_x \\ 0 & \text{if } T_y \leq T_x \end{cases}$$

As an integral:

$$\bar{A}_{x\overset{2}{y}} = \int_0^{\infty} v^t\, {}_tq_x\, {}_tp_y\, \mu_{y+t}\, dt$$

23.2 Since the two lives are the same age and have the same mortality, by symmetry we have:

$$A_{xx}^{\;2} = \tfrac{1}{2} A_{\overline{xx}} = \tfrac{1}{2}\left(A_x + A_x - A_{xx}\right) = \tfrac{1}{2}\left(0.4 + 0.4 - 0.6\right) = 0.1$$

Alternatively:

$$A_{xx}^{\;2} = A_x - A_{xx}^{\;1} = A_x - \tfrac{1}{2} A_{xx} = 0.4 - \tfrac{1}{2} \times 0.6 = 0.1$$

23.3 The probability can be expressed in integral form as follows:

$${}_{30}q_{50:60}^{\;2} = \int_0^{30} {}_tp_{50}\, \mu_{50+t}\, {}_tq_{60}\, dt$$

Now:

$$\mu_{50+t} = \frac{1}{50-t}$$

and:

$${}_tp_x = \exp\left(-\int_0^t \mu_{x+s}\, ds\right) = \exp\left(-\int_0^t \frac{1}{100-x-s}\, ds\right)$$

$$= \exp\Big[\ln\left(100-x-s\right)\Big]_0^t = \exp\left[\ln\left(\frac{100-x-t}{100-x}\right)\right]$$

$$= \frac{100-x-t}{100-x}$$

So:

$${}_tp_{50} = \frac{50-t}{50} \quad \text{and} \quad {}_tq_{60} = 1 - {}_tp_{60} = 1 - \frac{40-t}{40} = \frac{t}{40}$$

Substituting these factors into the integral gives:

$${}_{30}q_{50:60}^{\;2} = \int_0^{30} \left(\frac{50-t}{50} \times \frac{1}{50-t} \times \frac{t}{40}\right) dt = \int_0^{30} \frac{t}{2{,}000}\, dt = \left[\frac{t^2}{4{,}000}\right]_0^{30} = 0.225$$

23.4 (i) In terms of an integral, this probability is:

$$_{10}q_{xy}^{1}=\int_0^{10} {}_tp_x\,\mu_{x+t}\,{}_tp_y\,dt$$

Evaluating this gives:

$$_{10}q_{xy}^{1}=\int_0^{10} e^{-0.01t}\times 0.01\times e^{-0.02t}\,dt$$

$$=0.01\int_0^{10} e^{-0.03t}\,dt$$

$$=0.01\left[\frac{e^{-0.03t}}{-0.03}\right]_0^{10}$$

$$=\frac{0.01}{0.03}\left(1-e^{-0.3}\right)=0.08639$$

(ii) In terms of an integral, the expected present value of this reversionary annuity is:

$$\bar{a}_{y|x}=\int_0^{\infty} v^t\,{}_tp_x\,{}_tq_y\,dt=\int_0^{\infty} v^t\,{}_tp_x\,(1-{}_tp_y)\,dt$$

Evaluating this gives:

$$\bar{a}_{y|x}=\int_0^{\infty} e^{-0.04t}\,e^{-0.01t}\,(1-e^{-0.02t})\,dt$$

$$=\int_0^{\infty} e^{-0.05t}-e^{-0.07t}\,dt$$

$$=\left[\frac{e^{-0.05t}}{-0.05}-\frac{e^{-0.07t}}{-0.07}\right]_0^{\infty}$$

$$=\frac{1}{0.05}-\frac{1}{0.07}=5.7143$$

23.5 We can calculate this using the formula $a_{xy}^{(m)}\approx a_{xy}+\dfrac{m-1}{2m}$ with $m=4$:

$$a_{68:58}^{(4)}\approx a_{68:58}+\frac{3}{8}=\ddot{a}_{68:58}-1+\frac{3}{8}=11.849-1+\frac{3}{8}=11.224$$

Alternatively:

$$a_{68:58}^{(4)}=\ddot{a}_{68:58}^{(4)}-\frac{1}{4}\approx\ddot{a}_{68:58}-\frac{3}{8}-\frac{1}{4}=11.849-\frac{3}{8}-\frac{1}{4}=11.224$$

The relationship $a_{xy}^{(m)}=\ddot{a}_{xy}^{(m)}-\dfrac{1}{m}$ *holds because the only difference between the annuity payable in arrears and the annuity payable in advance is the very first payment of* $\dfrac{1}{m}$ *made at time 0.*

23.6 We have:

$$a^{(12)}_{70|60} = a^{(12)}_{60} - a^{(12)}_{70:60} \approx a_{60} + \frac{11}{24} - \left(a_{70:60} + \frac{11}{24}\right)$$

$$= a_{60} - a_{70:60} = \ddot{a}_{60} - 1 - \left(\ddot{a}_{70:60} - 1\right) = \ddot{a}_{60} - \ddot{a}_{70:60}$$

where the 60-year-old experiences female mortality and the 70-year-old experiences male mortality.

Taking values from the *Tables*, we find:

$$a^{(12)}_{70|60} = 16.652 - 10.978 = 5.674$$

23.7 This is a 20-year pure endowment based on the last survivor status. The EPV of the benefits from this policy can be written as:

$$50{,}000 A_{\overline{[40]:[44]}:\overline{20|}}^{\ \ \ \ \ \ \ \ \ \ \ 1}$$

This can be evaluated as follows:

$$50{,}000 A_{\overline{[40]:[44]}:\overline{20|}}^{\ \ \ \ \ \ \ \ \ \ \ 1} = 50{,}000 v^{20}\ {}_{20}p_{\overline{[40]:[44]}}$$

$$= 50{,}000 v^{20}\left(1 - {}_{20}q_{[40]}\ {}_{20}q_{[44]}\right)$$

$$= 50{,}000 v^{20}\left(1 - \left(1 - \frac{l_{60}}{l_{[40]}}\right)\left(1 - \frac{l_{64}}{l_{[44]}}\right)\right)$$

$$= 50{,}000(1.06)^{-20}\left(1 - \left(1 - \frac{9{,}287.2164}{9{,}854.3036}\right)\left(1 - \frac{8{,}934.8771}{9{,}811.4473}\right)\right)$$

$$= 15{,}510$$

23.8 (i) ***Present value random variable***

For the benefit to be paid, (x) must die between 5 years and 15 years after (y).

Letting Z denote the present value of the benefit, we have:

$$Z = \begin{cases} 15{,}000 v^{T_x} & \text{if } T_y + 5 < T_x < T_y + 15 \\ 0 & \text{otherwise} \end{cases}$$

[Total 2]

(ii) ***Integral expression for expected present value***

We can start by considering the expected present value of a benefit paid immediately on the death of (x) provided that (y) died at least 5 years earlier (*ie* ignoring the 15-year condition for the time being). This is:

$$15{,}000\int_5^\infty v^t\,{}_tp_x\,\mu_{x+t}\,{}_{t-5}q_y\,dt$$

The lower limit on this integral is 5, as the benefit cannot be paid during the first 5 years.

Similarly, the expected present value of a benefit paid immediately on the death of (x) provided that (y) died at least 15 years earlier is:

$$15{,}000\int_{15}^\infty v^t\,{}_tp_x\,\mu_{x+t}\,{}_{t-15}q_y\,dt$$

The benefit we require is the difference between these two (*ie* it is paid out provided that (y) died at least 5 years earlier but no more than 15 years earlier). So:

$$E[Z]=15{,}000\left\{\int_5^\infty v^t\,{}_tp_x\,\mu_{x+t}\,{}_{t-5}q_y\,dt-\int_{15}^\infty v^t\,{}_tp_x\,\mu_{x+t}\,{}_{t-15}q_y\,dt\right\}$$ [3]

Alternative solution

If (x) dies between time 0 and time 5, no benefit will be paid (as (y) cannot have died at least 5 years earlier).

If (x) dies between time 5 and time 15, the benefit is payable on the death of (x) at time t provided (y) is dead by time $t-5$. (In this case, (y) cannot have died more than 15 years before (x)). The expected present value of the benefit paid in this time interval is then:

$$15{,}000\int_5^{15} v^t\,{}_tp_x\,\mu_{x+t}\,{}_{t-5}q_y\,dt$$

If (x) dies after time 15, the benefit is payable on the death of (x) at time t provided (y) died between time $t-15$ and time $t-5$. The expected present value of the benefit paid in this time interval is then:

$$15{,}000\int_{15}^\infty v^t\,{}_tp_x\mu_{x+t}\left({}_{t-5}q_y-{}_{t-15}q_y\right)dt$$

The overall expected present value of the benefit can therefore be written as:

$$E[Z]=15{,}000\left\{\int_5^{15} v^t\, {}_tp_x\, \mu_{x+t}\, {}_{t-5}q_y dt+\int_{15}^{\infty} v^t\, {}_tp_x \mu_{x+t}\left({}_{t-5}q_y-{}_{t-15}q_y\right)dt\right\}$$

This simplifies to the same expression as before as follows:

$$E[Z]=15{,}000\left\{\int_5^{15} v^t\, {}_tp_x\, \mu_{x+t}\, {}_{t-5}q_y dt+\int_{15}^{\infty} v^t\, {}_tp_x \mu_{x+t}\, {}_{t-5}q_y dt-\int_{15}^{\infty} v^t\, {}_tp_x \mu_{x+t}\, {}_{t-15}q_y dt\right\}$$

$$=15{,}000\left\{\int_5^{\infty} v^t\, {}_tp_x\, \mu_{x+t}\, {}_{t-5}q_y dt-\int_{15}^{\infty} v^t\, {}_tp_x \mu_{x+t}\, {}_{t-15}q_y dt\right\}$$

[Total 3]

(iii) ***Proof***

Considering in the first integral in the expression:

$$E[Z]=15{,}000\left\{\int_5^{\infty} v^t\, {}_tp_x\, \mu_{x+t}\, {}_{t-5}q_y\, dt-\int_{15}^{\infty} v^t\, {}_tp_x\, \mu_{x+t}\, {}_{t-15}q_y\, dt\right\}$$

and making the substitution $t=s+5$, we see that:

$$\int_5^{\infty} v^t\, {}_tp_x\, \mu_{x+t}\, {}_{t-5}q_y\, dt=\int_0^{\infty} v^{s+5}\, {}_{s+5}p_x\, \mu_{x+s+5}\, {}_sq_y\, ds$$

$$=v^5\, {}_5p_x \int_0^{\infty} v^s\, {}_sp_{x+5}\, \mu_{x+s+5}\, {}_sq_y\, ds$$

$$=v^5\, {}_5p_x\, \bar{A}^{\,2}_{x+5:y} \qquad [1½]$$

Similarly:

$$\int_{15}^{\infty} v^t\, {}_tp_x\, \mu_{x+t}\, {}_{t-15}q_y\, dt=v^{15}\, {}_{15}p_x\, \bar{A}^{\,2}_{x+15:y} \qquad [½]$$

Hence:

$$E[Z]=15{,}000\left[v^5\, {}_5p_x\, \bar{A}^{\,2}_{x+5:y}-v^{15}\, {}_{15}p_x\, \bar{A}^{\,2}_{x+15:y}\right] \qquad [1]$$

[Total 3]

(iv) ***Premium payment term***

For premiums to be payable for this policy, we require:

- (x) to be alive (since the benefit is paid on (x)'s death), and [1]
- either (y) to be alive, or (y) to have died no more than 15 years ago. [1]

[Total 2]

23.9 (i) ***Monthly premium***

Let P denote the monthly premium. Then the equation of value is:

$$12P\ddot{a}^{(12)}_{65:62:\overline{5|}} = 10{,}000\bar{A}_{65:62}$$ [1]

Now:

$$\ddot{a}^{(12)}_{65:62:\overline{5|}} = \ddot{a}^{(12)}_{65:62} - v^5 \times \frac{l_{70}}{l_{65}} \times \frac{l_{67}}{l_{62}} \times \ddot{a}^{(12)}_{70:67}$$ [1]

$$\ddot{a}^{(12)}_{65:62} \approx \ddot{a}_{65:62} - \frac{11}{24} = 12.427 - \frac{11}{24} = 11.969$$ [½]

$$\ddot{a}^{(12)}_{70:67} \approx \ddot{a}_{70:67} - \frac{11}{24} = 10.233 - \frac{11}{24} = 9.775$$ [½]

and:

$$v^5 \times \frac{l_{70}}{l_{65}} \times \frac{l_{67}}{l_{62}} = 1.04^{-5} \times \frac{9{,}238.134}{9{,}647.797} \times \frac{9{,}605.483}{9{,}804.173} = 0.77108$$ [½]

So:

$$\ddot{a}^{(12)}_{65:62:\overline{5|}} \approx 11.969 - 0.77108 \times 9.775 = 4.432$$ [½]

We need to use the premium conversion relationship to determine the value of the benefit:

$$\bar{A}_{65:62} \approx (1+i)^{½} A_{65:62} = 1.04^{½}(1 - d\,\ddot{a}_{65:62}) = 1.04^{½}\left(1 - \frac{0.04}{1.04} \times 12.427\right) = 0.53238$$ [1]

So the monthly premium is:

$$P = \frac{10{,}000 \times 0.53238}{12 \times 4.432} = £100$$ [1]

[Total 6]

(ii) ***Prospective reserve after 3 years***

The reserve after three years will be:

$$_3V = 10{,}000\bar{A}_{68:65} - 12P\ddot{a}^{(12)}_{68:65:\overline{2}|}$$ [1]

where:

$$\ddot{a}^{(12)}_{68:65:\overline{2}|} = \ddot{a}^{(12)}_{68:65} - v^2 \times \frac{l_{70}}{l_{68}} \times \frac{l_{67}}{l_{65}} \times \ddot{a}^{(12)}_{70:67}$$ [½]

Now:

$$\ddot{a}^{(12)}_{68:65} \approx \ddot{a}_{68:65} - \frac{11}{24} = 11.112 - \frac{11}{24} = 10.654$$ [½]

$$\ddot{a}^{(12)}_{70:67} \approx 9.775 \text{ from (i)}$$

and:

$$v^2 \times \frac{l_{70}}{l_{68}} \times \frac{l_{67}}{l_{65}} = 1.04^{-2} \times \frac{9{,}238.134}{9{,}440.717} \times \frac{9{,}605.483}{9{,}703.708} = 0.89556$$ [½]

So:

$$\ddot{a}^{(12)}_{68:65:\overline{2}|} \approx 10.654 - 0.89556 \times 9.775 = 1.900$$ [½]

Also:

$$\bar{A}_{68:65} \approx (1+i)^{½} A_{68:65} = (1+i)^{½}\left(1 - d\,\ddot{a}_{68:65}\right) = 1.04^{½}\left(1 - \frac{0.04}{1.04} \times 11.112\right) = 0.58396$$ [½]

So the reserve at time 3 is:

$$10{,}000 \times 0.58396 - 1{,}200 \times 1.900 = £3{,}560$$ [½]

[Total 4]

23.10 Let P denote the annual premium. The expected present value of the premiums is:

$$P\ddot{a}_{60:64:\overline{10}|} = P\left(\ddot{a}_{60:64} - v^{10}\,{}_{10}p_{60}\,{}_{10}p_{64}\,\ddot{a}_{70:74}\right)$$ [1]

where the 60-year-old is subject to male mortality and the 64-year-old is subject to female mortality.

So, using the *Tables*:

$$P\ddot{a}_{60:64:\overline{10}|} = \left(13.325 - 1.04^{-10} \times \frac{9{,}238.134}{9{,}826.131} \times \frac{8{,}937.791}{9{,}742.640} \times 9.005\right)P = 8.078P$$ [1]

The expected present value of the expenses is:

$$750+0.03P\left(\ddot{a}_{60:64:\overline{10|}}-1\right)=750+0.03P\times 7.078=750+0.2123P \qquad [1]$$

The expected present value of the benefit payable on death is $50{,}000A^{\;1}_{\overset{\frown}{60:64}:\overline{10|}}$. We can evaluate this as follows, using premium conversion to find the endowment assurance value:

$$50{,}000A^{\;1}_{\overset{\frown}{60:64}:\overline{10|}}=50{,}000\left(A_{60:64:\overline{10|}}-A_{60:64:\overset{\;\;1}{\overline{10|}}}\right)$$

$$=50{,}000\left(1-d\,\ddot{a}_{60:64:\overline{10|}}-v^{10}\,{}_{10}p_{60}\,{}_{10}p_{64}\right) \qquad [1]$$

Using the *Tables*, and the value of the temporary annuity factor calculated earlier, the EPV of the death benefit is:

$$50{,}000\left(1-\frac{0.04}{1.04}\times 8.078-1.04^{-10}\times\frac{9{,}238.134}{9{,}826.131}\times\frac{8{,}937.791}{9{,}742.640}\right)=5{,}331.81 \qquad [1]$$

Alternatively, the EPV of the death benefit can be calculated as:

$$50{,}000A^{\;1}_{\overset{\frown}{60:64}:\overline{10|}}=50{,}000\left(A_{60:64}-v^{10}\,{}_{10}p_{60}\,{}_{10}p_{64}A_{70:74}\right)$$

$$=50{,}000\left(1-d\,\ddot{a}_{60:64}-v^{10}\,{}_{10}p_{60}\,{}_{10}p_{64}\left(1-d\,\ddot{a}_{70:74}\right)\right)$$

to obtain the same answer.

The deferred annuity is £10,000 *pa* while both lives are alive or £5,000 *pa* if only one life is alive. This is equivalent to each life receiving a single life annuity of £5,000 (since £10,000 will be paid in total if both are alive). [1]

So the expected present value of this benefit is:

$$5{,}000\left(v^{10}\,{}_{10}p_{60}\,\ddot{a}_{70}+v^{10}\,{}_{10}p_{64}\,\ddot{a}_{74}\right)$$

$$=5{,}000\times 1.04^{-10}\left(\frac{9{,}238.134}{9{,}826.131}\times 11.562+\frac{8{,}937.791}{9{,}742.640}\times 11.333\right)$$

$$=71{,}835.77 \qquad [1]$$

Setting the EPV of premiums equal to the EPV of the benefits and expenses gives:

$$8.078P=5{,}331.81+71{,}835.77+750+0.2123P$$

$$\Rightarrow\quad 7.866P=77{,}917.58$$

$$\Rightarrow\quad P=£9{,}906 \qquad [1]$$

[Total 8]

23.11 *This is an example of a 'Type 4' reversionary annuity, as described in Section 4.4.*

The expected present value of the spouse's benefit is:

$$25{,}000\left(a_{65|55} - v^{20}\,{}_{20}p_{55}\,a_{65|75}\right) \qquad [3]$$

The terms in this expression are:

$$a_{65|55} = a_{55}^{m} - a_{55:65}^{m\ f} = (\ddot{a}_{55}^{m} - 1) - (\ddot{a}_{55:65}^{m\ f} - 1) = \ddot{a}_{55}^{m} - \ddot{a}_{55:65}^{m\ f} = 17.364 - 13.880 = 3.484 \qquad [1]$$

$$a_{65|75} = a_{75}^{m} - a_{75:65}^{m\ f} = (\ddot{a}_{75}^{m} - 1) - (\ddot{a}_{75:65}^{m\ f} - 1) = \ddot{a}_{75}^{m} - \ddot{a}_{75:65}^{m\ f} = 9.456 - 8.833 = 0.623 \qquad [1]$$

and:

$$v^{20}\,{}_{20}p_{55} = 1.04^{-20} \times \frac{8{,}405.160}{9{,}904.805} = 0.38729 \qquad [1]$$

So the expected present value of the benefit is:

$$25{,}000\left(3.484 - 0.38729 \times 0.623\right) = £81{,}068 \qquad [1]$$

[Total 7]

To work out where to start with a question like this, it can help to imagine that the annuity benefit is payable continuously, and then write down and simplify an integral expression, as follows:

$$\int_0^\infty {}_tp_{65}\,\mu_{65+t}\,{}_tp_{55}\,v^t\,\bar{a}_{55+t:\overline{20|}}\,dt$$

$$= \int_0^\infty {}_tp_{65}\,\mu_{65+t}\,{}_tp_{55}\,v^t\left(\bar{a}_{55+t} - v^{20}\,{}_{20}p_{55+t}\,\bar{a}_{75+t}\right)dt$$

$$= \int_0^\infty {}_tp_{65}\,\mu_{65+t}\,{}_tp_{55}\,v^t\,\bar{a}_{55+t}\,dt - v^{20}\,{}_{20}p_{55}\int_0^\infty {}_tp_{65}\,\mu_{65+t}\,{}_tp_{75}\,v^t\,\bar{a}_{75+t}\,dt$$

$$= \bar{a}_{65|55} - v^{20}\,{}_{20}p_{55}\,\bar{a}_{65|75}$$

Converting the continuous annuities to annual annuities gives us the required formula:

$$a_{65|55} - v^{20}\,{}_{20}p_{55}\,a_{65|75}$$

End of Part 4

What next?

1. Briefly **review** the key areas of Part 4 and/or re-read the **summaries** at the end of Chapters 20 to 23.
2. Ensure you have attempted some of the **Practice Questions** at the end of each chapter in Part 4. If you don't have time to do them all, you could save the remainder for use as part of your revision.
3. Attempt **Assignment X4**.

24

Mortality profit

Syllabus objectives

6.3 Define and calculate, for a single policy or a portfolio of policies (as appropriate):

(a) death strain at risk

(b) expected death strain

(c) actual death strain

(d) mortality profit

for policies with death benefits payable immediately on death or at the end of the year of death; for policies paying annuity benefits at the start of the year or on survival to the end of the year; and for policies where single or non-single premiums are payable.

0 Introduction

In Section 5 of Chapter 21 we described the recursive relation between reserves and how an expression for the profit earned over a particular year could be derived from this.

In this chapter we look at that part of the profit earned during the year that is due to mortality, referred to as *mortality profit*.

In Chapter 21 it was shown that, if the experience exactly follows the reserve basis, then, on average, the income and outgo in each policy year are equal.

In this case 'outgo' includes the increase in reserves. When talking about outgo from the insurance company's point of view, we consider reserves as money for policyholders. Hence increase in reserves is a form of outgo.

If the experience does not follow the assumptions, then there will either be an excess of income over outgo (a profit, or surplus) or an excess of outgo over income (a loss or negative profit). Profits and losses may arise from any element of the reserve basis. For example:

1. **If the interest earned is greater than that assumed in the reserve, then the income will accumulate to more than the sum required to cover the cost of the benefits and the year-end reserve, giving an interest surplus.**

2. **If the policyholder decides to surrender his or her policy (that is, to cease paying premiums, and take some lump sum in respect of the future benefits already paid for) then the year-end outgo is not as assumed. If the lump sum is less than the reserve there will be a surrender profit. If no surrender benefit is paid, the profit will be equal to the reserve.**

3. **If the experienced mortality is heavier than that assumed in the basis, then there will be a profit or loss from mortality, depending on the nature of the contract. Where benefits are paid out on death, such as under a term assurance, lighter mortality than assumed will give rise to a profit. Where benefits are paid out on survival, such as under an annuity, then lighter mortality will give rise to a loss.**

So, if experience is not as assumed, profits or losses will arise. Exactly the same principle applies in pension schemes; surpluses and deficits arise because experience is not in line with the actuary's view of future experience.

Here we consider mortality profit only. We assume, therefore, that in all elements other than mortality, experienced rates follow the assumed rates exactly.

In practice, they do not; and each of the above will give rise to profits or losses. The impact of each element may be quantified. This procedure is known as analysis of surplus.

1 Mortality profit on a single policy

1.1 Death strain at risk (DSAR)

Consider a policy issued t years ago to a life then aged x, with sum assured S payable at the end of the year of death. Also, assume that no survival benefit is due if the life survives to $t+1$. (We will extend these ideas to death benefits payable immediately on death, and to survival benefits, in later sections). Let ${}_tV$ be the reserve at time t. Then we define the *death strain* in the policy year t to $t+1$ to be the random variable, DS, say,

$$\text{DS} = \begin{cases} 0 & \text{if the life survives to } t+1 \\ \left(S - {}_{t+1}V\right) & \text{if the life dies in the year } [t,\ t+1) \end{cases}$$

The maximum death strain, $\left(S - {}_{t+1}V\right)$ is called the *death strain at risk* or *DSAR*.

If we think of the reserve as money already set aside for the policyholder, the death strain at risk for the current policy year is the amount of extra money that the company would need to pay if the policyholder died during that policy year. The death strain at risk is sometimes also called the sum at risk.

The word *strain* is used loosely to mean a cost to the company. The reasoning behind the DSAR definition is seen more clearly if we rearrange the recursive relationship between ${}_tV$ and ${}_{t+1}V$.

Recall from Chapter 21 the general recursive relationship between gross premium reserves ${}_tV'$ and ${}_{t+1}V'$:

$$\left({}_tV' + G - e\right)\left(1+i\right) = q_{x+t}\left(S+f\right) + p_{x+t}\,{}_{t+1}V'$$

There is a similar recursive relationship that applies to net premium reserves, which involves the net premium P and ignores all expenses (e and f in the above formula).

Assuming level net premiums for simplicity:

$$\begin{aligned}\left({}_tV + P\right)\left(1+i\right) &= q_{x+t}\,S + p_{x+t}\,{}_{t+1}V \\ &= q_{x+t}\,S + (1-q_{x+t})\,{}_{t+1}V \\ &= {}_{t+1}V + q_{x+t}(S - {}_{t+1}V)\end{aligned}$$

In words, the reasoning is that for each policy we must pay out at least ${}_{t+1}V$ at the end of the year. In addition, if the policy becomes a claim during the year, with probability q_{x+t}, then we must pay out an extra sum of $\left(S - {}_{t+1}V\right)$ which is the DSAR. Note that q_{x+t} is the probability of dying in the year t to $t+1$, and therefore $x+t$ is the age at the *start* of the year.

1.2 Expected death strain (EDS)

The expected amount of the death strain is called the *expected death strain* (*EDS*). This is the amount that the life insurance company expects to pay in addition to the year-end reserve for the policy. The probability of claiming in the policy year t to $t+1$ is q_{x+t} so that:

$$\text{EDS} = q_{x+t}\left(S - {}_{t+1}V\right)$$

1.3 Actual death strain (ADS)

The *actual death strain* is simply the observed value at $t+1$ of the death strain random variable, that is:

$$\text{ADS} = \begin{cases} 0 & \text{if the life survived to } t+1 \\ \left(S - {}_{t+1}V\right) & \text{if the life died in the year } [t,\ t+1) \end{cases}$$

1.4 Mortality profit

The *mortality profit* is defined as:

Mortality profit = Expected Death Strain – Actual Death Strain

The EDS is the amount the company *expects* to pay out, in addition to the year-end reserve for a policy. The ADS is the amount it *actually* pays out, in addition to the year-end reserve. If it actually pays out less than it expected to pay, there will be a profit. If the actual strain is greater than the expected strain, there will be a loss.

2 Mortality profit on a portfolio of policies

We are often interested in analysing the experience of a group of similar policies. We use the term *portfolio of policies* to mean any group of policies. In this case we simply sum the EDS and the ADS over all the relevant policies. If all lives are the same age, and subject to the same mortality table, this gives:

$$\textbf{Total DSAR} = \sum_{\text{all policies}} \left(S - {}_{t+1}V\right)$$

$$\textbf{Total EDS} = \sum_{\text{all policies}} q_{x+t}\left(S - {}_{t+1}V\right)$$

$$= q_{x+t}\left(\sum_{\text{all policies}} \left(S - {}_{t+1}V\right)\right)$$

$$= q_{x+t}\left(\text{total DSAR}\right)$$

$$\textbf{Total ADS} = \sum_{\text{death claims}} \left(S - {}_{t+1}V\right)$$

$$\textbf{Mortality Profit} = \text{total EDS} - \text{total ADS}$$

If the policies are identical, then:

Total EDS = expected number of deaths $\times$ DSAR

Total ADS = actual number of deaths $\times$ DSAR

In many situations the DSAR of each individual policy is not known, but the total DSAR is simply the total sum assured less the total year-end reserve, and the total EDS is:

$$q_{x+t}\left(\text{total DSAR}\right)$$

In the above equations:

- the summation is over all policies that are in force at the *start* of the year
- the mortality rate q_{x+t} relates to the age of the policyholder(s) at the *start* of the year
- the reserves ${}_{t+1}V$ are calculated as at the *end* of the year.

Question

A life insurance company has a portfolio of 10,000 single premium one-year term assurances. For each policy, there is a sum assured of \$50,000 payable at the end of the year if the policyholder dies during the year. The company assumes that mortality will be 1% *pa*.

(i) Calculate the expected death strain for this portfolio.

(ii) Given that 89 people die during the year, calculate the actual death strain and hence the mortality profit or loss for this portfolio.

Solution

Since this is a one-year policy, no reserves will be required at the end of the year. The death strain at risk for each policy is therefore $50,000.

(i) The expected number of deaths is $10,000 \times 0.01 = 100$ and the expected death strain is $100 \times 50,000 = \$5m$.

(ii) The actual death strain is $89 \times 50,000 = \$4.45m$.

The mortality profit is the difference between these, *ie* $550,000.

The insurance company has made a profit here because the actual number of deaths (89) was less than the expected number (100).

3 Allowing for death benefits payable immediately

Where death benefits are payable immediately on death, in the calculation of the death strain we allow for interest between the time of payment and the end of the year of death. In this case, the death strain defined in Section 1.1 would become:

$$DS = \begin{cases} 0 & \text{if the life survives to } t+1 \\ \left(S(1+i)^{1/2} - {}_{t+1}V\right) & \text{if the life dies in the year } [t,\ t+1) \end{cases}$$

The death strain formula requires the value of the death benefit payment as at the *end* of the year of death. As the sum assured is paid out during the year, the value of this payment will increase with interest between the date of death and the end of the year of death. The above formula therefore assumes that death occurs half way through the year, on average.

Question

Explain how the formula for the death strain should be adjusted if the sum assured is paid two months after the actual date of death.

Solution

The death strain formula would change to:

$$DS = \begin{cases} 0 & \text{if the life survives to } t+1 \\ S\left(1+i\right)^{4/12} - {}_{t+1}V & \text{if the life dies in the year } [t,\ t+1) \end{cases}$$

This is because (assuming deaths occur half way through the year on average) the payment of the sum assured occurs 8/12 through the year, on average. So, in order to revalue the payment to the end of the year of death, we need to accumulate the payment with 4 months' interest.

Similar adjustments would be applied to the formulae in Sections 1.2, 1.3 and 2.

Question

Consider the following group of whole life assurance policies:

year of issue: 2011
number in force at the policy anniversary in 2016: 1,900
number in force at the policy anniversary in 2017: 1,867
exact age at the policy anniversary in 2016: 70
sum assured: 60,000 per policy, payable immediately on death
level premiums are payable annually in advance for the whole of life

Calculate the mortality profit for this group of policies for the policy year commencing at the policy anniversary in 2016, assuming death is the only cause of policy termination, and that the insurer holds net premium reserves for these contracts calculated assuming AM92 Ultimate mortality and 4% *pa* interest.

Solution

The death strain at risk for a single policy is calculated as:

$$DSAR = 60,000 \times (1+i)^{\frac{1}{2}} - {}_6V$$

where ${}_6V$ is the reserve held at the policy anniversary in 2017 (which is at exact duration 6).

Now:

$${}_6V = 60,000\bar{A}_{71} - P\ddot{a}_{71}$$

where:

$$P = \frac{60,000\bar{A}_{65}}{\ddot{a}_{65}} \approx \frac{60,000 \times 1.04^{\frac{1}{2}} \times 0.52786}{12.276} = 2,631.05$$

using $\bar{A}_x \approx (1+i)^{\frac{1}{2}} A_x$.

So:

$${}_6V \approx 60,000 \times 1.04^{\frac{1}{2}} \times A_{71} - 2,631.05\ddot{a}_{71}$$

$$= 60,000 \times 1.04^{\frac{1}{2}} \times 0.61548 - 2,631.05 \times 9.998 = 11,354.90$$

and the death strain at risk is:

$$DSAR = 60,000 \times 1.04^{\frac{1}{2}} - 11,354.90 = 49,833.33$$

The expected death strain for this group of policies is:

$$EDS = 1,900 \times q_{70} \times DSAR = 1,900 \times 0.024783 \times 49,833.33$$

$$= 2,346,537$$

During the policy year, 33 people died, so the actual death strain for the group of policies is:

$$ADS = 33 \times DSAR = 33 \times 49,833.33 = 1,644,500$$

This gives a mortality profit of:

$$EDS - ADS = £702,037$$

4 Allowing for survival benefits

Suppose the contract provides for a benefit at the end of policy year t to $t+1$. By convention, the expected present value of this will have been included in ${}_tV$ but will fall outside the computation of ${}_{t+1}V$. So, the survival benefit needs to be allowed for as an additional payment.

Let R be the benefit payable at the end of the policy year t to $t+1$ contingent on the survival of the policyholder. Assuming death benefits are paid at the end of the year of death, the recursive relationship between successive reserves is now:

$$\left({}_tV+P\right)(1+i)=q_{x+t}S+p_{x+t}({}_{t+1}V+R)$$

$$=q_{x+t}S+(1-q_{x+t})({}_{t+1}V+R)$$

$$={}_{t+1}V+R+q_{x+t}(S-({}_{t+1}V+R))$$

The DS is now:

$$\text{DS}=\begin{cases}0 & \text{if the life survives to } t+1\\ \left(S-\left({}_{t+1}V+R\right)\right) & \text{if the life dies in the year } [t,\ t+1)\end{cases}$$

as the office must pay out:

- **$\left({}_{t+1}V+R\right)$ for all policyholders, and an additional**
- **$S-\left({}_{t+1}V+R\right)$ for policies becoming claims by death.**

So the DSAR for a single policy is $S-\left({}_{t+1}V+R\right)$.

The expected death strain is then $q_{x+t}\left[S-\left({}_{t+1}V+R\right)\right]$; the actual death strain is 0 if the life survived the year and $S-\left({}_{t+1}V+R\right)$ if the life died during the year; the mortality profit is EDS − ADS, as before.

Question

Repeat the question at the end of Section 2, assuming that the survivors are paid a lump sum benefit of \$20,000 at the end of the year.

Solution

For policyholders who die during the year we need the sum assured of \$50,000 at the end of the year. So in the above formula $S=50{,}000$.

For policyholders who survive we need the endowment payment of \$20,000. So $R=20{,}000$.

As the reserve at the end of the year is zero, the death strain at risk is now:

$$50,000 - 20,000 = \$30,000$$

The expected death strain is:

$$0.01 \times 10,000 \times 30,000 = 3,000,000$$

and the actual death strain is:

$$89 \times 30,000 = 2,670,000$$

Therefore the mortality profit is:

$$3,000,000 - 2,670,000 = \$330,000\,.$$

In the next question, the only benefit payable is on survival to the end of the term of the contract.

Question

On 1 January 2009 a life insurance company issued a number of 30-year pure endowment contracts to lives then aged 35. Level premiums are payable annually in advance throughout the term of the contract or until earlier death. In each case, the only benefit is a sum assured of £20,000, payable on survival to the end of the term.

On 1 January 2017, 600 policies were still in force. During 2017, 3 policyholders died. Assuming that the company holds net premium policy reserves, calculate the profit or loss from mortality for calendar year 2017 in respect of this group of policies.

Basis:	Mortality:	AM92 Ultimate
	Interest:	4% *pa*

Solution

The reserve at 31 December 2017, *ie* at time 9, is:

$${}_9V = \text{EPV future benefits} - \text{EPV future premiums}$$

$$= 20,000\frac{D_{65}}{D_{44}} - P\ddot{a}_{44:\overline{21|}}$$

$$= 20,000 \times \frac{689.23}{1,747.41} - P \times 14.233$$

$$= 7,888.59 - 14.233P$$

In the above formula:

$$P=\frac{20{,}000\dfrac{D_{65}}{D_{35}}}{\ddot{a}_{35:\overline{30}|}}=\frac{20{,}000\times\dfrac{689.23}{2{,}507.40}}{17.629}=£311.85$$

So:

$${}_9V=£3{,}450.06$$

Since there is no benefit payable on death during the year, or on survival to the end of the year:

$$DSAR=-{}_9V=-£3{,}450.06$$

The expected death strain is:

$$EDS=600q_{43}\times DSAR=-600\times 0.001208\times 3{,}450.06=-£2{,}500.60$$

The actual death strain is:

$$ADS=3\times DSAR=-£10{,}350.17$$

So the mortality profit is:

$$-2{,}500.60-\left(-10{,}350.17\right)=£7{,}850$$

4.1 Annuities

In the case of an annuity of R *pa*, payable annually in arrears, with no death benefit, the DSAR would be $-\left({}_{t+1}V+R\right)$. In this case each death causes a negative strain or *release of reserves*.

Question

At the start of a particular year a life insurance company had a portfolio of 5,000 female pensioners, all aged exactly 60, who each receive an income of £10,000 per annum, paid annually in arrears.

The company holds net premium reserves, calculated using PFA92C20 mortality and 4% *pa* interest.

During that year, 9 pensioners died. Calculate the mortality profit or loss for that year.

Solution

For policyholders who die during the year, no funds are required at the end of the year.

The reserve required at the end of the year for each surviving policy plus the annuity payment due at that time is:

$$10,000a_{61}+10,000=10,000\ddot{a}_{61}=10,000\times 16.311=£163,110$$

So the death strain at risk is $0-163,110=-£163,110$.

From the *Tables*, $q_{60}=0.002058$. So the expected number of deaths during the year is $5,000q_{60}=10.29$.

So the EDS is $-£163,110\times 10.29=-£1,678,402$.

The ADS is $-£163,110\times 9=-£1,467,990$.

So the mortality profit is:

$$EDS-ADS=-1,678,402-(-1,467,990)=-£210,412$$

ie a loss of about £210,400. (The loss arises because fewer people died than expected.)

In the case of an annuity of R *pa*, payable annually in advance, with no death benefit, the DSAR would be $-{}_{t+1}V$ only. The annuity payment is made by all policies in force at the start of the year, and is not affected by whether or not the policyholder survives the year.

The annuity payment due at time $t+1$ is paid at the start of the *next* year, and is therefore included in ${}_{t+1}V$. If we took the DSAR to be $-\left({}_{t+1}V+R\right)$ for the current year we would be double-counting the annuity payment.

5 Allowing for different premium or annuity payment frequencies

The above formulae for the death strain and mortality profit are appropriate where premiums are either paid annually in advance or as a single payment at the outset of the policy.

Where premiums (or annuities) are paid more frequently than annually, the formulae for the death strain and mortality profit will be different, because the death or survival of the policyholder during the year will affect how many premiums are actually received, or how many annuity payments are actually made. These variations are beyond the scope of the CM1 syllabus.

6 Calculation of mortality profit for policies involving two lives

We shall now show how we can calculate the mortality profit for policies that each involve two lives. Essentially we can use the same methods as before (although the calculations can sometimes be more complex).

For life assurance policies where the death benefit is payable on the first death of two people, the calculations are fairly simple, and almost identical to those used for a single life policy. Let's look at an example.

Question

A company sells joint life whole of life assurances to male lives aged 60 exact and female lives aged 58 exact. The sum assured is £80,000, payable at the end of the year of death of the first life to die. Level premiums are payable annually in advance whilst the policy is in force.

(i) Calculate the annual premium for these policies.

Basis: Mortality: PMA92C20 for the male life and PFA92C20 for the female life

Interest: 4% *pa* effective

Expenses: Nil

A group of these policies was sold on 1 January 2006. On 1 January 2014, 200 of these policies were still in force. Of these policies, two made a claim during 2014.

(ii) Calculate the profit or loss from mortality for calendar year 2014, using the premium basis given above.

Solution

(i) ***Premium***

The premium equation is:

$$P\ddot{a}_{60:58} = 80{,}000 A_{60:58}$$

The annuity is tabulated ($\ddot{a}_{60:58} = 14.393$), and the assurance function is calculated using premium conversion:

$$A_{60:58} = 1 - d\,\ddot{a}_{60:58} = 1 - \frac{0.04}{1.04} \times 14.393 = 0.44642$$

So the premium is:

$$P = \frac{80{,}000 \times 0.44642}{14.393} = £2{,}481.33$$

(ii) ***Mortality profit or loss***

We first need to calculate the reserve needed for each policy on 31 December 2014, *ie* at time 9. At this point the lives will be aged 69 and 67, so the prospective reserve is:

$$_9V = 80{,}000 A_{69:67} - 2{,}481.33\,\ddot{a}_{69:67}$$

The joint life annuity is in the *Tables* ($\ddot{a}_{69:67} = 10.526$), and the assurance function is calculated using premium conversion:

$$A_{69:67} = 1 - d\,\ddot{a}_{69:67} = 1 - \frac{0.04}{1.04} \times 10.526 = 0.59515$$

So the reserve is:

$$_9V = 80{,}000 \times 0.59515 - 2{,}481.33 \times 10.526 = 21{,}493.83$$

So the death strain at risk is:

$$S - {}_9V = 80{,}000 - 21{,}493.83 = 58{,}506.17$$

The probability that a policy will become a claim during 2014 is the probability that at least one of the policyholders dies, which is one minus the probability that they both survive. Using the policyholders' ages at the *start* of 2014, the probability of a claim is:

$$1 - p_{68}^{m} \times p_{66}^{f} = 1 - \left(1 - 0.009930\right)\left(1 - 0.005467\right) = 0.015343$$

So the expected number of claims during the year from this cohort of policies is:

$$200 \times 0.015343 = 3.06854$$

and the actual number of claims is 2. So we can find the expected death strain (EDS) and the actual death strain (ADS):

$$EDS = 3.06854 \times 58{,}506.17 = 179{,}529$$

$$ADS = 2 \times 58{,}506.17 = 117{,}012$$

The mortality profit is the difference between these:

$$= 179{,}529 - 117{,}012 = £62{,}516$$

For policies where the death benefit is payable on the second death, the situation is more complex.

Consider a last survivor assurance where both lives are still alive at the start of the year. If both lives die during the year, the death strain at risk is, as usual, the sum assured less the reserve that would have been held at the end of the year if both lives had survived.

Suppose, however, that only one of the lives dies. There is still a mortality cost even though the sum assured has not been paid out, because the death of one of the policyholders means that a higher reserve will be needed at the end of the year compared to what it would have been if both people had survived. The death strain at risk in this case is therefore calculated as the amount by which the end-year reserve increases as a result of the particular death that occurs.

An example should help to make this clear.

Question

A company sells last survivor whole of life assurances to male lives aged 60 exact and female lives aged 58 exact. The sum assured is £80,000, payable at the end of the year of death of the second life to die. Level premiums (calculated using the basis below) of £1,237.61 are paid annually in advance whilst the policy is in force.

Basis: Mortality: PMA92C20 for the male life and PFA92C20 for the female life

Interest: 4% *pa* effective

Expenses: Nil

A group of these policies was sold on 1 January 2006. On 1 January 2014, there were 1,000 policies still in force where both lives were still alive. Of these policies, during 2014:

- there was one policy for which both lives died
- there were two policies where the female life died (but the male life did not)
- there were three policies where the male life died (but the female life did not).

Calculate the profit or loss from mortality for calendar year 2014, using the premium basis.

Solution

We first calculate the reserve at the end of 2014, assuming that both lives are still alive. Working prospectively, this is:

$$ {}_9V = 80{,}000 A_{\overline{69:67}} - 1{,}237.61\ddot{a}_{\overline{69:67}} $$

The last survivor annuity is calculated from the single and joint life annuities:

$$ \ddot{a}_{\overline{69:67}} = \ddot{a}_{69} + \ddot{a}_{67} - \ddot{a}_{69:67} = 11.988 + 14.111 - 10.526 = 15.573 $$

The last survivor assurance can be calculated using premium conversion:

$$ A_{\overline{69:67}} = 1 - d\,\ddot{a}_{\overline{69:67}} = 1 - \frac{0.04}{1.04} \times 15.573 = 0.40104 $$

So the reserve is:

$$ {}_9V = 80{,}000 \times 0.40104 - 1{,}237.61 \times 15.573 = £12{,}809.78 $$

We now consider separately the different events that occurred.

Both lives die

Consider a policy for which both lives die during 2014. If neither life had died, the reserve needed would have been £12,809.78. If both die, we need to pay the sum assured of £80,000. So the death strain at risk (DSAR) is the difference between these two figures:

$$DSAR = 80,000 - 12,809.78 = 67,190.22$$

The expected number of policies for which both lives die (using the ages at the start of 2014) is:

$$1,000 \times q_{68}^{m} \times q_{66}^{f} = 1,000 \times 0.009930 \times 0.005467 = 0.05429$$

So the expected death strain (EDS) for these policies is:

$$EDS = 0.05429 \times 67,190.22 = 3,647.58$$

The actual number of policies for which both lives died is one, so the actual death strain (ADS) is just 67,190.22. So the mortality profit (MP) for this group of policies is:

$$MP = EDS - ADS = 3,647.58 - 67,190.22 = -£63,542.65$$

(These figures are quite sensitive to rounding.)

Male life dies, female survives

If this event had not happened, so that neither life had died during the year, we would again have needed the last survivor reserve of £12,809.78. If the event does happen, we will need the single life reserve for the female life only:

$$\begin{aligned} {}_9V^{\text{female alive}} &= 80,000 A_{67}^{f} - 1,237.61\, \ddot{a}_{67}^{f} \\ &= 80,000 \times \left(1 - \frac{0.04}{1.04} \times 14.111\right) - 1,237.61 \times 14.111 = 19,117.62 \end{aligned}$$

So the DSAR for these policies is the difference between these two reserves:

$$DSAR = 19,117.62 - 12,809.78 = 6,307.85$$

The expected number of policies for which only the male dies is:

$$1,000 \times q_{68}^{m} \times p_{66}^{f} = 1,000 \times 0.009930 \times \left(1 - 0.005467\right) = 9.87571$$

So the EDS for these policies is:

$$EDS = 9.87571 \times 6,307.85 = 62,294.49$$

The actual number of policies for which only the male dies is three, so the ADS is:

$$ADS = 3 \times 6,307.85 = 18,923.54$$

So the MP for this group of policies is:

$$MP = EDS - ADS = 62{,}294.49 - 18{,}923.54 = £43{,}370.95$$

Female life dies, male life survives

Again, if this event had not happened during the year, we would have needed the last survivor reserve of £12,809.78. If the event does happen, we will need the single life reserve for the male life only:

$$\begin{aligned} {}_9V^{\text{male alive}} &= 80{,}000A_{69}^{m} - 1{,}237.61\,\ddot{a}_{69}^{m} \\ &= 80{,}000 \times \left(1 - \frac{0.04}{1.04} \times 11.988\right) - 1{,}237.61 \times 11.988 = 28{,}277.38 \end{aligned}$$

The DSAR for these policies is the difference between these two reserves:

$$DSAR = 28{,}277.38 - 12{,}809.78 = 15{,}467.60$$

The expected number of policies for which only the female dies is:

$$1{,}000 \times p_{68}^{m} \times q_{66}^{f} = 1{,}000 \times (1 - 0.009930) \times 0.005467 = 5.41271$$

So the EDS for these policies is:

$$EDS = 5.41271 \times 15{,}467.60 = 83{,}721.68$$

The actual number of policies for which only the female dies is two, so the ADS is:

$$ADS = 2 \times 15{,}467.60 = 30{,}935.20$$

So the MP for this group of policies is:

$$MP = EDS - ADS = 83{,}721.68 - 30{,}935.20 = £52{,}786.48$$

Adding together the mortality profit from the three groups of lives:

$$MP = -63{,}542.65 + 43{,}370.95 + 52{,}786.48 = £32{,}615$$

We can see that the calculation of the mortality profit for policies of this type can be quite time-consuming.

In real life, as well as the class of policy we saw in the calculation above (where both lives were alive at the start of 2014), we could have had two further classes of policy – those for which the male life had died before the start of 2014, and those for which the female life had died before the start of 2014. However, both these classes of policy are effectively single life policies by now, and so the mortality profit is calculated in the usual way for policies of this type.

Question

In the previous question, the company paid out more claims than expected (1 actual claim as opposed to the 0.05 policies that are expected to claim during the year).

Explain briefly why the insurance company has still made an overall profit from mortality during the year.

Solution

For the policies where both lives die, there was a higher claim rate than expected, so the mortality here is heavier than expected.

However, for the policies where only one life died, mortality was lighter than expected (we expected 9.88 policies with the male life only dying, but observed only 3; we expected 5.41 policies with the female life only dying, but observed only 2).

So this lighter than expected mortality is sufficient to enable us to make an overall mortality profit.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes

Chapter 24 Summary

Single life and joint life (first death) policies

In general:

$$\text{DSAR} = S - R - {}_{t+1}V$$

$$\text{Total EDS} = \sum_{\text{all policies}} q_{x+t} \times \text{DSAR}$$

$$\text{Total ADS} = \sum_{\text{all deaths}} (S - R - {}_{t+1}V)$$

$$\text{Mortality profit} = \text{Total EDS} - \text{Total ADS}$$

where:

S is the sum assured paid on death during the year (revalued to the end of the year)

R is the sum paid on survival to the end of the year

${}_{t+1}V$ is the reserve at the end of the year

$x+t$ is the age at the start of the year

the EDS is calculated by summing over all policies in force at the *start* of the year.

Last survivor policies

To calculate the mortality profit for a last survivor policy, we need the death strain at risk for the different classes of lives subject to the different possible death events:

Death event	***Death strain at risk***
Both lives die	$S - {}_{t+1}V^{\text{both alive}}$
x dies, y survives	${}_{t+1}V^{y\text{ alive}} - {}_{t+1}V^{\text{both alive}}$
y dies, x survives	${}_{t+1}V^{x\text{ alive}} - {}_{t+1}V^{\text{both alive}}$

The EDS then uses the relevant probability for the death event, and the ADS is worked out using the actual numbers of each event that have occurred.

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes

Chapter 24 Practice Questions

Exam style

All of the following questions are exam style.

24.1 On 1 January 2009, a life insurance company sold a number of 10-year pure endowment policies, each with a benefit amount of £40,000, to lives then aged 30. Level premiums are payable annually in advance.

(i) Calculate the annual premium. [3]

(ii) On 1 January 2010, there were 50 of these policies still in force. During 2010, one policyholder died. Calculate the company's mortality profit for 2010. [5]

Basis: Mortality: AM92 Select

Interest: 4% *pa* effective

Expenses: None [Total 8]

24.2 On 31 December 2009 a pension scheme had 100 members aged 75 exact, each eligible for a pension of £10,000 *pa*, payable annually on each 1 January. In addition, the members were entitled to a death benefit of £20,000 payable at the end of the year of death. No premiums were being paid in respect of these contracts after December 2009. Given that 4 of the lives died during 2010, calculate the mortality profit for these contracts for calendar year 2010 using the following basis:

Mortality: PFA92C20

Interest: 4% *pa*

Expenses: none [5]

24.3 (i) Express in the form of symbols, and also explain in words, the expressions 'death strain at risk' as it applies to a single policy, and the 'expected death strain' and 'actual death strain' for a group of policies. [3]

(ii) On 1 January 2001 a life insurance company issued a number of annual premium policies to a group of lives, each of whom was then aged exactly 45. All policies were for a term of 20 years and were of the following types:

(a) endowment assurances under which the sum assured was payable on survival to the end of the term or at the end of the year of earlier death

(b) term assurances under which the sum assured was payable only at the end of the year of death within the policy term

(c) pure endowments under which the only benefit payable is the sum assured on survival to the end of the policy term.

Assuming that there is no source of decrement other than death, calculate the profit or loss from mortality for the calendar year 2010 in respect of the policies issued to this group of lives, given the following information:

Type of policy	*Sums assured in force on 1 January 2010*	*Sums assured discontinued by death during 2010*
Endowment assurance	£600,000	£4,000
Term assurance	£200,000	£2,000
Pure endowment	£80,000	£500

Basis: Mortality: AM92 Ultimate

Interest: 4% *pa* effective

Expenses: None

[11]
[Total 14]

24.4 An insurance company issues a special single premium annuity contract, which pays £10,000 *pa* in arrears for 10 years. If the policyholder dies within the 10-year term the annuity payments cease, and a lump sum benefit is paid out immediately on death. The amount of the death benefit is calculated as:

$$100{,}000 - 10{,}000k$$

where k is the curtate duration of the policy at the time of death.

The policy cannot be terminated for any reason other than through death.

1,500 of these policies were issued during a particular year to lives who were all aged exactly 55 when they took out the policy. It is now more than four years since the most recent of these policies was issued.

The mortality experience to date of this group of policyholders is given as follows:

- number of policyholders receiving exactly 2 or fewer annuity payments = 8
- number of policyholders receiving exactly 3 annuity payments = 4
- number of policyholders receiving exactly 4 or more annuity payments = 1,488

Calculate the mortality profit earned for the insurance company in the fourth policy year of this block of business, on the following basis:

Mortality: AM92 Ultimate

Interest: 4% *pa*

Expenses: None [10]

24.5 Under a 10-year 'double endowment' assurance policy issued to a group of lives aged 50, a sum assured of £10,000 is payable at the end of the year of death and £20,000 is paid if the life survives to the maturity date. Premiums are payable annually in advance.

You are given the following information:

reserve at the start of the 8th year (per policy in force):	£12,951
number of policies in force at the start of the 8th year:	200
number of deaths during the 8th year:	3
annual premium (per policy)	£1,591

(i) Assuming that reserves are calculated according to the basis specified below, calculate the profit or loss arising from mortality in the 8th year. [5]

(ii) Comment on your results. [1]

Basis: Mortality: ELT15 (Males)

Interest: 4% *pa* effective

Expenses: None [Total 6]

24.6 On 1 January 2012, a life insurance company issued joint life whole life assurance policies. Each policy was issued to a male life aged 65 exact and a female life aged 60 exact. A sum assured of 75,000 is payable immediately on the death of the second of the lives to die.

Premiums of 1,395.11 are payable annually in advance for each policy while at least one of the lives is alive.

At the beginning of 2014, there were 5,997 policies in force. For all of these policies, both lives were still alive. During 2014, the following experience was observed:

- for 2 policies, both lives died
- for 12 policies, only the male life died
- for 8 policies, only the female life died.

Calculate, showing all your workings, the mortality profit or loss for the group of policies for the calendar year 2014.

Basis: Mortality: PMA92C20 for the male
PFA92C20 for the female

Interest: 4% per annum

Expenses: Ignore [10]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 24 Solutions

24.1 (i) ***Annual premium***

Let P denote the annual premium. Then:

$$\text{EPV premiums} = P\ddot{a}_{[30]:\overline{10}|} = P\left(\ddot{a}_{[30]} - \frac{D_{40}}{D_{[30]}}\ddot{a}_{40}\right)$$ [½]

From the *Tables*:

$$\frac{D_{40}}{D_{[30]}} = \frac{2,052.96}{3,059.68} = 0.67097$$ [½]

$$\ddot{a}_{[30]} = 21.837 \text{ and } \ddot{a}_{40} = 20.005$$ [½]

So:

$$\text{EPV premiums} = P(21.837 - 0.67097 \times 20.005) = 8.414P$$ [½]

Also:

$$\text{EPV benefits} = 40,000\frac{D_{40}}{D_{[30]}} = 26,838.89$$ [½]

So the annual premium is:

$$P = \frac{26,838.89}{8.414} = £3,189.71$$ [½]

[Total 3]

(ii) ***Mortality profit***

The reserve per policy in force at the end of 2010 is:

$${}_2V = 40,000\frac{D_{40}}{D_{32}} - 3,189.71\ddot{a}_{32:\overline{8}|}$$ [½]

From the *Tables*:

$$\frac{D_{40}}{D_{32}} = \frac{2,052.96}{2,825.89} = 0.72648$$ [½]

and:

$$\ddot{a}_{32:\overline{8}|} = \ddot{a}_{32} - \frac{D_{40}}{D_{32}}\ddot{a}_{40} = 21.520 - 0.72648 \times 20.005 = 6.987$$ [1]

So:

$${}_2V = 40,000 \times 0.72648 - 3,189.71 \times 6.987 = 6,773.71 \qquad [½]$$

There is no death benefit, so the death strain at risk (DSAR) for calendar year 2010 is:

$$DSAR = S - {}_2V = 0 - 6,773.71 = -6,773.71 \qquad [½]$$

The expected death strain (EDS) is:

$$EDS = 50q_{[30]+1} \times DSAR = 50 \times 0.000569 \times \left(-6,773.71\right) = -192.71 \qquad [1]$$

and the actual death strain (ADS) is:

$$ADS = 1 \times DSAR = -6,773.71 \qquad [½]$$

So the company's mortality profit for calendar year 2010 is:

$$EDS - ADS = -192.71 - \left(-6,773.71\right) = £6,581 \qquad [½]$$

[Total 5]

24.2 The death strain at risk (DSAR) for a single contract for calendar year 2010 is:

$$20,000 - {}_{31.12.10}V \qquad [½]$$

where:

$$\begin{aligned} {}_{31.12.10}V &= 10,000\ddot{a}_{76} + 20,000A_{76} \\ &= 10,000\ddot{a}_{76} + 20,000\left(1 - d\ddot{a}_{76}\right) \\ &= 10,000 \times 10.536 + 20,000\left(1 - \frac{0.04}{1.04} \times 10.536\right) \\ &= 117,255.38 \end{aligned} \qquad [2]$$

So the DSAR is:

$$20,000 - 117,255.38 = -97,255.38 \qquad [½]$$

The expected number of deaths during 2010 is:

$$100\,q_{75} = 100 \times 0.019478 = 1.9478 \qquad [½]$$

So the expected death strain is:

$$EDS = 100\,q_{75} \times DSAR = -189,434.04 \qquad [½]$$

The actual number of deaths during 2010 is 4. So the actual death strain is:

$$ADS = 4 \times DSAR = -389,021.54 \qquad [½]$$

Hence the mortality profit for the group of policies for 2010 is:

$$EDS - ADS = £199,587$$ [½]

[Total 5]

24.3 (i) ***Definitions***

The 'death strain at risk' for a policy for year $t+1$ ($t = 0,1,2,\dots$) is the excess of the sum assured (*ie* the value at time $t+1$ of all benefits payable on death during year $t+1$) over the end of year reserve. [1]

The 'expected death strain' for a group of policies for year $t+1$, is the total death strain that would be incurred in respect of all policies in force at the start of year $t+1$ if deaths conformed to the numbers expected.

$$\text{EDS for year } t+1 = \sum_{\substack{\text{policies in force}\\ \text{at start of year}}} q(S - {}_{t+1}V)$$ [1]

The 'actual death strain' for a group of policies for year $t+1$ is the total death strain incurred in respect of all claims actually arising during year $t+1$.

$$\text{ADS for year } t+1 = \sum_{\text{claims during year}} (S - {}_{t+1}V)$$ [1]

[Total 3]

(ii) ***Mortality profit***

The premiums per unit sum assured for the three types of policies can be found as follows:

$$P_a \ddot{a}_{45:\overline{20}|} = A_{45:\overline{20}|} \Rightarrow P_a = 0.46998 / 13.780 = 0.03411$$ [1]

$$P_b \ddot{a}_{45:\overline{20}|} = A^{1}_{45:\overline{20}|} \Rightarrow P_b = 0.05923 / 13.780 = 0.00430$$ [1]

$$P_c = P_a - P_b = 0.02981$$ [½]

The reserves at the end of the year per unit sum assured are:

$${}_{10}V_a = A_{55:\overline{10}|} - P_a \ddot{a}_{55:\overline{10}|} = 0.68388 - 0.03411 \times 8.219 = 0.4036$$ [1]

$${}_{10}V_b = A^{1}_{55:\overline{10}|} - P_b \ddot{a}_{55:\overline{10}|} = 0.06037 - 0.00430 \times 8.219 = 0.02504$$ [1]

$${}_{10}V_c = \frac{D_{65}}{D_{55}} - P_c \ddot{a}_{55:\overline{10}|} = 0.62351 - 0.02981 \times 8.219 = 0.3785$$ [1]

The total expected death strain is:

$$EDS = EDS_a + EDS_b + EDS_c$$

$$= q_{54}[600,000(1 - {}_{10}V_a) + 200,000(1 - {}_{10}V_b) + 80,000(0 - {}_{10}V_c)]$$

$$= 0.003976[600,000(1 - 0.4036) + 200,000(1 - 0.02504) + 80,000(-0.3785)]$$

$$= 2,080$$ [3]

The total actual death strain is:

$$ADS = ADS_a + ADS_b + ADS_c$$

$$= 4,000(1 - 0.4036) + 2,000(1 - 0.02504) + 500(-0.3785)$$

$$= 4,150$$ [2]

So there is a profit of $2,080 - 4,150 = -2,070$, *ie* a loss of £2,070. [½]

[Total 11]

24.4 If the policyholder dies in the fourth policy year, the curtate duration will equal 3. So the death strain at risk in the fourth policy year is:

$$DSAR = \left(100,000 - 3 \times 10,000\right) \times (1+i)^{½} - 10,000 - {}_4V$$ [2]

where ${}_4V$ is the reserve at the end of year 4. Now:

$${}_4V = 70,000 \bar{A}^{\ 1}_{59:\overline{6}|} - 10,000 \left(I\bar{A}\right)^{1}_{59:\overline{6}|} + 10,000 a_{59:\overline{6}|}$$ [1]

where:

$$\bar{A}^{\ 1}_{59:\overline{6}|} \approx (1+i)^{½} \left(A_{59} - \frac{D_{65}}{D_{59}} A_{65} \right)$$

$$= 1.04^{½} \times \left(0.44258 - \frac{689.23}{924.76} \times 0.52786 \right)$$

$$= 0.050136$$ [1]

$$\left(I\bar{A}\right)^{1}_{59:\overline{6}|} \approx (1+i)^{½} \left\{ (IA)_{59} - \frac{D_{65}}{D_{59}} \left[(IA)_{65} + 6A_{65} \right] \right\}$$

$$= 1.04^{½} \times \left(8.42588 - \frac{689.23}{924.76} \times \left[7.89442 + 6 \times 0.52786 \right] \right)$$

$$= 0.185205$$ [1½]

$$a_{59:\overline{6}|} = \left(\ddot{a}_{59} - 1 \right) - \frac{D_{65}}{D_{59}} \left(\ddot{a}_{65} - 1 \right) = 13.493 - \frac{689.23}{924.76} \times 11.276 = 5.089$$ [1]

So:

$${}_4V = 70,000 \times 0.050136 - 10,000 \times 0.185205 + 10,000 \times 5.089 = 52,546.66$$ [½]

Hence:

$$DSAR = 70,000 \times 1.04^{½} - 10,000 - 52,546.66 = 8,839.61$$ [½]

8 of the initial 1,500 policyholders received two or fewer annuity payments, which means that 8 policyholders died during the first three policy years. So there were 1,492 policies still in force at the start of the fourth policy year, when the policyholders were all aged 58. [½]

So the expected death strain was:

$$EDS = 1,492 \times q_{58} \times DSAR = 1,492 \times 0.006352 \times 8,839.61 = 83,775$$ [1]

4 policyholders received exactly 3 annuity payments, which means they must have died during the fourth policy year. So the actual death strain was:

$$ADS = 4 \times 8,839.61 = 35,358$$ [½]

and the mortality profit was:

$$EDS - ADS = 83,775 - 35,358 = £48,416$$ [½]

[Total 10]

24.5 (i) ***Mortality profit***

The reserve required (per policy) at the end of the 8th year can be found from the equation of equilibrium:

$$1.04 \times \left({}_7V + P\right) = q_{57} \times 10,000 + p_{57} \times {}_8V$$ [1]

Inserting the values gives:

$$1.04 \times (12,951 + 1,591) = 0.00995 \times 10,000 + 0.99005 \times {}_8V$$

So:

$${}_8V = 15,024.18 / 0.99005 = 15,175.17$$ [1]

The expected death strain is:

$$200q_{57}(10,000 - {}_8V) = 1.99(10,000 - 15,175.17) = -10,298.59$$ [1]

The actual death strain is:

$$3(10,000 - {}_8V) = 3(10,000 - 15,175.17) = -15,525.52$$ [1]

So the mortality profit for the year is:

$$EDS - ADS = -10{,}298.59 - (-15{,}525.52) = £5{,}227$$ [1]

[Total 5]

(ii) ***Comment***

In this case the reserve exceeds the death benefit, so the company makes a profit when people die. More people than expected died, so the result is a mortality profit. [1]

24.6 *This question is Subject CT5, April 2016, Question 11.*

We need the mortality profit for calendar year 2014, at the start of which all male and female policyholders were aged exactly 67 and 62 respectively.

End-year reserve required for the death strain at risk

Let $V_{\overline{(xy)}}$ denote the reserve at the end of 2014 (*ie* at time 3) if both policyholders are then still alive. This is the reserve required where no policyholder dies during the year, and is calculated as:

$$V_{\overline{(xy)}} = 75{,}000\,\bar{A}_{\overline{68:63}} - 1{,}395.11\,\ddot{a}_{\overline{68:63}}$$ [½]

where (68) and (63) indicate male and female lives respectively.

Now:

$$\ddot{a}_{\overline{68:63}} = \ddot{a}_{68} + \ddot{a}_{63} - \ddot{a}_{68:63} = 12.412 + 15.606 - 11.372 = 16.646$$ [½]

So:

$$\bar{A}_{\overline{68:63}} \approx 1.04^{½} \times \left(1 - d\,\ddot{a}_{\overline{68:63}}\right) = 1.04^{½} \times \left(1 - \frac{0.04}{1.04} \times 16.646\right) = 0.366894$$ [½]

$$\Rightarrow \qquad V_{\overline{(xy)}} \approx 75{,}000 \times 0.366894 - 1{,}395.11 \times 16.646 = 4{,}294.05$$

Mortality profit

Policies where both policyholders die during the year

The cost to the insurer of both policyholders dying during the year, accumulated to the end of the year, is:

$$75{,}000 \times 1.04^{½}$$

So the death strain at risk for this event is:

$$DSAR_{(\text{both})} = 75{,}000 \times 1.04^{½} - 4{,}294.05 = 72{,}191.24$$ [½]

The expected death strain, totalled over all 5,997 policies in force at the start of the year, is:

$$\begin{aligned} EDS_{(\text{both})} &= 5{,}997 \times q_{67} \times q_{62} \times DSAR_{(\text{both})} \\ &= 5{,}997 \times 0.008439 \times 0.002885 \times 72{,}191.24 \\ &= 10{,}540.36 \end{aligned}$$ [½]

where (67) and (62) denote male and female lives respectively.

The actual death strain is:

$$ADS_{(\text{both})} = 2 \times DSAR_{(\text{both})} = 2 \times 72{,}191.24 = 144{,}382.48$$ [½]

The mortality profit is then:

$$\begin{aligned} MP_{(\text{both})} &= EDS_{(\text{both})} - ADS_{(\text{both})} = 10{,}540.36 - 144{,}382.48 \\ &= -133{,}842.12 \end{aligned}$$ [½]

Policies where only the male dies during the year

The cost to the insurer of just the male life dying during the year, is the reserve at the end of the year where only the female is alive. Denoting $V_{(y)}$ to be this reserve, we have:

$$V_{(y)} = 75{,}000\,\bar{A}_{63} - 1{,}395.11\,\ddot{a}_{63}$$ [½]

Now:

$$\bar{A}_{63} \approx 1.04^{½} \times \left(1 - d\,\ddot{a}_{63}\right) = 1.04^{½} \times \left(1 - \frac{0.04}{1.04} \times 15.606\right) = 0.407686$$ [½]

So:

$$V_{(y)} \approx 75{,}000 \times 0.407686 - 1{,}395.11 \times 15.606 = 8{,}804.38$$

So the death strain at risk for this event is:

$$DSAR_{(\text{male only})} = 8{,}804.38 - 4{,}294.05 = 4{,}510.33$$ [½]

The expected death strain is:

$$\begin{aligned} EDS_{(\text{male only})} &= 5{,}997 \times q_{67} \times \left(1 - q_{62}\right) \times DSAR_{(\text{male only})} \\ &= 5{,}997 \times 0.008439 \times \left(1 - 0.002885\right) \times 4{,}510.33 \\ &= 227{,}603.13 \end{aligned}$$ [½]

The actual death strain is:

$$ADS_{\text{(male only)}} = 12 \times DSAR_{\text{(male only)}} = 12 \times 4,510.33 = 54,123.91 \qquad [½]$$

The mortality profit is then:

$$MP_{\text{(male only)}} = EDS_{\text{(male only)}} - ADS_{\text{(male only)}}$$

$$= 227,603.13 - 54,123.91 = 173,479.22 \qquad [½]$$

Policies where only the female dies during the year

The cost to the insurer of just the female life dying during the year, is the reserve at the end of the year where only the male is alive. Denoting $V_{(x)}$ to be this reserve, we have:

$$V_{(x)} = 75,000\,\bar{A}_{68} - 1,395.11\,\ddot{a}_{68} \qquad [½]$$

where:

$$\bar{A}_{68} \approx 1.04^{½} \times \left(1 - d\,\ddot{a}_{68}\right) = 1.04^{½} \times \left(1 - \frac{0.04}{1.04} \times 12.412\right) = 0.532965 \qquad [½]$$

So:

$$V_{(x)} \approx 75,000 \times 0.532965 - 1,395.11 \times 12.412 = 22,656.29$$

So the death strain at risk for this event is:

$$DSAR_{\text{(female only)}} = 22,656.29 - 4,294.05 = 18,362.23 \qquad [½]$$

The expected death strain is:

$$EDS_{\text{(female only)}} = 5,997 \times \left(1 - q_{67}\right) \times q_{62} \times DSAR_{\text{(female only)}}$$

$$= 5,997 \times \left(1 - 0.008439\right) \times 0.002885 \times 18,362.23$$

$$= 315,010.30 \qquad [½]$$

The actual death strain is:

$$ADS_{\text{(female only)}} = 8 \times DSAR_{\text{(female only)}} = 8 \times 18,362.23 = 146,897.85 \qquad [½]$$

The mortality profit is then:

$$MP_{\text{(female only)}} = EDS_{\text{(female only)}} - ADS_{\text{(female only)}}$$

$$= 315,010.30 - 146,897.85 = 168,112.45 \qquad [½]$$

So the total mortality profit for all possible death events is:

$$MP_{\text{(both)}} + MP_{\text{(male only)}} + MP_{\text{(female only)}}$$

$$= -133,842.12 + 173,479.22 + 168,112.45 = 207,750$$ [½]

[Total 10]

25

Competing risks

Syllabus objectives

5.2 Describe and illustrate methods of valuing cashflows that are contingent upon multiple transition events.

5.2.1 Define health insurance, and describe simple health insurance premium and benefit structures.

5.2.2 Explain how a cashflow, contingent upon multiple transition events, may be valued using a multiple state Markov model, in terms of the forces and probabilities of transition.

5.2.3 Construct formulae for the expected present values of cashflows that are contingent upon multiple transition events, including simple health insurance premiums and benefits, and calculate these in simple cases. Regular premiums and sickness benefits are payable continuously and assurance benefits are payable immediately on transition.

5.3 Describe and use methods of projecting and valuing expected cashflows that are contingent upon multiple decrement events.

5.3.1 Describe the construction and use of multiple decrement tables.

5.3.2 Define a multiple decrement model as a special case of a multiple state Markov model.

5.3.3 Derive dependent probabilities for a multiple decrement model in terms of given forces of transition, assuming forces of transition are constant over single years of age.

5.3.4 Derive forces of transition from given dependent probabilities, assuming forces of transition are constant over single years of age.

0 Introduction

So far we have considered contingencies where a life is exposed to death only. If we suppose that a life is subject to more than one transition, then the transitions are referred to as a set of *competing risks*. For example, a member of a pension scheme can, in order for the associated pension scheme benefits to be valued, be regarded as exposed to the competing risks of retirement and death.

In a similar way, a person with a health insurance policy who is in good health, can be considered as exposed to the competing risks of becoming sick and dying.

Some of the ideas in this chapter involve concepts that are also covered in Subject CS2.

1 Health insurance contracts

In the same way as insurance contracts exist that pay benefits contingent upon death or survival, so contracts also exist that pay benefits contingent upon the state of health of a person. In this case a policyholder can be considered to be subject to the competing risks of death and of becoming sick.

An *income protection* insurance contract pays an income to the policyholder while that policyholder is deemed as being 'sick' (with the definition of sickness being carefully specified in the policy conditions). If the policyholder recovers, the cover under the policy usually continues, so that subsequent bouts of qualifying sickness would merit further benefit payments.

Such policies are usually subject to a *deferred period* (*eg* 3 months) of continuous sickness that has to have elapsed before any benefits start to be paid, and during which no benefit is payable.

Premiums for these policies would normally be regular (*eg* monthly) and would typically be *waived* during periods of qualifying sickness. This means that premiums would not be paid at the same time as benefits are payable.

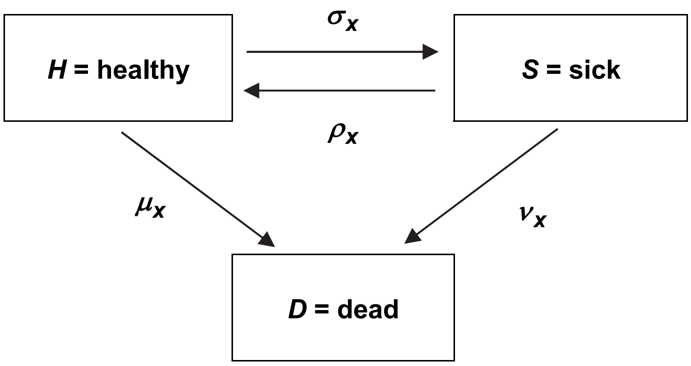
There are many varieties of insurance contracts that can be based on health-dependent contingencies. Other examples (covered in Subject SP1) include:

- ***critical illness* insurance, which normally pays a lump sum on diagnosis of a defined 'critical' illness (such as cancer); and**
- ***long-term care* insurance, which pays an income contingent upon the policyholder requiring long-term care, and hence supports the costs of receiving that care.**

2 Multiple state models

Multiple state models are well suited to valuing cashflows that are dependent on multiple transitions, such as of health insurance contracts. The model will be chosen to include the relevant states, and transitions between states, that are necessary to replicate the required cashflows for the contract concerned.

For example, for the simple income protection policy described in the previous section (with no deferred period), the general three-state healthy-sick-dead model would be suitable.



The labels on the arrows relate to the *transition intensities* from one state to another. A transition intensity can also be referred to as a *transition rate* or a *force of transition*, and so is the same type of quantity as the force of mortality that we met in earlier chapters.

Here, we use μ to refer specifically to the force of transition from the healthy state to the dead state (rather than the force of transition from 'alive' to 'dead', as is it used earlier in this course). The transition intensity from the *sick* state to the *dead* state is represented by ν (nu, the 13th letter of the Greek alphabet), not v (the 22nd letter of the English alphabet).

2.1 Notation

Let i and j denote any two different states. Define μ_x^{ij} to be the transition intensity from state i to state j at age x (so, for example, $\mu_x^{HS} = \sigma_x$ in the above model). Also define the related transition probability:

$$ {}_t p_x^{ij} = P[\text{in state } j \text{ at age } x+t \mid \text{ in state } i \text{ at age } x] $$

where now i and j need not be different.

For example, for the model shown above:

${}_t p_x^{HD}$ represents the probability that a life in the *healthy* state at age x will be in the *dead* state at age $x+t$. This probability encompasses all possible routes from *healthy* to *dead*, which may or may not include one or more visits to the *sick* state.

${}_t p_x^{SS}$ represents the probability that a life in the *sick* state at age x will be in the *sick* state at age $x+t$. This includes the probability that the life remained in the *sick* state throughout the period from x to $x+t$ and the probability that the life made one or more visits to the *healthy* state, returning on each occasion to the *sick* state.

The event whose probability is defined by the expression:

$$ {}_t p_x^{ij} = P\left[\text{in state } j \text{ at age } x+t \mid \text{ in state } i \text{ at age } x\right] $$

does not specify what must happen between age x and age $x+t$, however. In particular, if $i=j$, it does not require that the life remains in state i between these ages. So for any state i, also define the related transition probability:

$$ {}_t p_x^{\overline{ii}} = P\left[\text{in state } i \text{ from age } x \text{ to } x+t \mid \text{ in state } i \text{ at age } x\right] $$

This is sometimes referred to as the *occupancy probability*, as it relates to the probability of staying in (or occupying) state i from age x to age $x+t$.

If return to state i is impossible, then ${}_t p_x^{\overline{ii}} = {}_t p_x^{ii}$, but this is not true (for example) in the case of states H and S in the healthy-sick-dead model above.

Question

A life insurance company uses the three-state healthy-sick-dead model described above to calculate premiums for a 3-year sickness policy issued to healthy policyholders aged 60.

Let S_t denote the state occupied by the policyholder at age $60+t$, so that $S_0 = H$ and $S_t = H, S$ or D for $t = 1, 2, 3$.

The transition probabilities used by the insurer are defined in the following way:

$$ p_{60+t}^{jk} = P\left(S_{t+1} = k \mid S_t = j\right) $$

For $t = 0, 1, 2$, it is assumed that:

$$ p_{60+t}^{HH} = 0.9 \qquad p_{60+t}^{HS} = 0.08 $$

$$ p_{60+t}^{SH} = 0.7 \qquad p_{60+t}^{SS} = 0.25 $$

Calculate the probability that a new policyholder is:

(a) sick at exact age 62.

(b) dead at exact age 62.

Solution

(a) Since the policyholder is healthy at age 60, the probability that he is sick at age 62 is:

$$
\begin{aligned}
{}_2p_{60}^{HS} &= \left(p_{60}^{HH} \times p_{61}^{HS}\right) + \left(p_{60}^{HS} \times p_{61}^{SS}\right) \\
&= (0.9 \times 0.08) + (0.08 \times 0.25) \\
&= 0.092
\end{aligned}
$$

(b) The probability that the policyholder is dead at age 62 is:

$$
\begin{aligned}
{}_2p_{60}^{HD} &= p_{60}^{HD} + \left(p_{60}^{HH} \times p_{61}^{HD}\right) + \left(p_{60}^{HS} \times p_{61}^{SD}\right) \\
&= 0.02 + (0.9 \times 0.02) + (0.08 \times 0.05) \\
&= 0.042
\end{aligned}
$$

Note that $p_{60}^{HH} + p_{60}^{HS} + p_{60}^{HD} = 1$*, and a similar equation holds for lives starting from the sick state.*

In CM1, we assume that:

- **the above probabilities, as well as the transition intensities, are available**
- **the differential equation for ${}_tp_x^{\overline{ii}}$ has the closed form solution:**

$$
{}_tp_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j \neq i} \mu_{x+s}^{ij}\, ds\right)
$$

This result is particularly important in the construction of multiple decrement models.

Multiple decrement models are described in Section 3 of this chapter.

The differential equation for ${}_tp_x^{\overline{ii}}$ referred to above is derived in Subject CS2, but it is not required here. In this course, we will just use the solution provided.

In the formula for ${}_tp_x^{\overline{ii}}$, the term $\sum_{j \neq i} \mu_{x+s}^{ij}$ relates to the *total* force of transition out of state i at age $x+s$. If the forces of transition are constant over the period, and written as μ^{ij}, the formula for the occupancy probability simplifies to:

$$
{}_tp_x^{\overline{ii}} = \exp\left(-t \times \sum_{j \neq i} \mu^{ij}\right)
$$

Question

Using the three-state healthy-sick-dead model described above with transition intensities:

$$\sigma_{45+t} = 0.001t \qquad \mu_{45+t} = 0.002t$$

$$\rho_{45+t} = 0.002t^2 \qquad \nu_{45+t} = 0.01$$

calculate the probability that a life remains healthy from age 45 to age 50.

Solution

Using the form of the occupancy probability given above:

$$_5p_{45}^{\overline{HH}} = \exp\left(-\int_0^5 (\sigma_{45+t} + \mu_{45+t})\,dt\right) = \exp\left(-\int_0^5 0.003t\,dt\right)$$

Carrying out the integration gives:

$$_5p_{45}^{\overline{HH}} = \exp\left(-\left[0.0015t^2\right]_0^5\right) = e^{-0.0015\times 25} = e^{-0.0375} = 0.96319$$

2.2 Valuing continuous cashflows using multiple state models

Consider, for example, a healthy life who is subject to the competing risks of sickness and death. A multiple state model can be used to construct integral expressions for the EPVs of the following types of cashflows:

- **a lump sum paid immediately on transition from one state to another** (Type 1)
- **an income payable while occupying a particular state** (Type 2)**.**

Examples

The healthy-sick-dead model described in the previous section will be used.

1. **The EPV of a lump sum of 1 payable on death (whether directly from healthy or from having first become sick) of a healthy life currently aged x is:**

$$\int_0^\infty e^{-\delta t}\left({}_tp_x^{HH}\mu_{x+t} + {}_tp_x^{HS}\nu_{x+t}\right)dt$$

(assuming a constant force of interest δ).

This is a Type 1 cashflow. We can think about the integral as being built up as follows:

- The benefit of 1 is payable at time t if the policyholder dies at time t. This may be from the healthy state (in which case the policyholder is healthy at time t, and dies from this state at age $x+t$) or from the sick state (in which case the policyholder is sick at time t and dies from the sick state at age $x+t$).
- The benefit is then discounted back to time 0.
- The expression is then integrated over all possible points in time when death might occur.

2. The EPV of an annuity of 1 per annum payable continuously during sickness of a healthy life currently aged x is:

$$\int_0^\infty e^{-\delta t} \, {}_tp_x^{HS} \, dt$$

This is a Type 2 cashflow. Here we can think about a benefit being paid at time t provided that the life is sick at time t. So we need the probability that the currently healthy life is sick at time t. The benefit amount of 1 is then discounted back to time 0, and we integrate over all points in time at which a benefit could be paid.

3. The EPV of a premium of 1 per annum payable continuously, but waived during periods of sickness, by a healthy life currently aged x is:

$$\int_0^\infty e^{-\delta t} \, {}_tp_x^{HH} \, dt$$

This is also a Type 2 cashflow. The premium is payable at time t only by someone who is healthy at time t. So we need the probability that the currently healthy life is also healthy at time t, discounted back to time 0, and then integrated over all points in time during which the premium could be paid.

Question

Using the three-state healthy-sick-dead model and the notation defined in Section 2.1, write down an expression for the expected present value of each of the following sickness benefits for a healthy life aged 30.

(i) £3,000 *pa* payable continuously while sick, but ceasing at age 60.

(ii) £3,000 *pa* payable continuously throughout the first period of sickness only, but ceasing at age 60.

(iii) £3,000 *pa* payable continuously while sick provided that the life has been sick for at least one year. Again, any benefit ceases to be paid at age 60.

Solution

In all of these solutions v^t could be replaced by $e^{-\delta t}$.

(i) The expression for the EPV of this benefit is:

$$3{,}000\int_0^{30} v^t \, {}_tp_{30}^{HS} \, dt$$

This is very similar to Example 2 above, except that the upper limit on the integral is 30, as the payments continue for at most 30 years.

(ii) The expression for the EPV of this benefit is:

$$3{,}000\int_0^{30} v^t \, {}_tp_{30}^{\overline{HH}} \, \sigma_{30+t} \, \bar{a}^S_{30+t:\overline{30-t|}} \, dt$$

$$= 3{,}000\int_0^{30} v^t \, {}_tp_{30}^{\overline{HH}} \, \sigma_{30+t} \left(\int_0^{30-t} v^u \, {}_up_{30+t}^{\overline{SS}} \, du \right) dt$$

This can be reasoned as follows. The 'probability' that the policyholder falls sick for the first time at age $30+t$ *is* ${}_tp_{30}^{\overline{HH}} \, \sigma_{30+t}$ *(where the use of the occupancy probability ensures that this is the **first** transition to the sick state).*

Given that this has occurred, sickness benefits will be received continuously from age $30+t$ *until the policyholder recovers or reaches age 60, whichever is earlier. The expected value at time* t *of this benefit can be written as* $3{,}000\bar{a}^S_{30+t:\overline{30-t|}}$ *where the* S *superscript indicates that this annuity will be payable whilst the life remains in the sick state. The annuity can then be expressed in integral form to give* $3{,}000\int_0^{30-t} v^u \, {}_up_{30+t}^{\overline{SS}} \, du$.

Discounting back to time 0 and integrating over all possible values of t *gives the required result.*

(iii) The expression for the EPV of this benefit is:

$$3{,}000\int_0^{29} v^t \, {}_tp_{30}^{HH} \, \sigma_{30+t} \left(v \, p_{30+t}^{\overline{SS}} \, \bar{a}^S_{31+t:\overline{29-t|}} \right) dt$$

$$= 3{,}000\int_0^{29} v^t \, {}_tp_{30}^{HH} \, \sigma_{30+t} \left(\int_1^{30-t} v^u \, {}_up_{30+t}^{\overline{SS}} \, du \right) dt$$

This can be reasoned in a similar way. The 'probability' that the policyholder falls sick at time t *(ie age* $30+t$ *) is* ${}_tp_{30}^{HH} \, \sigma_{30+t}$ *(where we do not use an occupancy probability in this case, as the payments are made during **all** bouts of sickness, not just the first).*

Given that this occurs, if the policyholder stays sick for one year (ie until age $30+t+1$ *or age* $31+t$ *), then sickness benefits will be received continuously until the policyholder recovers or reaches age 60, whichever is earlier.*

The expected value at age $31+t$ *of this benefit can be written as* $3{,}000\bar{a}^{S}_{31+t:\overline{29-t|}}$ *, and the expected value at age* $30+t$ *(or time* t *) is* $3{,}000v\,p^{\overline{SS}}_{30+t}\,\bar{a}^{S}_{31+t:\overline{29-t|}}$*. Writing the annuity in integral form gives:*

$$3{,}000v\,p^{\overline{SS}}_{30+t}\int_0^{29-t} v^w\,{}_wp^{\overline{SS}}_{31+t}\,dw = 3{,}000\int_0^{29-t} v^{w+1}\,{}_{w+1}p^{\overline{SS}}_{30+t}\,dw$$

$$= 3{,}000\int_1^{30-t} v^u\,{}_up^{\overline{SS}}_{30+t}\,du$$

using the substitution $u=w+1$*. The final integral here reflects the fact that benefits will be received if the policyholder is sick at age* $30+t+u$*, where* $1\le u<30-t$*.*

Discounting back to time 0 and integrating over all possible values of t *gives the required result. If the policyholder falls sick after age 59, there will be no benefit payable, so the upper limit for t is 29.*

The actual evaluation of the above integrals may, in general, be done numerically.

For example, we could use a numerical technique such as the trapezium rule to approximate the value of an integral. The trapezium rule says that if we want to integrate a function $f(t)$ between the limits of $t=a$ and $t=b$, we divide the interval $[a,b]$ into n strips of equal length $h=\dfrac{b-a}{n}$ and use the formula:

$$\int_a^b f(t)dt \approx \frac{h}{2}\left[f(a)+2f(a+h)+2f(a+2h)+\cdots+2f(b-h)+f(b)\right]$$

Question

We consider a simple income protection insurance contract that pays at a rate of 20,000 per annum continuously while a policyholder is sick. The policy is issued to a healthy life aged 50 exact for a term of three years. Calculate an approximate EPV of the sickness benefit at outset on the following assumptions:

$${}_1p^{HS}_{50}=0.02 \qquad {}_2p^{HS}_{50}=0.04 \qquad {}_3p^{HS}_{50}=0.07$$

Interest of 3% per annum

(with symbols as defined in Section 2.1).

Solution

As an integral the EPV is:

$$\text{EPV}=\int_0^3 20{,}000e^{-\delta t}\,{}_tp^{HS}_{50}\,dt = 20{,}000\int_0^3 v^t\,{}_tp^{HS}_{50}\,dt$$

If we assume the function $v^t \, {}_tp_{50}^{HS}$ varies linearly over each policy year, we can approximate the integral using the trapezium rule:

$$\textbf{EPV} \simeq 20{,}000\left(\tfrac{1}{2}v^0 \, {}_0p_{50}^{HS} + v^1 \, {}_1p_{50}^{HS} + v^2 \, {}_2p_{50}^{HS} + \tfrac{1}{2}v^3 \, {}_3p_{50}^{HS}\right)$$

$$= 20{,}000\left(0 + \frac{0.02}{1.03} + \frac{0.04}{1.03^2} + \tfrac{1}{2} \times \frac{0.07}{1.03^3}\right)$$

$$= 1{,}783$$

(Other suitable approximations for the integral could be used).

The integral $\int_0^3 v^t \, {}_tp_{50}^{HS} dt$ has been evaluated using the form of the trapezium rule stated before the question with $h=1$, $a=0$, $b=3$ and $f(t)=v^t \, {}_tp_{50}^{HS}$.

2.3 Designing the multiple state model

An important actuarial skill is to be able to choose, or design, an appropriate model for a particular purpose or application. Here, we need to be able to design or select a multiple state model for the purpose of valuing cashflows, where the cashflows depend on life and/or health-dependent events.

Defining the model

A multiple state Markov model, of the type we have been using in this chapter, is fully defined by the following:

- the possible states that can be occupied
- the possible transitions that can be made between the states (*ie* all the arrows in the transition diagram)
- the values of the forces of transition between the states at each age.

Essentially, defining a multiple state Markov model involves drawing the transition diagram, indicating (and defining as necessary) the forces of transition that will apply on each arrow. So, for example, the transition diagram shown at the start of Section 2 above fully defines the multiple state model that is needed for valuing the cashflows covered in this chapter so far. In practice, of course, we will also need numerical values for all the transition rates in the model.

How to design a model

In order to decide on an appropriate model, we need to consider *which transitions affect the expected present value of the cashflows involved*. So, for example, the expected present value of the income benefits paid under a typical income protection policy (described in Section 1 of this chapter) will fairly obviously be affected by any:

- transition from healthy to sick (because this will cause payments to start)
- transition from sick to healthy (because this will cause payments to stop)
- transition from sick to dead (because this will cause payments to stop).

Question

State two other transitions that, in real life, might affect the expected present value of the sickness benefits for a new policyholder who is currently healthy. Assume that the policyholder is to pay regular premiums.

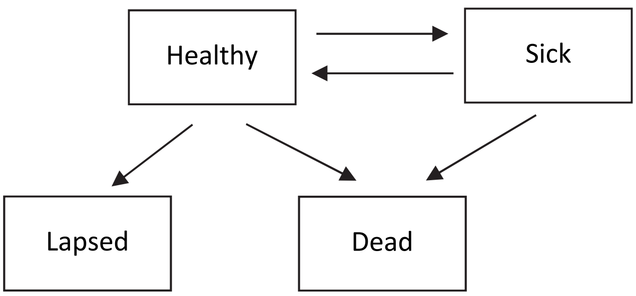
Solution

The expected present value of the sickness payments will also be affected by any:

- transition from healthy to dead
- transition from healthy to lapsed (*ie* where cover under the contract ceases as the policyholder has stopped paying premiums).

This is because both of these transitions would mean that the policyholder could not receive a sickness benefit in the future, reducing the expected present value of the sickness benefit payments.

So, in order to include all of the necessary transitions, the model will require four states (Healthy, Sick, Dead, Lapsed), with transitions between the states as shown in the following diagram:



To complete the model, we would also need to define:

$$\mu_x^{ij} = \text{force of transition from state } i \text{ to state } j \ (i \neq j) \text{ at exact age } x$$

Question

(i) Explain what would happen to the expected present value of the sickness payments if lapses were ignored in the model (and we just used the standard healthy-sick-dead model).

(ii) Give an example of a situation in which we could we justify ignoring lapses in practice.

Solution

(i) If we did not include lapses in the model, then the expected present value of the future benefit payments would be increased. This is because more policies would be assumed to stay in force to each future time point, which increases the chance of a sickness payment being made.

(ii) We might be justified in doing this if, for example, we wished to take a conservative (*ie* prudent) view of the expected present value of these benefit payments.

So, while it might be permissible to exclude lapses from the model in certain situations, *including* them allows us to value the sickness payments more accurately. That is, the model as defined above will enable us to perform a more *realistic* valuation of the sickness benefits.

Question

Explain whether adding a transition from lapsed to dead in the above model would increase the accuracy of the expected present value of the sickness benefit payments calculated.

Solution

Adding the extra transition from lapsed to dead into the model would make no difference to the expected present value of the sickness benefits calculated using the model. This is because the death of a lapsed policyholder does not affect the sickness benefit payments.

So introducing a transition from lapsed to dead would mean that the model was more complex than it needed to be for the desired purpose. Unnecessary complexity should always be avoided when designing a model, because, for example, the model may be slower and take longer to build, it may be harder to understand, be more prone to errors, require estimation of additional parameters – and so on. A model that contains no unnecessary features for its required purpose is described as *parsimonious*, and therefore parsimony is a desired feature of any model design.

In summary, an optimal design for a particular multiple state model will be achieved if we:

- include all transitions that materially affect the output from the model (*eg* the expected present value of some cashflows, a premium, or a reserve) so that we obtain the degree of accuracy we require
- exclude all transitions that do not materially affect the output from the model.

An example of this approach will be found in Section 3.1 below.

3 Multiple decrement models

A multiple decrement model is a multiple state model which has:

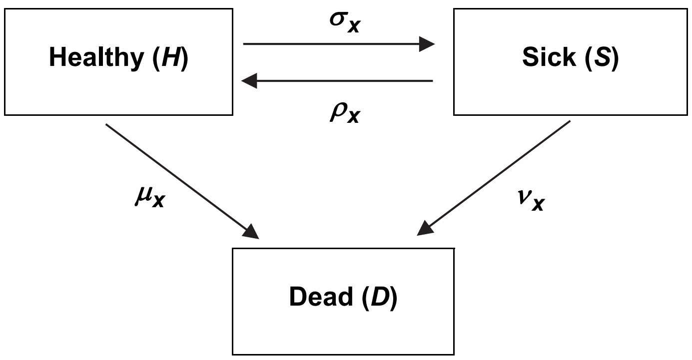
- **one active state, and**
- **one or more absorbing exit states.**

Many practical situations involving competing risks can be modelled adequately using this simplified model structure.

3.1 A simple example

We will use a simple example to illustrate the operation of multiple decrement models.

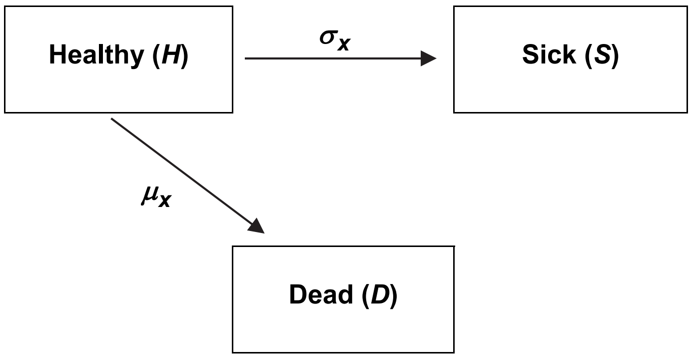
We begin with the general healthy-sick-dead multiple state model described in Section 2:



Suppose we have an insurance contract which only pays:

- **a once-only payment of X on transition from healthy to sick, or**
- **a once-only payment of Y on transition from healthy to dead.**

In this case, the following multiple decrement model should be sufficient to enable us to value these cashflows adequately:



In this multiple decrement model, H is the active state, and there are two absorbing exit states, S and D. The life H is subject to the competing risks of S and D.

3.2 Multiple decrement probabilities

In a multiple decrement model, we only need to define two types of probability.

The first is ${}_t(aq)_x^r$, which is defined as the *dependent* probability that an individual aged x in the active state will be removed from that state between ages x and $x+t$ by the decrement r. (By 'dependent', we mean *in the presence of all other risks of decrement in the population*). When $t=1$ this is written as $(aq)_x^r$.

The second is ${}_t(ap)_x$, which is defined as the dependent probability that an individual aged x in the active state will still be in the active state at age $x+t$. When $t=1$ this is written as $(ap)_x$.

For comparison, using multiple state notation, the dependent probabilities for our example would be written:

$$(aq)_x^s = {}_1p_x^{HS}$$

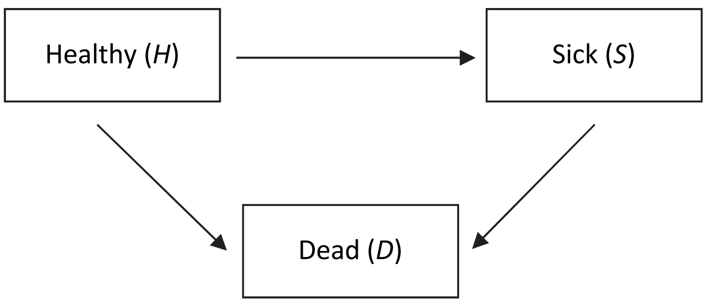
$$(aq)_x^d = {}_1p_x^{HD}$$

$$(ap)_x = {}_1p_x^{HH}$$

These equalities are *only* true in the case of the revised healthy-sick-dead model above. They are *not* true for the general healthy-sick-dead multiple state model described at the start of Section 3.1.

Question

Consider the following 3-state model:



Assuming that H is the active state, explain whether or not each of the following is true, and if not, state with reasons which of the two probabilities is the larger.

(a) $(aq)_x^s = {}_1p_x^{HS}$

(b) $(aq)_x^d = {}_1p_x^{HD}$

(c) $(ap)_x = {}_1p_x^{HH}$

Solution

(a) $(aq)_x^s = {}_1p_x^{HS}$

$(aq)_x^s$ is the probability that a healthy life aged x will leave the healthy state due to sickness during the next year. ${}_1p_x^{HS}$ is the probability that a life who is healthy at age x, will be sick at age $x+1$.

${}_1p_x^{HS}$ will be smaller than $(aq)_x^s$, because some of the lives who become sick during the year go on to die during the same year, and so are not present in State S at age $x+1$.

(b) $(aq)_x^d = {}_1p_x^{HD}$

$(aq)_x^d$ is the probability that a healthy life aged x will leave the healthy state due to death during the next year. ${}_1p_x^{HD}$ is the probability that a life who is healthy at age x, will be dead by age $x+1$.

${}_1p_x^{HD}$ will be larger than $(aq)_x^d$, because some of those who start the year healthy and who end the year dead, will have become sick and then died from sick during the same year. So ${}_1p_x^{HD}$ will include some lives who leave the healthy state through sickness, as well as all those who leave the healthy state directly through death.

(c) $(ap)_x = {}_1p_x^{HH}$

$(ap)_x$ is the probability that a healthy life aged x stays healthy until at least age $x+1$.

${}_1p_x^{HH}$ is the probability that a life who is healthy at age x, is also healthy at age $x+1$.

Because (in *this* model) it is impossible to return to State H once the state has been left, ${}_1p_x^{HH}$ also implies that the life remains in H for at least one year, and so these two probabilities are the same.

So, the relationship between the dependent probability notation (*eg* $(aq)_x^d$) and multiple state model probability notation (*eg* ${}_1p_x^{HD}$) depends on the precise form the model takes.

Question

Explain whether $(ap)_x = {}_1p_x^{HH}$ for the general healthy-sick-dead model defined at the start of Section 3.1.

Solution

As before, $(ap)_x$ means the probability of staying (continuously) healthy for at least one year, and ${}_1p_x^{HH}$ means the probability of a life, who is healthy at age x, being also healthy one year later. These probabilities are not the same in the general healthy-sick-dead model, as lives are able to leave healthy and go back again during the same year, which means that:

$$(ap)_x < {}_1p_x^{HH}$$

In this case, $(ap)_x$ is equal to the occupancy probability, ${}_1p_x^{\overline{HH}}$.

We also note that:

$$\boldsymbol{(ap)_x + (aq)_x^s + (aq)_x^d = 1}$$

and that we also write:

$$\boldsymbol{(aq)_x = (aq)_x^s + (aq)_x^d}$$

so that:

$$\boldsymbol{(ap)_x + (aq)_x = 1}$$

Question

Active members of a pension scheme are subject to the following probabilities of decrement at the given ages (where r and d stand for retirement and death, respectively).

Age x	$(aq)_x^r$	$(aq)_x^d$
60	0.1	0.03
61	0.2	0.04

Calculate the following probabilities, all relating to an active member who is currently exactly aged 60.

(i) The probability of retiring during the year of age 61 to 62.

(ii) The probability of dying as an active member before age 62 (*ie* without retiring first).

(iii) The probability of still being an active member at age 62.

Solution

(i) This is $(ap)_{60} \times (aq)^r_{61}$, where:

$$(ap)_{60} = 1 - (aq)^r_{60} - (aq)^d_{60} = 1 - 0.1 - 0.03 = 0.87$$

So the required probability is $0.87 \times 0.2 = 0.174$

This probability can also be written as ${}_{1|}(aq)^r_{60}$.

(ii) This probability is:

$$\begin{aligned} {}_2(aq)^d_{60} &= (aq)^d_{60} + (ap)_{60} \times (aq)^d_{61} \\ &= 0.03 + 0.87 \times 0.04 \\ &= 0.0648 \end{aligned}$$

(iii) This is:

$${}_2(ap)_{60} = (ap)_{60} \times (ap)_{61} = 0.87 \times [1 - 0.2 - 0.04] = 0.6612$$

It is also useful to consider the special case of a single decrement model, which only has one cause of decrement. For this we define ${}_tq^r_x$ to be the *independent* probability that an individual aged x in the active state will be removed from that state between ages x and $x+t$ by the decrement r. (By 'independent', we mean *when r is the only risk of decrement acting on the population*). When $t = 1$ this is written as q^r_x.

Independent probabilities of decrement assume that there are no other decrements operating on the population of interest. Dependent probabilities of decrement take into account the competing forces of decrement that operate on the population.

Question

A population is subject to two causes of decrement, death (d) and withdrawal (w). Explain whether the value of q^d_x would be larger or smaller than the value of $(aq)^d_x$.

Solution

q_x^d is the probability of a life aged x dying between ages x and $x+1$, where death is the only cause of decrement occurring.

$(aq)_x^d$ is the probability of a life aged x leaving the population directly through death, when withdrawals are also taking place. This probability is lower than it would be if death were the only decrement, because lives may withdraw before they die during the same year, *ie* $(aq)_x^d < q_x^d$.

3.3 Deriving probabilities from transition intensities

We can use the Kolmogorov forward differential equations to derive transition probabilities, as in the case of multiple state models. We note from Section 2.1 that, in the multiple state model, this produces the following general result:

$$ {}_t p_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j \neq i} \mu_{x+s}^{ij} \, ds \right) $$

The Kolmogorov differential equations are derived in Subject CS2.

In the case of the multiple decrement model, in which return to the active state is not possible, we have:

$$ {}_t(ap)_x = {}_t p_x^{HH} = {}_t p_x^{\overline{HH}} $$

Since our double decrement model has decrements of sickness and death, we have:

$$ {}_t(ap)_x = {}_t p_x^{\overline{HH}} = \exp\left[-\int_{s=0}^{t} \left(\mu_{x+s}^{HS} + \mu_{x+s}^{HD} \right) ds \right] = \exp\left[-\int_{s=0}^{t} \left(\sigma_{x+s} + \mu_{x+s} \right) ds \right] $$

Therefore, assuming constant transition intensities:

$$ {}_t(ap)_x = e^{-(\mu+\sigma)t} \qquad \textbf{(1)} $$

For the other probabilities, the differential equations (again assuming constant transition intensities) are:

$$ \frac{\partial}{\partial t} {}_t(aq)_x^s = \sigma \, {}_t(ap)_x = \sigma \, e^{-(\mu+\sigma)t} $$

and

$$ \frac{\partial}{\partial t} {}_t(aq)_x^d = \mu \, {}_t(ap)_x = \mu \, e^{-(\mu+\sigma)t} $$

These differential equations have the closed form solutions (with $t=1$):

$$(aq)_x^s = \frac{\sigma}{\mu+\sigma}\left(1-e^{-(\mu+\sigma)}\right) \qquad \textbf{(2)}$$

and

$$(aq)_x^d = \frac{\mu}{\mu+\sigma}\left(1-e^{-(\mu+\sigma)}\right) \qquad \textbf{(3)}$$

These solutions are obtained by integrating the differential equations with respect to t between the limits of $t=0$ and $t=1$. For example, integrating the first differential equation gives:

$$\left[{}_t(aq)_x^s\right]_0^1 = \int_0^1 \sigma e^{-(\mu+\sigma)t}\,dt$$

$$\Rightarrow (aq)_x^s - {}_0(aq)_x^s = \frac{-\sigma}{\mu+\sigma}\left[e^{-(\mu+\sigma)t}\right]_0^1$$

$$\Rightarrow (aq)_x^s = \frac{\sigma}{\mu+\sigma}\left[1-e^{-(\mu+\sigma)}\right]$$

since ${}_0(aq)_x^s = 0$.

Note that $(aq)_x^s$ is the product of:

- $1-e^{-(\mu+\sigma)} = 1-(ap)_x$, *ie* the probability that a life who is in the active state at the start of the year, is *not* in the active state at the end of the year, and
- $\dfrac{\sigma}{\mu+\sigma}$, *ie* the proportion of the total force acting on the life in the active state that relates to transitions to the sick state. This represents the conditional probability that the transition from the active state takes the life into the sick state, given that there is a transition out of the active state.

We now have formulae for the dependent probabilities in terms of transition intensities. So, if we can estimate the transition intensities, it is very easy to estimate the dependent probabilities using these formulae. The estimation of transition intensities is covered in CS2 – in CM1 we will be told what values or functions to use.

For the independent (single decrement) probabilities we obtain:

$$q_x^s = 1-e^{-\sigma} \qquad \textbf{(4)}$$

and:

$$q_x^d = 1-e^{-\mu}$$

The first of these is obtained by setting $\mu=0$ in (2) above (so that sickness is the *only* decrement operating, *ie* sickness is operating *independently*). The second of these is obtained by setting $\sigma=0$ in (3) above.

When $t \neq 1$, we have, for example:

$$_t(aq)_x^s = \frac{\sigma}{\mu+\sigma}\left(1-e^{-t(\mu+\sigma)}\right)$$

provided the transition intensities are constant over $[x, x+t]$ (and similar for other cases).

Question

A population of healthy people over the year of age 50 to 51 is subject to a constant force of decrement due to sickness of 0.08 per annum, and a constant force of mortality of 0.002 per annum.

Assuming that a double decrement model is used, calculate:

(i) the probability that a healthy person aged exactly 50 will still be healthy at exact age 51

(ii) the probability that a healthy person aged exactly 50 will leave the healthy population through death before exact age 51

(iii) the independent probability of a life aged exactly 50 dying before exact age 51.

Solution

(i) The probability of staying healthy is:

$$(ap)_{50} = e^{-(0.08+0.002)} = 0.921272$$

(ii) The dependent probability of leaving the population through death is:

$$(aq)_{50}^d = \frac{0.002}{0.082}\left(1-e^{-0.082}\right) = 0.001920$$

(iii) The independent probability of dying is:

$$q_x^d = 1-e^{-0.002} = 0.001998$$

Note that $(aq)_x^d < q_x^d$, as we would expect.

4 Multiple decrement tables

A multiple decrement table is a computational tool for dealing with a population subject to multiple decrements.

We introduce the following notation as an extension of the (single decrement) life table approach:

$$(al)_x = \text{active population at age } x$$

and $\alpha, \beta, \gamma, \ldots$ **the labels for the types of independent decrements to which the population is subject.**

Then the multiple decrement table is a numerical representation of the development of the population, such that:

$$\begin{aligned}(al)_{x+1} = (al)_x &- \text{number of lives removed between ages } x \text{ and } x+1 \\ &\quad \text{due to decrement } \alpha \\ &- \text{number of lives removed between ages } x \text{ and } x+1 \\ &\quad \text{due to decrement } \beta \\ &- \text{number of lives removed between ages } x \text{ and } x+1 \\ &\quad \text{due to decrement } \gamma \\ &- \cdots\end{aligned}$$

In general, as the decrements are assumed to operate independently, the number of lives removed due to decrement ' k **' will depend on the preceding population** $(al)_x$ ***as well as*** **the numbers removed by every other decrement other than** k **.**

We need to be clear about what the Core Reading means by 'independently' here. It means that the presence of multiple causes of decrement in a population does not affect the *forces* of decrement by each cause – *ie* that the forces of decrement are independent of each other. However, the *numbers* of decrements by any cause will certainly be affected by how many decrements from other causes occur.

We define the number of lives removed over the year of age due to decrement k as $(ad)_x^k$. Hence we have:

$$(aq)_x^k = \frac{(ad)_x^k}{(al)_x}$$

$$_n(aq)_x^k = \frac{(ad)_x^k + (ad)_{x+1}^k + \dots + (ad)_{x+n-1}^k}{(al)_x}$$

$$(ap)_x = \frac{(al)_{x+1}}{(al)_x}$$

$$_n(ap)_x = \frac{(al)_{x+n}}{(al)_x}$$

for $n = 0, 1, \dots$.

Question

You are given the following extract from a double decrement table:

Age x	$(al)_x$	$(aq)_x^d$	$(aq)_x^w$
50	100,000	175	2,490
51	97,335	180	2,160
52	94,995		

where d and w refer to death and withdrawal respectively.

Calculate:

(a) $(ap)_{51}$

(b) $(aq)_{51}^d$

(c) $_2(aq)_{50}^w$

(d) $_2(ap)_{50}$

Solution

(a) $(ap)_{51} = \dfrac{(al)_{52}}{(al)_{51}} = \dfrac{94,995}{97,335} = 0.975959$

(b) $(aq)_{51}^d = \dfrac{(ad)_{51}^d}{(al)_{51}} = \dfrac{180}{97,335} = 0.001849$

(c) $${}_{2}(aq)_{50}^{w} = \frac{(ad)_{50}^{w} + (ad)_{51}^{w}}{(al)_{50}} = \frac{2{,}490 + 2{,}160}{100{,}000} = 0.0465$$

(d) $${}_{2}(ap)_{50} = \frac{(al)_{52}}{(al)_{50}} = \frac{94{,}995}{100{,}000} = 0.94995$$

We can also use the table to calculate *deferred* dependent probabilities of the form:

$${}_{n|}(aq)_{x}^{k} = \frac{(ad)_{x+n}^{k}}{(al)_{x}}$$

This is the probability that an active life, currently aged x, leaves the population by cause k in the year of age $x+n$ to $x+n+1$.

The expression can be constructed from:

$${}_{n|}(aq)_{x}^{k} = {}_{n}(ap)_{x} \times (aq)_{x+n}^{k} = \frac{(al)_{x+n}}{(al)_{x}} \times \frac{(ad)_{x+n}^{k}}{(al)_{x+n}} = \frac{(ad)_{x+n}^{k}}{(al)_{x}}$$

Question

Using the multiple decrement table in the preceding question, calculate ${}_{1|}(aq)_{50}^{d}$, and state what this probability means in words.

Solution

The probability is:

$${}_{1|}(aq)_{50}^{d} = \frac{(ad)_{51}^{d}}{(al)_{50}} = \frac{180}{100{,}000} = 0.0018$$

This is the probability that a person aged exactly 50 leaves the population through death between exact ages 51 and 52.

It is also conventional to write the transition intensity (or *force of decrement*) due to cause k at age x in the multiple decrement model as $(a\mu)_{x}^{k}$.

4.1 Associated single decrement tables

For each of the causes in a multiple decrement table, it is possible to define a single decrement table which only involves a particular cause of decrement. In effect the other modes of decrement are assumed not to operate.

A life table, such as AM92 or ELT15, is an example of a single decrement table, where the only decrement is death. We can similarly construct other single decrement tables (at least in theory) where the decrement is just withdrawal, for example.

We define functions as in any single decrement table except that each symbol has a superscript indicating the mode of decrement that is being modelled.

The notation used is l_x^j, d_x^j, q_x^j, p_x^j, μ_x^j *etc* for mode of decrement j.

We've already met the independent probability q_x^j. The notation p_x^j ($=1-q_x^j$) represents the probability that a life aged exactly x remains in the population for at least one year, when cause j is the only way in which people can exit from the population.

The importance of single decrement tables in practice is that they are often used, at least as a starting point, in the construction of a multiple decrement table.

For example, we might wish to construct a double decrement table for endowment assurance policyholders incorporating decrements of death and withdrawal, in which the underlying (or 'independent') mortality basis is represented by a standard mortality table, such as AM92 Select. The probabilities q_x^d and p_x^d can be read off from the table, and then used to help construct the relevant dependent probabilities, as described below.

4.2 Relationships between single and multiple decrement tables

The linking assumption between the tables is:

$$(a\mu)_x^j = \mu_x^j \text{ for all } j \text{ and all } x$$

Normally we would expect the independent decrement probabilities and the dependent decrement probabilities not to be equal. For instance, $(aq)_x^d$ and q_x^d might not be equal because the dependent probability will be affected by the fact that some people will withdraw, or retire (if these are the other decrements operating) 'before they could die' in our population. So the fact that a lot of things could happen over the course of a year causes a problem.

If, however, we look at the transition intensities μ, then we are in effect looking at some infinitesimally small time interval. In that very small time interval there is only time for one decrement – *eg* death – and so the number of deaths occurring in that time interval will not be reduced by people withdrawing. Thus it is reasonable to assume that the independent intensity of death μ_x^d will be equal to the dependent intensity of death $(a\mu)_x^d$. The same reasoning will hold for any decrement.

The assumption is often called the 'independence of decrements'. One of its consequences is that if we observe the forces of decrement from cause j in two groups, one which includes all those who have not left the population as a result of decrement from cause i and the other which includes all those who have left the population as a result of decrement from cause i, then the observed forces of decrement from cause j in the two groups are the same. So the force of decrement from cause j is independent of the force from cause i.

For instance, if we have a population of pension scheme members where cause j is death and cause i is withdrawal, then 'independence of decrements' implies that:

'the mortality of active pension scheme members is equal to that of pension scheme members who have withdrawn.'

This means that any change in $(a\mu)_x^w$ would not affect $(a\mu)_x^d$.

So a consequence of the independence of decrements is that the decrements are non-selective – they do not alter the decrement experience of those 'left behind'. In practice this may not be true, especially if we are considering populations of life assurance policyholders, for example, where it is very likely that people who lapse or surrender their policies would have lower than average mortality. However, the independence assumption is required to make the theory more tractable.

There are a number of theoretical shortcomings of these assumptions but they are beyond the scope of this course and application of these assumptions provides us with a viable working model for actuaries in practice.

The sum of all the (dependent) forces of mortality is denoted by:

$$(a\mu)_x = \sum_{all\ j} (a\mu)_x^j$$

4.3 Constructing a multiple decrement table

To construct a multiple decrement table, we need to obtain the relevant dependent probabilities $(aq)_x^k$ at each age and for each cause of decrement k (as described in Section 4.4 below).

Once these are obtained we then choose a suitable starting age α for the table, and a suitable value for the radix $(al)_\alpha$.

Then construct:

$$(ad)_x^k = (al)_x \,(aq)_x^k$$

for all k, and

$$(al)_{x+1} = (al)_x - \sum_k (ad)_x^k$$

recursively for all $x = \alpha, \alpha+1, \ldots$

4.4 Obtaining dependent probabilities

The formulae in this section all require the independence of decrements assumption, defined in Section 4.2, to hold.

From the forces of decrement

The most logical starting point is to begin with the relevant forces of decrement, and assume these are constant over single years of age. Formulae such as (1), (2) and (3) of Section 3.3 can then be used to calculate the dependent probabilities directly.

Question

In a certain population, forces of decrement are assumed to be constant over individual years of age.

The following independent forces of decrement will be assumed for this population between the exact ages of 50 and 52:

Force of decrement for year of age commencing from exact age x

Age x	due to mortality	due to sickness
50	0.011	0.075
51	0.012	0.081

Construct a double decrement table including the two decrements of mortality and sickness, for this population between exact ages 50 and 52, assuming a radix of $(al)_{50} = 100,000$.

Solution

First we need the dependent probabilities of decrement by each cause at each age.

We will assume decrements are independent, so that the given forces can be assumed to apply when the two decrements occur together in the same population.

We will use μ_x^j to be the constant force of decrement due to cause j operating over the year of age x to $x+1$, where $j = d$ (death), s (sickness) .

From Section 3.3 we have:

$$(aq)_x^j = \frac{\mu_x^j}{\mu_x^d + \mu_x^s}\left(1 - e^{-(\mu_x^d + \mu_x^s)}\right)$$

We then obtain:

$$(aq)^d_{50} = \frac{0.011}{0.086}\left(1 - e^{-0.086}\right) = 0.010540$$

$$(aq)^s_{50} = \frac{0.075}{0.086}\left(1 - e^{-0.086}\right) = 0.071865$$

$$(aq)^d_{51} = \frac{0.012}{0.093}\left(1 - e^{-0.093}\right) = 0.011459$$

$$(aq)^s_{51} = \frac{0.081}{0.093}\left(1 - e^{-0.093}\right) = 0.077348$$

To construct the table, we use the radix of $(al)_{50} = 100{,}000$ and the formulae in Section 4.3, which give us:

Age x	$(al)_x$	$(ad)^d_x$	$(ad)^s_x$
50	100,000	1,054.03	7,186.55
51	91,759.42	1,051.46	7,097.37
52	83,610.59		

From an existing multiple decrement table

Here we will need to calculate the implied (constant) forces of decrement underlying the existing table. Taking formula (2) from Section 3.3 as an example (where we have decrements of sickness and mortality):

$$\boldsymbol{(aq)^s_x = \frac{\sigma}{\mu+\sigma}\left(1 - e^{-(\mu+\sigma)}\right)}$$

$$= \frac{\sigma}{\mu+\sigma}\left(1 - (ap)_x\right)$$

$$\boldsymbol{= \frac{\sigma}{\mu+\sigma}(aq)_x}$$

Rearranging:

$$\boldsymbol{\Rightarrow \sigma = \frac{(aq)^s_x}{(aq)_x}(\mu+\sigma) = \frac{(aq)^s_x}{(aq)_x}\left(-\ln(ap)_x\right)}$$

where:

$$(aq)_x = (aq)^s_x + (aq)^d_x = \sum_{\text{all } j}(aq)^j_x$$

Question

We wish to extend the multiple decrement table constructed in the previous question (incorporating the decrements of death (d) and sickness (s)) to include the decrement of withdrawal (w).

It is believed that the independent forces of withdrawal will conform to those underlying the withdrawal decrement in the multiple decrement table below:

Age x	$(al)_x$	$(aq)_x^d$	$(aq)_x^w$
50	100,000	175	2,490
51	97,335	180	2,160
52	94,995		

Calculate the first two lines of the triple decrement table (*ie* between ages 50 and 52) incorporating the three decrements d , s and w , assuming that the forces of sickness and mortality are unchanged.

Solution

We first need to calculate the independent forces of withdrawal from the table provided, using:

$$\mu_x^w = \frac{(aq)_x^w}{(aq)_x}\left[-\ln(ap)_x\right] = \frac{(ad)_x^w}{(ad)_x^d + (ad)_x^w}\left[-\ln\left(\frac{(al)_{x+1}}{(al)_x}\right)\right]$$

So, for $x = 50,51$:

$$\mu_{50}^w = \frac{2,490}{175+2,490}\left[-\ln\left(\frac{97,335}{100,000}\right)\right] = 0.025238$$

$$\mu_{51}^w = \frac{2,160}{180+2,160}\left[-\ln\left(\frac{94,995}{97,335}\right)\right] = 0.022463$$

Indicating the new triple-decrement functions with '*b*' rather than '*a*' prefixes, the new dependent probabilities for age x are:

$$(bq)_x^j = \frac{\mu_x^j}{\mu_x^d + \mu_x^s + \mu_x^w}\left(1 - e^{-(\mu_x^d + \mu_x^s + \mu_x^w)}\right) \qquad \text{for } j = d, s, w$$

So for $x = 50$, we have:

$$\mu_{50}^d + \mu_{50}^s + \mu_{50}^w = 0.011 + 0.075 + 0.025238 = 0.111238$$

$$1 - e^{-0.111238} = 0.105274$$

Then:

$$(bq)_{50}^{d} = \frac{0.011}{0.111238} \times 0.105274 = 0.010410$$

$$(bq)_{50}^{s} = \frac{0.075}{0.111238} \times 0.105274 = 0.070979$$

$$(bq)_{50}^{w} = \frac{0.025238}{0.111238} \times 0.105274 = 0.023885$$

For $x = 51$, we have:

$$\mu_{51}^{d} + \mu_{51}^{s} + \mu_{51}^{w} = 0.012 + 0.081 + 0.022463 = 0.115463$$

$$1 - e^{-0.115463} = 0.109046$$

Then:

$$(bq)_{51}^{d} = \frac{0.012}{0.115463} \times 0.109046 = 0.011333$$

$$(bq)_{51}^{s} = \frac{0.081}{0.115463} \times 0.109046 = 0.076499$$

$$(bq)_{51}^{w} = \frac{0.022463}{0.115463} \times 0.109046 = 0.021214$$

The first two lines of the triple decrement table are:

Age x	$(bl)_x$	$(bd)_x^d$	$(bd)_x^s$	$(bd)_x^w$
50	100,000	1,041.03	7,097.90	2,388.47
51	89,472.60	1,014.01	6,844.53	1,898.09
52	79,715.97			

From existing single decrement tables

Here we will need to calculate the implied (constant) forces of decrement underlying the existing single decrement tables. Taking formula (4) from Section 3.3 as an example:

$$\boldsymbol{q_x^s = 1 - e^{-\sigma}}$$

$$\boldsymbol{\Rightarrow \sigma = -\ln\left(1 - q_x^s\right)}$$

In all cases, the forces of decrement obtained may need to be adjusted before being applied to construct the dependent probabilities for any particular application.

Question

We wish to construct a double decrement table for mortality and sickness only.

The independent force of sickness for the year of age from 50 to 51 is 0.075 and the independent force of mortality for the same year of age is 80% of the force of mortality according to the ELT15 (Males) mortality table.

Assuming that all forces of decrement are constant over individual years of age, calculate the first line of this double decrement table, using a radix of $(al)_{50} = 100{,}000$.

Solution

The required force of mortality is found from:

$$\mu_{50}^{d} = 0.8\left[-\ln\left(1 - q_{50}^{d}\right)\right]$$

where q_{50}^{d} is the relevant probability of death in the ELT15 (Males) table. So:

$$\mu_{50}^{d} = 0.8\left[-\ln\left(1 - 0.00464\right)\right] = 0.003721$$

Calculating the dependent probabilities, we have:

$$(aq)_{50}^{d} = \frac{0.003721}{0.003721 + 0.075}\left(1 - e^{-(0.003721+0.075)}\right) = 0.003578$$

$$(aq)_{50}^{s} = \frac{0.075}{0.078721} \times \left(1 - e^{-0.078721}\right) = 0.072124$$

So the first line of this double decrement table is:

Age x	$(al)_x$	$(ad)_x^d$	$(ad)_x^s$
50	100,000	357.80	7,212.39
51	92,429.81		

These formulae can also be used to construct single decrement tables from existing multiple decrement tables, if desired. The key is always to define any existing tables in terms of their underlying forces of decrement, and then apply these forces to calculate the dependent or independent probabilities as required.

There is a direct relationship between the independent and dependent probabilities, which is sometimes useful.

If we have n decrements, labelled $j=1,2,...n$:

$$
\begin{aligned}
{}_t(ap)_x &= \exp\left[-\int_{s=0}^{t} (a\mu)_{x+s}\,ds\right] \\
&= \exp\left[-\int_{s=0}^{t} \sum_{j=1}^{n} (a\mu)_{x+s}^{j}\,ds\right] \\
&= \prod_{j=1}^{n} \exp\left[-\int_{s=0}^{t} (a\mu)_{x+s}^{j}\,ds\right]
\end{aligned}
$$

Assuming $(a\mu)_{x+s}^{j} = \mu_{x+s}^{j}$ for all j and s, as usual, then:

$$
\begin{aligned}
{}_t(ap)_x &= \prod_{j=1}^{n} \exp\left[-\int_{s=0}^{t} \mu_{x+s}^{j}\,ds\right] \\
&= \prod_{j=1}^{n} {}_t p_x^{j}
\end{aligned}
$$

So we can obtain the overall dependent probability of remaining in the active state, by multiplying the independent 'not-leaving' probabilities for all causes together.

4.5 Integral formulae for multiple decrement probabilities

We can also obtain expressions for multiple decrement probabilities without making the assumption that forces of decrement are constant over each year of age.

For example, with a time-varying sickness intensity, the differential equation for ${}_t(aq)_x^s$ in Section 3.3 would become:

$$\frac{\partial}{\partial t}\,{}_t(aq)_x^s = {}_t(ap)_x\,\sigma_{x+t} = {}_t(ap)_x\,(a\mu)_{x+t}^s$$

Integrating over $t=0$ to $t=1$ we obtain:

$${}_1(aq)_x^s - {}_0(aq)_x^s = \int_{t=0}^{1} {}_t(ap)_x\,(a\mu)_{x+t}^s\,dt$$

As ${}_0(aq)_x^s = 0$:

$$(aq)_x^s = \int_{t=0}^{1} {}_t(ap)_x\,(a\mu)_{x+t}^s\,dt$$

In general, we have:

$$(aq)_x^j = \int_{t=0}^{1} {}_t(ap)_x \, (a\mu)_{x+t}^j \, dt = \int_{t=0}^{1} {}_t(ap)_x \;\; \mu_{x+t}^j \, dt$$

assuming independence of decrements.

Such integrals for dependent probabilities are equivalent to the integral expression we met for the probability of death in Chapter 15:

$${}_t q_x = \int_0^t {}_s p_x \, \mu_{x+s} \, ds$$

5 Using multiple decrement tables to evaluate expected present values of cashflows

The approach to calculating EPVs of cashflows that are contingent on multiple decrements is similar to the single decrement case described in earlier chapters.

5.1 Continuous approach

Integrals can be used to express the expected present value of cashflows when these are continuously contingent on multiple decrements.

Question

An employee benefits package pays 50,000 immediately on the death of an employee while in employment. The employee population is assumed to be subject to multiple decrements, of which death in employment is one. All benefits (including death benefits) under the package cease on an employee's 70th birthday.

Write down an integral expression to represent the expected present value of the death benefits to an employee who is currently aged exactly 48.

Solution

The payment of 50,000 needs to be discounted back from the moment of death. Letting t denote the time of death measured from the current time (time 0), we have:

$$EPV = \int_{t=0}^{22} 50{,}000 v^t \, {}_t(ap)_{48} \, (a\mu)^d_{48+t} \, dt$$

We integrate from $t=0$ to $t=22$ because no benefit is payable after 22 years (which is when the employee attains age 70).

5.2 Discrete approach

As usual, the basic approach is summarised as:

$$EPV = \sum \{\text{cashflow}\} \times \{\text{discount factor}\} \times \{\text{probability}\}$$

where the summation is over all future payment periods. In the case of multiple decrements, the probability will be the relevant dependent probability that the cashflow will occur.

Question

An endowment assurance pays 10,000 in three years' time or immediately on the earlier death of a life aged 50 at entry. On surrender at any time, a surrender value equal to 75% of the premiums paid by the date of surrender (without interest) is payable. Surrender payments are assumed to occur immediately at the time of surrender. A level annual premium of 3,000 is paid at the start of each year.

Calculate the expected present value of the death, maturity and surrender benefits for a single policy at outset, using the following assumptions:

- mortality: AM92 Ultimate
- annual force of decrement due to surrender: 0.1 in year 1, and 0.05 in each of years 2 and 3
- interest: 3% per annum compound

State any further assumptions you make.

Solution

We will use a multiple decrement model with decrements of surrender and death, represented by s and d respectively.

Assuming that decrements occur half way through each year of age, on average, the EPV of the death and maturity benefits combined is:

$$EPV_{D,M} = 10{,}000\left\{v^{½}\ {}_{0|}(aq)^d_{50} + v^{1½}\ {}_{1|}(aq)^d_{50} + v^{2½}\ {}_{2|}(aq)^d_{50} + v^3\ {}_3(ap)_{50}\right\} \qquad (*)$$

$$= 10{,}000\left\{\frac{v^{½}(ad)^d_{50} + v^{1½}(ad)^d_{51} + v^{2½}(ad)^d_{52} + v^3(al)_{53}}{(al)_{50}}\right\} \qquad (**)$$

The EPV of the surrender benefits is:

$$EPV_S = 3{,}000 \times 0.75\left\{v^{½}\ {}_{0|}(aq)^s_{50} + 2v^{1½}\ {}_{1|}(aq)^s_{50} + 3v^{2½}\ {}_{2|}(aq)^s_{50}\right\} \qquad (*)$$

$$= 3{,}000 \times 0.75\left\{\frac{v^{½}(ad)^s_{50} + 2v^{1½}(ad)^s_{51} + 3v^{2½}(ad)^s_{52}}{(al)_{50}}\right\} \qquad (**)$$

We can either use the dependent probabilities directly, using (*), or we can construct a multiple decrement table and then use (**).

Calculating EPVs using probabilities directly

The required values are shown in the following table:

Age x Year t	50 1	51 2	52 3
μ^d	0.002511	0.002813	0.003157
μ^s	0.1	0.05	0.05
$(aq)_x$	0.097432	0.051443	0.051769
$(aq)_x^d$	0.002387	0.002740	0.003075
$(aq)_x^s$	0.095045	0.048703	0.048694
$(ap)_x$	0.902568	0.948557	0.948231
${}_t(ap)_x$	0.902568	0.856137	0.811816

where:

$$\mu^d = -\ln\left(1 - q_x^d\right)$$

$$(aq)_x = 1 - e^{-(\mu^d + \mu^s)}$$

$$(aq)_x^d = \frac{\mu^d}{\mu^d + \mu^s}(aq)_x$$

$$(aq)_x^s = (aq)_x - (aq)_x^d$$

$$(ap)_x = 1 - (aq)_x$$

$${}_t(ap)_x = \prod_{r=0}^{t-1}(ap)_{x+r}$$

Using (*):

$$EPV_{D,M} = 10{,}000\left\{\frac{0.002387}{1.03^{½}} + \frac{0.902568 \times 0.002740}{1.03^{1½}} + \frac{0.856137 \times 0.003075}{1.03^{2½}} + \frac{0.811816}{1.03^{3}}\right\}$$

$$= 7{,}500.89$$

$$EPV_S = 3{,}000 \times 0.75\left\{\frac{0.095045}{1.03^{½}} + \frac{2 \times 0.902568 \times 0.048703}{1.03^{1½}} + \frac{3 \times 0.856137 \times 0.048694}{1.03^{2½}}\right\}$$

$$= 661.30$$

Calculating EPVs using a multiple decrement table

Using (**) we first need to construct the multiple decrement table. We choose an arbitrary radix of $(al)_{50} = 100{,}000$:

Age x	$(al)_x$	$(ad)_x^d$	$(ad)_x^s$
50	100,000	238.7	9,504.5
51	90,256.8	247.3	4,395.8
52	85,613.7	263.3	4,168.9
53	81,181.5		

using:

$$(ad)_x^d = (al)_x \times (aq)_x^d$$

$$(ad)_x^s = (al)_x \times (aq)_x^s$$

$$(al)_{x+1} = (al)_x - (ad)_x^d - (ad)_x^s$$

Then:

$$EPV_{D,M} = \frac{10{,}000}{100{,}000}\left\{1.03^{-½} \times 238.7 + 1.03^{-1½} \times 247.3 + 1.03^{-2½} \times 263.3 + 1.03^{-3} \times 81{,}181.5\right\}$$

$$= 7{,}500.89$$

$$EPV_S = \frac{3{,}000 \times 0.75}{100{,}000}\left\{1.03^{-½} \times 9{,}504.5 + 2 \times 1.03^{-1½} \times 4{,}395.8 + 3 \times 1.03^{-2½} \times 4{,}168.9\right\}$$

$$= 661.30$$

as before.

This page has been left blank so that you can keep the chapter summaries together for revision purposes.

Chapter 25 Summary

Multiple state models

Definitions

$\mu_x^{ij} =$ force of transition (transition intensity) from state i to state j at exact age x

${}_tp_x^{ij} =$ probability of being in state j at age $x+t$ given in state i at age x

${}_tp_x^{\overline{ii}} =$ probability of staying continuously in state i between ages x and $x+t$ given in state i at age x

$$= \exp\left(-\int_0^t \sum_{j \neq i} \mu_{x+s}^{ij} ds\right)$$

Valuing cashflows

A lump sum benefit of S payable immediately on transition from state i to state j, for a life currently in state a, has expected present value:

$$S\int_0^\infty v^t {}_tp_x^{ai}\, \mu_{x+t}^{ij}\, dt$$

An income benefit of B *pa* payable continuously while in state j, for a life currently in state a, has expected present value:

$$B\int_0^\infty v^t {}_tp_x^{aj}\, dt$$

Integral expressions can be evaluated using numerical techniques, such as the trapezium rule.

Multiple decrement models

A multiple decrement model is appropriate if there is one active state, out of which transitions occur into one or more absorbing exit states only.

Dependent and independent probabilities

The dependent probability $(aq)_x^\alpha$ is the probability that a life aged x in a particular state will be removed from that state within a year by the decrement α, in the presence of all other decrements in the population.

The independent probability q_x^α is the probability that a life aged x in a particular state will be removed from that state within a year by the decrement α, where α is the only decrement acting on the population.

Results for the sickness-death (2 decrement) model

If the active state A is subject to decrements to exit states D and S, with constant forces of transition μ and σ over integer ages, then the independent probabilities are given by:

$$q_x^d = 1 - e^{-\mu} \qquad \text{and} \qquad q_x^s = 1 - e^{-\sigma}$$

The dependent probabilities are given by:

$$(aq)_x^d = p_x^{AD} = \frac{\mu}{\mu+\sigma}\left(1-e^{-(\mu+\sigma)}\right) \quad \text{and} \quad (aq)_x^s = p_x^{AS} = \frac{\sigma}{\mu+\sigma}\left(1-e^{-(\mu+\sigma)}\right)$$

The probability of remaining in the active state is:

$$(ap)_x = 1 - (aq)_x^d - (aq)_x^s = e^{-(\mu+\sigma)}$$

Given dependent probabilities, the forces of transition can be obtained as:

$$\mu = \frac{(aq)_x^d}{(aq)_x}\left(-\ln(ap)_x\right) \quad \text{and} \quad \sigma = \frac{(aq)_x^s}{(aq)_x}\left(-\ln(ap)_x\right)$$

where $(aq)_x = (aq)_x^d + (aq)_x^s$.

Independence of decrements

The 'independence of decrements' equation is $(a\mu)_x^j = \mu_x^j$ from which it follows that:

$$(ap)_x = p_x^1 p_x^2 p_x^3 \ldots p_x^m$$

where there are *m* causes of decrement $j = 1, 2, \ldots m$, and $p_x^j = 1 - q_x^j$.

The dependent probability expressed in integral form is:

$$(aq)_x^j = \int_0^1 {}_t(ap)_x \, (a\mu)_{x+t}^j \, dt = \int_0^1 {}_t(ap)_x \, \mu_{x+t}^j \, dt$$

(The second expression is true only if the independence of decrements assumption holds.)

Multiple decrement tables

Definitions

$(al)_x =$ expected number of active lives at exact age x

$(ad)_x^j =$ expected number of decrements due to cause j over the year of age x to $x+1$, given a radix of $(al)_\alpha$ lives active at age α.

Calculating probabilities

$$(aq)_x^j = \frac{(ad)_x^j}{(al)_x} \qquad {}_n(ap)_x = \frac{(al)_{x+n}}{(al)_x}$$

$${}_n(aq)_x^j = \frac{\sum_{r=0}^{n-1} (ad)_{x+r}^j}{(al)_x} \qquad {}_{n|}(aq)_x^j = \frac{(ad)_{x+n}^j}{(al)_x}$$

Construction of multiple decrement tables

$$(ad)_x^j = (al)_x \, (aq)_x^j \qquad (al)_{x+1} = (al)_x - \sum_{j=1}^{m} (ad)_x^j$$

where there are *m* causes of decrement $j = 1, 2, \ldots m$.

Valuing cashflows

Cashflows can be evaluated either using integrals (as for multiple state models), or using a discrete, annual summation approach:

$$EPV = \sum \{\text{cashflow}\} \times \{\text{discount factor}\} \times \{\text{probability}\}$$

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 25 Practice Questions

25.1 Determine which of the following assertions relating to a multiple decrement table are correct:

I The dependent probabilities of decrement can never exceed the corresponding independent probabilities.

II Forces of decrement can never exceed 1 in value.

III The total of all the decrement numbers summed over all ages equals the initial radix of the table.

25.2 The active male membership of a large pension scheme follows the experience of the multiple decrement table below.

Age x	Active lives $(al)_x$	Age retirements $(ad)_x^r$	Ill-health retirements $(ad)_x^i$	Withdrawals $(ad)_x^w$	Deaths $(ad)_x^d$
16	10,000	0	0	750	5
17	9,245	0	0	600	5
18	8,640	0	0	500	5
…	…	…	…	…	…
39	3,150	0	2	30	7
40	3,111	0	3	20	8
…	…	…	…	…	…
63	2,200	100	25	0	20
64	2,055	200	30	0	25
65	1,800	1,800	-	-	-

Assume that decrements occur continuously, except for age retirements at age 65, which all occur on the 65th birthday. Stating any other assumptions that you make, calculate the following:

(i) The probability that a man who joins the scheme on his 18th birthday, will retire for any reason after his 63rd birthday.

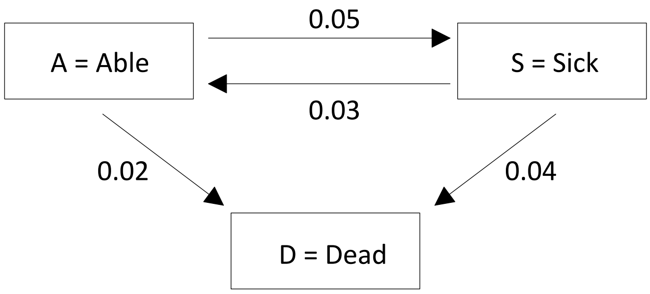
(ii) The independent probability of withdrawal between the ages of 17 and 18.

(iii) The expected present value of a lump sum retirement benefit of £10,000 payable on retirement at age 65, for a member now aged exactly 18, calculated using 4% *pa* interest.

(iv) The expected present value of a lump sum death benefit of £20,000 payable immediately on death while an active member if this occurs after the 63rd birthday, for a member now aged exactly 40, using 4% *pa* interest.

25.3 Exam style

An insurance company prices its sickness contracts using the three-state model and transition intensities shown below:



Level premiums of 2,500 *pa* are payable continuously while the policyholder is in the able state.

An able life aged 50 takes out a 15-year sickness contract that provides a 'no claim' benefit equal to 50% of the total premiums paid (without interest) if the life remains able for the full duration of the contract.

Calculate the expected present value of this 'no claim' benefit at outset, assuming that the force of interest is 6% *pa*. [3]

25.4 (i) Explain what is meant by a dependent probability of decrement and by an independent probability of decrement.

(ii) The following is an extract from a multiple decrement table subject to 3 modes of decrement, α, β and γ:

Age, x	$(al)_x$	$(ad)_x^\alpha$	$(ad)_x^\beta$	$(ad)_x^\gamma$
50	5,000	86	52	14
51	4,848	80	56	20

(a) Calculate the probability that a 50-year-old leaves the population through decrement γ between the ages of 51 and 52.

(b) Assuming that forces of decrement are constant between integer ages, calculate the independent probabilities q_{50}^α and ${}_{1|}q_{50}^\alpha$.

25.5 Exam style

A multiple decrement model involves three decrements a, b and c. Decrements a and b occur continuously over the year of age $(x, x+1)$, but decrement c occurs only at age $x+¼$. Also:

- the forces of decrement due to causes a and b are constant over the year of age $(x, x+1)$ and are equal to 0.03 and 0.01 per annum respectively
- the probability of decrement by cause c at exact age $x+¼$ is 0.06.

Calculate the value of $(aq)_x^a$. [6]

25.6 Exam style

A life insurance company issues sickness insurance policies to healthy individuals. Each policy pays the following benefits:

- an income of 6,000 per year payable continuously during all periods of temporary sickness (*ie* from which it is possible to recover), which is doubled during periods of permanent sickness (*ie* from which it is not possible to recover), with all sickness benefits ceasing at age 65
- on death at any time before age 65, a return of all premiums payable to date (including any waived premiums), without interest, payable immediately on death.

Level annual premiums are payable continuously to age 65 or to earlier death, except that they are waived (*ie* paid for by the insurer) during any period of sickness during that time.

(i) Draw and label a transition diagram that would be suitable for modelling this process for pricing purposes. [4]

(ii) Using the transition rates defined in your diagram, and probabilities of the form:

$$ {}_tp_x^{ij} = P\left(\text{person is in state } j \text{ at age } x+t \,\middle|\, \text{person is in state } i \text{ at age } x\right) $$

construct a formula, using integrals, for the annual premium for this contract (ignoring expenses), for a life aged exactly 45 at entry. [4]

[Total 8]

25.7 Exam style

The decrement table extract below is based on the historical experience of a very large multinational company's workforce:

	Number of employees	Deaths	Withdrawals
Age x	$(al)_x$	$(ad)_x^d$	$(ad)_x^w$
40	10,000	25	120
41	9,855	27	144
42	9,684		

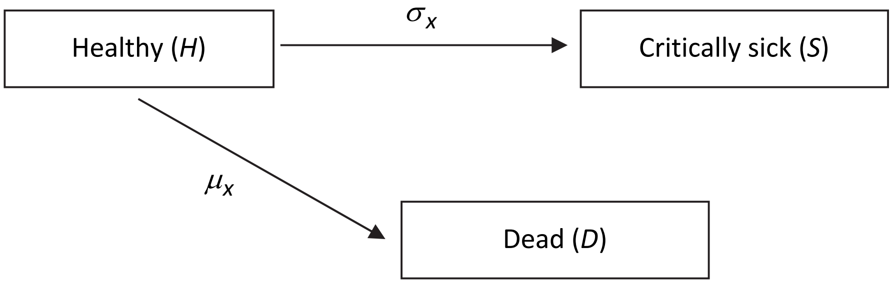
Recent changes in working conditions have resulted in an estimate that the annual independent force of withdrawal is now 75% of that previously used.

Calculate a revised table assuming no changes to the independent forces of mortality, stating your results to one decimal place. [7]

25.8 Exam style

An insurance company is considering the sale of a 'critical illness extra' term assurance policy. The critical illness benefit is £25,000, payable immediately on diagnosis of a critical illness within the 25-year term. The death benefit is £75,000, payable immediately on death during the term. Only one benefit is payable under any one policy and once the benefit has been paid, both the premiums and the cover cease. Annual premiums of £ P *pa* are payable continuously while the policy is in force.

The company assesses the profitability of the policy using the following multiple state model:



${}_tp_x^{ab}$ is defined as the probability that a life who is in state a at age x ($a = H, S, D$) is in state b at age $x + t$ ($t \geq 0$ and $b = H, S, D$).

(i) Suggest, with reasons, one group of customers the insurance company may wish to target in their marketing of this policy. [1]

(ii) Express in integral form, using the probabilities and the forces of transition, the expected present value of the profit from one such policy with an annual premium of £1,200 that has just been sold to a life aged exactly 50. [2]

After careful consideration, the company modifies the policy by changing both the death benefit and the critical illness benefit to be £50,000.

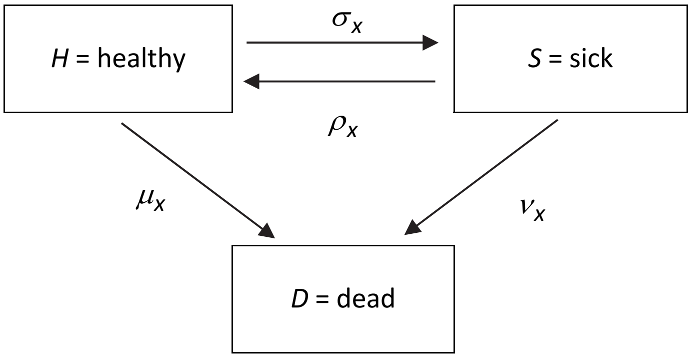
(iii) Explain how this modification could considerably reduce the cost of assessing claims. [2]

(iv) Given that $\mu_x = 0.0006$ and $\sigma_x = 0.0014$ for all $45 \leq x \leq 70$, and that the force of interest is 4% *pa*, calculate the expected present value of the benefits for the modified policy sold to a life aged exactly 45. [3]

[Total 8]

25.9 Exam style

A life insurance company uses a three-state model as shown below to calculate premiums for a two-year combined sickness and endowment assurance policy issued to healthy policyholders aged 58.



Premiums are payable at the start of each policy year and are waived if the policyholder is sick at the time the premium is due.

At the end of each policy year a benefit of £10,000 is payable if the life is then sick.

A sum assured of £15,000 is payable at the end of the year of death if this occurs during the term of the policy, or at the end of the term if the life is alive and has never claimed any sickness benefit.

S_t denotes the state occupied by the policyholder at age $58+t$, so that $S_0 = H$ (healthy) and $S_t = H, S$ or D (healthy, sick or dead) for $t = 1, 2$.

The transition probabilities used by the insurer are defined in the following way:

$$p_{58+t}^{jk} = P\left(S_{t+1} = k \mid S_t = j\right)$$

where for $t = 0, 1$:

$$p_{58+t}^{HD} = 0.02 \qquad p_{58+t}^{HS} = 0.1 \qquad p_{58+t}^{SD} = 0.05 \qquad p_{58+t}^{SS} = 0.09$$

Calculate the annual premium for this policy using the equivalence principle, based on the above transition probabilities and the following additional assumptions:

Interest: 3% *pa* effective

Initial expenses: £200 incurred at outset

Renewal expenses: £40 at time 1, whether the policyholder is healthy or sick

Claim expense: £30 at the date of payment of any benefit

[8]

25.10 A company provides the following benefits for its employees:

Exam style

- immediately on death in service, a lump sum of £20,000
- immediately on withdrawal from service (other than on death or in ill-health), a lump sum equal to £1,000 for each completed year of service
- immediately on leaving due to ill-health, a benefit of £5,000 *pa* payable monthly in advance for 5 years certain and then ceasing, and
- on survival in service to age 65, a pension of £2,000 *pa* for each complete year of service, payable monthly in advance from age 65 for 5 years certain and life thereafter.

The forces of decrement for the employees at each age, assumed to be constant over each year of age, are as follows:

Age x	μ_x^d	μ_x^i	μ_x^w
62	0.018	0.10	0.020
63	0.020	0.15	0.015
64	0.023	0.20	0.010

where μ_x^j is the (assumed constant) force of decrement by cause j over $(x, x+1)$, d represents death, i represents ill-health retirement and w represents withdrawal.

(i) Construct a multiple decrement table with radix $(al)_{62} = 100{,}000$ to show the numbers of deaths, ill-health retirements and withdrawals at ages 62, 63 and 64, and the number remaining in employment until age 65. [5]

(ii) Calculate the expected present value of each of the above benefits for a new entrant aged exactly 62. Assume that interest is 6% *pa* effective before retirement and 4% *pa* effective thereafter, and that mortality after retirement follows the PMA92C20 table. [10]

[Total 15]

Chapter 25 Solutions

25.1 I is true. For a dependent probability, the other decrements operate to reduce the exposure to the decrement of interest, so that a smaller proportion of lives are expected to leave by a particular cause than if the decrement was operating on its own.

II is false. Forces of decrement are rates of transition per unit time, and there is no restriction on their size (other than they cannot be negative). For example, the force of mortality exceeds 1 at high ages in the AM92 mortality table.

III is true. All the active lives at any age will eventually become a decrement by one cause or another at some future age.

25.2 (i) The probability that an 18-year-old will retire (voluntarily or through ill-health) after attaining age 63 is:

$$\frac{[(ad)^r_{63}+(ad)^r_{64}+(ad)^r_{65}]+[(ad)^i_{63}+(ad)^i_{64}]}{(al)_{18}}=\frac{(100+200+1{,}800)+(25+30)}{8{,}640}$$

$$=0.249421$$

(ii) Assuming the force of decrement due to withdrawal is constant over the year of age 17 to 18, the independent probability of withdrawal at age 17 can be calculated from:

$$q^w_{17}=1-e^{-\mu^w_{17}}$$

where μ^j_x is defined to be the (assumed constant) force of decrement due to cause j over the year of age $[x,x+1]$.

At age 17, there are two decrements operating (withdrawal and death). Again assuming constant forces of decrement over the year of age, we can find the value of $\bar{\mu}^w_{17}$ from:

$$\mu^w_{17}=\frac{(aq)^w_{17}}{(aq)_{17}}\left(-\ln[(ap)_{17}]\right)$$

$$=\frac{(ad)^w_{17}}{(ad)_{17}}\left(-\ln\left[\frac{(al)_{18}}{(al)_{17}}\right]\right)$$

$$=\frac{600}{605}\left(-\ln\left[\frac{8{,}640}{9{,}245}\right]\right)$$

$$=0.067121$$

Therefore:

$$q^w_{17}=1-e^{-0.067121}=0.064918$$

(iii) The expected present value is:

$$10{,}000 \times \frac{1{,}800}{8{,}640} \times v^{65-18} = £329.76$$

(iv) The expected present value (assuming deaths occur mid-year) is:

$$20{,}000 \times \left(\frac{20}{3{,}111} v^{63½-40} + \frac{25}{3{,}111} v^{64½-40} \right) = £112.64$$

25.3 If the policyholder remains in the able state for the full duration of the contract, 15 years of premiums will be paid, so the 'no claim' benefit will be:

$$0.5 \times 15 \times 2{,}500 = 18{,}750 \qquad [1]$$

The probability that the policyholder remains able for the full 15 years is:

$${}_{15}p_{50}^{\overline{AA}} = e^{-(0.02+0.05)\times 15} = e^{-1.05} \qquad [1]$$

So, the EPV of the 'no claim' benefit is:

$$18{,}750 \times e^{-0.06\times 15} \times {}_{15}p_{50}^{\overline{AA}} = 18{,}750 \times e^{-0.9} \times e^{-1.05} = 2{,}667.64 \qquad [1]$$

[Total 3]

25.4 (i) ***Dependent and independent probabilities***

A dependent probability of decrement takes into account the action of other decrements operating on the population. For example, the dependent probability $(aq)_x^{\alpha}$ is the probability that a life aged x will leave the active population through decrement α before age $x+1$, while all other decrements are operating.

An independent probability of decrement is a purely theoretical quantity that assumes there are no other decrements operating. For example, q_x^{α} is the probability that a life aged x will leave the active population through decrement α before age $x+1$, where α is the only decrement operating.

(ii)(a) ***Probability***

The probability that a 50-year-old member of the population leaves through decrement γ between the ages of 51 and 52 is:

$$\frac{(ad)_{51}^{\gamma}}{(al)_{50}} = \frac{20}{5{,}000} = 0.004$$

(ii)(b) ***Calculation of independent probabilities***

Since the forces of decrement are constant over each year of age, we have:

$$q_x^{\alpha} = 1 - e^{-\mu_x^{\alpha}}$$

where μ_x^j is defined to be the (assumed constant) force of decrement due to cause *j* over the year of age $[x, x+1]$.

We can find the value of μ_x^{α} from:

$$\mu_x^{\alpha} = \frac{(aq)_x^{\alpha}}{(aq)_x}\left(-\ln\left[(ap)_x\right]\right) = \frac{(ad)_x^{\alpha}}{(ad)_x}\left(-\ln\left[\frac{(al)_{x+1}}{(al)_x}\right]\right)$$

So:

$$\mu_{50}^{\alpha} = \frac{86}{86+52+14}\left(-\ln\left[\frac{4{,}848}{5{,}000}\right]\right) = 0.017467$$

and:

$$q_{50}^{\alpha} = 1 - e^{-0.017467} = 0.017315$$

Also:

$$\mu_{51}^{\alpha} = \frac{80}{80+56+20}\left(-\ln\left[\frac{4{,}848-(80+56+20)}{4{,}848}\right]\right) = 0.016773$$

so:

$$q_{51}^{\alpha} = 1 - e^{-0.016773} = 0.016633$$

From these we can then calculate:

$$_{1|}q_{50}^{\alpha} = \left(1 - q_{50}^{\alpha}\right) \times q_{51}^{\alpha} = (1 - 0.017315) \times 0.016633 = 0.016345$$

25.5 We will use notation such as $_t(ap)_x{}'$ and $_t(aq)_x{}'$ to represent dependent probabilities that *ignore* decrement c. We can then write:

$$(aq)_x^a = {}_{¼}(aq)_x^{a}{}' + {}_{¼}(ap)_x{}'\left(1 - q_{x+¼}^c\right){}_{¾}(aq)_{x+¼}^a{}' \qquad [2]$$

In order for the life to leave the population due to decrement a in the year of age $(x, x+1)$, it could either leave due to decrement a in the first quarter of the year, or, if it remains in the population until age $x+¼$, and does not leave due to decrement c at that age, it could leave due to decrement a in the last three quarters of the year.

Now for $0 \le r < 1$ and $0 < t \le 1-r$:

$$ {}_t(ap)_{x+r}{}' = \exp\left[-\left(\mu^a + \mu^b\right)t\right] = e^{-0.04t} \qquad [1]$$

and:

$$ {}_t(aq)^a_{x+r}{}' = \frac{\mu^a}{\mu^a + \mu^b}\left(1 - {}_t(ap)_{x+r}{}'\right) = \frac{0.03}{0.04} \times \left(1 - e^{-0.04t}\right) \qquad [1]$$

So:

$$ {}_{¼}(aq)^{a\,\prime}_x = 0.75\left(1 - e^{-0.04\times 0.25}\right) \qquad [½]$$

$$ {}_{¼}(ap)_x{}' = e^{-0.04\times 0.25} \qquad [½]$$

$$ {}_{¾}(aq)^a_{x+¼}{}' = 0.75\left(1 - e^{-0.04\times 0.75}\right) \qquad [½]$$

Therefore:

$$ (aq)^a_x = 0.75 \times \left(1 - e^{-0.01}\right) + e^{-0.01}\left(1 - 0.06\right) \times 0.75 \times \left(1 - e^{-0.03}\right) = 0.028091 \qquad [½]$$

[Total 6]

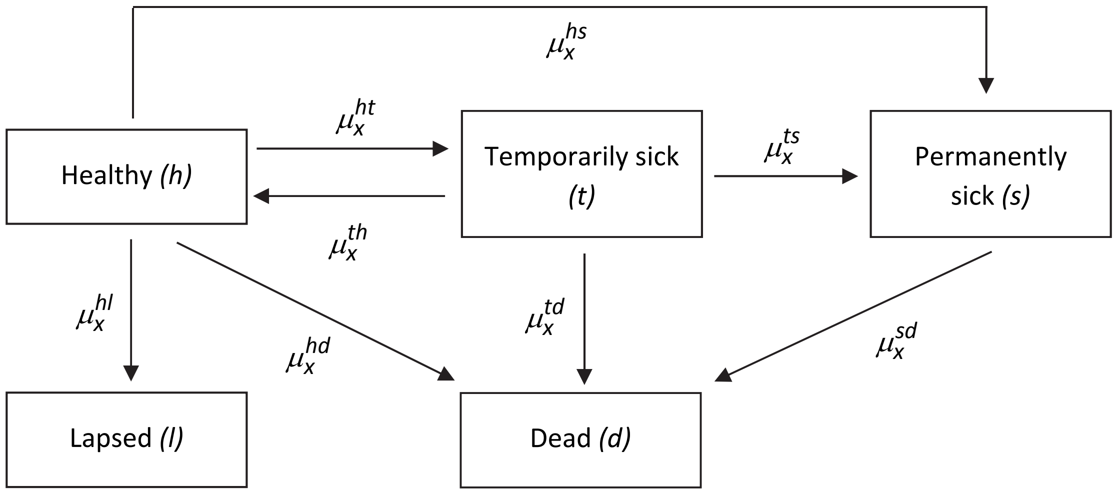
25.6 (i) ***Transition diagram***

Three key states to include will be healthy, sick and dead. However, we will need two separate sick states: a temporary (recoverable) sick state and a permanent (non-recoverable) sick state, as different benefit levels are paid in each. It would also be sensible to include lapses, as these policies are paid for by annual premiums so they can be lapsed if policyholders stop paying their premiums early.

We will also need to include the following transitions, each of which cause changes in payments:

- *healthy to temporary sickness and permanent sickness*
- *temporary sickness to healthy*
- *temporary sickness to permanent sickness*
- *healthy to lapsed*
- *healthy, temporary sickness, and permanent sickness, to dead.*

The following multiple state model would be appropriate:



[1½ *for including the correct states*]
[1½ *for including the correct transitions*]
[1 *for suitable labelling of transition rates*]
[Total 4]

(ii) ***Premium formula***

We will need to construct an equation of value:

$$EPV(\text{premiums}) = EPV(\text{benefits})$$

where P is the annual premium required (ignoring expenses).

(1) ***EPV of temporary sickness benefit***

This is:

$$6{,}000 \int_{r=0}^{20} v^r \, {}_rp_{45}^{ht} \, dr \qquad [1]$$

(2) ***EPV of permanent sickness benefit***

While permanently sick, 12,000 *pa* is paid. So the expected present value is:

$$12{,}000 \int_{r=0}^{20} v^r \, {}_rp_{45}^{hs} \, dr \qquad [½]$$

(3) ***EPV of death benefit***

On death at time r, a benefit of $P \times r$ would be payable, regardless of which state is then occupied (because the waived premiums are included in the benefit amount). As death can occur while in any of States *h*, *t*, or *s*, the expected present value is:

$$P \int_{r=0}^{20} r v^r \left[{}_r p_{45}^{hh} \mu_{45+r}^{hd} + {}_r p_{45}^{ht} \mu_{45+r}^{td} + {}_r p_{45}^{hs} \mu_{45+r}^{sd} \right] dr$$ [1½]

(4) ***EPV of premiums***

Premiums are only payable by healthy lives. So the expected present value is:

$$P \int_{r=0}^{20} v^r {}_r p_{45}^{hh} \, dr$$ [½]

So by setting $(1)+(2)+(3)=(4)$, we obtain:

$$P = \frac{6{,}000 \int_{r=0}^{20} v^r {}_r p_{45}^{ht} \, dr + 12{,}000 \int_{r=0}^{20} v^r {}_r p_{45}^{hs} \, dr}{\int_{r=0}^{20} v^r {}_r p_{45}^{hh} \, dr - \int_{r=0}^{20} r v^r \left[{}_r p_{45}^{hh} \mu_{45+r}^{hd} + {}_r p_{45}^{ht} \mu_{45+r}^{td} + {}_r p_{45}^{hs} \mu_{45+r}^{sd} \right] dr}$$ [½]

[Total 4]

Alternatively, we could use $e^{-\delta r}$ *in place of* v^r.

25.7 *This question is Subject CT5, April 2010, Question 10 (revised to make it consistent with the current syllabus).*

We first need to calculate the (dependent) forces of decrement from the existing multiple decrement table. Assuming forces of decrement are constant over individual years of age, and that independent and dependent forces are equal, we can use:

$$\mu_x^j = (a\mu)_x^j = \frac{(ad)_x^j}{(ad)_x^d + (ad)_x^w} \left[-\ln\left(\frac{(al)_{x+1}}{(al)_x} \right) \right]$$

Using this formula gives the following values:

Age x	μ_x^d	μ_x^w
40	0.002518	0.012088
41	0.002764	0.014740

[2]

Reducing the forces of withdrawal by 25% gives:

Age x	$^*\mu_x^d$	$^*\mu_x^w$
40	0.002518	0.009066
41	0.002764	0.011055

[1]

We now need to obtain the revised dependent probabilities, using:

$$^*(aq)_x^j = \frac{^*\mu_x^j}{^*\mu_x^d + {}^*\mu_x^w}\left(1 - e^{-(^*\mu_x^d + {}^*\mu_x^w)}\right)$$

This gives:

Age x	$^*(aq)_x^d$	$^*(aq)_x^w$
40	0.002504	0.009014
41	0.002745	0.010979

[2]

Finally using:

$$^*(al)_{40} = 10{,}000, \qquad ^*(ad)_x^j = {}^*(al)_x \times {}^*(aq)_x^j$$

$$^*(al)_{x+1} = {}^*(al)_x - {}^*(ad)_x^d - {}^*(ad)_x^w$$

we obtain the new table as:

Age x	$^*(al)_x$	$^*(ad)_x^d$	$^*(ad)_x^w$
40	10,000	25.0	90.1
41	9,884.8	27.1	108.5
42	9,749.2		

[2]
[Total 7]

25.8 (i) ***Possible customers***

This policy would be suitable for anybody with a family who has recently taken out a 25-year mortgage. [½]

If the policyholder dies or becomes critically ill, the benefit payment could be used to reduce the outstanding balance on the mortgage at a time of financial distress. [½]
[Total 1]

(ii) ***Expected present value of the profit***

The expected present value is:

$$1{,}200\int_0^{25} e^{-\delta t}\,{}_tp_{50}^{HH}dt - 75{,}000\int_0^{25} e^{-\delta t}\,{}_tp_{50}^{HH}\,\mu_{50+t}\,dt - 25{,}000\int_0^{25} e^{-\delta t}\,{}_tp_{50}^{HH}\,\sigma_{50+t}\,dt$$

$$= \int_0^{25} e^{-\delta t}\,{}_tp_{50}^{HH}\left(1{,}200 - 75{,}000\mu_{50+t} - 25{,}000\sigma_{50+t}\right)dt \qquad \text{[Total 2]}$$

In this model, since it is impossible to return to state H, ${}_tp_{50}^{HH} = {}_tp_{50}^{\overline{HH}}$.

(iii) ***Reducing the cost of underwriting claims***

The main problem with this policy in its original form is that the death benefit is three times larger than the critical illness benefit. Families of policyholders who become critically ill and then die may try (possibly fraudulently) to claim the death benefit instead of the critical illness benefit. [1]

The company would need to ensure that death claims are genuine in that the policyholder was not already critically ill when they died. This would involve running certain checks as part of the claims underwriting procedure and would cost money. [½]

If the death benefit is equal to the critical illness benefit then there is no need for these extra checks and hence costs. [½]

[Total 2]

(iv) ***Expected present value of the benefits***

The probability that a 45-year-old remains healthy for t years is:

$${}_tp_{45}^{HH} = \exp\left\{-\int_0^t\left(\mu_{45+s}+\sigma_{45+s}\right)ds\right\} = \exp\left\{-\int_0^t 0.002ds\right\} = \exp\left(-0.002t\right) \qquad [1]$$

The expected present value of the benefits is:

$$50{,}000\int_0^{25} e^{-\delta t}\,{}_tp_{45}^{HH}(\mu_{45+t}+\sigma_{45+t})dt = 50{,}000\int_0^{25} e^{-0.04t}e^{-0.002t}(0.0006+0.0014)dt$$

$$= 50{,}000\int_0^{25} 0.002e^{-0.042t}dt$$

$$= \frac{100}{0.042}\times\left[-e^{-0.042t}\right]_0^{25}$$

$$= £1{,}548 \qquad [2]$$

[Total 3]

25.9 *Expected present value of death benefit and death claim expenses*

This is:

$$15{,}030\times\left[p_{58}^{HD}v+\left(p_{58}^{HH}p_{59}^{HD}+p_{58}^{HS}p_{59}^{SD}\right)v^2\right]=15{,}030\times\left[\frac{0.02}{1.03}+\frac{0.88\times0.02+0.1\times0.05}{1.03^2}\right]$$

$$=612.02 \qquad [2]$$

where:

$$p_{58}^{HH}=1-p_{58}^{HS}-p_{58}^{HD}=1-0.1-0.02=0.88$$

Expected present value of sickness benefit and sickness claim expenses

This is:

$$10{,}030\times\left[p_{58}^{HS}v+\left(p_{58}^{HH}p_{59}^{HS}+p_{58}^{HS}p_{59}^{SS}\right)v^2\right]=10{,}030\times\left[\frac{0.1}{1.03}+\frac{0.88\times0.1+0.1\times0.09}{1.03^2}\right]$$

$$=1{,}890.85 \qquad [2]$$

Expected present value of maturity benefit and maturity claim expenses

This is:

$$15{,}030\,p_{58}^{HH}\,p_{59}^{HH}\,v^2=\frac{15{,}030\times0.88\times0.88}{1.03^2}=10{,}971.09 \qquad [1]$$

since $p_{58}^{HH}=p_{59}^{HH}$.

Expected present value of other expenses

This is:

$$200+40\left(p_{58}^{HH}+p_{58}^{HS}\right)v=200+\frac{40\left(0.88+0.1\right)}{1.03}=238.06 \qquad [1]$$

Expected present value of premiums

Using P for the annual premium, this is:

$$P\left[1+p_{58}^{HH}v\right]=P\left[1+\frac{0.88}{1.03}\right]=1.85437P \qquad [1]$$

Equating the EPV of the premiums with the EPV of the benefits and expenses, and solving for P, we obtain:

$$P=\frac{612.02+1{,}890.85+10{,}971.09+238.06}{1.85437}=£7{,}394.44 \qquad [1]$$

[Total 8]

25.10 (i) ***Multiple decrement table***

Since the forces of decrement are constant over each year of age, we can calculate the dependent probabilities of decrement using formulae of the form:

$$(aq)_x^d = \frac{\mu_x^d}{\mu_x^d + \mu_x^i + \mu_x^w}\left[1 - e^{-(\mu_x^d + \mu_x^i + \mu_x^w)}\right]$$

The dependent probabilities of decrement are then:

x	$(aq)_x^d$	$(aq)_x^i$	$(aq)_x^w$
62	0.01681	0.09341	0.01868
63	0.01826	0.13694	0.01369
64	0.02052	0.17841	0.00892

[3]

A multiple decrement table with radix $(al)_{62} = 100{,}000$ can then be created:

x	$(al)_x$	$(ad)_x^d$	$(ad)_x^i$	$(ad)_x^w$
62	100,000	1,681	9,341	1,868
63	87,110	1,591	11,929	1,193
64	72,397	1,486	12,916	646
65	57,349			

[2]
[Total 5]

(ii) ***Expected present value of benefits***

Death benefit

Assuming that death occurs, on average, halfway through the year, the expected present value of the death benefit is:

$$\frac{20{,}000}{(al)_{62}}\left(v^{½}(ad)_{62}^d + v^{1½}(ad)_{63}^d + v^{2½}(ad)_{64}^d\right) \quad [1½]$$

Evaluating this gives:

$$\frac{20{,}000}{100{,}000}\left(\frac{1{,}681}{1.06^{½}} + \frac{1{,}591}{1.06^{1½}} + \frac{1{,}486}{1.06^{2½}}\right) = £875 \quad [½]$$

Ill-health benefit

Assuming that ill-health retirement occurs, on average, halfway through the year, the expected present value of the ill-health retirement benefit is:

$$\frac{5{,}000}{(al)_{62}}\ddot{a}_{\overline{5}|}^{(12)}\left(v^{½}(ad)_{62}^{i}+v^{1½}(ad)_{63}^{i}+v^{2½}(ad)_{64}^{i}\right)$$ [1½]

Evaluating this, using $\ddot{a}_{\overline{5}|}^{(12)}=a_{\overline{5}|}\times\dfrac{i}{d^{(12)}}$ and values from the *Tables* at 4% (as this annuity is received after retirement), gives:

$$\frac{5{,}000\times4.4518\times1.021537}{100{,}000}\left(\frac{9{,}341}{1.06^{½}}+\frac{11{,}929}{1.06^{1½}}+\frac{12{,}916}{1.06^{2½}}\right)=£7{,}087$$ [1]

Withdrawal benefit

Assuming that withdrawal occurs, on average, halfway through the year, the expected present value of the withdrawal benefit is:

$$1{,}000v^{1½}\frac{(ad)_{63}^{w}}{(al)_{62}}+2{,}000v^{2½}\frac{(ad)_{64}^{w}}{(al)_{62}}=\frac{1{,}000}{100{,}000}\left(\frac{1{,}193}{1.06^{1½}}+\frac{2\times646}{1.06^{2½}}\right)=£22$$ [2]

Normal retirement benefit

The expected present value of the normal retirement benefit is:

$$3\times2{,}000\times v^{3}\times\frac{(al)_{65}}{(al)_{62}}\times\left(\ddot{a}_{\overline{5}|}^{(12)}+v^{5}\frac{l_{70}}{l_{65}}\ddot{a}_{70}^{(12)}\right)$$ [2]

Evaluating this gives:

$$\frac{6{,}000}{1.06^{3}}\times\frac{57{,}349}{100{,}000}\times\left[4.4518\times1.021537+\frac{1}{1.04^{5}}\times\frac{9{,}238.134}{9{,}647.797}\times\left(11.562-\frac{11}{24}\right)\right]$$

$$=£38{,}386$$ [1½]

[Total 10]

26

Unit-linked and accumulating with-profits contracts

Syllabus objectives

4.1 Define various assurance and annuity contracts.

4.1.3 Describe the operation of conventional unit-linked contracts, in which death benefits are expressed as combination of absolute amount and relative to a unit fund.

4.1.4 Describe the operation of accumulating with-profits contracts, in which benefits take the form of an accumulating fund of premiums, where either:

- the fund is defined in monetary terms, has no explicit charges, and is increased by the addition of regular guaranteed and bonus interest payments plus a terminal bonus; or
- the fund is defined in terms of the value of a unit fund, is subject to explicit charges, and is increased by regular bonus additions plus a terminal bonus (unitised with-profits).

In the case of unitised with-profits, the regular additions can take the form of (a) unit price increases (guaranteed and/or discretionary), or (b) allocations of additional units. In either case a guaranteed minimum monetary death benefit may be applied.

0 Introduction

In this chapter we describe policies for which the benefit takes the form of an accumulating fund of premiums, *ie* unit-linked (UL) and accumulating with-profits (AWP) contracts. Under UL contracts the insurance company has no discretion about the policy benefits that are paid but, under AWP, significant components of the benefit amounts are discretionary, following a similar rationale to conventional with-profits contracts, which were described in an earlier chapter. You should make sure you are familiar with this content before reading on.

The techniques required for valuing the cashflows for both these contract types are very different from the techniques used for the insurance contracts that we have looked at so far. These techniques, which involve the projection and discounting of future cashflows and profit flows, are described in later chapters.

1 Unit-linked contracts

Unit-linked assurances **(typically whole life or endowment) have benefits which are directly linked to the value of the underlying investments. Each policyholder receives the value of the units allocated to the policy. There is no pooling of investments or allocation of the pooled surplus.**

For example, policyholder X may choose to invest in a fund of domestic government securities, and policyholder Y may choose to invest in a fund of overseas equities. Policyholder X will experience the same return on their policy as all other policyholders investing in the domestic government securities fund over the same period, and policyholder Y will experience the same return as all other policyholders investing in the overseas equities fund over the same period. However, the returns obtained by X and Y will not affect each other.

As each premium is paid, a specified proportion (the 'allocation percentage') is invested in an investment fund chosen by the policyholder. The investment fund is divided into units which are priced continuously.

Suppose on 23rd April of a given year a particular fund of investments has a total market value of £50,000, and this fund is divided into 10,000 units. Every unit on a particular date has the same value (or 'price'), and so the price of a single unit on 23rd April is £5.00.

That is, in essence, the price of a unit at a particular time is calculated as the total market value of the investments held divided by the total number of units in issue.

Question

On the 24th April of the same year the total market value of the investments has increased to £50,756. There have been no cash transactions into or out of the fund between the two dates and so the fund still comprises 10,000 units.

(a) Calculate the price of each unit on 24th April, rounded to the nearest penny.

(b) A particular policy has been allocated 927 of these units and is to mature on 24th April. Calculate the maturity value that would be payable.

Solution

(a) ***Unit price on 24th April***

The unit price is equal to the total fund value divided by the number of units in issue on 24th April. This is:

$$\frac{£50,756}{10,000} = £5.08$$

(b) ***Maturity value***

At policy maturity the policyholder essentially exchanges their unit holdings for cash. The amount of cash received will be the total value of all the units that are cashed in. So the maturity value is:

$$927 \times 5.08 = £4,709.16$$

Once this policy has matured, the total fund value will reduce to:

$$£50,756 - £4,709.16 = £46,046.84$$

and the number of units in issue has reduced to:

$$10,000 - 927 = 9,073$$

So each unit's share of the fund remains equal to the current unit price, *ie*:

$$\frac{£46,046.84}{9,073} = £5.08$$

The reason for dividing the funds into units of equal value, in this way, is so that the returns credited to individual policies are as fair as possible. (As can be seen from our example, the fact that one policy has taken cash out of the fund does not affect the value of the remaining units, and so all the other policyholders who own units in the fund do not gain or lose any money as a result of the cash transaction.)

The calculation of the unit price from day to day is more complicated than this in practice. For example, there are usually *two* prices of each unit at the same time, the *bid price* (which is the cash-in value of each unit) and the *offer price* (which is the price that has to paid to purchase a unit in the fund). The difference between the two (with the offer price being greater than the bid price) is called the *bid-offer spread*, and this difference is money that the insurance company makes from each unit purchased and which helps cover its costs and enable it to make profits. Different approaches are also used for pricing units when the fund is expanding or contracting in size, but these details are beyond the scope of this course.

When each investment allocation is made, the number of units purchased by the policyholder is recorded. The value at the date of death or maturity of the cumulative number of units purchased is the sum assured under the policy.

The date of death or maturity would be the claim date of the policy and, in the absence of any guarantees or additional benefits, the benefit amount is simply the current value of the policyholders' allocated units at that point (usually just referred to as the current *unit fund* value).

Unit-linked policies may offer some guaranteed benefits. For example:

(1) on death during the policy term, the higher of a fixed sum assured or the value of units might be paid. This ensures that a significant benefit is paid out should the policyholder die early in the term.

(2) on survival to the maturity date of the policy, a minimum guaranteed sum assured, or a minimum average unit growth rate, may be applied. In either case, the maturity benefit is the higher of the guaranteed amounts (of sum assured or unit fund) and the actual unit fund value at the maturity date. This guarantee is to ensure that the policyholder avoids any difficulties arising from a particularly poor investment performance.

Death benefit guarantees are generally more common than maturity guarantees. In fact, policies with investment guarantees have proved very costly in the UK in the past.

In order to price and value unit-linked contracts, details of allocation percentages (usually specified in the policy) and an assumption about the future growth in the price of the units purchased are needed. The calculations involve projecting the expected profit flows on a year-on-year basis, and discounting these to obtain a measure of the expected profitability of the contract. The details of this are described in subsequent chapters.

So, with unit-linked contracts, the policyholder's basic entitlement is expressed in terms of units, which represent a portion of a fund or funds run by the life insurer. The value of these units moves in line with the performance of the fund.

Given this basic concept, the unit-linked idea can be used to provide many different types of product, for instance:

- a regular premium product offering a guaranteed sum assured on death, to give an endowment assurance;
- a single premium product offering a return of premium on death to give a pure savings bond; or
- a regular premium contract offering an annuity payment during periods of disability or unemployment.

Important terminology that we will use when discussing unit-linked contracts includes:

- unit account, or unit fund – this is the total value of the units in respect of the policy at any time.
- bid and offer price – as described above. The bid price is lower than the offer price (a 5% difference would be typical).
- charges – the company deducts money, either by cancelling units or by reducing the unit growth credited to the unit account on a periodic basis, for instance monthly. The company normally reserves the right to vary charges in the light of experience.

To illustrate these, consider a unit-linked endowment assurance policy, which has the following features and charges:

- Premiums are paid annually for ten years. 97% of every premium paid is used to buy units. The policyholder can choose between five different funds, *eg* the company's European Equities fund. There is a 5% bid/offer spread.
- There is a minimum guaranteed sum assured on death of £40,000 but no guaranteed minimum benefit on maturity.
- Every year the company deducts 1% from the bid value of the fund. This charge is made monthly in arrears (*ie* at the end of each month $1.01^{1/12}-1$ of the bid value of the units is deducted). This charge is called a fund management charge and is to pay for administration expenses. The company also deducts £50 *pa* from the policy's unit account. Again, this charge is made monthly in arrears. These charges may be varied in line with the company's experience.

Now suppose that in the first year the policyholder pays a premium of £1,500. Of this, 97% (= £1,455) is used to buy units. However, from this moment on, the fund is only worth 95% of £1,455 (*ie* £1,382.25) to the policyholder because of the 5% bid-offer spread. So the company only has to hold £1,382.25 worth of units, effectively taking the bid-offer spread from the policyholder at the moment the premium is paid.

Now let's imagine that the policyholder chooses to buy units entirely in the company's Mixed Fund, which is a mix of local and international bonds and equities. Then, over the next year, the fund grows by 12.3%.

So, by the end of the year, the unit account has (approximately) become:

$$1,382.25\times 1.123\times 0.99-50$$

(Actually a calculation of this type would have been done at least monthly, and possibly daily, so the above is slightly inaccurate.)

Contracts that are non-unit-linked are normally called *conventional*.

1.1 Unit funds and non-unit funds

The most important thing to bear in mind with unit-linked contracts is that we have two 'worlds' to keep track of: the *unit* world, and a *cash* (or *non-unit*) world. The policyholder pays premiums to acquire units, and the eventual benefit is normally denominated in these units, so we will need to keep track of the number of units bought by a policyholder, how they are growing, and what charges we are deducting from them.

However, the policyholder pays the life insurance company in real money. So we need to keep track of the cash not used to buy units, because that cash is a source of profit to the life insurance company. Conversely, if the policyholder dies there might be a cash denominated sum insured, and so we need to keep track of the cash outgo on claims. Another very significant cash outgo to consider is the company's expenses. These include expenses incurred in underwriting and maintaining the policy, as well as commission payments to whoever sold it.

Question

For a typical unit-linked contract, state:

(a) the different charges that might be used in the product, and

(b) the different non-unit cash outgoes that the company might have to pay.

Solution

(a) The charges include:

- money saved from the premium as a result of having an allocation rate of less than 100%
- bid/offer spread
- fund management charge
- expense charge (policy fee).

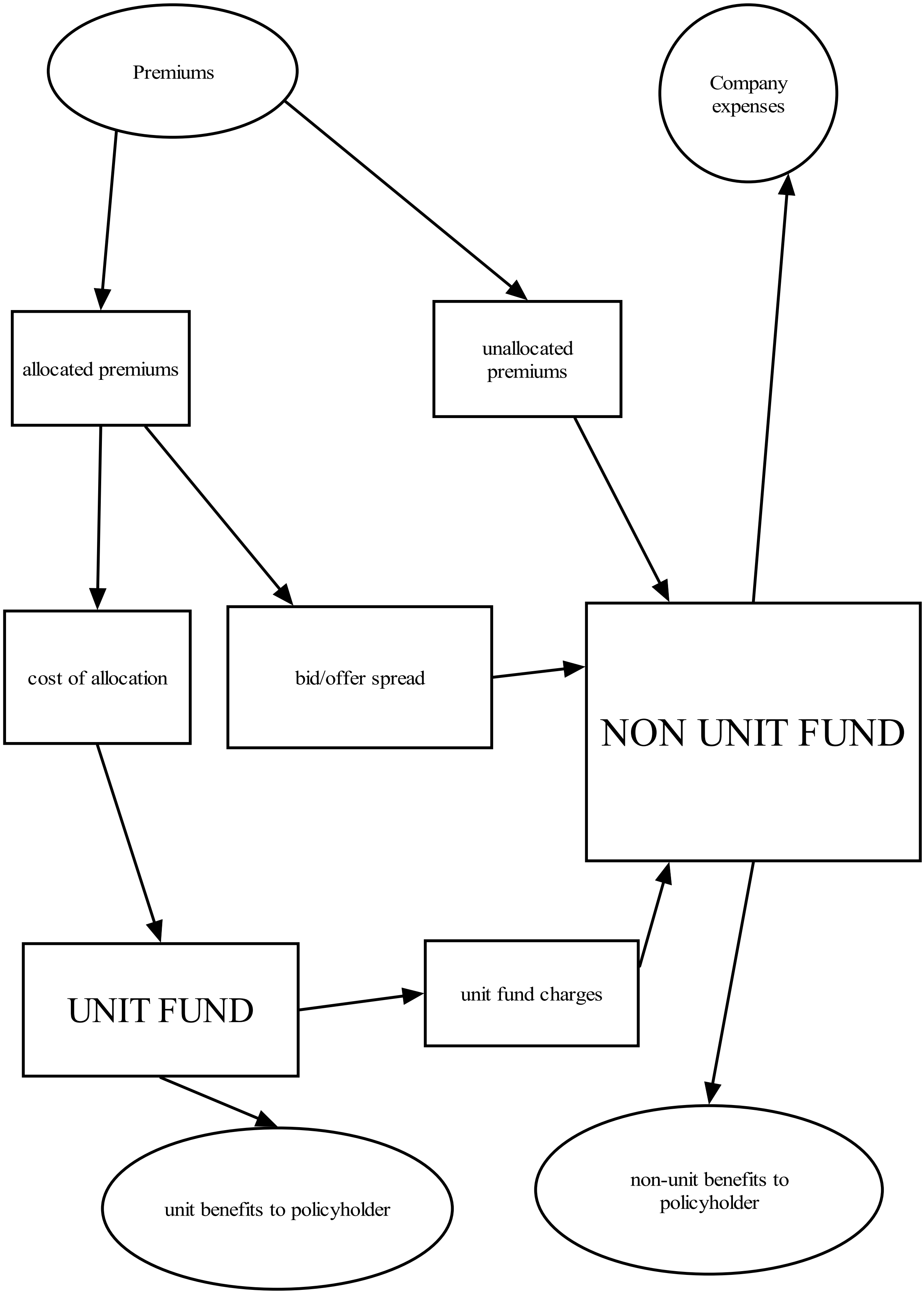
(b) The cash outgoes include:

- expenses
- maturity benefit (excess of any cash-denominated guaranteed maturity value over value of units)
- death benefit (excess of any cash-denominated guaranteed sum assured over value of units).

To clarify the situation, it is useful to consider things diagrammatically. Here, and at most other times in the course, we do not worry about splitting the policyholder's units into different funds, *eg* American Equities, Peruvian Goldmines, *etc*, but consider a generic unit fund.

It is also common to think of the cash world as a specific cash fund. We will refer to this as the *non-unit* fund.

The inter-relationship between the policyholder, the company, units and cash can be encapsulated in the following diagram.


Premiums
Company expenses
allocated premiums
unallocated premiums
cost of allocation
bid/offer spread
NON UNIT FUND
UNIT FUND
unit fund charges
unit benefits to policyholder
non-unit benefits to policyholder

Unit benefits are payable on surrender, death claim or maturity.

Non-unit benefits include, for instance, any sum insured payable on death in excess of the value of the units, or any guaranteed maturity value in excess of the value of the units.

Unit fund charges comprise:

- a fund management charge, for instance 0.5% *pa* of the unit fund in respect of fund management expenses,
- a policy fee (paid by cancellation of units) to cover other administration expenses, and
- a charge to cover the cost of providing any additional non-unit benefits, *eg* for any extra money paid out (over and above the unit fund value) when there is a guaranteed minimum death benefit.

One very important influence on the eventual experience of the policy is not shown in the diagram, because it occurs entirely 'within' the boxes. This is the fund growth experienced by the units, *eg* if the units are all invested in equities, it will be the combination of any capital appreciation of the equities, plus dividends received.

The insurance company is also likely to earn interest on any money it *holds* in the non-unit fund from time to time.

There are a number of important things to highlight in this diagram:

- The unit fund is worth only the *bid* value of the *allocated* premium – the rest of the premium goes to the non-unit fund.
- The unit fund charges are made in order to cover the expenses of fund management. However, charges and expenses are different cashflows (charges are part of the company's income while the expenses are part of its outgo) and are unlikely to be of the same amount.
- The profit or loss to the life company in each year is calculated as the difference between income (charges, unallocated premium, bid/offer spread) and outgo (expenses, non-unit benefits).
- The unit fund is what the policyholder sees – for instance, unit growth and all charges are communicated to the policyholder. The non-unit fund is what goes on within the company, and the policyholder does not see anything at this level.

Question

For the unit-linked endowment assurance shown on page 6 (details repeated below), calculate the unit fund value at the end of the first month of the policy (deducting all charges at the end of the month) for a 40-year-old policyholder, and calculate the total charges arising for the same period. Assume the unit fund has been growing at an annual rate of 12.3% over the first month (before the charges are deducted).

Policy details:

- Premiums of £1,500 are paid annually in advance.
- 97% of every premium paid is used to buy units.
- There is a 5% bid/offer spread.
- There is a minimum guaranteed sum assured on death of £40,000 but no guaranteed minimum benefit on maturity.
- A fund management charge payable at the rate of 1% *pa* is deducted at the end of each month from the bid value of the units.
- A policy fee payable at the rate of £50 *pa* is deducted at the end of each month by cancellation of units.

Solution

Premiums allocated are $97\% \times £1,500 = £1,455$.

We then deduct the bid/offer spread from this, so the value of units allocated is:

$$(1-5\%) \times £1,455 = £1,382.25$$

Growth over the month takes this up to $£1,382.25 \times 1.123^{\frac{1}{12}} = £1,395.68$.

We have the following charges:

- cash policy fee of: $\dfrac{£50}{12} = £4.17$
- management charge of $£1,395.68 \times \left(1.01^{\frac{1}{12}} - 1\right) = £1.16$.

So charges total £5.33 and the fund at the end of the month reduces to £1,390.35.

In the next question we consider what might happen to the *non*-unit fund in the same time period.

Question

The company's expenses in respect of this policy in the first month were 55% of the annual premium plus £178, and on average the mortality experience of all such policyholders was 58% of AM92 Ultimate.

Death claim payments are made at the end of the month, after all charges have been deducted.

Calculate the profit or loss to the company for the first month (ignoring any interest in the non-unit fund and assuming that the proportion of policyholders dying during the first month is one-twelfth of the annual proportion).

Solution

The policyholder pays a premium of £1,500 at the start of the first year. We saw in the solution to the previous question that the value of the allocated units is £1,382.25. The remainder of the premium, *ie* £117.75, goes into the non-unit fund.

Expenses are $0.55 \times 1{,}500 + 178 = £1{,}003$. Interest on the non-unit fund is to be ignored.

Since the value of the unit fund at the end of the first month is only £1,390.35, the guaranteed minimum benefit of £40,000 is paid out on death. The unit fund value will go towards paying the death benefit, but if the policyholder dies during the month an additional amount of $£40{,}000 - £1{,}390.35$ needs to be paid out of the non-unit fund to cover the shortfall.

To calculate the average actual cost of the death benefit we need to multiply the non-unit amount by the proportion of the policyholders dying over the first month. This is one-twelfth of 58% of the value of q_{40} in the AM92 Ultimate table (0.00937), and so the average actual death cost is therefore:

$$\left(40{,}000 - 1{,}390.35\right) \times \tfrac{1}{12} \times 0.58 \times 0.000937 = £1.75 \,.$$

From the solution to the previous question, the charges from the unit fund at the end of the month total £5.33.

So the profit is $117.75 - 1{,}003 - 1.75 + 5.33 = -£881.67$.

2 Accumulating with-profits contracts

These contracts originated in the UK, and now form almost all of the new with-profits business sold by UK insurers at the present time.

2.1 Definition

Under an accumulating with-profits (AWP) contract, the basic benefit takes the form of an accumulating fund of premiums (like a unit-linked policy, described in Section 1). If the accumulating fund at time t is denoted by F_t, the simplest form of an AWP contract follows the following recursive formula:

$$F_t = (F_{t-1} + P)(1 + b_t)$$

This example assumes that annual premiums of P are payable at the start of each year. b_t is the annual *bonus interest* declared for year t.

In this particular example, the 'bonus' interest is the *only* interest that is credited to the policy during the year.

As in the case of the regular reversionary bonus described for conventional with-profits contracts, this is a discretionary amount determined by the insurance company each year.

So the word 'bonus' is used here to refer to this interest being a discretionary payment, rather than as some addition to any other interest that might have been earned.

The bonus will reflect both the returns achieved on the underlying assets over the period plus any additional profits made on the contract in this time. As it is discretionary, it does not exactly reflect these amounts, and in practice the insurer tends to smooth out the variations in returns and profits achieved from year to year to produce a bonus interest rate that is more stable over time than the underlying asset returns, for example. A key feature of the regular bonus interest is that it cannot be negative, whereas for certain asset types (*eg* equity portfolios) actual returns can be negative.

Question

A man pays a premium of £7,000 at the start of each year under an accumulating with-profits contract. Calculate the fund value of the policy after 3 years if the insurer declares annual bonus interest rates as follows:

year 1: 2.3%

year 2: 2.6%

year 3: 2.5%

Solution

The fund value after 3 years is calculated recursively as follows:

$$F_1 = 7{,}000 \times 1.023 = £7{,}161$$

$$F_2 = (7{,}161 + 7{,}000) \times 1.026 = £14{,}529.19$$

$$F_3 = (14{,}529.19 + 7{,}000) \times 1.025 = £22{,}067.42$$

Sometimes, as was often the case in the UK in the past, part of the bonus interest would be guaranteed. One way of including a guaranteed bonus interest rate of g per annum is shown in the following recursive formula:

$$F_t = \left(F_{t-1} + P\right)(1+g)(1+b_t)$$

An alternative approach is just to guarantee that the value for b_t in any given year cannot be lower than g.

It is unusual for any guaranteed rate to be applied to AWP in modern conditions (other than the degenerate case where $g = 0$).

This is because investment returns in general are currently too low to enable insurance companies to offer non-zero guaranteed rates without risking significant losses. However, interest rates have been much higher historically, and many of the early AWP contracts included significant guarantees. For example, in the 1980s guarantees of 3% and 4% *pa* were not uncommon.

As with conventional with-profits, the regular bonuses under AWP can be reduced so as to retain profit for subsequent deferred payment as a terminal bonus. The contractual benefit under an AWP policy (payable on death or maturity as appropriate) could then be defined as:

$$B_t = F_t + T_t$$

where T_t is the amount of terminal bonus payable on a claim at time t. The purpose and rationale for paying terminal bonus is the same under AWP as it is for conventional with-profits.

Apart from the terminal bonus component, these simple AWP contracts operate in a very similar way to a deposit account administered by a bank.

Question

A man currently aged exactly 42 wishes to provide himself with a pension of around £25,000 *pa* on his retirement at age 67. He intends to purchase an accumulating with-profits endowment policy that will mature on his 67th birthday, and which he hopes will provide enough funds at retirement to purchase the required pension.

Calculate the annual premium that would provide the required expected amount of pension based on the following assumptions:

- premiums are level and are paid at the start of each year throughout the duration of the AWP contract, which has no explicit charges
- the insurance company declares an annual bonus interest rate of 3.5% *pa* throughout the duration of the AWP contract
- terminal bonus is ignored
- the annuity is to be paid monthly in advance for the whole of life from age 67, without guarantee
- the insurance company projects that it will use the following annuity basis to convert cash into annuity payments at the time of retirement:

Interest:	4% *pa*
Mortality:	PMA92C20 with a 7-year deduction from the age
Expenses:	Ignored

Solution

The fund required at age 67 to produce an annuity of £25,000 *pa*, payable monthly for the whole of life, is:

$$25{,}000\,\ddot{a}_{67-7}^{(12)} \approx 25{,}000 \times \left(\ddot{a}_{60} - \tfrac{11}{24}\right)$$

$$= 25{,}000 \times \left(15.632 - \tfrac{11}{24}\right)$$

$$= 379{,}342$$

This fund needs to equal the accumulated amount of the premiums, which are paid annually in advance over the preceding 25 years and which will be accumulated at the bonus interest rate of 3.5% *pa* over this period. Hence the premium P required to provide the fund under the AWP policy, using the given assumptions, satisfies:

$$P\,\ddot{s}_{\overline{25}|}^{@3.5\%} = 379{,}342$$

where:

$$\ddot{s}_{\overline{25}|}^{@3.5\%} = \frac{1.035^{25} - 1}{0.035/1.035} = 40.31310$$

Hence:

$$P = \frac{379{,}342}{40.31310} = £9{,}410 \ pa$$

2.2 Unitised (accumulating) with-profits contracts

Many companies that sell AWP administer the contract in unitised form (called *unitised with-profits* (UWP)). The policyholder is allocated units, and the fund value at any time for any policy is equal to the number of units held multiplied by the current price (or value) of each unit at that time.

In this way, UWP operates in a very similar way to unit-linked contracts. A key difference is the way the unit price is calculated. Two example possibilities are:

Method (1) **the unit price allows for guaranteed bonus interest increases only; the discretionary bonus is credited to the policy by awarding additional (bonus) units from time to time**

Method (2) **the unit price allows for both guaranteed and bonus interest increases.**

In both cases, it would be normal for unit prices to be changing on an effectively continuous basis (*eg* daily). The company would declare its regular bonus interest rate in advance, so that interest would accrue to policies at the equivalent daily rate.

2.3 Charges and benefits under UWP

The unitised nature of UWP means that it is easy to make allocations or deductions from the policyholder's fund at any time. As a result, much more complex product designs have been developed (often mirroring the unit-linked products that might be issued by the same insurance company). For UWP, insurers typically make explicit deductions for expense (and other) charges, as appropriate, in the same way as for unit-linked policies.

As for unit-linked policies, there will often be a minimum monetary benefit paid on death. For example, if a minimum death benefit of S was to be applied, then the death benefit payable on death at time t would be calculated as:

$$\max\left[S,\ B_t\right]=\max\left[S,\ F_t+T_t\right]$$

A minimum sum assured could also be applied at maturity in theory, but this is uncommon in practice.

A regular charge would then typically be deducted from the policyholder's fund to pay for the cost of providing the additional death (and/or maturity) benefit.

The charge for the minimum death benefit would be taken regularly, possibly annually but more likely on a monthly basis, and it would be proportional to the expected cost of providing the extra death benefit over the year (or month).

So the charge at policy duration t might be calculated as:

$$q'_{x+t}\times\max\left\{S-F_t\,,0\right\}$$

where q'_{x+t} is the probability of the policyholder (aged $x+t$ at duration t) dying over the next year (or month), S is the guaranteed sum assured payable on death, and F_t is the fund value at time t. This is also how the charge would be calculated for a unit-linked contract.

Question

Two UWP contracts (*A* and *B*) were issued on the same day, both for the same annual premium of £5,000 and with the same term to maturity of 20 years. Policy *A* had a minimum death benefit of £50,000, while Policy *B* had no minimum death benefit. The policies were otherwise identical.

Explain the differences that would be expected between the two policies in terms of the likely amounts of their:

(a) death benefits in 5 years' time

(b) death benefits in 15 years' time

(c) maturity benefits.

Solution

(a) ***Death benefits in 5 years' time***

For Policy *A*, in 5 years' time the fund value will have accumulated due to the addition of 5 premiums of £5,000 each, plus some bonus interest less some charges. There might also be a small terminal bonus component at this stage. The total is therefore likely to be somewhat higher than $5 \times £5{,}000 = £25{,}000$, but almost certainly nowhere near as high as the minimum sum assured of £50,000. The death benefit under Policy *A* will therefore be £50,000 at this time.

As Policy *B* pays out the fund plus any terminal bonus at the time of death, the death benefit under this policy will be significantly smaller than for Policy *A* after 5 years.

(b) ***Death benefits in 15 years' time***

After 15 years, 15 annual premiums of £5,000 will have been paid under each policy – amounting to £75,000 (ignoring interest and charges) – so the fund value itself will be greater than the minimum death benefit. The death benefit under Policy *A* should therefore be somewhat higher than £75,000, after bonus interest has been added, charges deducted, and any terminal bonus added at the time of claim.

At first sight the death benefit under policy *B* should be the same, as the minimum death benefit does not apply. However, it is probable that the insurer will have made appropriate charges on Policy *A* to cover the additional cost of providing the minimum death benefit during the early years. These additional charges will have caused the fund value to grow slightly more slowly in the case of Policy *A*, and so by time 15 the fund value (and therefore the death benefit) for Policy *B* is likely to be slightly higher than for Policy *A*.

(c) ***Maturity benefits***

The policies mature after 20 years. There is no minimum maturity value, so both will equal their respective fund values plus any terminal bonuses paid. The situation will therefore be similar to (b), *ie* if Policy *A* has incurred charges for the minimum death benefit, then the maturity value for Policy *B* will be slightly greater than that for Policy *A*.

2.4 Comparison between UWP and the simple AWP designs

With the simple AWP design (described in Section 2.1), the bonus interest would distribute profits net of all expenses and other costs incurred. In this way it is similar to the with-profits approach that is embodied in conventional with-profits contracts, described in a previous chapter.

In the case of UWP designs (described in Sections 2.2 and 2.3), explicit charges are made to cover the various expense and other costs incurred for the policy. The bonus rates declared would then be closely related to the rates of return obtained on the underlying assets only, smoothed (and possibly deferred) over time as usual. These contracts then fit in well with the unit-linked products that the same insurers might be offering.

It should be noted that AWP (and UWP) essentially provides an accumulating fund approach to with-profits. Many individual variations on the basic design are possible and it is, therefore, impossible to document them all in this course. Students should be aware of the basic approach and main variations described above.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 26 Summary

Unit-linked contracts

With unit-linked contracts the policyholder's basic entitlement is expressed in terms of units, which represent a portion of a fund or funds of investments managed by the life insurer. This entitlement is referred to as the *unit fund* of the policy. The unit fund value moves in line with the performance of the backing investments. Given this basic concept, the unit-linked idea can be used to provide many different types of product, for instance:

- a regular premium product offering a guaranteed sum assured on death, to give an endowment assurance;
- a single premium product offering a return of premium on death to give a pure savings bond; or
- a regular premium contract offering an annuity payment during periods of disability or unemployment.

The *non-unit fund* represents the accumulation of all cashflows paid in that are not used to buy units, less all cashflows paid out that have not arisen from the cancellation of units. As such it represents the accumulation of the company's profits from the policy at any time.

Accumulating with-profits (AWP)

The basic benefit is an accumulating fund of premiums with discretionary interest rates. The accumulation follows the recursive relation:

$$F_t = \left(F_{t-1} + p\right)(1+g)(1+b_t)$$

where g is the guaranteed annual interest and b_t is the bonus annual interest for year t.

On death or survival, the benefits can be further increased by a terminal bonus.

AWP contracts can be unitised (UWP) or non-unitised. For UWP, guaranteed interest is factored into the unit price. The bonus interest can be factored into the unit price also, or can be allocated by creating new units. Policies may include a guaranteed minimum death benefit. UWP is usually subject to explicit charges, to cover expenses and any additional death benefit cost.

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 26 Practice Questions

26.1 Describe the main features of a unit-linked policy.

26.2 Explain the terms 'unit fund' and 'non-unit fund' in the context of a unit-linked life assurance contract, listing the various items that make up the non-unit fund.

26.3 A woman now aged exactly 64 has paid £20,000 a year into an accumulating with-profits contract at the start of each of the last four years.

Exam style

The policy has incurred the following charges:

- £1,000 deducted at the start of year 1
- £100 deducted at the start of each subsequent year.

The following rates of regular bonus interest have been applied:

Year t	1	2	3	4
Bonus interest b_t	2.9%	3.1%	3.2%	3.4%

Additionally, there is a terminal bonus on contractual claim, currently payable at the rate of $0.015 \times (t-1)$ of the fund value, where t is the number of years the policy has been in force at the time of claim.

The policy is now maturing, and the woman is using all of the maturity proceeds to buy a level annuity from the insurance company. The annuity will be payable monthly in advance for a minimum of 5 years and for the whole of life thereafter.

Calculate the monthly amount of annuity that the woman will receive, if the insurance company uses the following basis in its annuity pricing:

Mortality: PFA92C20 with a 3-year age deduction

Interest: 4% *pa*

Expenses: £400 initial plus 0.35% of each annuity payment. [7]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 26 Solutions

26.1 Features of unit-linked policies:

- Benefits are directly linked to the value of the underlying investment.
- The benefit payable in respect of each policy depends on the value of the units allocated to that policy.
- Every time the policyholder pays a premium, part of it (the allocated premium) is invested on the policyholder's behalf in a fund chosen by the policyholder. The remainder goes into the company's non-unit fund.
- The investment fund is divided into units, which are priced continuously.
- Most companies use a bid/offer spread to help cover expenses and contribute to profit. The policyholder buys units at the offer price and sells them back to the company at the bid price. The bid price is usually about 5% lower than the offer price.
- Every time the policyholder pays a premium, the number of units purchased is recorded. When the policy matures or a claim is made, the value of the cumulative number of units purchased is available to be paid out as the benefit payment.
- The company will deduct money from the unit account on a periodic basis, *eg* monthly. This is to cover expenses and the cost of providing insurance. The charges are usually variable, and can be modified in the light of the company's experience.
- There may be a minimum guaranteed sum assured to protect the policyholder against poor investment performance, or to provide some benefit in the event of an early death.
- The most common types of unit-linked assurance are whole life and endowment assurances.

26.2 The unit fund is the amount held in units on behalf of the policyholder at any time.

It may not necessarily be the amount that the policyholder is entitled to at that time. For example, if the policy is surrendered, the policyholder may receive only a proportion of the full bid value of the units, and on death there may be a guaranteed minimum sum assured which means that more than the unit fund value might be paid.

On death, maturity or surrender, the units held will be used to pay the benefit. Any excess/ shortfall in the unit fund will give rise to a positive/negative cashflow in the non-unit fund.

The amount of money in the non-unit fund is the net result of the life office's (non-unit) cashflows.

These cashflows arise from the following sources:

- premium less cost of allocation, *ie* the difference between the premium paid by the policyholder and the amount invested in the unit fund on the policyholder's behalf
- expenses incurred by the life office
- interest earned/charged on the non-unit fund

- management charges taken from the unit fund
- extra death or maturity costs (if the benefit payable on death or maturity is greater than the value of the units held at the time of death or maturity)
- profit on surrender (if the benefit payable on surrender is less than the value of the units held at the time of surrender).

26.3 First we need to calculate the maturity benefit of the accumulating with-profits policy. The fund at maturity can be calculated recursively using the formula:

$$F_0 = 0$$

$$F_t = (F_{t-1} + P - E_t)(1 + b_t) \qquad t = 1, 2, 3, 4$$

where:

$F_t =$ fund value at time t

$P =$ premium

$E_t =$ expense charge at start of year t

$b_t =$ bonus interest for year t

This produces the fund values shown in the following table:

Year	Premium	Expense charge	Bonus interest rate	Fund at end of year
1	20,000	1,000	2.9%	19,551.00
2	20,000	100	3.1%	40,673.98
3	20,000	100	3.2%	62,512.35
4	20,000	100	3.4%	85,214.37

[2]

The terminal bonus rate at maturity is $0.015 \times 3 = 0.045$, so the total maturity value is:

$$85{,}214.37 \times 1.045 = 89{,}049.01$$ [1]

Using the equivalence principle, the monthly annuity payment (of X) can be obtained from the following equation, where the annuity factor is calculated for a life aged 61 in the *Tables* (being the age at maturity of 64 less the assumed three-year age reduction):

$$89{,}049.01 = 400 + (12X) \times \ddot{a}^{(12)}_{61:\overline{5}|} + (12X) \times 0.0035 \times \ddot{a}^{(12)}_{61:\overline{5}|}$$

$$\Rightarrow X = \frac{89{,}049.01 - 400}{12 \times 1.0035 \times \ddot{a}^{(12)}_{61:\overline{5}|}}$$ [1½]

Now:

$$\ddot{a}^{(12)}_{61:\overline{5}|} = \ddot{a}^{(12)}_{\overline{5}|} + v^5 \, {}_5p_{61} \, \ddot{a}^{(12)}_{66}$$

$$\approx \frac{1-v^5}{d^{(12)}} + v^5 \frac{l_{66}}{l_{61}} \left(\ddot{a}_{66} - \frac{11}{24} \right)$$

$$= \frac{1-1.04^{-5}}{0.039157} + 1.04^{-5} \times \frac{9,658.285}{9,828.163} \times \left(14.494 - \frac{11}{24} \right)$$

$$= 15.88456$$ [2]

and so the monthly amount of the annuity is:

$$X = \frac{89,049.01 - 400}{12 \times 1.0035 \times 15.88456} = £463.45$$ [½]

[Total 7]

27

Profit testing

Syllabus objectives

6.4 Project expected future cashflows for whole life, endowment and term assurances, annuities, unit-linked contracts, and conventional/unitised with-profits contracts, incorporating multiple decrement models as appropriate.

6.4.1 Profit test life insurance contracts of the types listed above and determine the profit vector, the profit signature, the net present value and the profit margin.

6.4.2 Show how a profit test can be used to price a product, and use a profit test to calculate a premium for life insurance contracts of the types listed above.

0 Introduction

In this chapter we introduce the technique of profit testing. This is the process of projecting the income and outgo emerging from a policy, and discounting the results. The results can then be used for various different purposes, such as setting the premium for a life policy that will give us our required level of profitability.

We shall see in the next chapter how we can also use profit tests to set reserves, and various other applications.

1 Evaluating expected cashflows for various contract types

Profit testing begins with the projection of the expected cashflows of a hypothetical policy. We first need to decide what units of time to use – for instance, whether we are going to consider every month of the policy's lifetime, or only every year.

The standard approach is to divide the total duration of a contract into a series of consecutive non-overlapping time periods. The length of each time period is chosen so that it is reasonable to make simple assumptions about the cashflows within each period, *eg* funds earn a constant rate of interest during the period, a particular cashflow accrues uniformly during the period. These assumptions allow the expected cashflows during the period to be evaluated.

We consider in great detail below exactly what these assumptions are.

The arithmetic of these calculations is usually most straightforward when the expected cashflows per contract in force at the start of the time period are calculated.

This turns out to be simpler than calculating the expected cashflows per contract in force at policy inception.

In practice, time periods will be short where there are many rapidly changing cashflows, *eg* the start of a contract, and longer where there are fewer cashflows.

So for instance we might project monthly for the first year of a policy's lifetime, and then yearly thereafter. In fact, given the extent of computational power now easily available, it may be simpler to just carry on with a monthly breakdown for the duration of the policy.

Question

Explain why cashflows are often changing rapidly in the first year of a contract, to the extent that a yearly breakdown could be very inaccurate.

Solution

We would expect the pattern of expenses to be very uneven over the course of the first year: big initial expenses on the first day, then small admin expenses spread over the year.

The expected cashflows, both positive and negative, are used to construct a projected revenue account (per contract in force at the start of the period) for each time period. The balancing item in the projected revenue account is the profit emerging at the end of the time period.

The revenue account for a life company is:

(+) premiums

(–) expenses

(+) investment income

(–) benefit payouts (claims, maturity, surrender values)

(–) increase in reserves

= *profit gross of tax*

(–) tax

= *profit net of tax*

Here the investment income is the income on reserves, plus interest on the balance of the cashflow itself; we exclude the investment income on the life company's other assets unconnected with the policies under consideration.

All that we are trying to do is to construct such a revenue account in respect of an individual (and probably hypothetical) policy for some given contract type. If we are doing this for a unit-linked or unitised with-profits (UWP) contract, then our projections get more complicated, and include separate projections of the unit fund and the non-unit fund. We shall consider an example of this later.

For some contracts there is only one in-force status and so only one projected revenue account is needed. For example, for a life assurance contract, it is in force when the policyholder is alive. For other contracts there is more than one in-force status. For example, for a disability insurance contract the policyholder can be alive and not receiving disability benefits or alive and receiving disability benefits. Projections for these contracts will require separate revenue accounts for each in-force status. Details of this are beyond the scope of this subject.

To calculate the individual elements of the projected revenue account for a policy, we require a basis, *ie* a set of assumptions (as described in Chapter 20).

Very often, the basis for these projected cashflows is a realistic estimate of expected future experience (as opposed to the more prudent estimates we might use for reserving). We consider the question of different bases in more detail in the next chapter.

In order to calculate the expected cashflows the following information is needed:

- **premiums received and their times of payment**
- **expected expenses (from the basis) and their times of payment**
- **contingent benefits payable under the contract, *eg* death benefit, annuity payment, survival benefit for endowment, difference between guaranteed sum assured and value of unit fund for unit-linked endowment**
- **other benefits payable under the contract, *eg* surrender values**
- **other expected cash payments, *eg* taxes**
- **other expected cash receipts, *eg* management charges levied on a unit fund, and**
- **the reserves required for a contract, usually at the beginning and end of the time period, calculated using the valuation basis**

together with the different probabilities of the various events leading to the payment of particular cash amounts. Any balance on the expected revenue account during the time period will be invested, and an assumption about the rate of investment return is needed. This allows the expected investment income during the period to be calculated and credited at the end of the period.

1.1 Example 1: Conventional whole life assurance

The contract is issued to a select life aged x and has a sum assured of S secured by level annual premiums of P payable in advance. The premium basis assumes initial expenses of I and renewal expenses of e. The valuation basis requires reserves of $S._tV$ for an in force policy with sum assured S at policy duration t. The basis assumes that invested funds earn an effective rate i. The surrender value basis determines that a surrender value $(SV)_t$ will be paid to policies surrendered at policy duration t.

Note the difference between $(SV)_t$, where '*S*' stands for surrender and $S._tV$, where '*S*' stands for the sum assured.

The probabilities of events are determined from a multiple decrement table with decrements of death, d, and surrender, w, having dependent probabilities at age, x, of $(aq)_x^d$ and $(aq)_x^w$.

The following shows the expected profit calculation for all policy years except the first year. In this formula time t is the beginning of the policy year, and time $t+1$ is the end of that year. This profit is the what we would expect each year in relation to a policy that is *in force* at the start of that year, *ie* for a policy that is in force at exact time t .

Income		
Premiums	P **(from data)**	
Interest on reserves	$i.S._tV$	
Interest on balances	$(P-e)i$	**(*)**
Expenditure		
Expenses	e **(from data)**	**(*)**
Expected surrender value	$(aq)^w_{[x]+t}.(SV)_{t+1}$	
Expected death claims	$(aq)^d_{[x]+t}.S$	
Transfer to reserves	$(ap)_{[x]+t}.S._{t+1}V - S._tV$	
Profit	**Balancing item**	

This assumes that expenses occur at the start of each time period, and death claims and surrender values are paid at the end of each time period.

The 'transfer to reserves' item needs some explanation. At the start of the year the insurance company holds reserves of amount $S._tV$. This is money that is currently 'in the bank' for a policy that is in force at this point. When we reach the end of the year, we need to account for the fact that the company is *required* to hold reserves of $S._{t+1}V$ for each policy that is then in force. The increase in reserve between the start and the end of the year represents additional money that the company will have to set aside (*ie* transfer) from other income, *ie* it will be a deduction from the profit earned during the year. However, not all of the policies that start out the year will still be in force at the end of the year, and we will only need to hold reserves for policies that are still in force. So, we multiply the reserve at the end of the year by the probability of the policy staying in force during the year (*ie* $(ap)_{[x]+t}$) and so the item in the profit calculation is the *expected* cost of the profit transfer per policy in force at the start of the year.

Combining the items together and rearranging them slightly, we can show the expected profit arising for the time period $[t, t+1]$, for $t = 1, 2, \ldots$, as:

$$P - e + (P-e)i - (aq)^d_{[x]+t}.S - (aq)^w_{[x]+t}.(SV)_{t+1} + S._tV + i.S._tV - (ap)_{[x]+t}.S._{t+1}V$$

This is the formula we are going to use later when we start doing some example calculations.

In the first policy year (*ie* for $t = 0$), the renewal expenses e are replaced by the initial expenses I at (*) above, and in the above formula.

Question

In the first policy year, the reserve at the start of the year requires the value of ${}_0V$. Explain what the value of ${}_0V$ would be, and hence write down a formula for the expected profit arising in the first policy year.

Solution

According to the reserve notation used in this section of Core Reading, ${}_0V$ is the reserve per unit of sum assured for a policy in force at the very start of the policy, *ie* just before the first premium is paid. The aim of the calculation is to work out the expected profit in the first policy year, and so it is sensible to assume that the insurance company has assets of zero (*ie* has no money) in relation to this policy at this point. Hence we assume ${}_0V = 0$.

The expected profit in the first policy year is therefore:

$$P - I + (P - I)i - (aq)^d_{[x]}.S - (aq)^w_{[x]}.(SV)_1 - (ap)_{[x]}.S.{}_1V$$

Temporary assurances and endowment assurances follow a similar approach.

1.2 Example 2: Conventional endowment assurance

Suppose a life insurance company sells a 5-year regular-premium endowment assurance policy to a 55-year old male. The sum insured is £10,000 payable at the end of year of death. Initial expenses are 50% of annual premium, renewal expenses are 5% of subsequent premiums. Premiums are payable annually in advance.

There is a surrender benefit payable equal to a return of premiums paid, with no interest. This is paid at the end of the year of withdrawal.

The company is required to hold net premium reserves, calculated ignoring surrenders.

We shall now calculate the projected yearly cashflows per policy in force at the start of each year, using the following bases.

For pricing:	AM92 Ultimate mortality, 4% *pa* interest, expenses as above and ignoring surrenders, using the equivalence principle
For reserving:	Interest and mortality as per pricing

For future cashflow projection: Interest and expenses as per pricing, dependent surrender and mortality probabilities as in the table below.

Age x	$(aq)_x^d$	$(aq)_x^w$
55	0.005	0.1
56	0.006	0.05
57	0.007	0.05
58	0.008	0.01
59	0.009	0

We first need to calculate the annual premium payable, P. This satisfies the equation

$$P\ddot{a}_{55:\overline{5|}} = 10{,}000 A_{55:\overline{5|}} + 0.5P + 0.05P\left(\ddot{a}_{55:\overline{5|}} - 1\right)$$

So:

$$P = \frac{10{,}000 A_{55:\overline{5|}}}{0.95\ddot{a}_{55:\overline{5|}} - 0.45} = \frac{10{,}000 \times 0.82365}{0.95 \times 4.585 - 0.45} = £2{,}108.81$$

We now need to work out the reserves that the company must hold over the term of the policy. We are told the company holds net premium reserves, and for this policy type we can use the formula given on page 37 of the *Tables*, remembering to include a factor of 10,000 for the sum assured. So the reserve at time t is:

$$_tV = 10{,}000\left(1 - \frac{\ddot{a}_{55+t:\overline{5-t|}}}{\ddot{a}_{55:\overline{5|}}}\right)$$

So we have:

$$_0V = 0$$

$$_1V = 10{,}000\left(1 - \frac{\ddot{a}_{56:\overline{4|}}}{\ddot{a}_{55:\overline{5|}}}\right) = 10{,}000\left(1 - \frac{3.745}{4.585}\right) = 1{,}832.06$$

$$_2V = 10{,}000\left(1 - \frac{\ddot{a}_{57:\overline{3|}}}{\ddot{a}_{55:\overline{5|}}}\right) = 10{,}000\left(1 - \frac{2.870}{4.585}\right) = 3{,}740.46$$

and so on.

This gives the following development of reserves.

Year t	Reserve at start of year, ${}_{t-1}V$
1	0
2	1,832.06
3	3,740.46
4	5,736.10
5	7,818.97

We now need to calculate the expected cashflows for a policy in force at the start of each year. For instance, for the first year we have:

Premium paid in of	2,108.81	
Expenses	–1,054.41	from 0.5 × 2,108.81
Interest on other cashflow	42.18	from $(P-e)i$
Expected death claims	–50	from $10,000(aq)_{55}^{d}$
Expected surrenders	–210.88	from $P(aq)_{55}^{w}$
Expected maturities	0	
Expected Increase in reserves	–1,639.69	

The expected increase in reserve (or 'transfer to reserve' as it is called in Section 1.1) is calculated as follows. At the start of the year we have no reserve. This also means there is no interest earned on the reserve. At the end of the year we need 1,832.06 per policy still in force. The probability that a policy in force at the start of Year 1 is still in force at the end of Year 1 is:

$$(ap)_{55} = 1-(aq)_{55}^{d}-(aq)_{55}^{w} = 0.895$$

So the expected cost of increasing the reserve is:

$$1,832.06 \times 0.895 - 0 \times 1.04 = 1,639.69$$

Altogether this gives an expected profit at the end of Year 1 of:

$$2,108.81 - 1,054.41 + 42.18 - 50 - 210.88 - 1,639.69 = -803.99$$

We can carry on and do the same over the remaining years of the policy's term, as shown in the table below (where outgoing cashflows are shown as negative values):

Year	Premium	Expense	Interest	Expected claim cost	Expected surrender cost	Expected cost of increase in reserves	Expected profit per policy in force at start of year
1	2,108.81	–1,054.41	42.18	–50	–210.88	–1,639.69	–803.99
2	2,108.81	–105.44	80.13	–60	–210.88	–1,625.65	186.97
3	2,108.81	–105.44	80.13	–70	–316.32	–1,519.06	178.12
4	2,108.81	–105.44	80.13	–80	–84.35	–1,712.68	206.47
5	2,108.81	–105.44	80.13	–10,000	0	8,131.73	215.23

where the expected claim cost includes the cost of maturing policies.

Question

(i) Verify the entries for Year 3 of this policy.

(ii) For Year 5, verify the entry for the expected cost of increase in reserves.

Solution

(i) ***Year 3 calculations***

The premium is £2,108.81.

Expenses are 5% of premium = –105.44.

Interest is $4\% \times (2{,}108.81 - 105.44) = 80.13$.

Expected claim cost is $-10{,}000(aq)^d_{57} = -10{,}000 \times 0.007 = -70$.

Expected surrender cost is $-3P(aq)^w_{57} = -3 \times 2{,}108.81 \times 0.05 = -316.32$.

The reserve at the start of Year 3 is 3,740.46. This earns interest at the rate of 4% *pa* to become 3,890.08 by the end of Year 3. At the end of Year 3, we need a reserve per policy in force of:

$$_3V = 10{,}000\left(1 - \frac{\ddot{a}_{58:\overline{2|}}}{\ddot{a}_{55:\overline{5|}}}\right) = 10{,}000 \times \left(1 - \frac{1.955}{4.585}\right) = 5{,}736.10$$

The probability that a policy in force at the start of Year 3 is still in force at the end of Year 3 is $(ap)_{57} = 1-(aq)_{57}^{d} - (aq)_{57}^{w} = 0.943$. So the expected cost of the increase in reserves is $5{,}736.10 \times 0.943 - 3{,}890.08 = 1{,}519.06$. (This is a cost to the company so it is shown as a negative entry in the table of cashflows.)

The expected profit emerging at the end of Year 3 per policy in force at the start of Year 3 is then:

$$2{,}108.81 - 105.44 + 80.13 - 70 - 316.32 - 1{,}519.06 = £178.12$$

(ii) ***Year 5 calculations***

At the start of Year 5 we have reserves of 7,818.97 per policy in force. This earns interest at the rate of 4% *pa* to become 8,131.73 by the end of Year 5. We don't need to hold any reserve at time 5 (once we have paid the benefits) because the policies are finished. So there is a release of reserves of £8,131.73, which is shown as a positive cashflow in the table.

The entries in the table for other policy years are calculated similarly.

Question

Explain why there is a big loss in the first year, despite the fact that the expenses 'experienced' were the same as those used to price the policy.

Solution

It's not how we *priced* the contract that matters here, but how we calculated the *reserves* that we have used. If we had used a gross premium reserve, we would have deducted the future gross premiums in the reserve calculations, which would have made the reserve at the end of Year 1 much smaller than 1,832, for example. This would have offset the large loss in Year 1 caused by the large expenses in Year 1.

However, instead we have a used a net premium reserve. In a net premium reserve we only deduct the future *net* premiums in the reserve calculation. Net premiums are significantly smaller than gross premiums, which makes the reserve larger. This bigger reserve, combined with the high initial expenses that occur in Year 1, leads to the large loss that we have seen in that year.

1.3 Example 3: Unit-linked endowment assurance

The contract is issued to a life aged x and has a sum assured equal to the bid value, at the time of death, of the units purchased, subject to a minimum guaranteed sum assured of S. It is secured by level annual premiums of P, of which $a_t\%$ is allocated to the unit fund at the start of policy year t at the offer price.

Recall that the bid price is the price at which units are bought back from the policyholder or, in other words, the price at which the insured redeems units; the bid value is the number of units multiplied by the bid price. The offer price is the price at which units are sold to the policyholder or, in other words, the price at which the insured purchases units.

The $a_t\%$ is an arbitrary number, not to be confused with an annuity value

The premium basis assumes initial expenses of I and renewal expenses of e per annum incurred at the beginning of each of the second and subsequent policy years. Unit reserves are held in the unit fund, and no allowance is made for reserves in the non-unit (cash) fund.

By unit reserve, here we mean the bid value of units. We shall see in the next chapter that it is sometimes necessary to hold not only unit reserves but also non-unit reserves, *ie* a cash-denominated reserve as a contingency against future negative (non-unit) cashflows. In this example no such reserves are required.

Investments in the non-unit fund are assumed to grow at $i\%$ *pa*. Suppose the unit fund projections show a fund value of F_t at policy duration t after management charges at a rate $k_t\%$ *pa* have been paid to the non-unit fund. The bid price (sale price) of units is $(1-\lambda)\times$ offer price, *ie* there is a bid-offer spread of $100\lambda\%$. F_t is evaluated at the bid price.

The annual management charge paid from the unit fund to the non-unit fund together with the bid-offer spread (*ie* units are sold to policyholders at a price greater than their underlying value) are analogous to charges covering the level annual expenses for a traditional assurance policy.

They are analogous in that the main way the company covers level annual expenses with unit-linked contracts is via these two charges. However, the fund management charge has a very different 'shape' to what we see with conventional contracts because, as the fund grows, the fund management charge will grow. This growth is very significant for regular premium contracts as the fund (and hence the charge) increases significantly every time a premium is paid.

The charge of $(1-a_t\%)$, which results from the allocation percentage imposed by the company, can be used to cover both initial expenses (with a very low a_1) and renewal expenses (with a sufficiently low a_t for $t > 1$).

On death, the bid value of the unit fund, after management charges, is paid at the end of the year of death, subject to the minimum guaranteed sum assured of S.

The following formulae look complex, but applying them to numerical examples is comparatively intuitive. So don't worry too much about the details of this on first reading; instead, study carefully the numerical example and questions that follow.

The expected cashflows to and from the non-unit fund in policy year t can be evaluated, where $t=1$ denotes the first policy year.

Income

Premiums not allocated to unit fund	$\left(1-\dfrac{a_t}{100}\right)P$
Bid-offer spread	$\lambda\dfrac{a_t}{100}P$
Management charge on the unit fund (taken at the year-end)	$\dfrac{F_{t+1}}{\left(1-\dfrac{k_t}{100}\right)}\left(\dfrac{k_t}{100}\right)$
	(from unit fund projections)
Investment income on balances	$\left\{\left(1-\dfrac{a_t}{100}+\dfrac{\lambda a_t}{100}\right)P-I\right\}i \qquad t=1$
	$\left\{\left(1-\dfrac{a_t}{100}+\dfrac{\lambda a_t}{100}\right)P-e\right\}i \qquad t>1$

Expenditure

Expenses	e **(from data) (or I when $t=1$)**
Expected cost of death claims	$(S-F_{t+1})q_{x+t}$ **if positive**
Profit	**Balancing item**

The fact that no non-unit reserves were deemed necessary means that there is no 'interest on reserves' element in the income section.

It is important to realise that the above gives us only the cashflows in the non-unit fund. It will normally be necessary to calculate the projected unit fund first, so as to calculate the F_{t+1} values needed for several of the above items.

We shall see how this works in the following numerical example.

A life insurance company is studying the profitability of a 5-year unit-linked endowment assurance contract. Details are as follows:

Age at issue	50	
Annual premium	£2,000	
Benefit	The greater of the bid value of units and £5,000 (paid at maturity or at the end of the policy year of earlier death)	
Allocation rate	First year:	60%
	Other years:	98%
Bid/offer spread	5%	

Management charge	1% *pa* (deducted at end of year)	
Unit growth rate	6% *pa*	
Interest for non-unit fund	6% *pa*	
Mortality	AM92 Ultimate	
Expenses	Initial	£1,150
	Renewal	£75 at the start of the second year, subsequently inflating at 4% *pa*

We shall calculate the expected profit or loss on the non-unit fund in each year, per policy in force at the start of each year.

We first need to project the size of the unit fund over the term of the policy.

Question

Calculate the value of the unit fund at the end of the first year.

Solution

At the start of Year 1 the allocated premium is $2,000 \times 0.6 = 1,200$. The bid value of the allocated premium is then $1,200 \times 0.95 = 1,140$. (Alternatively we could just calculate this as $2,000 \times 0.6 \times 0.95 = 1,140$.) This amount is referred to as the 'cost of allocation'.

The fund at end of year, before deduction of the management charge, is $1,140 \times 1.06 = 1,208.40$.

The management charge is $-1\% \times 1,208.40 = -12.08$.

The fund after deduction of management charge is $1,208.40 - 12.08 = 1,196.32$ (this can alternatively be calculated as 99% of 1,208.40).

We can repeat the logic described above to project the value of the unit fund to the end of the five-year term (where outgoing cashflows are shown as negative entries):

Year	Prem rec'd	Prem all'd	Cost of all'n	Fund after all'n	Fund before mgt charge	Mgt charge	Fund at year end
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	2,000	1,200	1,140	1,140.00	1,208.40	–12.08	1,196.32
2	2,000	1,960	1,862	3,058.32	3,241.81	–32.42	3,209.40
3	2,000	1,960	1,862	5,071.40	5,375.68	–53.76	5,321.92
4	2,000	1,960	1,862	7,183.92	7,614.96	–76.15	7,538.81
5	2,000	1,960	1,862	9,400.81	9,964.86	–99.65	9,865.21

The entries in each column are calculated as follows:

(3) premium allocated = premium received × allocation percentage

(4) cost of allocation = premium allocated (3) × (1 – bid/offer spread)

(5) fund after allocation = fund at end of previous year (8) + cost of allocation (4)

(6) fund at the end of the year before deduction of management charge = fund after allocation (5) × (1 + unit growth rate)

(7) fund management charge = fund on day 364 (6) × management charge

(8) fund at year end after deduction of management charge = fund before deduction (6) – management charge (7).

Question

Verify the entries for Year 3.

Solution

At the start of the year the allocated premium is $2{,}000 \times 0.98 = 1{,}960$.

The bid value of the allocated premium is $1{,}960 \times 0.95 = 1{,}862$.

The fund after allocation is $1{,}862 + 3{,}209.40 = 5{,}071.40$.

The fund at end of year, before management charge, is $5{,}071.40 \times 1.06 = 5{,}375.68$.

The management charge is 1% of this , *ie* – 53.76.

The fund after deduction of this management charge is:

$$5{,}375.68 - 53.76 = 5{,}321.92$$

Having done this, we can start to calculate the non-unit cashflows. For instance, in the first year we have the following elements of outgo:

Expenses – 1,150

Expected death cost $-q_{50}\left(sum\ assured - unit\ \ fund\right) = 0.002508 \times \left(5{,}000 - 1{,}196.32\right)$

= – 9.54

Question

Calculate the other expected cashflows in the non-unit fund in this first year, remembering to allow for interest and the fund management charge. Hence calculate the expected profit in the non-unit fund in the first policy year.

Solution

We have the following items of income in the non-unit fund:

- at the start of the year we have the remainder of the premium, after the cost of allocating the premium has been deducted, *ie* $2{,}000-1{,}140=860$, and
- at the end of the year the fund management charge of 12.08 (from the unit fund calculations).

Interest on the non-unit fund is negative in the first year due to the effect of expenses:

$$(860-1{,}150)\times 6\%=-17.40$$

The total of all these elements is:

$$860+12.08-17.40=854.68$$

The expected profit in the non-unit fund at the end of the first year is then:

$$854.68-1{,}150-9.54=-£304.86$$

We can do the same for every year of the contract, to determine the profit at the end of each year per policy in force at the start of that year.

One area where we need to be slightly careful is in calculating the expected death cost, because if the value of the units goes above the guaranteed sum insured then the death cost will be zero.

We find the following development of non-unit cashflows. It helps to think chronologically through the company's cashflows when constructing tables like this.

Year *t*	Premium less cost of all'n	Expenses	Interest	Expected death cost	Mgt charge	Expected profit
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	860	–1,150.00	–17.40	–9.54	12.08	–304.86
2	138	–75.00	3.78	–5.03	32.42	94.17
3	138	–78.00	3.60	0.00	53.76	117.36
4	138	–81.12	3.41	0.00	76.15	136.44
5	138	–84.36	3.22	0.00	99.65	156.51

The entries in each column are calculated as follows:

(2) premium less cost of allocation is (2) – (4) in unit fund table (also referred to as the 'profit on allocation')

(3) expenses from the set of assumptions

(4) interest as $\big((2)+(3)\big)i$ (*ie* interest earned on the profit on allocation less the expenses incurred)

(5) expected death cost as $q_{50+t-1} \times \max\big(5{,}000-(8)\,\text{in unit fund table},\, 0\big)$

(6) management charge from (7) in unit fund table

(7) expected profit as the sum of all the above.

Be careful about signs here. From the point of view of the company (*ie* the non-unit fund's point of view), the management charge is a positive item because it is something deducted from the unit fund (the 'policyholder's fund') – although it was a negative when we were considering it in the context of the unit fund.

One way of thinking about all of these items is 'if it's good for profits, it must be positive' – so a management charge must be positive, because if the company increases the management charge then profits go up. Similarly the death cost must be negative: if there are more deaths, company profits go down.

1.4 Example 4: Single premium unitised with-profits contract

Unitised with-profits contracts operate in a similar way to unit-linked contracts, in that the benefits are expressed as an accumulating (unit) fund of premiums and explicit charges are deducted to cover expenses and other costs. Non-unit cashflows can therefore be projected in the same way as for unit-linked contracts, and used in profit testing as described in Section 2 below.

For this example we assume the contract has a five-year term, paid for by a single premium. The allocation rate is 100% and there is no bid-offer spread. The only charges under the contract are a policy fee equal to 0.5% of the bid value of the unit fund, deducted at the end of each year by cancelling units, and a penalty on surrender calculated as a percentage of the unit fund value depending on policy duration as follows:

After 1 year: 2.4%

After 2 years: 1.8%

After 3 years: 1.2%

After 4 years: 0.6%

The unit price increases each year according to the insurer's declared bonus interest rate. The full value of the unit fund, plus any discretionary terminal bonus, is paid out on maturity or at the end of the year of earlier death. Surrender is permitted only at the end of each year, when the fund value less the above surrender penalty is payable.

We will assume that, over the long term, all profits earned from investment returns are fully distributed to policyholders through the bonus payments (we explain the significance of this assumption at the end of this section, below).

We shall calculate the projected profit for each policy year based on the following assumptions:

Age at entry: 45 exact

Single premium: £20,000

Mortality: $q_x = 0.002$ for $x = 45, 46, ..., 49$

Surrender rate: 5% of all policies in force at the end of each of Years 1-4

Bonus interest rate: 3.5% *pa*

Terminal bonus rate: Nil

Non-unit interest: 2% *pa*

Initial expenses: £300

Renewal expenses: £25 at the start of Year 2, and thereafter at the start of each subsequent year, inflating at 2.5% *pa*

Claim expenses: £100 per death, surrender, or maturity

Non-unit reserves: £50 per policy in force at the start of Year 5

The first thing we need to do is to project the unit fund values at the end of each year allowing for the deduction of the policy fee. At the end of the first year, the fund value (before deduction of the policy fee) is the single premium plus one year's bonus:

$$20,000 \times 1.035 = 20,700$$

The policy fee at the end of the year is:

$$-20,700 \times 0.005 = -103.50$$

This means that the fund remaining at the end of the year is:

$$20,700 - 103.50 = 20,596.50$$

Working through the other years in a similar way, we obtain the following table of projected values:

Year t	Fund at end of year before deduction of policy fee (A)	Policy fee (B)	Fund at end of year after deduction of policy fee (C)
1	20,700	–103.50	20,596.50
2	21,317.38	–106.59	21,210.79
3	21,953.17	–109.77	21,843.40
4	22,607.92	–113.04	22,494.88
5	23,282.20	–116.41	23,165.79

where, for years $t = 2, 3, 4, 5$:

$$(A)_t = (C)_{t-1} \times 1.035$$

$$(B) = (A) \times 0.005$$

$$(C) = (A) + (B)$$

Next we need to work out the surrender penalties. These are equal to the end-year fund values (after charges) multiplied by the appropriate percentage rates:

Year t	Fund at end of year	Surrender penalty
1	20,596.50	494.32
2	21,210.79	381.79
3	21,843.40	262.12
4	22,494.88	134.97
5	23,165.79	0

We will also need the dependent probabilities of decrement by death and surrender. Because surrenders take place at the end of each year, the dependent probability of dying is the same as the independent probability, *ie* equal to 0.002 each year. The dependent surrender probability is:

$$\left(1 - 0.002\right) \times 0.05 = 0.0499$$

as we are told that 5% of the *end* of year in-force policies surrender each year.

The dependent probability of a policy staying in force over any particular year is then:

$$1 - 0.002 - 0.0499 = 0.9481$$

(or alternatively this can be calculated as $\left(1 - 0.002\right) \times \left(1 - 0.05\right) = 0.9481$).

The non-unit cashflows can now be calculated. For the first four years of the contract, these are as follows:

Year t	Initial and renewal expenses (1)	Interest (2)	Policy fee (3)	Expected surrender profit (4)	Expected claim expenses (5)	Expected non-unit cashflow (6)
1	–300	–6	103.5	24.67	–5.19	–183.02
2	–25	–0.5	106.59	19.05	–5.19	94.95
3	–25.62	–0.51	109.77	13.08	–5.19	91.53
4	–26.27	–0.53	113.04	6.73	–5.19	87.78
5						

where:

$$(1)_t = (1)_{t-1} \times 1.025 \quad \left(3 \le t \le 5\right)$$

$$(2) = (1) \times 0.02$$

$$(3) = \text{from table of fund calculations}$$

$$(4) = \{\text{surrender penalty}\} \times 0.0499$$

$$(5) = -100 \times \left(0.002 + 0.0499\right) \quad \left(1 \le t \le 4\right)$$

$$(6) = (1) + (2) + (3) + (4) + (5)$$

Question

Calculate the table entries for Year 5.

Solution

The expenses in Year 5 are the Year 4 expenses increased by inflation at 2.5%. The amount is therefore:

$$-26.27 \times 1.025 = -26.93$$

The interest is equal to the interest lost over the year due to the expenses incurred at the start of the year. Its amount is:

$$-26.93 \times 0.02 = -0.54$$

The policy fee income is read from the fund calculation table, and for Year 5 this is 116.41.

As there is no surrender penalty (and no-one surrenders!) then the expected surrender profit from the surrender penalty is zero.

The claim expenses of 100 are paid out on all policies that become claims during the year. All policies in force at the start of Year 5 will ultimately claim at the end of the year – either by surviving and receiving the maturity benefit, or by dying during the year and receiving the death benefit. So the expected amount of claim expenses is –100.

The expected cashflow for Year 5 is then calculated by summing across all of the above items:

$$= -26.93 - 0.54 + 116.41 - 100 = -11.06$$

The completed cashflow table now reads as follows:

Year t	Initial and renewal expenses (1)	Interest (2)	Policy fee (3)	Expected surrender profit (4)	Expected claim expenses (5)	Expected non-unit cashflow (6)
1	–300	–6	103.5	24.67	–5.19	–183.02
2	–25	–0.5	106.59	19.05	–5.19	94.95
3	–25.62	–0.51	109.77	13.08	–5.19	91.53
4	–26.27	–0.53	113.04	6.73	–5.19	87.78
5	–26.93	–0.54	116.41	0	–100	–11.06

Having worked out the expected cashflows, we now need to calculate the expected profit for each year, taking into account the effect of the non-unit reserves.

For this policy, we are told that a non-unit reserve of 50 is required per policy in force at the start of Year 5 (and also therefore per policy in force at the *end* of Year 4); at all other times no non-unit reserve is to be held.

This will affect the expected profit in Year 4. For a policy in force at the start of Year 4, the insurer expects a cashflow of 87.78 at the end of the year. However, the insurer now has to set aside reserves of 50 for each policy that's remained in force through to the end of the year. The probability of doing this is 0.9481 (from above).

So the expected profit for Year 4 (per policy in force at the start of that year) reduces to:

$$87.78-50\times 0.9481=40.38$$

Question

Calculate the expected profit at the end of Year 5, per policy in force at the start of Year 5, allowing for the reserve of 50 held at the start of that year.

Solution

The expected profit for Year 5 will equal:

$$\{\text{expected cashflow}\}+\{\text{reserve at start of year}\}+\{\text{interest on reserve}\}$$
$$-\{\text{expected cost of reserve at end of year}\}$$

Putting in the values (and noting that the reserve at the end of the year is zero) we obtain:

$$-11.06+50+0.02\times 50-0=39.94$$

As there are no reserves held at any time in Years 1-3, the expected profits for these years are all equal to the expected cashflows shown in the table above.

Non-unit reserves, such as in this example, are required for unit-linked and unitised with-profits policies wherever a future negative cashflow is expected.

Question

Given that there is an expected negative cashflow of 11.06, it might seem logical to hold the smallest reserve possible to cover this expected cost, *ie* 11.06 rather than the 50 actually held. Suggest a reason why the insurer might use the higher figure.

Solution

All reserves need to be prudent, in order for there to be a high probability that the liabilities (future outgo) will be covered. The insurer will therefore project its cashflows on more cautious assumptions leading to a somewhat higher negative cashflow than 11.06 in Year 5, and as a result may decide that a reserve of 50 is necessary to cover this.

The reason for profit-testing unitised with-profits contracts

As explained in an earlier chapter, the idea of a with-profits contract is to return to the policyholder the profits earned by the insurer on the policy over the policy term. One approach for unitised with-profits is to earmark the investment profits for policyholders (via the unit fund), and deduct explicit charges to cover the insurer's non-unit expenses and other costs, as in the example we have just looked at. By projecting the future non-unit profits, the insurer can check that its charges are on track to cover these outgoes.

This was why in the above example we assumed that all the investment profits were distributed to policyholders. An exam question may or may not state this assumption explicitly, but it generally would be implied, as the main point of profit testing these contracts in this way is to 'test' the adequacy of the charges in covering the non-unit liabilities.

2 Profit tests for annual premium contracts

Having now considered how to project the revenue accounts for a policy, we see how to use that projection.

The first step in the profit testing of a contract is the construction of the projected revenue accounts for the non-unit (cash) fund for each policy year.

For some contracts other funds, *eg* unit fund for unit-linked assurances, reserves for traditional assurances, provide cashflows to the non-unit fund.

For conventional (or 'traditional') products, the whole profit test is really a 'non-unit fund projection', where one of the elements is the interest on reserves.

In such cases these funds will need to be projected so that the expected cashflows to the non-unit fund can be determined. These calculations will require data items about the contract, *eg* proportion of premium allocated to purchase of units, bid-offer spread in unit prices; and assumptions which form a basis for the calculations, *eg* growth rate of unit fund, mortality and interest rate basis used to calculate required reserves.

These expected cashflows, together with the direct expected cashflows into and out of the non-unit fund, are the components of the projected revenue account. The calculation of the direct expected cashflows will also require data items about the contract, *eg* initial and renewal expenses; and assumptions to form a basis, *eg* mortality of policyholders, rate of return earned on non-unit fund.

Profit vector

The vector of balancing items in the projected revenue accounts for each policy year is called the profit vector; $(PRO)_t$, $t = 1, 2, 3, \ldots$. The profit vector gives the expected profit at the end of each policy year per policy in force at the beginning of that policy year.

For example, the conventional endowment assurance policy studied in Section 1.2 gave the following profit vector:

$$(-803.99, 186.97, 178.12, 206.47, 215.23)$$

Profit signature

The vector of expected profits per policy issued is called the profit signature. This is obtained by multiplying the profit vector by the probability of a policy remaining in force from policy duration 0 to policy duration $t-1$, *ie*:

$$(PS)_t = {}_{t-1}(ap)_x \, (PRO)_t$$

where $(PS)_t$ denotes the t th entry in the profit signature and $(PRO)_t$ denotes the t th entry in the profit vector.

Summarising the above, we have the very important distinction:

- profit vector = vector whose entries represent the expected profit at the end of each year per policy in force at the start of that year
- profit signature = vector whose entries represent the expected profit at the end of each year per policy in force at inception.

For example, for the conventional endowment assurance policy studied in Section 1.2, we found the second element of the profit vector to be £186.97.

To obtain the corresponding entry in the profit signature, we multiply the entry in the profit vector by the probability that the policy is still in force at the start of Year 2. This is:

$$186.97 \times (ap)_{55}$$

Question

Use the relevant table of dependent decrement probabilities from Section 1.2 (reproduced below) to calculate the required probability, and hence calculate the value of the second entry in the profit signature.

Age x	$(aq)_x^d$	$(aq)_x^w$
55	0.005	0.1
56	0.006	0.05
57	0.007	0.05
58	0.008	0.01
59	0.009	0

Solution

The probability that a policy is still in force at the start of Year 2 is:

$${}_1(ap)_{55} = 1 - (aq)_{55}^d - (aq)_{55}^w = 1 - 0.005 - 0.1 = 0.895$$

So the second entry in the profit signature is:

$$186.97 \times {}_1(ap)_{55} = 186.97 \times 0.895 = 167.34$$

The other elements of the profit signature can be obtained in the same way: by multiplying the profit vector element for a given year by the *probability of a policy remaining in force from policy inception to the start of that year*.

Question

Calculate the other elements of the profit signature for the conventional endowment insurance policy studied in Section 1.2.

Solution

The first element of the profit signature is unchanged from the profit vector, because the amount (-803.99) is already the amount per policy in force at the beginning of year 1, *ie* at issue.

For Year 3 we need:

$$(PS)_3 = (PRO)_3 \times {}_2(ap)_{55}$$

We calculate $(ap)_{55}$, $(ap)_{56}$, *etc* from:

$$(ap)_x = 1 - (aq)_x^d - (aq)_x^w$$

(using the dependent probabilities given in Section 1.2); and ${}_2(ap)_{55}$ (for example) from the cumulative probability:

$${}_2(ap)_{55} = (ap)_{55}\,(ap)_{56}$$

The required probabilities are:

Year t	$(ap)_{54+t}$	${}_{t-1}(ap)_{55}$
1	0.895	1
2	0.944	0.895
3	0.943	0.8449
4	0.982	0.7967
5	not required	0.7824

(For example, $0.7967 = 0.895 \times 0.944 \times 0.943$.)

The t th entry of the profit signature is obtained by multiplying the t th entry of the profit vector by ${}_{t-1}(ap)_{55}$:

Year t	Profit in year t	${}_{t-1}(ap)_{55}$	Profit signature
1	–803.99	1	–803.99
2	186.97	0.895	167.34
3	178.12	0.8449	150.49
4	206.47	0.7967	164.49
5	215.23	0.7824	168.40

So the profit signature is (–803.99, 167.34, 150.49, 164.49, 168.40).

The vector representing the profit signature $(PS)_t$; $t = 1, 2, 3, \ldots$ can be displayed graphically to illustrate the way in which profits are expected to emerge over the lifetime of the policy. However, it is difficult to compare this information for different policies when there is a need to evaluate alternative designs for a product (policy) or to decide which of several different possible policies is the most profitable. Decisions like this are usually made easier by summarising each profit signature as a single figure. We describe three such summary measures below.

2.1 Summary measures of profit

Summary measures usually involve determining the present values of the expected cashflows. In some cases, this requires an assumption about the discount rate. This rate is chosen to equal the cost of capital plus a risk premium, and is called the risk discount rate, i_d .

The cost of capital is the rate at which funds can be borrowed, or the rate of return such funds would earn if invested elsewhere, (*ie* the 'opportunity cost').

The risk premium reflects the risks and uncertainties surrounding the cashflows to and from the policy.

Writing a policy can be thought of as an investment by the shareholders of the company, because they supply the capital to make good the shortfall between the premium income and the outgo of expenses and the setting up of reserves. If the shareholders are providing capital, then they expect a return on that capital appropriate to the riskiness of their investment. It is much riskier investing in life insurance business than buying government bonds, because more things could go wrong.

To allow for the extra risk, we add a margin to the investment returns on relatively risk-free assets such as government bonds, and then price the product using the resulting risk discount rate. This will then give us premiums that contain an adequate allowance for the risk.

We can now find the price of the product by projecting cashflows, and then varying the premium amount until we meet the profit criterion (as described below) which has been calculated using our risk discount rate.

Net present value (NPV)

This is the present value of the profit signature determined using the risk discount rate.

$$\text{NPV} = \sum_{t=1}^{t=\infty} (1+i_d)^{-t} (PS)_t$$

The NPV can be interpreted as the EPV of the future profits from the policy, for a single policy as at the start date of the contract.

Question

Calculate the net present value of the conventional endowment assurance policy studied in Section 1.2, using a risk discount rate of 7%.

Solution

Discounting the profit signature at 7%, we obtain:

Year t	Profit signature	Discount factor v^t	Discounted profit
1	–803.99	0.9346	–751.41
2	167.34	0.8734	146.16
3	150.49	0.8163	122.84
4	164.49	0.7629	125.49
5	168.40	0.7130	120.07
Total			–236.85

So the net present value of profits is –£236.85.

The expected profits for each year include all interest earned (on cashflows and reserves) during the year. So the profits are essentially already accumulated with interest to the end of each year, and all we have to do is to discount *from* the end of each year.

Question

Give two possible reasons why this net present value is negative.

Solution

One reason is that the pricing of the contract did not allow for withdrawals, and on early withdrawal the company will lose out as it has not recouped the initial expenses of the policy.

A second reason (which makes the answer more negative than it already is) is that we have valued profits using a risk discount rate that is much higher than the 4% interest rate used in calculating the reserves for the product. The effect of such differences is discussed in more detail in the next chapter.

Profit margin

This is the NPV expressed as a percentage of the EPV of the premium income. If the premium paid at the beginning of the t th policy year is P_t, this is:

$$\frac{\sum_{t=1}^{t=\infty}(1+i_d)^{-t}\,(PS)_t}{\sum_{t=1}^{t=\infty}(1+i_d)^{-(t-1)}\;{}_{t-1}(ap)_x\;P_t}$$

In other words, the profit margin is:

$$\frac{\text{NPV}}{\text{EPV premiums}}$$

where the risk discount rate is used to do the discounting in both the numerator and the denominator.

Question

Calculate the profit margin of the conventional endowment assurance policy studied in Section 1.2 , using a risk discount rate of 7%.

Solution

The discounted present value of the premiums is £8,059.11, calculated as in the table below. Note that, because premiums are payable in advance, the discount factors are based on the duration at the *start* of each year.

Year t	Premium	In force probability ${}_{t-1}(ap)_{55}$	Discount factor v^{t-1}	Discounted premium
1	2,108.81	1	1	2,108.81
2	2,108.81	0.895	0.93458	1,763.91
3	2,108.81	0.84488	0.87344	1,556.20
4	2,108.81	0.79672	0.81630	1,371.49
5	2,108.81	0.78238	0.76290	1,258.70
Total				8,059.11

So the profit margin is $\dfrac{-236.85}{8{,}059.11} = -2.9\%$.

The internal rate of return (IRR)

This has been described earlier in this subject in relation to non-contingent cashflows, but it can also be calculated for contingent cashflows, as here. It is defined as the discount rate that would make the NPV of the contract equal to zero.

The internal rate of return does not always exist. It does exist where the profit signature has a single financing phase, *ie* where the first profit flow is negative and the profit flows in all subsequent years are positive. This situation commonly occurs, however, and so the IRR is often a useful measure in practice.

Suppose that the profit signature on a five year life insurance policy is:

$$(-300,\ 100, 100, 100, 100)$$

The internal rate of return is the discount rate (r) such that:

$$\frac{-300}{1+r}+\frac{100}{(1+r)^2}+\frac{100}{(1+r)^3}+\frac{100}{(1+r)^4}+\frac{100}{(1+r)^5}=0$$

$$\Rightarrow -300+100a_{\overline{4}|}=0$$

$$\Rightarrow a_{\overline{4}|}=\frac{300}{100}$$

$$\Rightarrow \frac{1-(1+r)^{-4}}{r}=3$$

By trial and error, the value of r that satisfies this equation is 12.6%, so this is the IRR in this case.

This is the kind of calculation that is easy to do using a spreadsheet or other suitable computer software, and also using Table mode on many calculators.

Question

A policy with a 3-year term has a profit vector of $(-428,\ 72,\ 560)$. The policyholder is aged 55 exact at entry. Calculate the internal rate of return for this policy, assuming AM92 Select mortality.

Solution

First of all, we need to calculate the profit signature, by multiplying each element of the profit vector by the probability of the policyholder being in force at the start of each year. The vector of probabilities is:

$$\left(1,\ {}_1p_{[55]},\ {}_2p_{[55]}\right)$$

$$=\left(1,\ \frac{l_{[55]+1}}{l_{[55]}},\ \frac{l_{57}}{l_{[55]}}\right)$$

$$=\left(1,\ \frac{9{,}513.9375}{9{,}545.9929},\ \frac{9{,}467.2906}{9{,}545.9929}\right)$$

$$=\left(1,\ 0.996642,\ 0.991755\right)$$

Multiplying the profit vector by the probabilities we get the profit signature to be:

$$\left(-428\times 1,\ 72\times 0.996642,\ 560\times 0.991755\right)=\left(-428,\ 71.76,\ 555.38\right)$$

The IRR is then the rate of discount, r, that satisfies:

$$\frac{-428}{1+r}+\frac{71.76}{(1+r)^2}+\frac{555.38}{(1+r)^3}=0 \quad \Rightarrow \quad -428\times(1+r)^2+71.76\times(1+r)+555.38=0$$

Using the quadratic formula, we can solve for $1+r$:

$$1+r=\frac{-71.76\pm\sqrt{71.76^2-4\times(-428)\times 555.38}}{2\times(-428)}$$

for which the positive root is:

$$1+r=1.2260 \quad \Rightarrow \quad r=0.2260$$

That is, the IRR is 22.60% *pa*.

3 Profit testing using the present value random variable

3.1 Introduction

In this section we describe an alternative approach to calculating the expected present value of the future profits when profit testing. It is not the method usually used in practice but could come up in exam questions.

So far we have calculated the expected values of each year's future profit, and then discounted them to obtain the expected *present* value of the profits. These values sum to give the net present value (NPV), as we saw in Section 2.1 above.

As explained in Section 2.1, the NPV is the same as the expected present value (EPV) of the future profits. So, to arrive at the NPV, we could alternatively do our discounting first, and then take the expectation afterwards. We have seen this approach used for calculating EPVs almost everywhere else in this course.

3.2 Example

Let's see how this works for the 5-year conventional endowment assurance example first shown in Section 1.2. The first thing to do is to write down the present value as a random variable. This is not a trivial task.

To do this we consider what the present value would be on the occurrence of each possible event (contingency) that could happen to the policy during the potential 5 years of the policy. For example, the policyholder could die in the first policy year, in which case the policy would terminate after just one year with the payment of a death claim at the end of the year. If this occurred, then the profit for the policy would be calculated as:

Premium*	Expenses*	Interest*	Claim cost	Profit
2,108.81	– 1,054.41	42.18	– 10,000	– 8,903.42

* *as shown in the example in Section 1.2*.

We here deduct 10,000 for the death claim. This is because we are calculating the profit *if* the policyholder dies in Year 1, in which case the full sum assured is paid out at the end of the year.

The present value of the profit (discounting at the risk discount rate of 7% *pa*), *given* (*ie* if) the policy terminates by death in the first year, is then:

$$\{PV|\text{Dies in first year}\} = \frac{-8{,}903.42}{1.07} = -8{,}320.95$$

Question

Calculate the present value of the profit assuming the policy is surrendered in Year 1.

Solution

The profit flows are the same as before, except instead of the death claim at the end of the year, the surrender value is paid out. The Year 1 surrender value is 2,108.81 (equal to the total premium paid to date), and so the profit is:

Premium	Expenses	Interest	Claim cost	Profit
2,108.81	– 1,054.41	42.18	– 2,108.81	– 1,012.23

So the present value of the profit, if the policy surrenders in Year 1, is:

$$\left\{ \text{PV} \middle| \text{Surrenders in first year} \right\} = \frac{-1{,}012.23}{1.07} = -946.01$$

Next we calculate the present value of the profit if the policyholder dies in the second year. So we calculate:

Yr	Premium	Expense	Int	Reserve at start of year	Interest on reserve	Claim cost	Reserve at end of year
1	2,108.81	–1,054.41	42.18	0	0	0	–1,832.06
2	2,108.81	–105.44	80.13	1,832.06	73.28	–10,000	0

In the first year, we have to set up the reserve of 1,832.06 out of cashflows, so this is deducted from the Year 1 profit.

In Year 2, assets of 1,832.06 are available and, along with interest, contribute to the profit made in Year 2.

Summing each row gives the profit for each year. We then discount the two profit flows at the risk discount rate, and sum to give the total present value. The Year 1 figure is divided by 1.07 as it emerges at the end of Year 1, and the Year 2 value is divided by 1.07^2 as it emerges at the end of Year 2. So we obtain:

Yr	Profit	Present value
1	–735.48	–687.36
2	–6,011.16	–5,250.38
		–5,937.74

Continuing in this way for all possible future events, we arrive at the following table, which defines the probability distribution of the present value random variable for a single policy at outset.

Event	Present value of profit | Event occurs
Dies in Year 1	–8,320.95
Surrenders in Year 1	–946.01
Dies in Year 2	–5,937.74
Surrenders in Year 2	–887.18
Dies in Year 3	–3,757.17
Surrenders in Year 3	–758.44
Dies in Year 4	–1,764.93
Surrenders in Year 4	–571.18
Dies in Year 5	52.43
Stays in force to end of term	52.43

Question

Verify the present value of the profit assuming:

(a) the policy is surrendered in Year 3

(b) the policy stays in force to the end of the term.

Solution

(a) ***Present value if the policy is surrendered in year 3***

The present values of the profits for each of the three years, assuming the policy is surrendered in the third year, are shown in the following table:

Yr	Premium – Expense + Interest	Reserve at start of year	Interest on reserve	Claim cost	Reserve at end of year	Profit	Present value
1	1,096.58	0	0	0	–1,832.06	–735.48	–687.36
2	2,083.50	1,832.06	73.28	0	–3,740.46	248.39	216.95
3	2,083.50	3,740.46	149.62	–6,326.43	0	–352.85	–288.03
						Total	–758.44

In this case, the surrender value (equal to three times the annual amount of premium) is paid out at the end of Year 3, at which point the policy terminates.

The present values in the last column are equal to the profits for each year discounted at 7% *pa* to the start of the policy.

(b) ***Present value if the policy stays in force until the end of the term***

The present values of the profits for each of the five years, assuming the policy stays in force until the end of the term, are shown in the following table:

Yr	Premium – Expense + Interest	Reserve at start of year	Interest on reserve	Claim cost	Reserve at end of year	Profit	Present value
1	1,096.58	0	0	0	–1,832.06	–735.48	–687.36
2	2,083.50	1,832.06	73.28	0	–3,740.46	248.39	216.95
3	2,083.50	3,740.46	149.62	0	–5,736.10	237.48	193.86
4	2,083.50	5,736.10	229.44	0	–7.818.97	230.08	175.53
5	2,083.50	7,818.97	312.76	–10,000	0	215.23	153.46
						Total	52.43

In this case the policy pays out the maturity benefit of 10,000 at the end of the five-year period.

We now calculate the EPV as:

$$EPV = \sum_{\text{All events}} \left\{ \text{Present value of profit} \middle| \text{Event occurs} \right\} \times P\left(\text{Event occurs}\right) \quad (*)$$

So we need to calculate the probability that each event occurs. For this we need deferred dependent probabilities of the form:

$$_{n|}(aq)^k_{55} = {_n}(ap)_{55}\,(aq)^k_{55+n} \qquad \text{for } k = d, w$$

The required calculations are shown in the following table:

n	$_n(ap)_{55}$	$(aq)^d_{55+n}$	$(aq)^w_{55+n}$	$_{n\|}(aq)^d_{55}$	$_{n\|}(aq)^w_{55}$
0	1	0.005	0.1	0.005	0.1
1	0.895	0.006	0.05	0.00537	0.04475
2	0.84488	0.007	0.05	0.005914	0.042244
3	0.796722	0.008	0.01	0.006374	0.007967
4	0.782381	0.009	0	0.007041	0
5	0.775339				

Question

(i) Verify the probability that a policy issued at age 55 will terminate by death in the third policy year.

(ii) Calculate the expected present value of the profit using (*) above.

Solution

(i) ***Probability that policy will terminate by death in third year***

This probability is ${}_{2|}(aq)^d_{55} = {}_2(ap)_{55}\,(aq)^d_{57}$, where:

$$ {}_2(ap)_{55} = \left[1-(aq)^d_{55}-(aq)^w_{55}\right]\times\left[1-(aq)^d_{56}-(aq)^w_{56}\right] $$

$$ =(1-0.005-0.1)\times(1-0.006-0.05)=0.84488 $$

So:

$$ {}_{2|}(aq)^d_{55} = 0.84488\times 0.007 = 0.005914 $$

(ii) ***Expected present value of profit***

The required calculations are shown in the following table:

Event	Present value of profit | Event occurs	Probability	EPV
D Yr 1	–8,320.95	0.005	–41.60
S Yr 1	–946.01	0.1	–94.60
D Yr 2	–5,937.74	0.00537	–31.89
S Yr 2	–887.18	0.04475	–39.70
D Yr 3	–3,757.17	0.005914	–22.22
S Yr 3	–758.44	0.042244	–32.04
D Yr 4	–1,764.93	0.006374	–11.25
S Yr 4	–571.18	0.007967	–4.55
D Yr 5	52.43	0.007041	0.37
Matures	52.43	0.775339	40.65
		Total	–236.83

This is the same (apart from rounding differences) as the value obtained for the net present value using the standard profit-testing approach shown in Section 2.1.

4 Pricing using a profit test

The actual profitability is not known until each respective contract terminates and we know the actual experience. However, we can calculate the expected profitability using criteria set out in Section 4.1 below. To do this we need details of the premium, any charges and other relevant data items relating to the contract. A projection basis is also needed.

When products are designed, the expected level of profit will usually be specified as an objective and the features of the product can be set to achieve this objective. For conventional contracts, the benefits and attaching terms and conditions are usually specified in advance, therefore the key variable to determine when setting the profit objective is the level and pattern of premium payments. For unit-linked and unitised with-profits contracts the key variable is usually the charges, such as premium allocation rate and the management charge.

4.1 Profit criterion

The objective specified for the expected level of profit is termed the 'profit criterion'.

Careful choice of a profit criterion is central to the actuarial management of the company selling the products. It is common for those marketing and selling the products to receive part of their salary in the form of a 'productivity' bonus, *eg* commission which is a percentage of the total premiums for the policies sold. If the profit criterion chosen is directly related to this 'productivity' bonus, the company's profits will be maximised if the salesforce maximises its income. Such considerations are important in choosing the profit criterion to be used.

Examples of the profit criterion are:

NPV = 40% of Initial Sales Commission

Profit Margin = 3% of EPV of premium income

For conventional products, the profit test is completed using a spreadsheet or similar software, and the premiums are varied until the required 'target' *ie* NPV, level of profit margin, IRR, is achieved. The premium or price of the product has been determined using a profit test.

For unit-linked and unitised with-profits products, the management charges are varied to try to achieve an acceptable charging structure (in comparison with other products in the market) which satisfies the profit criterion.

It is common practice to investigate the sensitivity of this profit to variation in a number of factors such as the key features of the product design and/or the assumptions made in the projection basis. This is done by performing a 'sensitivity test' whereby each proposed premium or charging structure will be input into the model and the assumptions varied to determine the impact on the expected profit. The objective is to design a product which is robust and exhibits minimal variation in the expected profit to feasible changes in the data and the assumptions used. Sensitivity tests are covered in the next chapter.

So the steps in profit testing can be summarised as follows:

- decide on the structure of the product, *eg* single premium unit-linked deferred annuity
- build a model to project cashflows for the product
- choose some specimen policies – what age, sex, level of cover do we expect?
- decide on a risk discount rate and profit criterion
- choose a basis – probably our best estimate – of all important parameters required for the profit test, *eg* unit growth, levels of withdrawals, *etc*
- decide on some 'first draft' premiums (conventional product) / charges (unit-linked product)
- profit test our specimen policies using these premiums
- vary the premiums / charges until our profit criterion is met, remembering that premiums need to be acceptable in the market
- sensitivity test by varying key parameters (*eg* investment return, mortality, expense inflation) in the cashflow projection, to check that our product design is sufficiently resilient to adverse changes in future experience
- keep varying premiums and, if necessary, features of the product design until we have a product that:
  - meets the profitability criterion,
  - is marketable, and
  - is resilient to adverse future experience.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 27 Summary

Projecting profit flows – conventional products

Per policy in force at start of each year, project expected amounts of:

(+) premiums

(–) expenses

(+) investment income

(–) benefit payouts (death, maturity, surrender)

(–) increase in reserves*

= profit vector

* Increase in reserves =

$$\{\text{probability of staying in force over the year}\} \times \{\text{reserve per policy at end of year}\}$$
$$-\{\text{reserve at start of year}\} - \{\text{interest on start year reserve}\}$$

Projecting cashflows – unit-linked and unitised with-profits

Project unit fund at end of each year as:

(+) Premium $\times$ allocation rate $\times(1-\text{bid-offer spread})$ *

(+) unit fund from end of previous year

(+) expected unit fund growth

(–) management charge

= fund at end of year after charges

* 'Cost of allocation'

Per policy in force at start of each year, project expected amounts of:

(+) premium less cost of allocation

(–) expenses

(+) interest on non-unit fund

(–) non-unit benefit costs

(+) non-unit surrender profit

(+) management charge transferred from unit fund

(+) other charges from unit fund, if applicable

= non-unit cashflow vector (= profit vector in absence of any non-unit reserves)

The effect of non-unit reserves on the profit vector is covered in the next chapter.

Summary measures of profit

Profit signature = vector whose entries represent the expected profit emerging at the end of year per policy in force at inception.

Net present value (NPV) = total present value of profit signature, discounted at risk discount rate.

Profit margin = $\dfrac{NPV}{EPV\ premiums}$

Internal rate of return (IRR) = discount rate that makes the NPV equal to zero.

We determine the risk discount rate as:

risk discount rate = cost of capital + risk premium

where the cost of capital is the return that can be obtained by investing money without risk, and the risk premium reflects the degree of riskiness associated with the product.

Using present value random variable approach

Alternatively, the NPV can be calculated using:

$$NPV = \sum_{\text{All events}} \left\{ \text{Present value of profit} \middle| \text{Event occurs} \right\} \times P\left(\text{Event occurs}\right)$$

Profit testing

We can set the premiums for a product to give a desired level of profitability by projecting cashflows under a certain set of assumptions, deciding on a risk discount rate and profit criterion, and then varying the premium amount until the profit criterion is satisfied.

Chapter 27 Practice Questions

27.1 An actuary is profit testing a 15-year endowment assurance policy. The sum assured is £25,000 payable on survival or at the end of the year of earlier death. On surrender, a return of premiums is paid without interest at the end of the year of surrender.

A level premium of £1,500 *pa* is payable annually in advance.

For a policy in force at the start of the eighth year the remaining details are as follows:

	(£)
Renewal expenses	35
Claim expenses on death or surrender	75
Reserve at the start of year, ${}_7V$	8,000
Reserve at end of year per survivor, ${}_8V$	9,300
Rate of interest	8% *pa*
Dependent probability of death during 8th year	0.02
Dependent probability of surrender during 8th year	0.05

Calculate the profit expected to emerge at the end of the eighth year, per policy in force at the start of that year.

27.2 A unit-linked policy issued to lives aged 50 has a minimum death benefit of £3,000 (payable at the end of the year). Write down an expression for the expected death cost in the non-unit fund for Year 2 for a policy in force at the start of the year, expressed in terms of F_i, the size of the unit fund at the end of year i.

27.3 Amit, aged 60, invests £100 at the beginning of each month in an account earning interest at the rate of 1% per month. Amit requires a guaranteed amount of £3,000 at the end of the month of his death. To provide this guarantee, he buys a decreasing term assurance with a sum assured payable at the end of the month following death equal to the difference between the balance in the account and £3,000. The office premium for the assurance is £10 per month. The office incurs initial expenses of £25 and renewal expenses of £5 per month. The mortality basis for premium calculations is AM92 Ultimate and a uniform distribution of deaths over each year of age is assumed.

Determine the expected net outgo for the 18th month of the assurance contract as at policy outset. (Ignore interest earned by the life office.)

27.4 A life insurance company sells five-year-term, single-premium, unit-linked policies each for a premium of £10,000. There is no bid/offer spread and the allocation percentage is 100%. The only charge is a 2% annual management charge. The maturity, death and surrender benefits are equal to the value of the units at maturity, or at the end of the year of death or surrender, as appropriate, after deduction of the annual management charge in each case.

(i) Assuming unit growth of 9% *pa*, calculate the value of the units at the start and end of each year after deduction of the management charge, and the amount of management charge each year.

(ii) Calculate the net present value of the contract assuming:

- Commission of 5% of the premium
- Initial expenses of £50
- Annual renewal expenses of £20 in the 1st year, inflating at 5% *pa*
- Independent probability of mortality is 0.5% at each age
- Independent probability of surrender is 5% at each age
- Non-unit fund interest rate is 9% *pa*
- Risk discount rate 12% *pa*

The company holds unit reserves equal to the full value of the units (after deduction of annual management charge) and zero non-unit reserves.

You may assume that expenses are incurred at the start of the year and that death and surrender payments are made at the end of the year.

27.5 A life insurance company issues five-year without profit endowment assurances for an annual premium of £3,600 and a sum assured of £20,000 payable on maturity or at the end of the year of death if earlier.

The company uses the following assumptions for profit testing:

Year	Mortality probability	Surrender probability	Expenses at start year per policy	Reserves at end of year per policy	Surrender value at end of year per policy
1	0.01	0.05	£750	£3,100	£2,800
2	0.01	0.05	£15	£6,800	£6,250
3	0.01	0.05	£15	£10,900	£10,000
4	0.01	0.05	£15	£15,300	£14,500
5	0.01	0	£15	–	–

Surrenders occur only at the end of a year immediately before a premium is paid. The surrender probabilities shown in the table above are applied to the number of policies in force at each year-end. The company assumes it will earn 6% *pa* on its investments.

(i) Set out the column headings and the formulae you would use to calculate the profit arising each year per policy in force at the beginning of the year.

(ii) Calculate (to the nearest 1%) the internal rate of return obtained by the company.

27.6 Exam style

A life office is planning to issue a new series of three-year unit linked endowment policies. Two designs of policy are under consideration, both having level annual premiums of £1,000.

Type A: 85% of the first year's premium and 101% of each subsequent premium is invested in units. On surrender the bid value of the units allocated is paid.

Type B: 95% of each premium is invested in units. On surrender the bid value of the units allocated is paid less a penalty of 10% of the total premiums outstanding under the policy.

There is a bid/offer spread in unit values, the bid price being 95% of the offer price. A fund management charge of 1% of the value of the policyholder's fund is deducted at the end of each policy year.

The death benefit, which is payable at the end of the year of death, and the maturity value are equal to the bid value of the units allocated. Surrenders are assumed to take place at the end of the year.

The office's expenses in respect of the policy are £100 at the start of the first year and £30 at the start of the second and third years.

The office holds unit reserves equal to the bid value of the units and zero non-unit reserves.

The dependent probability of mortality at each age is assumed to be 1% and the dependent probability of surrender at each duration is 5%.

The non-unit fund is assumed to grow at the rate of 7½% *pa*.

(i) Calculate the unit fund values at the end of each year assuming that the growth in unit value is 7½% *pa* and hence calculate the estimated maturity proceeds for each policy type. [6]

(ii) Calculate the net present value of the profit that is expected to arise under each policy type, using a discount rate of 10% *pa*. [8]

[Total 14]

27.7 A two-year insurance policy has the following particulars:

- annual premium of 520
- death benefit of 15,000, paid at the end of the year of death
- surrender value in Year 1 of 420, paid at the end of the year
- age at entry 50
- reserve at the end of Year 1 of 510 per policy then in force

The policy is to be profit tested using the following assumptions:

- mortality AM92 Ultimate
- 15% of policies surviving to the end of Year 1 surrender at that time
- expenses of 250 and 25 at the start of Years 1 and 2 respectively
- return on assets of 2% *pa*
- risk discount rate 5% *pa*

(i) Calculate the present value of the profit for this policy at outset, using the risk discount rate, assuming the policyholder:

(a) dies in Year 1

(b) surrenders in Year 1

(c) dies in Year 2.

(ii) The present value of the profit from the policy, assuming the policyholder neither dies in the first two years nor surrenders in the first year, is 250.

Calculate the net present value of the policy.

Chapter 27 Solutions

27.1 The expected profit is:

$$\begin{aligned}&(8{,}000+1{,}500-35)\times 1.08\\&-(8\times 1{,}500+75)\times 0.05-(25{,}000+75)\times 0.02\\&-9{,}300\times 0.93\\&=£467.95\end{aligned}$$

27.2 For a policy in force at the start of Year 2, the probability of a death claim at the end of Year 2 is q_{51} (as policyholders were aged 50 at issue).

If the unit fund value at the end of Year 2 is greater than 3,000, then the death benefit is the bid value of the units and there is no cashflow from the non-unit fund. However, if the unit fund value at the end of Year 2 is less than 3,000, then a benefit of 3,000 is payable if the policyholder dies in Year 2, and the shortfall must come from the non-unit fund. This shortfall is $3{,}000-F_2$.

So the expected death cost in the non-unit fund is:

$$\max\{0,\ 3{,}000-F_2\}\times q_{51}$$

27.3 The cashflows occurring at the end of the 18th month are:

(1) a premium of £10 paid if Amit is alive at the beginning of the month

(2) renewal expenses of £5 paid if Amit is alive at the beginning of the month

(3) a sum assured of $3{,}000-100\ddot{s}_{\overline{18}|}^{@1\%}$ paid if Amit dies in the 18th month.

The probability that Amit survives to the beginning of the 18th month is:

$$\frac{l_{61\frac{5}{12}}}{l_{60}}$$

Assuming deaths are uniformly distributed over the year of age we can calculate the value of l_x at the non-integer age using linear interpolation between the values at adjacent integer ages. So the required probability is equal to:

$$\frac{\left(\frac{7}{12}l_{61}+\frac{5}{12}l_{62}\right)}{l_{60}}=\frac{\left(\frac{7}{12}\times 9{,}212.7143+\frac{5}{12}\times 9{,}129.7170\right)}{9{,}287.2164}=0.98825$$

The probability that Amit dies in the 18th month is:

$$p_{60}\,{}_{\frac{1}{12}}q_{61}=\frac{1}{12}\times\frac{d_{61}}{l_{60}}=\frac{82.9973}{12\times 9{,}287.2164}=0.0007447$$

The sum assured required on death in the 18th month is 3,000 less the accumulated fund value at that point. The accumulated fund value is:

$$100\ddot{s}_{\overline{18}|} \text{ calculated at } i = 1\%$$

$$= 100 \times \left(\frac{1.01^{18} - 1}{0.01} \right) \times 1.01 = 1,981.09$$

So the expected net outgo is:

$$(3,000 - 1,981.09) \times 0.0007447 - (10 - 5) \times 0.98825 = -£4.18$$

27.4 (i) ***Unit reserves***

The following table shows the figures required:

Year	Value of units start of year	Value of units at end of year before mgmt charge	Mgmt charge	Value of units at end of year after mgmt charge
1	10,000.00	10,900.00	–218.00	10,682.00
2	10,682.00	11,643.38	–232.87	11,410.51
3	11,410.51	12,437.46	–248.75	12,188.71
4	12,188.71	13,285.69	–265.71	13,019.98
5	13,019.98	14,191.78	–283.84	13,907.94

(ii) ***Net present value***

The following table shows the calculation of the profit signature.

Year	Expenses	Interest	Mgmt charge	Profit vector	Prob in force	Profit signature
1	–570.00	–51.30	218.00	–403.30	1.000	–403.30
2	–21.00	–1.89	232.87	209.98	0.9453	198.48
3	–22.05	–1.98	248.75	224.71	0.8935	200.78
4	–23.15	–2.08	265.71	240.48	0.8446	203.10
5	–24.31	–2.19	283.84	257.34	0.7983	205.44

where probability in force at start of year t = $(0.95 \times 0.995)^{t-1}$.

There are no expected costs from deaths, surrenders or at maturity as the benefit level is equal to the unit fund value in all cases.

The net present value of the contract at a risk discount rate of 12% is:

$$-\frac{403.30}{1.12}+\frac{198.48}{1.12^2}+\frac{200.78}{1.12^3}+\frac{203.10}{1.12^4}+\frac{205.44}{1.12^5}=186.69$$

27.5 (i) ***Column headings***

The column headings and formulae required are:

(t)	Year	$t=1,2,\ldots,5$
(1)	Premium received	£3,600
(2)	Expenses	£750 (Year 1)
		£15 (Years 2-5)
(3)	Interest earned	$0.06\times[(1)-(2)]$
(4)	Expected death cost	$0.01\times 20{,}000$
(5)	Expected maturity cost	$0.99\times 20{,}000$ (Year 5 only)
(6)	Expected surrender cost	$0.05\times 0.99\times SV$ (Years 1-4 only)
	(The surrender value SV is a data item.)	
(7)	Expected in force cashflow	$(1)-(2)+(3)-(4)-(5)-(6)$
(8)	Expected increase in reserves	$0.95\times 0.99\times {}_tV-{}_{t-1}V$
	(The reserve at the end of Year 5 is zero.)	
(9)	Interest on reserves	$0.06\times {}_{t-1}V$
(10)	Profit vector	$(7)-(8)+(9)$

We have adopted the convention here that the figures in the expenses and claims columns are shown as positive entries. If you have shown these as negative numbers and adjusted the signs in the other column definitions accordingly, that is an equally valid approach.

(ii) ***Internal rate of return***

The calculations required to determine the profit signature are set out in the tables below, which include two extra columns:

(11)	Probability in force	$0.95^{t-1}\times 0.99^{t-1}$
(12)	Profit signature	$(10)\times(11)$

We then have:

Year	Premiums	Expenses	Interest	Expected death cost	Expected maturity cost	Expected surrender cost
t	(1)	(2)	(3)	(4)	(5)	(6)
1	3,600	750	171	200	–	138.60
2	3,600	15	215.10	200	–	309.38
3	3,600	15	215.10	200	–	495.00
4	3,600	15	215.10	200	–	717.75
5	3,600	15	215.10	200	19,800	–

Year	Expected in force cashflow	Expected increase in reserves	Interest on reserves	Profit vector	Prob in force	Profit signature
t	(7)	(8)	(9)	(10)	(11)	(12)
1	2,682.40	2,915.55	0.00	–233.15	1.0000	–233.15
2	3,290.73	3,295.40	186.00	181.33	0.9405	170.54
3	3,105.10	3,451.45	408.00	61.65	0.8845	54.53
4	2,882.35	3,489.65	654.00	46.70	0.8319	38.85
5	–16,199.90	–15,300	918.00	18.10	0.7824	14.16

The internal rate of return is the interest rate that satisfies the equation:

$$-233.15v + 170.54v^2 + 54.53v^3 + 38.85v^4 + 14.16v^5 = 0$$

By trial and error:

$$i = 11\% \quad LHS = 2.24$$

$$i = 12\% \quad LHS = -0.68$$

So the internal rate of return is approximately 11.8%, or 12% to the nearest percent.

27.6 (i) ***Unit fund values and maturity proceeds***

The calculations of the build up of each fund are set out in the tables below:

TYPE A							
Policy year	Prem	Prem all'd	Cost of all'n	Fund after all'n	Fund before charge	Annual charge	Fund at year end
1	1,000.00	850.00	807.50	0.00	868.06	–8.68	859.38
2	1,000.00	1,010.00	959.50	859.38	1,955.30	–19.55	1,935.75
3	1,000.00	1,010.00	959.50	1,935.75	3,112.39	–31.12	3,081.27

So the estimated maturity value for policy Type A is £3,081.27. [3]

TYPE B							
Policy year	Prem	Prem all'd	Cost of all'n	Fund after all'n	Fund before charge	Annual charge	Fund at year end
1	1,000.00	950.00	902.50	0.00	970.19	–9.70	960.49
2	1,000.00	950.00	902.50	960.49	2,002.71	–20.03	1,982.68
3	1,000.00	950.00	902.50	1,982.68	3,101.57	–31.02	3,070.55

So the estimated maturity value for policy Type B is £3,070.55. [3]

(ii) ***Net present value***

The calculations of the net present values of the profits are set out in the tables below:

TYPE A					
Policy year	Profit on allocation	Expenses	Non-unit interest	Annual charge	Profit vector
1	192.50	–100.00	6.94	8.68	108.12
2	40.50	–30.00	0.79	19.55	30.84
3	40.50	–30.00	0.79	31.12	42.41

[2]

Since the dependent probabilities of mortality and withdrawal are 1% and 5%, the probability of remaining in force until the end of the year is $1-0.01-0.05=0.94$. [1]

So the net present value for policy Type A is:

$$NPV_A = 108.12v + 30.84v^2 \times 0.94 + 42.41v^3 \times 0.94^2 = £150.40$$ [1]

TYPE B						
Policy year	Profit on allocation	Expenses	Non-unit interest	Expected surrender profit	Annual charge	Profit vector
1	97.50	–100.00	–0.19	10.00	9.70	17.01
2	97.50	–30.00	5.06	5.00	20.03	97.59
3	97.50	–30.00	5.06	0.00	31.02	103.58

[3]

The expected surrender profit is equal to the amount of surrender penalty recovered on a surrender, multiplied by the dependent probability of surrender occurring during the year. So, for example, at the end of Year 1 the total premiums outstanding is £2,000, so the surrender penalty is 10% of this, which is £200. The dependent probability of surrender in Year 1 is 5%, and so the expected surrender profit at the end of Year 1 is:

$$0.05 \times 200 = £10$$

The values in other years are calculated similarly.

So the net present value for policy Type B is:

$$NPV_B = 17.01v + 97.59v^2 \times 0.94 + 103.58v^3 \times 0.94^2 = £160.03$$ [1]

27.7 (i) ***Present values of profits***

(a) If the policyholder dies in Year 1, the present value of the profit (*PV*) is:

$$PV_a = \frac{(P-e_1)(1+i)-S}{1+r} = \frac{(520-250)\times 1.02 - 15{,}000}{1.05} = \frac{275.4 - 15{,}000}{1.05} = -14{,}023.43$$

(b) If the policyholder surrenders at the end of Year 1, the *PV* is:

$$PV_b = \frac{(P-e)(1+i)-SV}{1+r} = \frac{275.4-420}{1.05} = -137.71$$

(c) If the policyholder dies in Year 2, the *PV* is:

$$PV_c = \frac{(P-e_1)(1+i) - {}_1V}{1+r} + \frac{({}_1V + P - e_2)(1+i) - S}{(1+r)^2}$$

$$= \frac{275.4-510}{1.05} + \frac{(510+520-25)\times 1.02 - 15{,}000}{1.05^2}$$

$$= -12{,}899.07$$

(ii) ***Net present value***

For this we will need the probabilities of the four possible outcomes.

The probabilities of events (a), (b), and (c), denoted by P_a, P_b, and P_c, are:

$$P_a = q_{50} = 0.002508$$

$$P_b = \left(1 - q_{50}\right) \times 0.15 = 0.997492 \times 0.15 = 0.149624$$

$$P_c = \left(1 - q_{50}\right) \times \left(1 - 0.15\right) \times q_{51} = 0.997492 \times 0.85 \times 0.002809 = 0.002382$$

The probability of none of these events occurring (which is the probability that the policy matures at time 2) is:

$$P' = 1 - P_a - P_b - P_c = 1 - 0.002508 - 0.149624 - 0.002382 = 0.845487$$

So the net present value is:

$$NPV = -14{,}023.43 \times 0.002508 - 137.71 \times 0.149624 - 12{,}899.07 \times 0.002382 + 250 \times 0.845487$$

$$= 124.87$$

28

Reserving aspects of profit testing

Syllabus objectives

6.4 Project expected future cashflows for whole life, endowment and term assurances, annuities, unit-linked contracts, and conventional/unitised with-profits contracts, incorporating multiple decrement models as appropriate.

6.4.3 Show how gross premium reserves can be computed using the above cashflow projection model and included as part of profit testing.

6.5 Show how, for unit-linked contracts, non-unit reserves can be established to eliminate ('zeroise') future negative cashflows, using a profit test model.

0 Introduction

In the previous chapter, we saw how to project the cashflows for policies. In this chapter, we describe how such techniques can be used to set reserves for both unit-linked and conventional contracts, and how changes in the reserving and pricing assumptions affect profit.

1 Pricing and reserving bases

In this section we consider the different bases that may be used in the financial management of a life insurance contract.

The profit test, as described in the previous chapter, needs many assumptions to be made in order to compute the expected future profits of a contract for comparison with a stated profit criterion.

The assumptions will, in the first instance, be the insurer's best estimate of expected future experience. This can be termed the *experience basis*.

Many life insurance companies see this as an acceptable basis for pricing because of the implicit allowance for risk in the pricing basis, through the risk discount rate. In fact, competition may force a company into marketing a 'loss leader'.

Assuming that the chosen risk discount rate reflects fully the uncertainties in the assumptions, no further margins would be taken.

Thus, we arrive at the *pricing basis* for the contract, *ie* the insurer's realistic expected outlook that it chooses to use in setting premiums and/or charges.

In practice, the actual profits arising in the future will depend on the actual future experience values of the items for which assumptions have been made. The insurer must consider two implications of this:

(i) How will profits be affected by the actual values for assumptions turning out to be different from the pricing basis?; and

(ii) Might the actual experience give rise to a need for additional finance?

The answer to both questions lies in re-running the profit test with a different set of assumptions from the pricing basis.

To answer question (i), typically the insurer will choose a variety of different assumptions in order to determine how quickly the expected future profit changes on varying any particular assumption. Such alternative bases represent *sensitivity test assumptions*. Such 'sensitivity tests' can give the insurer an understanding of how profits might be increased as well as how they might decrease, or even become losses. The results of these tests may indicate ways in which a product might be re-designed to minimise the volatility of expected profits. Any redesign would need to be profit tested itself, so this process can be iterative.

A similar approach could be taken to answer question (ii), but in this case the alternative assumptions would concentrate just on situations where profits would be reduced to the extent that external finance, or capital, would be required. These external capital requirements must then be supported by *reserves* held by the insurer (see Sections 2 and 3 below). The profit test can be re-run for a range of scenarios which give rise to varying levels of reserves. The more the assumptions diverge from the pricing basis, the greater the required reserves will become, and it becomes necessary to choose a single set of assumptions for which the reserves would provide adequate protection to policyholders without being beyond the means of an insurer's finances.

This is one of the most important, and difficult, aspects of life insurance company management.

Consider the following two extremes:

- A company whose reserving basis is too optimistic runs a significant risk of paying out too much profit to its capital providers. The result could be insufficient assets in future to meet liabilities, and the company would become insolvent.
- A company whose reserving basis is too pessimistic will be holding extremely large reserves. Large reserves require large amounts of capital from the shareholders (or with-profits policyholders). The profits from the business then have to be higher in order to provide the required rate of return on that capital, which means greater cost to the customer through having to pay higher premiums or charges.

Policyholders will be prepared to pay higher premiums up to a point, in return for the increased security of the fund (and therefore obtaining greater certainty that the company will meet its future obligations to them). On the other hand, if the cost of capital is too great, the company will lose customers and could go out of business.

The actuary therefore has to determine the level of reserve that leaves the company with an acceptably low probability (or risk) of insolvency occurring in the future, whilst at the same time imposing a cost of capital on the company that the policyholders are willing to pay for. The result is that the reserving basis will be prudent. A significantly more pessimistic basis than 'best estimate' will be assumed, but it will not be beyond the realms of reasonable possibility (for example, we would not assume that all the company's life assurance policyholders will die on the day after the valuation date!).

This single *valuation (or reserving) basis* is set by an insurer's actuary to ensure that an adequate assessment of the reserves is made. In practice, the valuation basis chosen will have to satisfy any local legislation and professional guidance which exists to protect the interests of policyholders. Detail is beyond the scope of this course.

Once the actuary has decided on an appropriate level of reserves to be held by the insurer, the profit test would be finally re-run, still on the original pricing basis, but with the cashflows paid into reserves modelled explicitly. This may mean a reassessment of premiums, benefits, and charges with consequential reassessment of valuation bases. The approach of pricing a contract can therefore be iterative.

2 Calculating reserves for unit-linked contracts

In this section we see how cashflow projections can be used to set reserves. This is particularly important for unit-linked products, where it is the only way of determining appropriate reserves, but we shall also see that we can apply the same methodology to conventional products. We first take another look at reserves in order to put things in perspective.

2.1 Reserves revisited

We have already come across the idea of a reserve as being a sum of money that a life insurance company puts aside to meet future liabilities. Now that we have discussed cashflow projections, we can look at reserving in a different light.

Let's consider what might happen if we project the cashflows for a policy, where no reserves are established, *ie* if we just look at the cash income (premiums and investment return) less cash outgo (expenses and claims) for each year. For instance, for a regular premium endowment assurance of term *n* years we would expect something like the following pattern of cashflows:

- in Year 1 small and positive, or possibly negative, due to initial expenses
- in Years 2 to $n-1$ positive due to premiums exceeding outgo
- in Year *n* very negative due to maturity payout.

For an example consider the endowment assurance we looked at in the previous chapter. The policy is a 5-year regular-premium endowment assurance, with sum assured £10,000 payable on survival or at the end of the year of earlier death. Premiums of £2,108.81 are payable annually in advance, and on surrender a return of premiums is paid, without interest, at end of the year of surrender.

Using appropriate assumptions for interest, mortality, surrender and expenses, we obtain the following yearly cashflows:

Year	Premium	Expense	Interest	Expected claim cost	Expected surrender cost	Expected cashflow per policy in force at start of year
1	2,108.81	–1,054.41	42.18	–50	–210.88	835.70
2	2,108.81	–105.44	80.13	–60	–210.88	1,812.62
3	2,108.81	–105.44	80.13	–70	–316.32	1,697.18
4	2,108.81	–105.44	80.13	–80	–84.35	1,979.15
5	2,108.81	–105.44	80.13	–10,000	0	–7,916.50

where the expected cashflow values are calculated by summing over all the columns for each year. So, the insurer needs to put aside money early on to provide for the large negative cashflow in the last year.

We have already seen how to calculate the required amount of reserve prospectively, by valuing the stream of future benefit payments and future expenses and deducting future premiums. In other words we take each item of the expected cashflow and sum (with discounting and allowing for the probabilities of remaining in force) over all future years. We would get the same answer by summing each year's cashflow (again with discounting and allowing for the probabilities of remaining in force) and multiplying by –1, as we now explain below.

The calculation of a reserve involves summing all the elements in the following box, with suitable discounting for interest and allowing for staying in force:

Year	**Discounted premium (positive)**	**…**	**Discounted expenses (negative)**	**Discounted claims (negative)**	***etc* …**	**Sum = discounted cashflow**
1	$+P_1$		$-e_1$	$-C_1$		EPV at time 0 of premiums, expenses, *etc* in year 1
…						
t	$+P_t$		$-e_t$	$-C_t$		EPV at time 0 of premiums, expenses, *etc* in year *t*
…						
Sum	Sum of discounted premiums		Sum of discounted expenses			Sum of sums × –1 = reserve

We can either sum the rows first and then sum the row totals, or sum the columns first and then sum the column totals. We then multiply by -1 so as to convert from:

'EPV of future income less outgo'

to:

'EPV of future outgo less income'

which is what we require in order to produce a (prospective) reserve value. So we can take our cashflow projections – these are our row totals above – and use these to calculate suitable reserves.

We illustrate how this can be done for conventional contracts in Section 3. In the rest of this section we look at the situation for unit-linked contracts. This is slightly different from the simple summation implied above.

2.2 Calculating reserves for unit-linked contracts

Unit-linked contracts require a unit reserve, which is equal to the unit fund value at any particular time, and a non-unit reserve. The calculation of the non-unit reserve follows the procedure described below, which is sometimes referred to as *zeroising negative casflows*.

We have already come across the idea that the reserve for a unit-linked contract should be the bid value of the units. However, this will clearly not be adequate if we expect future negative non-unit cashflows. In that case we need to set up a cash, or non-unit, reserve to fund for those future negative cashflows. We do this using the projected cashflows.

It is a principle of prudent financial management that, once sold and funded at the outset, a product should be self-supporting. This implies that the profit signature has a single negative value (funds are provided by the insurance company) at policy duration zero. This is often termed 'a single financing phase at the outset'.

Many unit-linked products naturally produce profit signatures which have a single financing phase. However some products, particularly those with substantial expected outgo at later policy durations, can give profit signatures which have more than one financing phase. In such cases these later negative non-unit cashflows (financing phases) should be reduced to zero by establishing reserves in the non-unit fund at earlier durations. These reserves are funded by reducing earlier positive non-unit cashflows. Good financial management dictates that these reserves should be established as late as possible during the term of the contract.

So we find the latest negative cashflow, set up a reserve the year before to fund for that negative, and if necessary carry on working back until we have no negative cashflows.

Suppose we have a four-year unit-linked policy with expected non-unit cashflows of:

(10, 15, –5, 8)

per policy in force at the start of the year. The expected cashflow at the end of Year 4 is positive, so no reserve is required at the start of Year 4 (*ie* at time 3). However, the expected cashflow at the end of Year 3 is negative, so we need to set up a reserve at the start of Year 3 (*ie* at time 2) to zeroise this. For simplicity, we will ignore interest and mortality for now.

If we set up a reserve of 5 at the end of the second year, then the new cashflow in the third year will be 5 + (–5) = 0. However, setting up the reserve of 5 in the second year costs 5 at the end of that second year. So the second year's cashflow is reduced by 5, from 15 to 10.

So our pattern of expected cashflows is now (10, 10, 0, 8).

Question

It might be argued that we don't need a reserve at the start of Year 3, because we are expecting more than enough money in Year 4 to make up the loss. That is:

$$\{\text{Reserve at time 2}\} = 5 - 8 = -3$$

and so we'd assume a reserve of zero for prudence. Explain the flaws in this argument.

Solution

One problem is that the positive cashflow comes after the negative cashflow, so if the company does not have any reserves available in Year 3, it might find it physically impossible to pay out all its claims and expenses at the required time.

A second and ultimately more serious problem is that the fourth policy year may never happen (because the policy may lapse at the start of Year 4). The insurer is therefore potentially permanently short of money (and, in the scenario we have just described, might become completely bankrupt).

So, when working out non-unit reserves for unit-linked contracts, it is necessary to *ignore all positive cashflows that occur after the last negative cashflow*.

General approach

A policy has a non-unit cashflow vector (profit vector without non-unit reserves) of $(\text{NUCF})_t$; $t = 1, 2, 3, \ldots$ determined using the methods described in the previous chapter. Non-unit reserves (reserves in the cash fund) are to be set up so that there is only a single financing phase. The reserves to be established at policy duration t are ${}_tV$.

The reserving basis interest rate is i_s, and the probability of a policy staying in force for one year at age x is $(ap)_x$. After establishing non-unit reserves the profit vector is $(\text{PRO})_t$, $t = 1, 2, 3, \ldots$.

So, in the example at the start of this section, (10, 15, –5, 8) is the cashflow vector, and (10, 10, 0, 8) is the profit vector.

The equation of value at the end of policy year t, for cashflows in policy year t, per policy in force at time $t-1$, is:

$$(\text{NUCF})_t + {}_{t-1}V(1+i_s) - (ap)_{x+t-1}\,{}_tV = (\text{PRO})_t$$

In this equation, the entry in the profit vector corresponding to the end of year t is equal to:

- the non-unit cashflow at time t
- plus the accumulated amount of the reserve that was set up at time $t-1$
- minus the cost of setting up the reserve at time t for each policy that's remained in force over the policy year.

The process of establishing reserves begins at the greatest duration t for which $(\text{NUCF})_t$ is negative. Let this be duration $t = m$. Non-unit reserves will not be required at durations $t \geq m$ because during these policy years the product is expected to be self-financing. Hence we know that ${}_tV = 0$ for $t \geq m$.

For policy year m we can write:

$$\left(\mathbf{NUCF}\right)_m + {}_{m-1}V\left(1+i_s\right) - (ap)_{x+m-1} \times 0 = \left(\mathbf{PRO}\right)_m$$

where $\left(\mathrm{NUCF}\right)_m < 0$ and we wish to choose ${}_{m-1}V$ so that $\left(\mathrm{PRO}\right)_m = 0$.

Thus ${}_{m-1}V$ should be chosen to be:

$${}_{m-1}V = -\frac{(\mathbf{NUCF})_m}{(1+i_s)} \qquad (*)$$

$\left(\mathrm{NUCF}\right)_m < 0$ has been 'turned into' $\left(\mathrm{PRO}\right)_m = 0$; the expected cashflows have been zeroised.

So we set up a reserve at time $m-1$ (the beginning of the year m). This reserve, when accumulated at the rate of interest i_s assumed in the basis, is exactly enough to cancel out the negative cashflow at the end of year m. So the profit at the end of year m is now 0.

However, setting up a reserve at time $m-1$ will now have an impact on the non-unit profit at time $m-1$.

The reserve ${}_{m-1}V$ will be established at policy duration $m-1$ out of the funds available at duration $m-1$. A reserve of ${}_{m-1}V$ is required for every policyholder alive at the start of the mth policy year. The non-unit cashflow $\left(\mathrm{NUCF}\right)_{m-1}$ at time $m-1$ is $\left(\mathrm{NUCF}\right)_{m-1}$ for every policyholder alive at the start of the $(m-1)$th policy year. If we take the required reserve ${}_{m-1}V$ out of this cashflow, then the adjusted cashflow $(NUCF)'_{m-1}$ is given by:

$$(\mathbf{NUCF})'_{m-1} = (\mathbf{NUCF})_{m-1} - (ap)_{x+m-2}\ {}_{m-1}V$$

We use the probability $(ap)_{x+m-2}$ because we are interested in the probability of the policy staying in force to age $x+m-1$ from a year before that age, *ie* from age $x+m-2$ to age $x+m-1$.

If $(\mathrm{NUCF})_{m-1} < 0$, then $(\mathrm{NUCF})'_{m-1}$ will be negative. However, if $(\mathrm{NUCF})_{m-1} > 0$, then $(\mathrm{NUCF})'_{m-1}$ may be positive or negative. If $(\mathrm{NUCF})'_{m-1} > 0$ then:

$$(\mathbf{PRO})_{m-1} = (\mathbf{NUCF})'_{m-1}$$

So if the non-unit cashflow is positive after setting up the reserve, then the entry in the profit vector is equal to this cashflow. However, if the non-unit cashflow is now negative, we have to zeroise this as well.

If $(\mathrm{NUCF})'_{m-1} < 0$, then we repeat the process establishing non-unit reserves ${}_{m-2}V$ at policy duration $m-2$.

So we have:

$$(\mathbf{NUCF})'_{m-1} + {}_{m-2}V(1+i_s) = (\mathbf{PRO})_{m-1}$$

and choose ${}_{m-2}V$ so that $(PRO)_{m-1} = 0$, *ie*:

$${}_{m-2}V = -\frac{(\mathbf{NUCF})'_{m-1}}{(1+i_s)}$$

The process then repeats as from (*) above, and continues in this way until all the necessary reserves at durations $t = 1, 2, 3, \ldots, m-1$ have been obtained.

Let's look at an example. Suppose we have a five-year unit-linked policy. Assuming no non-unit reserves are held, the expected non-unit cashflows emerging at the end of each year, per policy in force at the start of that year, are:

$$(-60.20, -20.50, -17.00, 50.13, 85.75)$$

We shall calculate the reserves required if negative cashflows other than in Year 1 are to be eliminated, and hence obtain the revised profit vector allowing for the reserves. We'll assume that reserves earn interest at a rate of 5% per annum, and we'll ignore mortality.

Here we have negative cashflows in Years 2 and 3 that need to be dealt with. If the company sets up reserves at the end of Years 1 and 2 that can release:

- £20.50 at the end of Year 2
- £17.00 at the end of Year 3

then the negative cashflows at the end of Years 2 and 3 will be matched exactly by a positive cashflow from reserves, and the profit vector will show a zero entry for these two years.

No reserves are required after Year 3 since there are no losses after Year 3.

So, we require a reserve at the start of Year 3, ${}_2V$, such that:

$${}_2V = \frac{17.00}{1.05} = 16.19$$

Setting up this reserve turns the cashflow of –17.00 at the end of Year 3 into a profit of 0, and (since we are ignoring mortality) it turns the cashflow at the end of Year 2 into an 'adjusted cashflow' of:

$$-20.50 - 16.19 = -36.69$$

We now need to set up a reserve at time 1 to zeroise this negative. The reserve at time 1 is assumed to earn interest at the rate of 5% *pa*, so its accumulated value at time 2 is ${}_1V \times 1.05$. In order to eliminate the adjusted cashflow of –36.69 at time 2, we require:

$${}_1V = \frac{36.69}{1.05} = 34.94$$

This corresponds to the approach set out in the above Core Reading because we are ignoring mortality (and other decrements), *ie* setting $(ap)_{x+m-2} = 1$.

No other reserves are required, since negative cashflows except in Year 1 have been eliminated. So, the reserves required are:

$${}_1V = 34.94, \ {}_2V = 16.19$$

and $${}_0V = {}_3V = {}_4V = {}_5V = 0$$

The net effect is that the loss in Year 1 is increased by the amount of the reserve set up, and the losses in Years 2 and 3 have been zeroised, so the profit vector is:

$$(-95.14, \ 0, \ 0, \ 50.13, 85.75)$$

Question

Under a five-year unit-linked policy, and assuming no non-unit reserves are held, the expected non-unit cashflows emerging at the end of each year, per policy in force at the start of that year, are:

$$\left(-10, \ -20, \ 5, \ -15, \ 40\right)$$

Calculate the non-unit reserves that should be set up to zeroise the negative cashflows, and give the profit vector. Assume 6% *pa* interest, and ignore mortality.

Solution

We do not need to set up a reserve at the start of Year 5 since the expected cashflow for the 5th policy year is positive.

We want to zeroise the –15 in Year 4. So we need a reserve in place at the start of that year, *ie* at time 3, equal to:

$${}_3V = \frac{15}{1.06} = 14.15$$

The Year 3 adjusted cashflow is $5 - 14.15 = -9.15$ and so this now needs to be eliminated.

We therefore need a reserve at the start of Year 3 (*ie* at time 2) of:

$${}_2V = \frac{9.15}{1.06} = 8.63$$

This means the adjusted cashflow for Year 2 is $-20 - 8.63 = -28.63$, so we need a reserve at the start of Year 2 (*ie* at time 1) of:

$${}_1V = \frac{28.63}{1.06} = 27.01$$

This changes the year 1 cashflow to $-10-27.01=-37.01$.

So the profit vector is (–37.01, 0, 0, 0, 40).

In the above example and question we have ignored mortality. However, we normally need to take mortality into account in determining the reserves to be established.

To show this, we'll take the same example as above, which had in-force expected cashflows (before reserves) of $(-60.20, -20.50, -17.00, 50.13, 85.75)$. We'll again calculate the non-unit reserves and the profit vector, but this time we'll assume that the policy is issued to lives aged 55 with mortality assumed to be such that:

$$q_{55+t} = 0.01 + 0.001t \quad \text{for } t = 0, 1, \ldots, 4$$

We will assume 5% *pa* interest, as before.

The reserve required at the start of Year 3 (time 2) does not change, *ie* we still have:

$${}_2V = 16.19$$

since we still need to reduce the expected loss of 17.00 at the end of Year 3 to zero for each policy that was in force at the start of that year (*ie* at time 2), allowing for interest at 5%.

Now let's consider the reserve we need at time 1, *ie* ${}_1V$. This is the reserve that needs to be held *per policy in force at time 1*. This reserve has to cover:

(1) the expected cash outgo from this in-force policy, occurring at the end of Year 2

(2) the required reserves that need to be carried over to the third policy year, which will be needed *only for those policies that are still in force at the end of Year 2*, *ie* for the proportion surviving from time 1 to time 2.

To assess this, we work out the adjusted cashflow at time 2, per policy in force at time 1.

This is:

$$\begin{aligned}(NUCF)'_2 &= -20.50 - {}_2V \times p_{56} \\ &= -20.50 - 16.19 \times (1 - q_{56}) \\ &= 16.19 \times 0.989 \\ &= -36.51\end{aligned}$$

In other words, the total expected cash *outgo* at the end of Year 2, per policy in force at the *start* of Year 2, is $+36.51$, and it is this amount that the reserve for each policy that starts the year needs to cover. So, to eliminate this loss, we need:

$${}_1V = \frac{36.51}{1.05} = 34.77$$

Holding this reserve (and then using it to cover the year's adjusted cashflow) will now result in a zero expected profit for Year 2.

The other reserves are zero as before.

Finally, we turn to Year 1. The ${}_1V$ reserves now become part of the cash outgo for this year. However, the profit vector element for this year is defined as the expected profit per policy in force at the start of *that* year, *ie* as at time 0. As the ${}_1V$ reserves are only required for the survivors of Year 1, the expected adjusted cashflow at the end of Year 1 is:

$$(NUCF)'_1 = -60.20 - {}_1V \times p_{55} = -60.20 - 34.77 \times 0.99 = -94.62$$

which is also the profit for that year. So, the revised profit vector is:

$$(-94.62, 0, 0, 50.13, \ 85.75)$$

Question

The in-force expected non-unit cashflows for a five-year unit-linked contract taken out by a person aged exactly 50 are $(-10, \ -20, \ 5, \ -15, \ 40)$. Calculate the non-unit reserves required to zeroise any negative cashflows other than those occurring in the first policy year. Assume AM92 Ultimate mortality and 6% *pa* interest.

Solution

We do not need to set up a reserve at the start of Year 5 since the expected cashflow for the 5th policy year is positive.

We want to zeroise the cashflow of –15 in Year 4. So we need a reserve in place at the start of that year, *ie* at time 3, equal to:

$${}_3V = \frac{15}{1.06} = 14.15$$

Setting up this reserve at time 3 affects the expected non-unit cashflow at the end of the third policy year. The probability that a policy is still in force at the end of the third year given that it was in force at the start of the third year is p_{52}. So the (adjusted) expected cashflow at the end of Year 3, per policy in force at the start of Year 3, is now:

$$5 - {}_3V \times p_{52} = 5 - 14.15 \times 0.996848 = -9.11$$

Since this is negative, we need to set up a reserve at the start of Year 3 to zeroise it. We require:

$${}_2V \times 1.06 = 9.11$$

So ${}_2V = 8.59$.

We now repeat this process for Year 2. Setting up the reserve of 8.59 at time 2 affects the expected non-unit cashflow at the end of the second policy year. The probability that a policy is still in force at the end of the second year given that it was in force at the start of the second year is p_{51}. So the (adjusted) expected cashflow at the end of Year 2, per policy in force at the start of Year 2, is:

$$-20 - {}_2V \times p_{51} = -20 - 8.59 \times 0.997191 = -28.57$$

Since this is negative, we need to set up a reserve at the start of Year 2 to zeroise it. So we require:

$${}_1V \times 1.06 = 28.57 \quad \Rightarrow \quad {}_1V = 26.95$$

(We have not needed to use a survival probability for Year 4 in any of these calculations because there was no reserve required at the end of Year 4.)

In some cases, the vector of non-unit cashflows may show several runs of negative entries. In this case, the above method is repeated as many times as necessary until all the negative entries have been eliminated.

For example, suppose the in-force expected cashflows for a 6-year policy issued to lives aged x is:

$$(-131.53\,, -70.11\,, 25.00\,, -20.15\,, 55.74\,, 157.91)$$

We shall calculate the reserves required and the resulting profit vector if negative cashflows after Year 1 are to be zeroised. We shall assume non-unit reserves earn interest at 6% *pa* and that the probability of death during any year is 0.01.

No reserves will be required after Year 4 as the expected non-unit cashflows in Years 5 and 6 are positive. To zeroise the negative cashflow in Year 4, we require a reserve at the start of Year 4 such that:

$${}_3V \times 1.06 = 20.15 \quad \Rightarrow \quad {}_3V = 19.01$$

The expected cashflow in the previous year is +25.00. This is sufficient to set up the reserve. Allowing for the reserve, the expected profit emerging at the end of Year 3 per policy in force at the start of Year 3 is:

$$25.00 - 0.99 \times 19.01 = 6.18$$

We now need to zeroise the negative cashflow in Year 2. This will require a reserve at the start of Year 2 such that:

$${}_1V = \frac{70.11}{1.06} = 66.14$$

The expected profit for Year 1 then becomes:

$$-131.53 - 0.99 \times 66.14 = -197.01$$

So the profit vector is:

$$(-197.01\,,\ 0\,,\ 6.18\,,\ 0\,,\ 55.74\,,\ 157.91)$$

Question

Calculate the reserves required for the policy in the above example if the cashflow in Year 3 is 15.00 instead of 25.00.

Solution

As before, we have a reserve at the end of Year 3 of 19.01. So the adjusted cashflow in Year 3 becomes $15-0.99\times 19.01=-3.82$. We will need to set up a reserve at the start of Year 3 in respect of this negative result.

The start of Year 3 reserve is then:

$${}_2V=\frac{3.82}{1.06}=3.60\,.$$

Then the expected profit at the end of Year 2 per policy in force at the start of Year 2 becomes:

$$-70.11-{}_2V\times p_{x+1}=-70.11-3.60\times 0.99=-73.68$$

So we need a reserve at the start of Year 2 of:

$${}_1V=\frac{73.68}{1.06}=69.51$$

Summary of the method

To recap all of the above, here is a summary of the steps required to zeroise negative cashflows.

Step 1

Starting from the last negative entry in the vector of non-unit cashflows (in Year m say), calculate ${}_{m-1}V$ as:

$${}_{m-1}V = \frac{-\{\text{Cashflow in Year } m\}}{1+i}$$

Step 2

Calculate the adjusted expected non-unit cashflow in the previous year, as:

$$\{\text{Adjusted cashflow in Year } m-1\} = \{\text{Cashflow in Year } m-1\} - {}_{m-1}V \times (ap)_{x+m-2}$$

using the value of ${}_{m-1}V$ found from Step 1, and where the (ap) probability is the probability of staying in force over Year $m-1$. Any surrenders, as well as mortality, will need to be taken into account in this probability.

Step 3

If the adjusted cashflow from Step 2 is negative, then find ${}_{m-2}V$ from:

$${}_{m-2}V = \frac{-\{\text{Adjusted cashflow in Year } m-1\}}{1+i}$$

using the adjusted cashflow value found from Step 2, and go to Step 4.

Step 4

Carry on working backwards through the profit vector by repeated application of Steps (3) and (4) until either Year 1 is reached, or a positive value for the adjusted cashflow is obtained.

If Year 1 is reached, the profit for Year 1 is equal to the adjusted cashflow for Year 1, *ie*:

$$\{\text{Cashflow in Year } 1\} - {}_1V \times (ap)_x$$

If a positive adjusted cashflow is obtained, (in Year k say, $k > 1$), the process stops for this run of negative entries, the profit for Year k is equal to the adjusted cashflow for Year k, and the reserve at the start of the year (*ie* ${}_{k-1}V$) is equal to zero.

Step 5

Repeat Steps 1 to 4 for any other runs of negative entries.

Incorporating non-unit reserves into the profit test

The assumptions (of interest and mortality – and indeed expenses) that we use for calculating the non-unit reserves will be the insurer's valuation (or reserving) basis that was referred to towards the end of Section 1. In other words, we calculate the non-unit reserves that zeroise the expected negative cashflows that arise *when using the reserving basis*.

These reserves so calculated will then be put into the company's pricing profit-test model, as also described in that section.

Question

In an earlier example, we looked at a five-year unit-linked contract which had in-force expected cashflows (before reserves) of $(-60.20, -20.50, -17.00, 50.13, 85.75)$. Assuming 5% *pa* interest and mortality probabilities:

$$q_{55+t} = 0.01 + 0.001t \quad \text{for } t = 0, 1, \ldots, 4$$

it was found that this policy required non-unit reserves of ${}_1V = 34.77$ and ${}_2V = 16.19$. These reserves resulted in a profit vector of $(-94.62, 0, 0, 50.13, \ 85.75)$, when calculated on the same basis.

Assume that the assumptions that were used for these calculations were more prudent than the insurer's current best estimate basis.

Without doing any calculations, explain how the profit vector produced would be expected to change, when we run the company's pricing profit-test model using best estimate assumptions and incorporating the above reserves.

Solution

The profit test basis will be less cautious than the reserving basis. This means we will have some or all of the following:

- lower expected expenses and expense inflation
- higher investment returns
- lower expected claim costs

than before. These will in turn lead to having:

- higher expected profits from expenses and expense inflation
- higher investment profits
- higher mortality profits

than before. As this will apply to every year of the projection, all the profits will be larger (*ie* become more positive or less negative) than they were when calculated using the reserving basis.

This means the expected profits in Years 2 and 3 in this example will now be *greater* than zero, not equal to zero. So, while our non-unit reserves will zeroise profits in certain years on the *reserving* basis, our *best estimates* of profits in those years will be *greater* than zero.

When answering profit-testing questions where non-unit reserves are required, these reserves should be calculated on the basis specified for these in the question.

3 Calculating reserves for conventional contracts using a profit test

In the previous section we saw how to determine non-unit reserves for unit-linked products by working backwards from the last negative cashflow. We can apply exactly the same methodology to conventional contracts.

A profit test can also be used to determine the reserves for a conventional (*ie* non unit-linked) policy. We illustrate the procedures by using a without-profit endowment assurance with a term of n years, a sum assured of S payable at the end of the year of death or on survival to the end of the term, and a surrender value payable at the end of year t of U_t, which is secured by a level annual premium of P.

A basis is required for the projection of the cashflows and for calculating the required reserves. This will consist of an interest rate i, dependent probabilities of death and surrender $(aq)^d_{x+t}$ and $(aq)^s_{x+t}$, dependent probabilities of remaining in force ${}_t(ap)_x$, and expenses per policy in force at time t of e_t.

$(CF)_t$, the expected cashflow at time t per policy in force at time $t-1$, ignoring reserves, is:

$$(CF)_t = \left(P - e_{t-1}\right)(1+i) - S\,(aq)^d_{x+t-1} - U_t\,(aq)^s_{x+t-1} \qquad t = 1, 2, 3, \ldots, n-1$$

$$(CF)_n = \left(P - e_{n-1}\right)(1+i) - S$$

assuming there is no surrender value paid at the end of the last policy year.

Technically it is not possible to surrender an annual premium endowment assurance in the last policy year, because all the premiums will have been paid by then and the policy will simply terminate as a maturity, receiving the full sum assured in benefit payment.

These cashflows will usually be positive for earlier years of a contract and negative during the later years. For example a five-year endowment assurance with a sum assured of 1,000 might have cashflows:

t	1	2	3	4	5
$(CF)_t$	156.39	187.41	186.33	185.14	–803.17

If the contract is to be self-funding, then reserves must be established using the earlier positive cashflows. These reserves should be sufficient to pay the later expected negative cashflows. This requirement is exactly analogous to the need to establish reserves in the non-unit fund for a unit-linked contract. Reserves can be established for conventional assurances using the same procedures as those used to establish non-unit reserves.

Let $(CF)_t$ denote the expected cashflow at the end of Year t, and let $(PRO)_t$ denote the t th entry in the profit vector. Calculations begin at the longest policy duration, m, at which there is a negative cashflow.

For policy year m, we write:

$$(\mathrm{CF})_m + {}_{m-1}V(1+i) - (ap)_{x+m-1} \times 0 = (\mathrm{PRO})_m$$

where $(\mathrm{CF})_m < 0$ and we wish to choose ${}_{m-1}V$ so that $(\mathrm{PRO})_m = 0$.

This requires that ${}_{m-1}V$ is chosen to be:

$$_{m-1}V = \frac{-(CF)_m}{(1+i_r)} \qquad (^{**})$$

We set up this reserve from $(\mathrm{CF})_{m-1}$ and determine the adjusted cashflow:

$$(CF)'_{m-1} = (\mathrm{CF})_{m-1} - (ap)_{x+m-2}\ {}_{m-1}V$$

If $(CF)'_{m-1} > 0$, then ${}_{m-2}V = 0$. If $(CF)'_{m-1} < 0$, then we calculate:

$$_{m-2}V = \frac{-(\mathrm{CF})'_{m-1}}{(1+i)}$$

The process then repeats as from (**) above, and continues in this way until all the necessary reserves at durations $t = 1,\ 2,\ 3,...,\ m-1$ have been obtained.

For conventional assurances it is usually the case that reserves are needed at all policy durations. So the calculation begins with $(\mathrm{CF})_n$ and concludes with $(\mathrm{CF})_1$.

For the example cashflows shown above, if the reserving basis is 3% *pa* interest, mortality follows the AM92 Ultimate for a life assumed to be aged 55 exact at entry, and surrenders are ignored, we obtain:

t	1	2	3	4	5
$(\mathrm{CF})_t$	**156.39**	**187.41**	**186.33**	**185.14**	**–803.17**
${}_tV$	**177.20**	**371.79**	**572.51**	**779.78**	**0**

These figures are calculated as follows. First we need a reserve at time 4 to cover the negative cashflow at time 5. We simply discount the negative cashflow at time 5 over one year, and multiply by -1. So:

$$_4V = \frac{803.17}{1.03} = 779.78$$

Next we need to work out the expected cashflow at time 4 allowing for the reserve. This is:

$$(CF)'_4 = 185.14 - 779.78\,p_{58} = 185.14 - 779.78 \times (1 - 0.006352) = -589.68$$

We only need reserves at the end of Year 4 for those policyholders that have *survived* Year 4, so we multiply the required reserve by p_{58} (the probability of surviving Year 4). Remember that all of these end-year cashflows are expressed per policy in force at the *start* of each year.

As $(CF)'_4$ is negative, we need a reserve at the start of Year 4 of:

$$_3V = \frac{589.68}{1.03} = 572.51$$

The other figures are obtained by continuing this process until $(CF)'_t$ becomes positive or, as in this case, we get to the start date of the policy.

When a policy has a cashflow pattern like this one, where only the final cashflow is negative, calculating reserves using:

1. the EPV future outgo – EPV future income formula
2. discounted future cashflows
3. zeroisation of future negative cashflows

all give the same result.

This is also true when the cashflows become increasingly negative over time, as in the case of a term assurance. However, as we discussed in Section 2.2, when a policy produces some negative cashflows but the last cashflow is positive, zeroisation of future negative cashflows gives a different answer to the other two methods and is preferable on the grounds of prudence.

As before, these reserves are then put back into the profit testing model used to price the contract, as described in Section 1 above.

So we:

- calculate reserves on a prudent basis,
- calculate cashflows on a realistic basis, then allow for the prudent reserves so as to give the projected profits for the profit test,
- play around with premiums and product design until we achieve our profitability criterion using these 'realistic experience but prudent reserves' cashflows.

Clearly we cannot derive prudent reserves from realistic (*ie* best estimate) cashflows. We go into more detail on the interaction between a realistic experience basis and a prudent reserving basis in the next section of this chapter.

4 Effect of pricing and reserving bases on a profit test

The writing of each contract represents an investment of an insurer's capital. It has been seen from above that the expected profit from a contract is the present value of the projected profit test net cashflows at the risk discount rate.

By 'net' cashflows we mean the values of the profit vector, as described earlier.

The chosen pricing and valuation assumptions will each have an impact on the expected profit.

Remember by 'valuation assumptions' we mean the assumptions used to calculate the reserves.

For example, the assumed investment return in the pricing basis would normally, by definition of the risk discount rate, be less than that risk discount rate. A lower assumption will lead to lower present values of profits.

A valuation basis which gives rise to positive reserves will normally reduce the present value of profits from a contract. This is because the reserves reduce otherwise positive cashflows and are then invested in lower risk assets whose rate of investment return is expected to be less than the risk discount rate. Over the whole term of a contract, and assuming that experience turns out to be in line with the assumed pricing basis, stronger (*ie* larger) reserves will not reduce the overall aggregate size of the cashflows, but they will delay the cashflows so that they emerge later than they would have done using a weaker valuation basis, hence reducing the present value of profits.

In the above description of the profit test calculations, we have assumed that reserves are set up at the end of the policy year, and this is conventionally how profit tests are carried out. However, it is more prudent to assume that reserves are set up at the beginning of the policy year as soon as a premium is paid. In this case, the usual assumption that reserves will earn less than the risk discount rate gives rise to a slightly smaller profit. In practice, insurance companies calculate cashflows monthly rather than annually, and therefore the convention of end-of-period reserves does not lead to a material overstatement of profit.

A more general approach which can be used to discount cashflows, given sufficient computing power, would be to discount cashflows as soon as they arise, rather than first accumulating them to the end of policy years at an assumed rate of interest. This approach would be more prudent for all negative cashflows including reserve cashflows, such as expenses paid.

For example, if we had an initial expense of 100, accumulating this negative cashflow to the end of the year at 8% and then discounting back at 12% gives:

$$100 \times \frac{1.08}{1.12} = 96.43$$

So it is more prudent to account for the negative cashflow of 100 as soon as it arises rather than accumulate to the end of the policy year and discount.

We can illustrate the above concepts by considering a very simple single premium 5-year pure endowment in a zero-mortality zero-expense world. The sum assured is 1,000.

Suppose we have the following bases for the various rates of interest:

Pricing 6% *pa*

Experience 7% *pa* (on cash and reserves)

Reserving 5% *pa*

Risk discount rate 9% *pa*

So this means we use 6% *pa* interest when calculating the premium, 5% *pa* interest when calculating the amounts of the reserves, 7% *pa* interest when calculating the interest earned on the premium and reserves, and 9% *pa* interest to discount the projected profits.

Expenses, mortality and withdrawals are zero for all bases (pricing, reserving and expected experience).

So the premium is $1{,}000v^5$ at 6% giving 747.26.

The projected cashflows, showing them both before and after reserving, are shown in the tables below.

Year	Premium	Interest	Benefit outgo	Cashflow (before reserves)
1	747.26	52.31	0	799.57
2	–	0	0	0
3	–	0	0	0
4	–	0	0	0
5	–	0	–1,000	–1,000

Year	End of year reserve	Interest earned on reserves	Cost of increasing reserves (ignoring interest on reserves)	Profit vector
1	822.70	0	–822.70	–23.14
2	863.84	57.59	–41.14	16.45
3	907.03	60.47	–43.19	17.28
4	952.38	63.49	–45.35	18.14
5	0.00	66.67	952.38	19.05
Total			0.00	47.78

Since there is no mortality, the cost of increasing the reserve at the end of Year 1 is calculated as ${}_1V - {}_0V = {}_1V$. Since this is a cost to the company, it is shown as a negative entry in the table above. The other entries in this column are calculated in a similar way.

Question

(i) Verify the entries above for the second year.

(ii) Calculate the net present value of the profits for this policy.

Solution

(i) ***Verifying the Year 2 entries***

There is no premium, so no interest on the premium. There is no benefit either. So the only cashflows come from the reserves.

At the start of the year, the company holds a reserve of 822.70 per policy in force. Interest on reserves is 7%, which gives interest of 57.59.

The reserve required at the end of the year is $1{,}000v^3$. Valuing at 5%, this is 863.84. Since there is no mortality, the cost of increasing the reserve is:

863.84 – 822.70 = 41.14

So profit is 57.59 – 41.14 = 16.45.

(ii) ***Net present value***

Net present value of profits is

$$-23.14v + 16.45v^2 + \cdots + 19.05v^5 = 31.19$$

discounting at 9%.

In this example, the value of the profit in the first policy year is negative. This means selling one of these policies will require an injection of extra money equal to this amount, to avoid the company becoming insolvent, and this required extra money is referred to as the *new business strain* of the contract.

Extra money such as this has to be provided from the company's capital resources.

Question

(i) Explain why the new business strain arises with this contract, given that there are no expenses.

(ii) State the one feature in the basis that tells us that the total non-discounted profits will be positive.

Solution

(i) ***Reason for new business strain***

New business strain has arisen because the reserving basis is stronger (*ie* more pessimistic) than the pricing basis. The premium contains 747.26 in respect of the value of future benefits, while the reserves we require to set up, after receiving the premium, have a present value of $\frac{822.70}{1.05} = 783.53$ at the start of the year.

(ii) ***Reason for positive total profit***

It is because the experience investment income assumption is greater than the pricing assumption.

Now let's consider the effect of strengthening the reserving basis. So, if we were to calculate reserves using 3% *pa* interest, for example, the cashflows before reserves are unchanged from before, and the projected profits become:

Year	End of year reserve	Interest on reserves	Cost of increasing reserves	Profit vector
1	888.49	0	–888.49	–88.92
2	915.14	62.19	–26.65	35.54
3	942.60	64.06	–27.45	36.61
4	970.87	65.98	–28.28	37.70
5	0.00	67.96	970.87	38.83
Total			0.00	59.76

The net present value of these profits is now 28.55. The total amount of profits is slightly higher because we have set up bigger reserves and those reserves bring in interest – but that interest is lower than the risk discount rate, so the effect is a reduction in the *value* of profits.

We see that the profitability of the contract has gone down due to the increase in the gap between the investment return earned on cash and reserves, and the required rate of return.

We could also have increased that gap by increasing the risk discount rate – this would have reduced the net present value of profits.

We can see also that strengthening the reserving basis has increased the new business strain.

5 Setting out the calculations

We have used several different ways of showing our calculations in this and in the previous chapter, *eg* sometimes showing the interest on the reserves separately and sometimes including it within the 'increase in reserves' column; or sometimes showing every reserve-related item in separate columns (*ie* 'reserve at start of year' + 'interest on reserves' – 'expected cost of reserve at end of year'). There is no fixed rule as to which of these is 'best' to use, as they all achieve the same thing and result in the same answers. Different people find different approaches more intuitive and easier to remember or use, and all (correct) methods are equally acceptable in the exam.

Chapter 28 Summary

Calculating reserves using cashflow projections

We can use cashflow projections to calculate reserves by:

- identifying negative cashflows (starting from the end of the projection) in year t,
- calculating the reserves needed at the end of the previous year $t-1$ to fund such negative cashflows,
- paying for such a reserve from the cashflow of year $t-1$,
- and if the cashflow of year $t-1$ is now negative, repeat the above process.

We need to do this with unit-linked contracts to ensure that policies in force will not require further financing by the life company.

We can use the same technique with conventional contracts, as an alternative to using a prospective gross premium formula approach.

Different bases

A basis is a set of assumptions about quantities such as future investment returns, mortality rates, surrender rates and expenses.

Assumptions that are our best estimates of the future give an experience basis.

The basis we use to set premiums is called the pricing basis and can be the same as the experience basis, or slightly more prudent, or more risky.

The basis we use to determine reserves – the valuation or reserving basis – is generally much more prudent than the experience basis. This will defer the emergence of profits from the contract. If the required rate of return is greater than the investment return on reserves, this will reduce the profitability of the contract.

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 28 Practice Questions

28.1 Explain why a life insurance company might need to set up non-unit reserves in respect of a unit-linked life assurance contract.

28.2 The following table shows (in £'s) a profit testing calculation (with some of the entries missing) for three-year endowment assurance contracts issued to lives aged exactly 57 with a sum assured of £5,000 payable at the end of the year of death. Outgo terms are shown as negative entries.

Year	Premium	Expenses	Interest	Expected cost of death and maturity claims	Expected cost of increasing reserves (*)	Profit vector
1	1,530	–50	?	?	?	–51
2	1,530	?	?	?	?	21
3	1,530	?	?	?	?	45

The mortality probability at each age is 1%. The rate of interest earned on cashflows and reserves is 6%. Reserves are calculated using an interest rate of 4%. The reserves are zero at the start and end of the contract. The interest earned on the reserve in the third year is £195.

(i) Complete the table.

(ii) Calculate the internal rate of return.

(iii) Explain the effect that changing to a weaker reserving basis would have on the internal rate of return.

(iv) Calculate the net present value using a risk discount rate of 7%.

(v) Explain the effect that changing to a weaker reserving basis would have on the net present value.

(*) *Allowing for interest earned on reserves.*

28.3 A profit test for unit-linked policies issued to lives aged 60 has been carried out. The expected non-unit cashflows before setting up non-unit reserves are as follows:

Year	Expected non-unit cashflows per policy in force at start of year
1	–30
2	–12
3	–6
4	20
5	30

Write down an expression for the expected loss at the end of Year 1, after zeroisation of all negative cashflows.

28.4 A conventional 3-year endowment assurance is issued to a life aged exactly 56. The details are:

- sum assured 10,000 payable after 3 years or at the end of the year of death, if earlier
- surrender value equal to the return of premiums without interest, less 400, at the end of the year of surrender
- annual premium 3,250 paid at the start of each year.

The company calculates its reserves for this contract on the following basis:

Expenses: 30 at the start of Year 2

32 at the start of Year 3

Surrender probabilities: 4% of policies in force at the end of Year 2 only

Mortality: AM92 Ultimate

Interest: 2% *pa*

(i) Calculate the prospective gross premium reserves required per policy in force at the start of Years 2 and 3, according to the above basis, using a cashflow projection approach.

(ii) Calculate the expected profit arising in the second year per policy in force at the start of Year 2, assuming the following profit test experience basis:

Expenses: as reserving basis

Surrenders: as reserving basis

Reserves: as calculated in part (i)

Mortality: 75% of AM92 Select from policy outset

Interest: 4% *pa*

(iii) Explain why the expected profit calculated in part (ii) is not zero.

28.5 Exam style

A life insurance company issues a number of 3-year term assurance contracts to lives aged exactly 60. The sum assured under each contract is £200,000, payable immediately on death. Premiums are payable annually in advance for the term of the policy, ceasing on earlier death.

The company carries out profit tests for these contracts using the following assumptions:

Initial expenses:	£200 plus 35% of the first year's premium
Renewal expenses:	£25 plus 3% of the annual premium, incurred at the beginning of the second and subsequent years
Mortality:	AM92 Ultimate
Investment return:	7% per annum
Risk discount rate:	15% per annum
Reserves:	One year's office premium

(i) Show that the office premium, to the nearest pound, is £2,610, if the net present value of the profit is 25% of the office premium. [10]

(ii) Calculate the expected in-force cashflows if the company holds zero reserves throughout the contract, using a premium of exactly £2,610. [2]

(iii) Explain why the company might not hold reserves for this contract and the impact on profit if it did not hold any reserves. [3]

[Total 15]

28.6 Exam style

A 5-year unit-linked endowment assurance is issued to a male aged exactly 55. The expected year-end cashflows in the non-unit fund, $(NUCF)_t$ $(t = 1, .. , 5)$ per policy in force at the start of Year t are:

Year t	1	2	3	4	5
$(NUCF)_t$	–200	+20	+45	–60	+480

You are given:

Independent probabilities of mortality:	AM92 Select
Independent probabilities of withdrawal:	0.1 for Years 1 and 2
	0.05 for Years 3 and 4
	0 for Year 5

Withdrawals can occur at any time over the policy year.

(i) Calculate the net present value of profit at a risk discount rate of 10% *pa* assuming that the company holds no non-unit reserves. [4]

The rate of interest earned on non-unit reserves is assumed to be 8% *pa*.

(ii) (a) Calculate the reserves that are required at times $t = 1, .., 4$ in order to zeroise future negative cashflows.

(b) Calculate the net present value of the policy assuming that the company holds the non-unit reserves calculated in (i)(a). [6]

(iii) Without carrying out any more calculations, explain the effect on the net present value if non-unit reserves earned interest at the rate of 10% *pa*. [2]

[Total 12]

28.7 Exam style

A special endowment policy pays a sum assured of £20,000 to a life who is currently aged exactly 57 after three years or at the end of the year of earlier death.

Annual reversionary bonuses are declared at the end of each policy year, and an additional terminal bonus is payable at maturity only.

Policies may be surrendered only at the end of each policy year. On surrender, the policyholder receives a return of premiums with interest calculated at the rate of 3% per annum.

A level premium of £8,000 is paid at the start of each year.

The premium basis is as follows:

Interest:		7% per annum
Mortality:		AM92 Select
Surrender rates:		15% of all policies in force at the end of year 1
		5% of all policies in force at the end of year 2
Reversionary bonuses:		6% per annum compound
Terminal bonus:		10% of all other benefits payable at maturity
Expenses:	Initial	£500
	Renewal	£30 at start of year 2
		£35 at start of year 3
	Termination	£100 per termination (death, surrender or maturity)
Reserves:		Net premium reserves, using AM92 Ultimate mortality and 4% per annum interest

(i) Calculate the profit signature for this policy according to the premium basis. [14]

(ii) By accumulating the elements of the profit signature to the maturity date, explain briefly whether you think the company expects to declare the bonus rates it has assumed in its premium basis, assuming all the other assumptions in the basis are realistic. [2]

[Total 16]

28.8 Exam style

Craig, aged 40, buys a four-year unit-linked endowment policy under which level annual premiums of £1,000 are payable. 75% of the first premium and 105% of each subsequent premium is invested in units. There is a bid/offer spread in unit values, the bid price being 95% of the offer price.

A fund management charge of 0.75% of the value of the policyholder's fund is deducted at the end of each policy year.

The death benefit, which is payable at the end of the year of death, is £3,000 or the bid value of the units if greater. The maturity value is equal to the bid value of the units.

The insurance company incurs expenses of £150 at the start of the first year, £75 at the start of the second year, and £25 at the start of each of the third and fourth years.

The mortality probability (q_x) is assumed to be 0.01 at each age and withdrawals may be ignored.

(i) Assuming that the growth in the unit value is 5% *pa*, the non-unit interest rate is 5% *pa*, and the insurance company holds unit reserves equal to the value of units and zero non-unit reserves, calculate the expected profit emerging in each policy year. [10]

(ii) Calculate the revised profit emerging each year assuming that the office sets up non-unit reserves to ensure that the expected profit emerging in the second and subsequent policy years is non-negative. [5]

[Total 15]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 28 Solutions

28.1 A life office might set up reserves in the non-unit fund if the overall cashflow in any year other than Year 1 would otherwise be negative. It is desirable that the policy be self-funding after Year 1.

Reserves are set up early in the contract so that money can be released from the reserves as required to eliminate the negative cashflow. The regulatory authorities, *eg* the Financial Conduct Authority in the UK, may insist on this. Setting up reserves will result in larger losses (or smaller profits) in the early years of the contract, but the life office will not expect to have to find extra capital to support the policies later on.

28.2 (i) ***Completed table***

The completed table is:

Year	Premiums	Expenses	Interest	Expected cost of claims	Expected cost of increasing reserves	Profit vector
1	*1,530*	*–50*	88.80 (1)	–50 (2)	–1,569.80 (3)	*–51.00*
2	*1,530*	–13.30 (4)	91.00 (6)	–50 (2)	–1,536.70 (4)	*21.00*
3	*1,530*	–20.57 (5)	90.57 (6)	–5,000(2)	3,445.00	*45.00*

(Pre-calculated figures are shown in italics.) The missing figures can be derived using the following steps (which are indicated in brackets beside the figures in the table).

(1) Interest in the first year is $(1{,}530-50)\times 0.06$.

(2) The expected cost of claims in Years 1 and 2 is $5{,}000\times 0.01$. The expected cost of claims in Year 3 is 5,000 since all policies in force at the start of Year 3 will receive a benefit of 5,000 at the end of Year 3. These figures are shown as negative entries in the table as they are outgo for the insurer.

(3) Let e_t denote the amount of the expense paid at time t and PRO_t be the profit vector term at time t. If the policyholder is aged x at time 0, then the recursive formula tells us that:

$$\left({}_tV+P-e_t\right)(1+i)={}_{t+1}V\,p_{x+t}+S\,q_{x+t}+PRO_{t+1}\quad \text{for } t=0,1\,.$$

This can also be written as:

$$PRO_{t+1}=(P-e_t)(1+i)-\underbrace{\left({}_{t+1}V\,p_{x+t}-{}_tV(1+i)\right)}_{\substack{\text{expected cost of increasing}\\ \text{the reserve at time } t+1}}-S\,q_{x+t}$$

So for Year 1, we have:

$$-51=(1,530-50)\times 1.06-\underbrace{\left({}_1V\times 0.99-0\right)}_{\substack{\text{expected cost of}\\ \text{increasing the}\\ \text{reserve at time 1}}}-5,000\times 0.01$$

and hence the expected cost of increasing the reserve at the end of Year 1 is 1,569.80 (which is shown in the table as a negative since it represents a cost to the insurer) …

… and:

$${}_1V=\frac{1,569.80}{0.99}=1,585.66$$

(4) For Year 2, the recursive formula is:

$$21=(1,530-e_1)\times 1.06-\underbrace{\left({}_2V\times 0.99-1,585.66\times 1.06\right)}_{\substack{\text{expected cost of}\\ \text{increasing the}\\ \text{reserve at time 2}}}-5,000\times 0.01$$

But we know that the interest earned on the reserve in the third year is 195, so:

$${}_2V\times 0.06=195\Rightarrow {}_2V=3,250$$

Hence the expected cost of increasing the reserve at the end of the second year is:

$$3,250\times 0.99-1,585.66\times 1.06=1,536.70$$

and:

$$e_1=1,530-\frac{21+50+1,536.70}{1.06}=13.30$$

(5) For Year 3, the recursive formula is:

$$45=(1,530-e_2)\times 1.06-\underbrace{\left({}_3V\times 0.99-{}_2V\times 1.06\right)}_{\substack{\text{expected cost of}\\ \text{increasing the}\\ \text{reserve at time 3}}}-5,000$$

But ${}_3V=0$, so the expected cost of increasing the reserve at time 3 is

$$-{}_2V\times 1.06=-3,250\times 1.06=-3,445$$

This is negative because the reserves that have been built up over the term of the policy are released at time 3 to cover the benefits.

We can summarise the expected cost of increasing reserves calculations in a table:

Year	Reserve at start of year	Interest	Expected reserve at end of year	Expected cost of increasing reserves
1	0	0	1,569.80	–1,569.80
2	1,585.66	95.14	3,217.50	– 1,536.70
3	3,250.00	195.00	0	3,445.00

So we have:

$$e_2 = 1,530 - \frac{45 + 5,000 - 3,445}{1.06} = 20.57$$

(6) So, finally, the interest earned on the fund in Years 2 and 3 is:

$$(1,530 - 13.30) \times 0.06 = 91.00 \quad \text{for Year 2}$$

$$(1,530 - 20.57) \times 0.06 = 90.57 \quad \text{for Year 3}$$

(ii) ***Internal rate of return***

The internal rate of return is the interest rate that satisfies the equation:

$$-51v + 21 \times 0.99v^2 + 45 \times 0.99^2 v^3 = 0$$

(where the 0.99 factors represent the probability of surviving each year).

Dividing through by v gives:

$$-51 + 20.79v + 44.10v^2 = 0$$

For the quadratic equation $ax^2 + bx + c = 0$, we know that the roots are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Applying this quadratic formula gives (ignoring the negative root):

$$v = 0.8652 \Rightarrow i = \frac{1}{0.8652} - 1 = 15.6\%$$

(iii) ***Effect on internal rate of return***

The internal rate of return will increase.

This is because the weaker reserving basis reduces the amounts of the reserves, causing profits to be released sooner, and this will increase their expected present value.

(iv) ***Net present value***

The net present value calculated using a risk discount rate of 7% is:

$$NPV = -51v + 20.79v^2 + 44.10v^3 = -47.66 + 18.16 + 36.00 = £6.50$$

(v) ***Effect on net present value***

The net present value will increase.

This is because the weaker reserving basis reduces the amounts of the reserves, causing profits to be released sooner, and this will increase their expected present value.

28.3 The latest negative cashflow is at the end of Year 3. So first we need a non-unit reserve at the start of Year 3 of:

$${}_2V = 6v$$

where v is calculated at the valuation (reserving) rate of interest.

Setting up the reserve at the start of Year 3 produces an adjusted expected non-unit cashflow at the end of Year 2 of:

$$-12 - {}_2V \times p_{61} = -(12 + 6vp_{61})$$

The non-unit reserve at the start of Year 2 required to zeroise this is:

$${}_1V = -(12 + 6vp_{61})v = 12v + 6v^2 p_{61}$$

So the expected loss at the end of Year 1 is:

$$L = 30 + {}_1V \times p_{60} = 30 + \left(12v + 6v^2 p_{61}\right)p_{60} = 30 + 12vp_{60} + 6v^2\,{}_2p_{60}$$

28.4 (i) ***Reserves at the start of Years 2 and 3***

As we only need reserves at the start of Years 2 and 3, we only need to project cashflows for those last two years. The calculations are shown in the following table:

Year	Premium	Expenses	Interest (1)	Expected claim cost	Expected surrender cost	Cashflow
2	3,250	–30	64.40	–56.50 (2)	–242.62 (3)	2,985.28
3	3,250	–32	64.36	–10,000	0	–6,717.64

where:

$$(1) = (\{\text{premium}\} - \{\text{expenses}\}) \times 0.02$$

$$(2) = -10{,}000 \times q_{57} = -10{,}000 \times 0.005650 = -56.50$$

$$(3) = -(3{,}250 \times 2 - 400) \times (1 - 0.005650) \times 0.04 = -242.62$$

The reserve at the start of Year 3 is:

$$ {}_2V = \frac{6,717.64}{1.02} = 6,585.92 $$

The reserve at the start of Year 2 is:

$$ {}_1V = \frac{(1-0.005650)\times(1-0.04)\times 6,585.92 - 2,985.28}{1.02} = 3,236.75 $$

(ii) ***Expected profit at end of Year 2***

The expected cashflow at the end of Year 2, per policy in force at the start of the year, is:

$$ \{\text{premium}\} - \{\text{expenses}\} + \{\text{interest}\} - \{\text{expected death cost}\} - \{\text{expected surrender cost}\} $$

$$ = 3,250 - 30 + (3,250 - 30)\times 0.04 - 0.75q_{[56]+1}\times 10,000 $$

$$ -\left(1-0.75q_{[56]+1}\right)\times 0.04\times\left(2\times 3,250 - 400\right) $$

$$ = 3,220 + 128.80 - 0.75\times 0.005507\times 10,000 - (1-0.75\times 0.005507)\times 0.04\times 6,100 $$

$$ = 3,064.51 $$

Allowing for the start and end of year reserves, the expected profit at the end of Year 2, per policy in force at the start of the year, is:

$$ \{\text{expected cashflow}\} + \{\text{reserve at start of year}\} + \{\text{interest on reserve}\} $$

$$ -\{\text{expected cost of end-year resserve}\} $$

$$ = 3,064.51 + {}_1V + {}_1V\times 0.04 - (ap)_{[56]+1}\times {}_2V $$

$$ = 3,064.51 + 3,236.75\times 1.04 - (1-0.75\times 0.005507)\times(1-0.04)\times 6,585.92 $$

$$ = 134.35 $$

(iii) ***Why the profit is not zero***

If the projection assumptions had been the same as the reserving basis, then an expected profit of exactly zero would have been calculated for Year 2. However, the expected future experience has:

- lower mortality rates than the reserving basis, which reduces the expected death cost
- a higher interest rate on cashflows and reserves, which will increase investment income

both of which will increase the expected profit, which will therefore be greater than zero.

28.5 (i) ***Office premium***

The table for the profit test is as follows:

Year	Premium	Expenses	Interest	Expected death cost
1	P	$-200-0.35P$	$0.07(0.65P-200)$	$-1{,}659.60$
2	P	$-25-0.03P$	$0.07(0.97P-25)$	$-1{,}863.80$
3	P	$-25-0.03P$	$0.07(0.97P-25)$	$-2{,}091.99$
		[½]	[1]	[1]

Year	Expected end of year cashflow	Expected cost of increase in reserves	Profit vector
1	$0.6955P-1{,}873.60$	$-0.9920P$	$-0.2965P-1{,}873.60$
2	$1.0379P-1{,}890.55$	$0.0790P$	$1.1169P-1{,}890.60$
3	$1.0379P-2{,}118.74$	$1.07P$	$2.1079P-2{,}118.74$
	[1]	[1½]	[½]

Year	Probability in force	Profit signature
1	1	$-0.2965P-1{,}873.60$
2	0.991978	$1.1079P-1{,}875.38$
3	0.983041	$2.0722P-2{,}082.81$
	[1]	[1]

The expected death cost in Year t is calculated as:

$$200{,}000\times q_{60+t-1}\times 1.07^{½}$$

This assumes the immediate death payment occurs on average half way through the year, and so earns half a year's interest by the end of the year.

So the expected net present value of the profit is:

$$\frac{-0.2965P-1{,}873.60}{1.15}+\frac{1.1079P-1{,}875.38}{1.15^2}+\frac{2.0722P-2{,}082.81}{1.15^3}$$

$$=1.9424P-4{,}416.76$$ [1½]

Setting this equal to $0.25P$ and solving for P gives $P=£2{,}610$ to the nearest £1. [1]

[Total 10]

(ii) ***Cashflows, ignoring reserves***

The cashflows are given in the table below:

Year	Premium	Expenses	Interest	Expected death cost	Expected end of year cashflow
1	2,610	–1,113.5	104.76	–1,659.60	–58.34
2	2,610	–103.3	175.47	–1,863.80	818.37
3	2,610	–103.3	175.47	–2,091.99	590.18
		[½]	[½]		[1]

[Total 2]

(iii) ***Why reserves are not needed and impact of holding reserves***

As the table above shows, the policy is self-funding after the first year. [½]

So there is no need to hold reserves to cover future outgo. [½]

If the company didn't hold reserves, this would accelerate the emergence of profit. [1]

Since the risk discount rate is higher than the investment return, not holding reserves would increase the net present value of the profit. [1]

[Total 3]

28.6 (i) ***Net present value assuming no non-unit reserves***

The independent probabilities of mortality are:

$$q_{55}^d = q_{[55]} = 0.003358$$

$$q_{56}^d = q_{[55]+1} = 0.004903$$

$$q_{57}^d = q_{57} = 0.005650$$

$$q_{58}^d = q_{58} = 0.006352$$ [½]

So we have:

$p_{55}^d = 0.996642$ $p_{55}^w = 0.9$

$p_{56}^d = 0.995097$ $p_{56}^w = 0.9$

$p_{57}^d = 0.994350$ $p_{57}^w = 0.95$

$p_{58}^d = 0.993648$ $p_{58}^w = 0.95$ [1]

and:

$$(ap)_{55} = p_{55}^{d} \times p_{55}^{w} = 0.896978$$

$$(ap)_{56} = p_{56}^{d} \times p_{56}^{w} = 0.895587$$

$$(ap)_{57} = p_{57}^{d} \times p_{57}^{w} = 0.944633$$

$$(ap)_{58} = p_{58}^{d} \times p_{58}^{w} = 0.943966$$ [1]

It follows that:

$${}_2(ap)_{55} = (ap)_{55} \times (ap)_{56} = 0.803322$$

$${}_3(ap)_{55} = {}_2(ap)_{55} \times (ap)_{57} = 0.758844$$

$${}_4(ap)_{55} = {}_3(ap)_{55} \times (ap)_{58} = 0.716323$$ [½]

So the net present value of the profit from the contract is:

$$-\frac{200}{1.1} + \frac{20 \times 0.896978}{1.1^2} + \frac{45 \times 0.803322}{1.1^3} - \frac{60 \times 0.758844}{1.1^4} + \frac{480 \times 0.716323}{1.1^5} = 42.56$$ [1]

[Total 4]

(ii)(a) ***Reserves required to zeroise negative cashflows***

We do not need a reserve at time 4 since the cashflow at time 5 is positive. So:

$${}_4V = 0$$ [½]

We do need a reserve at time 3 since the cashflow at time 4 is negative. We require:

$${}_3V(1+i) = 60 \Rightarrow {}_3V = \frac{60}{1.08} = 55.56$$ [½]

The non-unit cashflow at time 3 then becomes:

$$(NUCF)'_3 = 45 - {}_3V\,(ap)_{57} = 45 - 55.56 \times 0.944633 = -7.48$$ [1]

We now need a reserve at time 2 to zeroise this negative:

$${}_2V(1+i) = 7.48 \Rightarrow {}_3V = \frac{7.48}{1.08} = 6.93$$ [½]

The non-unit cashflow at time 2 then becomes:

$$(NUCF)'_2 = 20 - {}_2V\,(ap)_{56} = 20 - 6.93 \times 0.895587 = 13.80$$ [1]

Since this is positive, we do not need a reserve at time 1, *ie*:

$${}_1V = 0$$ [½]

(ii)(b) ***Net present value assuming non-unit reserves are set up***

The profit vector is the vector of non-unit cashflows after the reserves have been set up. So for this policy the profit vector is:

$$\left(-200,\ \ 13.80,\ \ 0,\ \ 0,\ \ 480\right)$$ [½]

The net present value is then:

$$-\frac{200}{1.1}+\frac{13.80\times 0.896978}{1.1^2}+\frac{480\times 0.716323}{1.1^5}=41.91$$ [1½]

[Total 6]

(iii) ***If non-unit reserves earned 10% pa interest***

Holding reserves delays the emergence of profit. [½]

However, if the rate of interest earned on the reserves is 10%, then we are accumulating and discounting at the same rate. In this case, delaying the emergence of profit will have no effect on the net present value for the contract. [1]

So the net present value would be 42.56 as in (i). [½]

[Total 2]

28.7 (i) ***Profit signature***

The calculations are shown in the following tables.

Year t	Premium (1)	Expenses (2)	Interest (3)	Mortality probability (4)	Death benefit + termination expenses (5)	Expected death cost (6)
1	8,000	–500	525.00	0.004171	20,100	–83.84
2	8,000	–30	557.90	0.006180	21,300	–131.63
3	8,000	–35	557.55	0.007140	22,572	–161.16

[3]

Year t	Dependent surrender probability (7)	Surrender value + terminat'n expenses (8)	Expected surrender cost (9)	Maturity value + terminat'n expenses (10)	Survival probability (11)	Expected maturity cost (12)
1	0.149374	8,340.00	–1,245.78	0	0.846455	0
2	0.049691	16,827.20	–836.16	0	0.944129	0
3	0	0	0	26,302.35	0.992860	–26,114.55

[6]

Year t	Cashflow (13)	Reserve at start of year (14)	Interest on reserve (15)	Expected cost of end yr reserve (16)	Profit vector (17)	Survival probability (18)
1	6,695.38	0	0	–6,336.14	359.24	1
2	7,560.11	7,485.50	523.98	–14,547.34	1,022.25	0.846455
3	–17,753.16	15,408.21	1,078.57	0	–1,266.38	0.799163

[4½]

Year t	Profit signature (19)
1	359.24
2	865.29
3	–1,012.04

[½]

Key to tables:

$(3)=\left[(1)+(2)\right]\times 0.07$ (as (2) is a deduction)

$(5)_t = 20{,}000\left(1.06\right)^{t-1}+100$

$(6)=-(4)\times(5)$

$(7)=\left[1-(4)\right]\times\left[\text{surrender probability}\right]$

$(8)_1 = 8{,}000\times 1.03+100$

$(8)_2 = 8{,}000\left(1.03^2+1.03\right)+100$

$(9)=-(7)\times(8)$

$(10)_3 = 20{,}000\times 1.06^3\times 1.1+100$

$(11)=1-(4)-(7)$

$(12)=-(10)\times(11)$

$(13)=(1)+(2)+(3)+(6)+(9)+(12)$

Column (14) shows the net premium reserve at time $t-1$. The net premium for a with-profits policy is calculated ignoring all bonuses and expenses, and so in this case is equal to:

$$20,000\frac{A^{4\%}_{57:\overline{3}|}}{\ddot{a}^{4\%}_{57:\overline{3}|}}$$

The reserve is then the EPV of the guaranteed sum assured plus reversionary bonuses declared up to time $t-1$, less the EPV of the future net premiums. So:

$$(14)_t = 20,000\times 1.06^{t-1}\,A^{4\%}_{57+t-1:\overline{3-(t-1)}|} - 20,000\frac{A^{4\%}_{57:\overline{3}|}}{\ddot{a}^{4\%}_{57:\overline{3}|}}\ddot{a}^{4\%}_{57+t-1:\overline{3-(t-1)}|}$$

$(15) = (14)\times 0.07$

$(16)_t = -(14)_{t+1}\times(11)_t$

$(17) = (13)+(14)+(15)+(16)$

$(18)_1 = 1$

$(18)_t = (18)_{t-1}\times(11)_{t-1},\ t>1$

$(19) = (17)\times(18)$ [Total 14]

(ii) ***Affordability of future bonuses***

Assuming 7% *pa* investment return, the first two years' cashflows accumulate to the following value at the end of Year 3:

$$359.24\times 1.07^2 + 865.29\times 1.07 = £1,337.15$$ [1]

The outgo in the third year is £1,012.04. So the company can afford all the assumed bonuses during the policy with an additional profit of £325.11 at the end of the term. So, given these assumptions, it would be quite likely for the company to pay higher bonuses than these (though it depends upon how much profit is required for any shareholders). [1]

[Total 2]

28.8 (i) ***Emerging profit***

The tables required for the calculations are shown below:

Unit fund

Policy year	Premium allocated	Cost of allocation	Plus fund b/f	Fund before charge	Annual charge	Fund c/f
1	750.00	712.50	712.50	748.13	–5.61	742.52
2	1,050.00	997.50	1,740.02	1,827.02	–13.70	1,813.32
3	1,050.00	997.50	2,810.82	2,951.36	–22.14	2,929.22
4	1,050.00	997.50	3,926.72	4,123.06	–30.92	4,092.14

[5]

Non-unit fund

Policy year	Premium minus cost of allocation	Expenses	Non unit interest	Annual charge	Expected death cost	Expected in-force cashflow
1	287.50	–150.00	6.88	5.61	–22.57	127.42
2	2.50	–75.00	–3.63	13.70	–11.87	–74.30
3	2.50	–25.00	–1.13	22.14	–0.71	–2.20
4	2.50	–25.00	–1.13	30.92	0.00	7.29

[5]

The expected death cost is calculated as:

$$-\max\left[3{,}000-\{\text{unit fund at end of year}\},0\right]\times\{\text{probability of dying}\}$$

Eg in Year 1 we calculate:

$$-\max\left[3{,}000-742.52,0\right]\times 0.01=-2{,}257.48\times 0.01=-22.57$$

[Total 10]

(ii) ***Revised profit***

First of all we calculate the non-unit reserves.

The expected cashflow at the end of Year 4 is positive, so we do not need to hold a reserve at the start of year 4, *ie* ${}_3V=0$. [½]

At the end of Year 3, there is an expected cashflow of –2.20 per policy in force at the start of that year. To zeroise this, we need to hold a reserve of:

$$_2V = \frac{2.20}{1.05} = 2.10 \qquad [½]$$

Holding this reserve has an effect on the expected cashflow at the end of Year 2. The adjusted cashflow is:

$$-74.30 - {}_2V \times p_{41} = -74.30 - 2.10 \times 0.99 = -76.38 \qquad [1]$$

To zeroise this, we need a non-unit reserve at the start of Year 2 equal to:

$$_1V = \frac{76.38}{1.05} = 72.74 \qquad [½]$$

Applying these reserves will give the following expected profits, per policy in force at the start of each year:

Year 1

The expected profit is equal to the expected cashflow less the expected cost of setting up the end-year reserve. This is:

$$127.42 - {}_1V \times p_{40} = 127.42 - 72.74 \times 0.99 = 55.41 \qquad [1]$$

Years 2 and 3

The negative cashflows in these two years have both been zeroised by setting up the reserves. So the expected profit for both of these years is zero. [½]

Year 4

There are no reserves held at the start or end of Year 4, and so the expected profit is equal to the expected cashflow for this year, *ie* 7.29. [½]

The profit vector is therefore:

$$\left(55.41, 0, 0, 7.29\right) \qquad [½]$$

[Total 5]

End of Part 5

What next?

1. Briefly **review** the key areas of Part 5 and/or re-read the **summaries** at the end of Chapters 24 to 28.
2. Ensure you have attempted some of the **Practice Questions** at the end of each chapter in Part 5. If you don't have time to do them all, you could save the remainder for use as part of your revision.
3. Attempt **Assignment X5**.

And finally ...

Good luck!

Subject CM1: Assignment X1

2019 Examinations

Time allowed: 2¾ hours

Instructions to the candidate

1. *Please:*

   – *attempt all of the questions, as far as possible under exam conditions*

   – ***begin your answer to each question on a new page***

   – ***leave at least 2cm margin on all borders***

   – *write in black ink using a medium-sized nib because we will be unable to mark illegible scripts*

   – *note that assignment marking is not included in the price of the course materials. Please purchase Series Marking or a Marking Voucher before submitting your script.*

   – *note that we only accept the current version of assignments for marking, ie you can only submit this assignment in the sessions leading to the 2019 exams.*

2. *Please **do not**:*

   – *use headed paper*

   – *use highlighting in your script.*

At the end of the assignment

If your script is being marked by ActEd, please follow the instructions on the reverse of this page.

In addition to this paper, you should have available actuarial tables and an electronic calculator.

Submission for marking

You should aim to submit this script for marking by the recommended submission date. The recommended and deadline dates for submission of this assignment are listed on the summary page at the back of this pack and on our website at **www.ActEd.co.uk**.

Scripts received after the deadline date will not be marked, unless you are using a Marking Voucher. *It is your responsibility to ensure that scripts reach ActEd in good time*. If you are using Marking Vouchers, then please make sure that your script reaches us by the Marking Voucher deadline date to give us enough time to mark and return the script before the exam.

When submitting your script, please:

- complete the cover sheet, including the checklist
- scan your script, cover sheet (and Marking Voucher if applicable) and save as a pdf document, then email it to: ActEdMarking@bpp.com
- **do not submit a photograph of your script**
- **do not include the question paper in the scan.**

In addition, please note the following:

- Please title the email to ensure that the subject and assignment are clear *eg* 'CM1 Assignment X1 No. 12345', inserting your ActEd Student Number for 12345.
- The assignment should be scanned the **right way up** (so that it can be read normally without rotation) and as a single document. We cannot accept individual files for each page.
- Please set the resolution so that the script is legible and the resulting PDF **is less than 4 MB** in size.
- Do not protect the PDF in any way (otherwise the marker cannot return the script to ActEd, which causes delays).
- Please include the 'feedback from marker' sheet when scanning.
- Before emailing to ActEd, please check that your scanned assignment includes all pages and conforms to the above.

Subject CM1: Assignment X1

2019 Examinations

Please complete the following information:

Name:

ActEd Student Number (see Note below):

Note: Your ActEd Student Number is printed on all personal correspondence from ActEd. Quoting it will help us to process your scripts quickly. If you do not know your ActEd Student Number, please email us at ActEd@bpp.com.

Your ActEd Student Number is not the same as your IFoA Actuarial Reference Number or ARN.

Number of following pages: _______

Please put a tick in this box if you have solutions and a cross if you do not: ☐

Please tick here if you are allowed extra time or other special conditions in the profession's exams (if you wish to share this information): ☐

Time to do assignment (see Note below): _____ hrs _____ mins

Under exam conditions (delete as applicable): yes / nearly / no

Note: If you take more than 2¾ hours, you should indicate how much you completed within this exam time so that the marker can provide useful feedback on your progress.

Score and grade for this assignment (to be completed by marker):

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	Q13	Q14	Q15	Q16	**Total**	
__ 1	__ 2	__ 3	__ 3	__ 3	__ 4	__ 4	__ 4	__ 5	__ 5	__ 5	__ 6	__ 7	__ 7	__ 7	__ 14	__ 80	=_____%

Grade: A B C D E **Marker's initials:** ________

Please tick the following checklist so that your script can be marked quickly. Have you:

[ ] Checked that you are using the latest version of the assignments, *ie* 2019 for the sessions leading to the 2019 exams?

[ ] Written your full name in the box above?

[ ] Completed your ActEd Student Number in the box above?

[ ] Recorded your attempt conditions?

[ ] Numbered all pages of your script (excluding this cover sheet)?

[ ] Written the total number of pages (excluding the cover sheet) in the space above?

[ ] Included your Marking Voucher or ordered Series X Marking?

Please follow the instructions on the previous page when submitting your script for marking.

Feedback from marker

Notes on marker's section

The main objective of marking is to provide specific advice on how to improve your chances of success in the exam. The most useful aspect of the marking is the comments the marker makes throughout the script, however you will also be given a percentage score and the band into which that score falls. Each assignment tests only part of the course and hence does not give a complete indication of your likely overall success in the exam. However it provides a good indicator of your understanding of the material tested and the progress you are making with your studies:

A = Excellent progress B = Good progress C = Average progress
D = Below average progress E = Well below average progress

Please note that you can provide feedback on the marking of this assignment at:

www.ActEd.co.uk/marking

X1.1 Calculate the present value of a perpetuity paying £50 *pa* in arrears at an annual effective rate of interest of 6%. [1]

X1.2 Calculate the length of time it will take £800 to accumulate to £1,000 at a simple rate of interest of 4% *pa*. [2]

X1.3 Explain the main differences between a deterministic model and a stochastic model. [3]

X1.4 A 91-day government bill is discounted at a simple rate of discount of 10% *pa*. Calculate the annual effective rate of interest earned on this investment. [3]

X1.5 Outline the three key forms of data analysis. [3]

X1.6 An annuity of \$300 *pa* is paid annually in advance for seven years, followed by \$100 *pa* paid quarterly in arrears for a further five years. The rate of interest is 6% *pa* convertible half-yearly.

Determine the accumulated amount at the end of twelve years. [4]

X1.7 An investor, who has a sum of £10,000 to invest, wishes to purchase an annuity certain with a term of 10 years. Calculate the amount of the payments that can be provided if the annuity takes each of the following forms (assuming interest of 8% *pa* effective):

(i) a level annuity payable monthly in arrears [2]

(ii) a level annuity due payable half-yearly, commencing in 2 years' time. [2]

[Total 4]

X1.8 Explain when you would use real and money rates of interest. Give an example of when each rate of interest would be used. [4]

X1.9 (i) Calculate the nominal annual rate of discount convertible quarterly equivalent to a nominal rate of interest of 10% *pa* convertible quarterly. [2]

(ii) A single investment of £500 is accumulated at a nominal rate of discount of 6% *pa* convertible half-yearly for 1 year, followed by a nominal rate of interest of 6% *pa* convertible every 4 months for 1 year. Calculate the accumulated amount of this investment after 2 years. [3]

[Total 5]

X1.10 List the factors that should be considered when assessing the suitability of an actuarial model for its purpose. [5]

X1.11 For each of the following calculate the equivalent effective annual rate of interest:

(i) an effective rate of interest of 12.7% paid every 2 years [1]

(ii) an effective rate of discount of 5.75% *pa* [1]

(iii) a force of interest of ½% per month [1]

(iv) a nominal rate of discount of 6% *pa* convertible quarterly [1]

(v) a nominal rate of interest of 14% *pa* convertible every 2 years. [1]

[Total 5]

X1.12 An investor receives payments half-yearly in arrears for 20 years. The first payment is £250, and each payment is 2% higher than the previous one.

The interest rate is 6% *pa* effective for the first 10 years and 4% *pa* effective for the final 10 years.

Calculate, showing all workings, the present value of the payments. [6]

X1.13 The force of interest, $\delta(t)$, is a function of time and at time t, measured in years, is given by:

$$\delta(t) = 0.03 - 0.005t + 0.001t^2 \qquad 0 \le t \le 10$$

(i) Calculate the equivalent constant force of interest per annum for the period $t = 0$ to $t = 10$. [3]

(ii) Calculate, showing all workings, the accumulated value at time $t = 7$ of an investment of £250 at time $t = 0$ plus a further investment of £150 at time $t = 5$. [4]

[Total 7]

X1.14 (i) Prove that:

$$(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i}$$ [3]

(ii) An annuity payable monthly in arrears has a first payment of £300, with subsequent payments decreasing by £10 each month, until a final payment of £70 is made in two years' time.

Calculate the present value of the payments from this annuity using an effective rate of interest of 6% *pa.* [4]

[Total 7]

X1.15 In return for a fixed initial deposit, an investor receives a continuously payable annuity for a term of 15 years. The annual rate of payment is 50 in the first year, and the rate of payment increases in each subsequent year.

The investor can select either:

- Option 1: the rate of payment increases by 2 at the end of each year,
- Option 2: the rate of payment increases by 3% *pa* compound at the end of each year.

Determine which option would provide the better deal for the investor at an annual effective interest rate of 7%. [7]

X1.16 The force of interest $\delta(t)$ is a function of time, and at any time t, measured in years, is given by the formula:

$$\delta(t)=\begin{cases} 0.04+0.005t & 0\le t<6 \\ 0.16-0.015t & 6\le t<8 \\ 0.04 & 8\le t \end{cases}$$

(i) Derive expressions in terms of t for the accumulated amount at time t of an investment of 1 at time 0. [6]

(ii) (a) Calculate the value at time 0 of £100 due at time 9.

(b) Calculate the annual effective rate of discount implied by the transaction in (a). [3]

(iii) A continuous payment stream is received at a rate of $45e^{0.01t}$ units per annum between time 10 and time 15. Calculate, showing all workings, the present value at time 4 of this payment stream. [5]

[Total 14]

END OF PAPER

Subject CM1: Assignment X2

2019 Examinations

Time allowed: 2¾ hours

Instructions to the candidate

1. *Please:*

   - *attempt all of the questions, as far as possible under exam conditions*
   - ***begin your answer to each question on a new page***
   - ***leave at least 2cm margin on all borders***
   - *write in black ink using a medium-sized nib because we will be unable to mark illegible scripts*
   - *note that assignment marking is not included in the price of the course materials. Please purchase Series Marking or a Marking Voucher before submitting your script.*
   - *note that we only accept the current version of assignments for marking, ie you can only submit this assignment in the sessions leading to the 2019 exams.*

2. *Please* ***do not****:*

   - *use headed paper*
   - *use highlighting in your script.*

At the end of the assignment

If your script is being marked by ActEd, please follow the instructions on the reverse of this page.

In addition to this paper, you should have available actuarial tables and an electronic calculator.

Submission for marking

You should aim to submit this script for marking by the recommended submission date. The recommended and deadline dates for submission of this assignment are listed on the summary page at the back of this pack and on our website at **www.ActEd.co.uk**.

Scripts received after the deadline date will not be marked, unless you are using a Marking Voucher. *It is your responsibility to ensure that scripts reach ActEd in good time*. If you are using Marking Vouchers, then please make sure that your script reaches us by the Marking Voucher deadline date to give us enough time to mark and return the script before the exam.

When submitting your script, please:

- complete the cover sheet, including the checklist
- scan your script, cover sheet (and Marking Voucher if applicable) and save as a pdf document, then email it to: ActEdMarking@bpp.com
- **do not submit a photograph of your script**
- **do not include the question paper in the scan.**

In addition, please note the following:

- Please title the email to ensure that the subject and assignment are clear *eg* 'CM1 Assignment X2 No. 12345', inserting your ActEd Student Number for 12345.
- The assignment should be scanned the **right way up** (so that it can be read normally without rotation) and as a single document. We cannot accept individual files for each page.
- Please set the resolution so that the script is legible and the resulting PDF **is less than 4 MB** in size.
- Do not protect the PDF in any way (otherwise the marker cannot return the script to ActEd, which causes delays).
- Please include the 'feedback from marker' sheet when scanning.
- Before emailing to ActEd, please check that your scanned assignment includes all pages and conforms to the above.

Subject CM1: Assignment X2

2019 Examinations

Please complete the following information:

Name:

ActEd Student Number (see Note below):

Note: Your ActEd Student Number is printed on all personal correspondence from ActEd. Quoting it will help us to process your scripts quickly. If you do not know your ActEd Student Number, please email us at ActEd@bpp.com.

Your ActEd Student Number is not the same as your IFoA Actuarial Reference Number or ARN.

Number of following pages: _______

Please put a tick in this box if you have solutions and a cross if you do not: ☐

Please tick here if you are allowed extra time or other special conditions in the profession's exams (if you wish to share this information): ☐

Time to do assignment
(see Note below): _____ hrs _____ mins

Under exam conditions
(delete as applicable): yes / nearly / no

Note: If you take more than 2¾ hours, you should indicate how much you completed within this exam time so that the marker can provide useful feedback on your progress.

Score and grade for this assignment (to be completed by marker):

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	**Total**	
— 4	— 4	— 5	— 5	— 7	— 9	— 12	— 13	— 21	— 80	=_____%

Grade: A B C D E **Marker's initials:** _________

Please tick the following checklist so that your script can be marked quickly. Have you:

[ ] Checked that you are using the latest version of the assignments, *ie* 2019 for the sessions leading to the 2019 exams?
[ ] Written your full name in the box above?
[ ] Completed your ActEd Student Number in the box above?
[ ] Recorded your attempt conditions?
[ ] Numbered all pages of your script (excluding this cover sheet)?
[ ] Written the total number of pages (excluding the cover sheet) in the space above?
[ ] Included your Marking Voucher or ordered Series X Marking?
[ ] Rated your X1 marker at **www.ActEd.co.uk/marking**?

Please follow the instructions on the previous page when submitting your script for marking.

Feedback from marker

Notes on marker's section

The main objective of marking is to provide specific advice on how to improve your chances of success in the exam. The most useful aspect of the marking is the comments the marker makes throughout the script, however you will also be given a percentage score and the band into which that score falls. Each assignment tests only part of the course and hence does not give a complete indication of your likely overall success in the exam. However it provides a good indicator of your understanding of the material tested and the progress you are making with your studies:

A = Excellent progress B = Good progress C = Average progress
D = Below average progress E = Well below average progress

Please note that you can provide feedback on the marking of this assignment at:

www.ActEd.co.uk/marking

X2.1 A borrower has agreed to repay a loan of £20,000 with payments of £427.90 made monthly in arrears for 5 years.

Calculate, showing all workings, the APR charged on the loan. [4]

X2.2 An ordinary share pays half-yearly dividends. The next dividend is expected to be 25p per share and is due in exactly 2 months' time. It is expected that subsequent dividends will grow at a compound rate of 1.5% per half-year and that the rate of inflation will be 3% *pa* effective. Dividends are expected to be received in perpetuity.

Calculate the price an investor should pay for this share in order to obtain a real rate of return of 2% *pa* effective. [4]

X2.3 An investor, who is liable to income tax at 20% but is not liable to capital gains tax, wishes to earn a net effective rate of return of at least 5% per annum. A bond that pays coupons half-yearly in arrear at a rate of 6.25% per annum has just been issued. The bond will be redeemed at par on a coupon date between 10 and 15 years after the date of issue, inclusive. The date of redemption is at the option of the borrower.

Determine the maximum price that the investor is willing to pay for £100 nominal of the bond. [5]

X2.4 A loan of £3,000 is repayable over 5 years by level quarterly instalments of £170.07, calculated using a rate of interest of 5% *pa* effective.

(i) Calculate the capital content of the sixth repayment. [2]

(ii) Calculate how much interest is paid in the second year. [3]
[Total 5]

X2.5 An investor finances a project with an initial payment of £10,000. Income from the project is received half-yearly in arrears for the next 10 years. The income is £1,500 *pa* for the first 5 years, and £3,000 *pa* for the final 5 years.

Using an effective rate of interest of 8% *pa*, calculate:

(i) the net present value [2]

(ii) the discounted payback period. [5]
[Total 7]

X2.6 An index-linked bond was issued on 1 January 2010 with a term of 7 years. Coupons were payable annually and the redemption value before indexing was £105 per £100 nominal. Coupons and redemption payments were indexed by reference to the value of an inflation index with a time lag of 6 months. The annual nominal coupon rate on the bond was 4% and a coupon of £4.31 per £100 nominal was paid on 1 January 2014.

A tax-exempt investor purchased £100 nominal of the bond on 1 July 2014 and held it to redemption. The inflation index was as follows:

Date	***Index value***
1 July 2013	125.0
1 January 2014	127.1
1 July 2014	128.2
1 January 2015	130.9
1 July 2015	131.3
1 January 2016	132.6
1 July 2016	134.8
1 January 2017	136.0

(i) Show that the value of the inflation index on 1 July 2009 was 116.0. [1]

(ii) Calculate the amounts of the coupon payments and the redemption payment received by the investor who purchased the bond on 1 July 2014. [4]

(iii) Determine the price paid by the investor on 1 July 2014, given that he achieved a real return of 5% *pa* effective from owning the bond. [4]

[Total 9]

X2.7 A loan is to be repaid by a series of instalments payable annually in arrears for 20 years. The first instalment is £1,400 and payments increase thereafter by £300 per year. Repayments are calculated using an annual effective interest rate of 7%.

(i) Calculate the amount of the loan. [3]

(ii) Calculate the capital outstanding immediately after the third instalment has been paid. [2]

(iii) Explain your answer to (ii). [2]

Immediately after the third instalment has been paid, it is decided to restructure the loan, so that level payments are made quarterly in arrears for the remaining term of the loan. The interest rate on the restructured loan is 9% *pa* convertible half-yearly.

(iv) Calculate the amount of the quarterly payment. [3]

(v) Calculate the total interest paid over the whole term of the loan. [2]

[Total 12]

X2.8 One year ago, a bond was issued with a coupon rate of 14% *pa*, payable half-yearly in arrears. The bond will be redeemed at £110% in nine years' time.

The bond was issued at a price such that an investor subject to income tax at 35%, but not subject to capital gains tax, would obtain a net yield of 9.5% *pa*.

(i) Calculate, showing all workings, the issue price for £100 nominal. [3]

The investor has now decided to sell the bond and has found a potential buyer, who is subject to income tax at 25% and capital gains tax at 35%, and who is prepared to buy the bond provided that he obtains a net yield of at least 10% *pa*.

(ii) Calculate, showing all workings, the best price (per £100 nominal) the original investor can expect to obtain from the potential buyer. [5]

(iii) Calculate the net running yield obtained by the buyer. [1]

(iv) Determine the net yield that will be obtained by the original investor if the bond is sold to the buyer at the price determined in (ii). [4]

[Total 13]

X2.9 (i) Explain what is meant by the following terms:

(a) equation of value

(b) discounted payback period from an investment project. [4]

(ii) An insurance company is considering setting up a branch in a country in which it has not previously operated. The company is aware that access to capital may become difficult in twelve years' time. It therefore has two decision criteria. The cashflows from the project must provide an internal rate of return greater than 9% per annum effective and the discounted payback period at a rate of interest of 7% per annum effective must be less than twelve years.

The following cashflows are generated in the development and operation of the branch.

Cash Outflows

Between the present time and the opening of the branch in three years' time the insurance company will spend $£1.5m$ per annum on research, development and the marketing of products. This outlay is assumed to be a constant continuous payment stream. The rent on the branch building will be $£0.3m$ per annum paid quarterly in advance for twelve years starting in three years' time. Staff costs are assumed to be $£1m$ in the first year, $£1.05m$ in the second year, rising by 5% per annum each year thereafter. Staff costs are assumed to be incurred at the beginning of each year starting in three years' time and assumed to be incurred for 12 years.

Cash Inflows

The company expects the sale of products to produce a net income at a rate of $£1m$ per annum for the first three years after the branch opens rising to $£1.9m$ per annum in the next three years and to $£2.5m$ for the following six years. This net income is assumed to be received continuously throughout each year. The company expects to be able to sell the branch operation 15 years from the present time for $£8m$.

Determine which, if any, of the decision criteria the project fulfils. [17]
[Total 21]

END OF PAPER

Subject CM1: Assignment X3

2019 Examinations

Time allowed: 2¾ hours

Instructions to the candidate

1. *Please:*

   – *attempt all of the questions, as far as possible under exam conditions*

   – ***begin your answer to each question on a new page***

   – ***leave at least 2cm margin on all borders***

   – *write in black ink using a medium-sized nib because we will be unable to mark illegible scripts*

   – *note that assignment marking is not included in the price of the course materials. Please purchase Series Marking or a Marking Voucher before submitting your script.*

   – *note that we only accept the current version of assignments for marking, ie you can only submit this assignment in the sessions leading to the 2019 exams.*

2. *Please **do not**:*

   – *use headed paper*

   – *use highlighting in your script.*

At the end of the assignment

If your script is being marked by ActEd, please follow the instructions on the reverse of this page.

In addition to this paper, you should have available actuarial tables and an electronic calculator.

Submission for marking

You should aim to submit this script for marking by the recommended submission date. The recommended and deadline dates for submission of this assignment are listed on the summary page at the back of this pack and on our website at **www.ActEd.co.uk**.

Scripts received after the deadline date will not be marked, unless you are using a Marking Voucher. *It is your responsibility to ensure that scripts reach ActEd in good time*. If you are using Marking Vouchers, then please make sure that your script reaches us by the Marking Voucher deadline date to give us enough time to mark and return the script before the exam.

When submitting your script, please:

- complete the cover sheet, including the checklist
- scan your script, cover sheet (and Marking Voucher if applicable) and save as a pdf document, then email it to: ActEdMarking@bpp.com
- **do not submit a photograph of your script**
- **do not include the question paper in the scan.**

In addition, please note the following:

- Please title the email to ensure that the subject and assignment are clear *eg* 'CM1 Assignment X3 No. 12345', inserting your ActEd Student Number for 12345.
- The assignment should be scanned the **right way up** (so that it can be read normally without rotation) and as a single document. We cannot accept individual files for each page.
- Please set the resolution so that the script is legible and the resulting PDF **is less than 4 MB** in size.
- Do not protect the PDF in any way (otherwise the marker cannot return the script to ActEd, which causes delays).
- Please include the 'feedback from marker' sheet when scanning.
- Before emailing to ActEd, please check that your scanned assignment includes all pages and conforms to the above.

Subject CM1: Assignment X3

2019 Examinations

Please complete the following information:

Name:

ActEd Student Number (see Note below):

Note: Your ActEd Student Number is printed on all personal correspondence from ActEd. Quoting it will help us to process your scripts quickly. If you do not know your ActEd Student Number, please email us at ActEd@bpp.com.

Your ActEd Student Number is not the same as your IFoA Actuarial Reference Number or ARN.

Number of following pages: _______

Please put a tick in this box if you have solutions and a cross if you do not: ☐

Please tick here if you are allowed extra time or other special conditions in the profession's exams (if you wish to share this information): ☐

Time to do assignment
(see Note below): _____ hrs _____ mins

Under exam conditions
(delete as applicable): yes / nearly / no

Note: If you take more than 2¾ hours, you should indicate how much you completed within this exam time so that the marker can provide useful feedback on your progress.

Score and grade for this assignment (to be completed by marker):

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	Q13	Q14	**Total**	
__ 2	__ 2	__ 3	__ 4	__ 4	__ 5	__ 5	__ 6	__ 6	__ 7	__ 7	__ 9	__ 9	__ 11	__ 80	=_____%

Grade: A B C D E **Marker's initials:** _________

Please tick the following checklist so that your script can be marked quickly. Have you:

[ ] Checked that you are using the latest version of the assignments, *ie* 2019 for the sessions leading to the 2019 exams?
[ ] Written your full name in the box above?
[ ] Completed your ActEd Student Number in the box above?
[ ] Recorded your attempt conditions?
[ ] Numbered all pages of your script (excluding this cover sheet)?
[ ] Written the total number of pages (excluding the cover sheet) in the space above?
[ ] Included your Marking Voucher or ordered Series X Marking?
[ ] Rated your X2 marker at **www.ActEd.co.uk/marking**?

Please follow the instructions on the previous page when submitting your script for marking.

Feedback from marker

Notes on marker's section

The main objective of marking is to provide specific advice on how to improve your chances of success in the exam. The most useful aspect of the marking is the comments the marker makes throughout the script, however you will also be given a percentage score and the band into which that score falls. Each assignment tests only part of the course and hence does not give a complete indication of your likely overall success in the exam. However it provides a good indicator of your understanding of the material tested and the progress you are making with your studies:

A = Excellent progress B = Good progress C = Average progress
D = Below average progress E = Well below average progress

Please note that you can provide feedback on the marking of this assignment at:

www.ActEd.co.uk/marking

X3.1 At time $t = 0$, the 2-year spot rate is 4% *pa* effective, the 3-year spot rate is 5% *pa* effective and the 4-year spot rate is 6% *pa* effective.

Calculate the 2-year continuous-time forward rate from time $t = 2$. [2]

X3.2 A two-year term assurance policy is issued to a life aged 75 exact. The benefit amount is 100,000 if the life dies in the first year, and 200,000 if the life dies in the second year. Benefits are payable at the end of the year of death.

Calculate the expected present value of the benefits from this policy. [2]

Basis:	Mortality:	PFA92C20
	Interest:	5% *pa* effective

X3.3 A life insurance company issues an annuity to a life aged 60 exact, under which the annual payment is 17,000. The annuity is payable monthly in advance and is guaranteed to be paid for a period of 10 years and for the whole of life thereafter.

Calculate the expected present value of the annuity payments. [3]

Basis:	Mortality:	AM92 Ultimate
	Interest:	6% *pa* effective

X3.4 Calculate, showing all working and using an effective annual interest rate of 10%, the present value, volatility and convexity of a portfolio that provides payments of £1,000 in 5 years' time and £5,000 in 10 years' time. [4]

X3.5 A life insurance company issues a deferred annuity contract to a life aged exactly 40. Premiums of £4,500 are payable annually in advance for 25 years or until earlier death.

The policy provides the following benefits:

- On survival to age 65, an annuity of £15,000 *pa* is payable continuously for the whole of the policyholder's life.
- In the event of the policyholder's death during the deferment period, a lump sum is payable at the end of the year of death equal to the total amount of premiums paid to date, without interest.

Calculate the total expected present value of benefits from this policy. [4]

Basis	Mortality:	AM92 Select before age 65
		PMA92C20 after age 65
	Interest:	4% *pa* effective

X3.6 A man aged exactly 42 purchases a whole life assurance policy with a sum assured of 5,000 payable immediately on death.

(i) Write down an expression for the random variable representing the present value of the benefits from this policy. [1]

(ii) Show that the variance of this random variable is $5{,}000^2\left({}^2\bar{A}_{42}-(\bar{A}_{42})^2\right)$. [2]

(iii) Calculate the variance of the present value of the benefits from this assurance policy, assuming AM92 Ultimate mortality and 4% *pa* interest. [2]

[Total 5]

X3.7 A policyholder aged 65 exact buys a whole life annuity that provides payments at the end of each policy year. The first payment is £10,000, and subsequent payments increase by 3% *pa* compound.

Let X denote the present value random variable for this annuity.

Interest is assumed to be 3% *pa* effective and mortality is assumed to follow the AM92 Ultimate table.

(i) Derive an expression for X and simplify this as much as possible. [2]

(ii) Calculate $E(X)$. [1]

(iii) Calculate the probability that the present value of the benefit received by the policyholder is greater than £250,000. [2]

[Total 5]

X3.8 Estimate ${}_2p_{63.25}$ assuming ELT15 (Males) mortality at integer ages and:

(a) a uniform distribution of deaths between integer ages

(b) a constant force of mortality between integer ages. [6]

X3.9 (i) Explain what is meant by the following notation, and calculate its value using AM92 mortality:

$${}_{3|}q_{[55]+1}$$ [2]

(ii) Calculate the value of the following symbols, using AM92 mortality and an interest rate of 6% *pa* effective:

(a) $\ddot{a}^{(4)}_{[40]:\overline{5}|}$

(b) $(I\bar{a})_{70:\overline{10}|}$ [4]

[Total 6]

X3.10 In 20 years' time a sum of 20,000 is to be divided equally amongst the survivors of two independent lives now aged 30 and 40 and a charitable trust.

Determine the expected present value and the variance of the present value of the amount due to the charitable trust. [7]

Basis: Mortality: AM92 Ultimate

Interest: 4% *pa* effective

X3.11 (i) Describe the four different methods of allocating bonuses to conventional with-profits contracts. [3]

A life insurance company issues a 15-year with-profits endowment assurance policy to a life aged 50. The sum assured of 50,000 plus declared reversionary bonuses is payable on survival to the end of the term or immediately on earlier death.

The company assumes that it will award compound bonuses at a rate of 1.92308% *pa* at the end of each policy year (*ie* the death benefit does not include any bonus relating to the policy year of death).

(ii) Calculate the expected present value of the benefits from this policy. [4]

Basis: Mortality: AM92 Ultimate

Interest: 6% *pa* effective

[Total 7]

X3.12 The liabilities of a fund consist of two lump sum payments due at known times in the future. The second lump sum is due for payment 5 years after the first and is twice the amount of the first.

The total present value and the discounted mean term of the liabilities, both calculated using an interest rate of 6% *pa* effective, are £75,000 and 8 years, respectively.

(i) Determine the timing and amounts of the liability payments. [6]

The assets of the fund consist of a single zero-coupon bond that will mature 8 years from now with a redemption payment of £119,540.

(ii) Explain what Redington's theory of immunisation tells you about this fund's portfolio. [3]

[Total 9]

X3.13 An economist's model of interest rates indicates that the n-year spot rate of interest is given by the formula $0.1\left(1+e^{-0.2n}\right)^{-1}$.

(i) Sketch the yield curve based on this formula, indicating clearly the values of the immediate spot rate and the limiting yield on long-dated stocks. [3]

(ii) Explain briefly the shape of this yield curve by reference to the liquidity preference theory. [2]

(iii) Assuming that the economist's model is correct, calculate, showing all workings:

(a) the price per £100 nominal of a bond, purchased now, paying coupons of 6% annually in arrears and redeemable at par in 3 years' time

(b) the 3-year par yield. [4]

[Total 9]

X3.14 Let K denote the curtate future lifetime random variable of a life aged exactly x.

(i) Describe the benefit whose present value random variable is:

$$W = \begin{cases} 10{,}000\ddot{a}_{\overline{K+1}|} & \text{if } K < 10 \\ 10{,}000\ddot{a}_{\overline{10}|} & \text{if } K \geq 10 \end{cases}$$ [1]

(ii) Prove the premium conversion formula:

$$A_{x:\overline{n}|} = 1 - d\,\ddot{a}_{x:\overline{n}|}$$ [2]

(iii) Calculate the expected present value and the standard deviation of the present value of the benefit in (i), assuming:

- a force of interest of 0.04 *pa*
- the life is subject to a constant force of mortality of 0.02 *pa*. [8]

[Total 11]

END OF PAPER

Subject CM1: Assignment X4

2019 Examinations

Time allowed: 3¼ hours

Instructions to the candidate

1. *Please:*

   – *attempt all of the questions, as far as possible under exam conditions*

   – ***begin your answer to each question on a new page***

   – ***leave at least 2cm margin on all borders***

   – *write in black ink using a medium-sized nib because we will be unable to mark illegible scripts*

   – *note that assignment marking is not included in the price of the course materials. Please purchase Series Marking or a Marking Voucher before submitting your script.*

   – *note that we only accept the current version of assignments for marking, ie you can only submit this assignment in the sessions leading to the 2019 exams.*

2. *Please **do not**:*

   – *use headed paper*

   – *use highlighting in your script.*

At the end of the assignment

If your script is being marked by ActEd, please follow the instructions on the reverse of this page.

In addition to this paper, you should have available actuarial tables and an electronic calculator.

Submission for marking

You should aim to submit this script for marking by the recommended submission date. The recommended and deadline dates for submission of this assignment are listed on the summary page at the back of this pack and on our website at **www.ActEd.co.uk**.

Scripts received after the deadline date will not be marked, unless you are using a Marking Voucher. *It is your responsibility to ensure that scripts reach ActEd in good time*. If you are using Marking Vouchers, then please make sure that your script reaches us by the Marking Voucher deadline date to give us enough time to mark and return the script before the exam.

When submitting your script, please:

- complete the cover sheet, including the checklist
- scan your script, cover sheet (and Marking Voucher if applicable) and save as a pdf document, then email it to: ActEdMarking@bpp.com
- **do not submit a photograph of your script**
- **do not include the question paper in the scan.**

In addition, please note the following:

- Please title the email to ensure that the subject and assignment are clear *eg* 'CM1 Assignment X4 No. 12345', inserting your ActEd Student Number for 12345.
- The assignment should be scanned the **right way up** (so that it can be read normally without rotation) and as a single document. We cannot accept individual files for each page.
- Please set the resolution so that the script is legible and the resulting PDF **is less than 4 MB** in size.
- Do not protect the PDF in any way (otherwise the marker cannot return the script to ActEd, which causes delays).
- Please include the 'feedback from marker' sheet when scanning.
- Before emailing to ActEd, please check that your scanned assignment includes all pages and conforms to the above.

Subject CM1: Assignment X4

2019 Examinations

Please complete the following information:

Name:

ActEd Student Number (see Note below):

Note: Your ActEd Student Number is printed on all personal correspondence from ActEd. Quoting it will help us to process your scripts quickly. If you do not know your ActEd Student Number, please email us at ActEd@bpp.com.

Your ActEd Student Number is not the same as your IFoA Actuarial Reference Number or ARN.

Number of following pages: _______

Please put a tick in this box if you have solutions and a cross if you do not: ☐

Please tick here if you are allowed extra time or other special conditions in the profession's exams (if you wish to share this information): ☐

Time to do assignment
(see Note below): _____ hrs _____ mins

Under exam conditions
(delete as applicable): yes / nearly / no

Note: If you take more than 3¼ hours, you should indicate how much you completed within this exam time so that the marker can provide useful feedback on your progress.

Score and grade for this assignment (to be completed by marker):

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	Q13	Q14	Total	
— 2	— 4	— 4	— 5	— 5	— 5	— 5	— 6	— 6	— 9	— 10	— 10	— 11	— 18	— 100	=_____%

Grade: A B C D E **Marker's initials:** _________

Please tick the following checklist so that your script can be marked quickly. Have you:

[ ] Checked that you are using the latest version of the assignments, *ie* 2019 for the sessions leading to the 2019 exams?
[ ] Written your full name in the box above?
[ ] Completed your ActEd Student Number in the box above?
[ ] Recorded your attempt conditions?
[ ] Numbered all pages of your script (excluding this cover sheet)?
[ ] Written the total number of pages (excluding the cover sheet) in the space above?
[ ] Included your Marking Voucher or ordered Series X Marking?
[ ] Rated your X3 marker at **www.ActEd.co.uk/marking**?

Please follow the instructions on the previous page when submitting your script for marking.

Feedback from marker

Notes on marker's section

The main objective of marking is to provide specific advice on how to improve your chances of success in the exam. The most useful aspect of the marking is the comments the marker makes throughout the script, however you will also be given a percentage score and the band into which that score falls. Each assignment tests only part of the course and hence does not give a complete indication of your likely overall success in the exam. However it provides a good indicator of your understanding of the material tested and the progress you are making with your studies:

A = Excellent progress B = Good progress C = Average progress
D = Below average progress E = Well below average progress

Please note that you can provide feedback on the marking of this assignment at:

www.ActEd.co.uk/marking

X4.1 A 5-year endowment assurance is issued to a life aged 60. The policy pays a benefit of £20,000 on maturity or at the end of the year of earlier death. Level premiums are payable annually in advance while the policy is in force.

Calculate the net premium reserve at the end of the second policy year using AM92 Ultimate mortality and an effective annual interest rate of 6%. [2]

X4.2 A 65-year-old male buys a one-year term assurance policy with a sum assured of 200,000 paid immediately on death. Assuming that the force of mortality is constant between integer ages, calculate the single premium that he should pay for this policy.

Basis:	Mortality:	ELT15 (Males)
	Interest:	$\delta = 2.5\%$ *pa*
	Expenses:	none

[4]

X4.3 Calculate ${}_{5|4}q^{\;1}_{[61]:[61]}$ where both lives are subject to AM92 mortality. [4]

X4.4 Define $\ddot{a}^{(12)}_{60:50:\overline{20}|}$ fully in words and calculate its value using an interest rate of 4% *pa* effective, and PMA92C20 and PFA92C20 mortality for the two lives respectively. [5]

X4.5 A joint life annuity of 1 *pa* is payable continuously to lives currently aged x and y while both lives are alive. The present value of the annuity payments is expressed as a random variable, in terms of the joint future lifetime of x and y.

Derive and simplify as far as possible expressions for the expected present value and the variance of the present value of the annuity. [5]

X4.6 A life insurance company issues whole life assurance policies with a sum assured of £100,000 payable at the end of the year of death to lives aged exactly 35. Level premiums are charged annually in advance while the policy is in force.

Calculate the minimum premium the office could charge in order that the probability of making a loss on any one policy would be 1% or less. [5]

Basis:	Mortality:	AM92 Select
	Interest:	6% *pa* effective
	Expenses:	5% of each premium

X4.7 A population is subject to the force of mortality at age x of $\mu_x = e^{0.0002x} - 1$.

Calculate the probability that a life now aged 20 exact:

(i) survives to age 70 exact [2]

(ii) dies between ages 60 exact and 70 exact. [3]
[Total 5]

X4.8 An n-year term assurance with a sum assured of 1 payable at the end of the year of death is issued to a life aged x. Level premiums are payable annually in advance throughout the term of the policy or until the policyholder's earlier death. The premium includes an initial expense loading of I, and a renewal expense loading of e at the start of each policy year, including the first.

(i) Write down expressions, in terms of standard actuarial functions, for:

(a) the gross premium

(b) the prospective gross premium reserve at the end of the t th policy year (where $t < n$)

(c) the retrospective gross premium reserve at the end of the t th policy year (where $t < n$). [3]

(ii) Hence show that, if all three of the expressions in (i) are calculated on the same basis, the prospective and retrospective gross premium reserves are equal. [3]
[Total 6]

X4.9 A policyholder, aged exactly 50, takes out a 15-year term assurance where the sum assured is £10,000 for the first 5 years and £15,000 thereafter. The sum assured is payable at the end of the year of death.

Level premiums are payable annually in advance for at most 10 years while the policy is in force.

Calculate the annual premium. [6]

Basis:	Mortality:	AM92 Select	
	Interest:	4% *pa* effective	
	Expenses:	Initial:	25% of the first premium
		Renewal:	5% of the second and subsequent premiums

X4.10 A life insurance company issues an annuity policy to two lives each aged 60 exact in return for a single premium. Under the policy, an annuity of £10,000 *pa* is payable annually in advance while at least one of the lives is alive.

(i) Write down an expression for the net future loss random variable at the outset for this policy. [2]

(ii) Calculate the single premium, using the equivalence principle.

Basis:	Mortality:	PMA92C20 for the first life, PFA92C20 for the second life
	Interest:	4% *pa* effective
	Expenses:	Ignore

[3]

(iii) Calculate the standard deviation of the net future loss random variable at the outset for this policy, using the basis in part (ii).

You are given that $\ddot{a}_{\overline{60:60}} = 11.957$ at a rate of interest 8.16% *pa*. [4]

[Total 9]

X4.11 An index-linked deferred annuity is issued to a life currently aged exactly 51. The first annuity payment is made at exact age 65 and continues at annual intervals thereafter until the death of the policyholder. The benefit level, which at the outset of the policy is 20,000 *pa*, increases annually in line with inflation, both while in payment and in deferment.

Level premiums are paid annually in advance until age 65 or the earlier death of the policyholder.

On death before age 65, all premiums paid to the date of death are returned without interest, paid at the end of the year of death.

Calculate the annual premium. [10]

Basis:	Mortality:		AM92 Select
	Interest:		4% *pa* effective
	Inflation:		4% *pa* compound
	Expenses:	Initial:	300
		Renewal:	3% of each premium excluding the first
		Claim:	150 on death before age 65 (payable at the same time as the death claim), inflating at 4% *pa* from policy outset plus 0.25% of each annuity payment

X4.12 A life insurance company issues an annuity contract to a man aged 65 exact and his wife aged 62 exact. Under the contract, an annuity of £20,000 *pa* is guaranteed payable for a period of 5 years and thereafter during the lifetime of the man. On the man's death, an annuity of £10,000 *pa* is payable to his wife, if she is then alive. This annuity commences on the monthly payment date next following, or coincident with, the date of his death or from the 5th policy anniversary, if later, and is payable for the lifetime of his wife. Annuities are payable monthly in advance.

Calculate the single premium required for the contract. [10]

Basis:	Mortality:	PMA92C20 for the male and PFA92C20 for the female
	Interest:	4% *pa* effective
	Expenses:	None

X4.13 A life insurance company issued a with-profits whole life policy to a life aged 20 exact, on 1 July 2015. Under the policy, the basic sum assured of £100,000 and attaching bonuses are payable immediately on death. The company declares simple reversionary bonuses at the start of each year. Level premiums are payable annually in advance under the policy.

(i) Give an expression for the gross future loss random variable under the policy at the outset. Define symbols where necessary. [3]

(ii) Calculate the annual premium, using the equivalence principle.

Basis:	Mortality:		AM92 Select
	Interest:		6% *pa* effective
	Bonus loading:		3% *pa* simple
	Expenses:	Initial:	£200
		Renewal:	5% of each premium payable in the second and subsequent years

Assume bonus entitlement is earned immediately on payment of premium. [4]

(iii) On 30 June 2018 the policy is still in force. A total of £10,000 has been declared as a simple bonus to date on the policy.

The company calculates reserves for the policy using a gross premium prospective basis, with the following assumptions:

Mortality:	AM92 Ultimate
Interest:	4% *pa* effective
Bonus loading:	4% *pa* simple
Renewal expenses:	5% of each premium

Calculate the reserve for the policy as at 30 June 2018. [4]

[Total 11]

X4.14 A special term assurance policy is such that a sum of £20,000 is payable if a life (x) dies within a 20-year period. The sum assured is payable immediately on (x)'s death if another life (y) dies before (x). However, if (y) is alive at the time of (x)'s death, payment of the sum assured is deferred until the end of the 20-year period. A continuous level premium is payable.

(i) State, with reasons, the appropriate annuity factor that should be used to calculate the expected present value of the premiums, as at policy outset. [3]

(ii) Calculate the annual rate of premium payable, assuming a constant annual force of interest of 0.05 throughout, and that both lives are subject to the same constant annual force of mortality of 0.005 at all ages. Ignore expenses. [9]

(iii) Assuming the same interest and mortality basis as in (ii), and that no benefit has yet been paid out under the policy, calculate the prospective reserve on this policy after exactly 4 years under each of the following scenarios:

(a) only life (x) is alive at that time

(b) only life (y) is alive at that time

(c) both lives are dead by that time. [5]

(iv) Comment briefly on the differences between the answers you have obtained in (iii). [1]

[Total 18]

END OF PAPER

Subject CM1: Assignment X5

2019 Examinations

Time allowed: 3¼ hours

Instructions to the candidate

1. *Please:*

   – *attempt all of the questions, as far as possible under exam conditions*

   – ***begin your answer to each question on a new page***

   – ***leave at least 2cm margin on all borders***

   – *write in black ink using a medium-sized nib because we will be unable to mark illegible scripts*

   – *note that assignment marking is not included in the price of the course materials. Please purchase Series Marking or a Marking Voucher before submitting your script.*

   – *note that we only accept the current version of assignments for marking, ie you can only submit this assignment in the sessions leading to the 2019 exams.*

2. *Please **do not**:*

   – *use headed paper*

   – *use highlighting in your script.*

At the end of the assignment

If your script is being marked by ActEd, please follow the instructions on the reverse of this page.

In addition to this paper, you should have available actuarial tables and an electronic calculator.

Submission for marking

You should aim to submit this script for marking by the recommended submission date. The recommended and deadline dates for submission of this assignment are listed on the summary page at the back of this pack and on our website at **www.ActEd.co.uk**.

Scripts received after the deadline date will not be marked, unless you are using a Marking Voucher. *It is your responsibility to ensure that scripts reach ActEd in good time*. If you are using Marking Vouchers, then please make sure that your script reaches us by the Marking Voucher deadline date to give us enough time to mark and return the script before the exam.

When submitting your script, please:

- complete the cover sheet, including the checklist
- scan your script, cover sheet (and Marking Voucher if applicable) and save as a pdf document, then email it to: ActEdMarking@bpp.com
- **do not submit a photograph of your script**
- **do not include the question paper in the scan.**

In addition, please note the following:

- Please title the email to ensure that the subject and assignment are clear *eg* 'CM1 Assignment X5 No. 12345', inserting your ActEd Student Number for 12345.
- The assignment should be scanned the **right way up** (so that it can be read normally without rotation) and as a single document. We cannot accept individual files for each page.
- Please set the resolution so that the script is legible and the resulting PDF **is less than 4 MB** in size.
- Do not protect the PDF in any way (otherwise the marker cannot return the script to ActEd, which causes delays).
- Please include the 'feedback from marker' sheet when scanning.
- Before emailing to ActEd, please check that your scanned assignment includes all pages and conforms to the above.

Subject CM1: Assignment X5

2019 Examinations

Please complete the following information:

Name:

ActEd Student Number (see Note below):

Note: Your ActEd Student Number is printed on all personal correspondence from ActEd. Quoting it will help us to process your scripts quickly. If you do not know your ActEd Student Number, please email us at ActEd@bpp.com.

Your ActEd Student Number is not the same as your IFoA Actuarial Reference Number or ARN.

Number of following pages: _______

Please put a tick in this box if you have solutions and a cross if you do not: [ ]

Please tick here if you are allowed extra time or other special conditions in the profession's exams (if you wish to share this information): [ ]

Time to do assignment
(see Note below): _____ hrs _____ mins

Under exam conditions
(delete as applicable): yes / nearly / no

Note: If you take more than 3¼ hours, you should indicate how much you completed within this exam time so that the marker can provide useful feedback on your progress.

Score and grade for this assignment (to be completed by marker):

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	Total	
__ 3	__ 4	__ 4	__ 6	__ 7	__ 7	__ 9	__ 9	__ 11	__ 12	__ 13	__ 15	__ 100	=_____%

Grade: A B C D E **Marker's initials:** ________

Please tick the following checklist so that your script can be marked quickly. Have you:

[ ] Checked that you are using the latest version of the assignments, *ie* 2019 for the sessions leading to the 2019 exams?
[ ] Written your full name in the box above?
[ ] Completed your ActEd Student Number in the box above?
[ ] Recorded your attempt conditions?
[ ] Numbered all pages of your script (excluding this cover sheet)?
[ ] Written the total number of pages (excluding the cover sheet) in the space above?
[ ] Included your Marking Voucher or ordered Series X Marking?
[ ] Rated your X4 marker at **www.ActEd.co.uk/marking**?

Please follow the instructions on the previous page when submitting your script for marking.

Feedback from marker

Notes on marker's section

The main objective of marking is to provide specific advice on how to improve your chances of success in the exam. The most useful aspect of the marking is the comments the marker makes throughout the script, however you will also be given a percentage score and the band into which that score falls. Each assignment tests only part of the course and hence does not give a complete indication of your likely overall success in the exam. However it provides a good indicator of your understanding of the material tested and the progress you are making with your studies:

A = Excellent progress B = Good progress C = Average progress
D = Below average progress E = Well below average progress

Please note that you can provide feedback on the marking of this assignment at:

www.ActEd.co.uk/marking

X5.1 A 10-year endowment assurance policy has a sum assured of £12,000 payable on survival or at the end of the year of earlier death. If the policy is surrendered, the policyholder will receive a return of premiums without interest. Surrenders can occur only at the end of a policy year.

A level premium of £1,100 *pa* is payable annually in advance.

For a policy in force at the start of the fifth year you are given the following details:

	(£)
Renewal expenses	40
Claim expenses on death or surrender	100
Reserve at the start of year	5,000
Reserve at end of year (per policy still in force)	6,500
Rate of interest	8% *pa* effective
Dependent probability of death	0.01
Dependent probability of surrender	0.07

Calculate the profit expected to emerge at the end of the fifth year, per policy in force at the start of that year. [3]

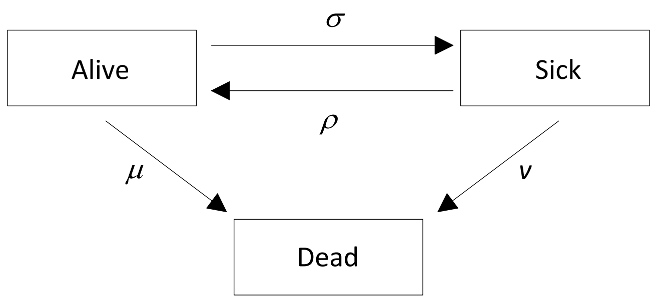
X5.2 A 4-year conventional endowment assurance policy issued to lives aged exactly 61 has a sum assured of £10,000. The profit signature, calculated assuming AM92 Ultimate mortality and making no allowance for surrenders, is (–100, –20, 80, 140). Reserves have been calculated on a net premium basis using 6% *pa* interest and AM92 Ultimate mortality.

The calculations are modified to allow for 10% of policies in force at the end of the first year to be surrendered with a surrender value of £1,500.

(i) Calculate the revised profit in the first year. [3]

(ii) Comment on the impact on the profit signature in years 2 to 4. [1]
[Total 4]

X5.3 A three-state transition model is shown in the following diagram:



Assume that the transition intensities are constant at all ages with $\mu = 0.02\ pa$, $\nu = 0.04\ pa$, $\rho = 0.01\ pa$ and $\sigma = 0.05\ pa$.

Calculate the expected present value of a sickness benefit of £2,000 *pa* paid continuously to a sick life now aged 40 exact, for this period of sickness only, discounted using an interest rate of 4% *pa* effective and payable to a maximum age of 60 exact. [4]

X5.4 (i) Outline the main features of a (non-unitised) accumulating with-profits contract. [4]

(ii) You are given the following details about a unitised with-profits contract:

- fund value on 11th March: £65,292
- monthly premium (payable on first day of month): £600
- annual bonus interest rate for the calendar year: 4.25%
- monthly policy fee (payable on fifteenth day of month): £3

The bonus interest is credited to the policy on a daily basis, by increasing the unit price at the appropriate daily effective interest rate.

Assuming there are no other charges on the contract, calculate the fund value for this policy as at 6th April of the same year. You should assume that there are 365 days in this calendar year. [2]

[Total 6]

X5.5 A unit-linked policy has the following profit vector:

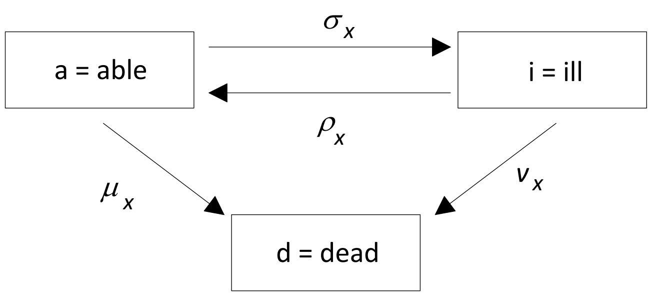
Year	*In-force profit*
1	–25
2	–12
3	–6
4	25
5	35

(i) Calculate the reserves required, in order to zeroise the losses occurring at the end of years 2 and 3. Assume a rate of accumulation of 8% *pa* effective, and that $q_x = 0.01$ at each age. [2]

(ii) If the risk discount rate used is 10% *pa* effective, determine the net present value of the profits before and after zeroisation and state with reasons which of these figures you would expect to be greater. [5]

[Total 7]

X5.6 Consider the following three-state illness-death model:



Let ${}_tp_x^{jk}$ denote the probability that a life in state j at age x will be in state k at age $x+t$, and let ${}_tp_x^{\overline{jj}}$ denote the probability that a life in state j at age x will remain in state j for at least t years. Given a constant force of interest of δ *pa*, write down integral expressions for the expected present value of each of the following benefits payable to a policyholder who is currently aged exactly 50 and in the able state:

(i) a benefit of £50,000 payable immediately on death, provided that death occurs within the next 10 years [2]

(ii) a benefit of £50,000 payable immediately on death, provided that death occurs within the next 10 years and the life has been sick for at least a year at the time of death [3]

(iii) a sickness benefit of £5,000 *pa* payable continuously throughout any period of sickness, ceasing at age 60. [2]

[Total 7]

X5.7 A life office issues 3-year term assurance policies to a group of lives aged 62 exact. Each policy has a sum assured of 150,000, payable at the end of the year of death. Premiums are payable annually in advance, ceasing on earlier death.

The life office calculates reserves on a net premium basis using an interest rate of 4% *pa* effective, and AM92 Ultimate mortality.

The life office makes the following additional assumptions when carrying out a profit test:

Mortality:	AM92 Select
Expenses:	Initial: 400
	Renewal: 50 at the start of the second and subsequent policy years
Interest rate:	6% *pa* effective on investments
Risk discount rate:	9% *pa* effective

Carry out a profit test to determine the premium that will produce a net present value of zero on the above basis. [9]

X5.8 On 1 January 2014 an insurer issued a block of 25-year annual premium endowment policies that pay £120,000 at maturity, or £60,000 at the end of the year of earlier death, to lives aged exactly 65.

The premium basis assumed 4% *pa* effective interest, AM92 Select mortality and allowed for an initial expense of £200 and renewal expenses of 1% of each subsequent premium. The annual premium, calculated on the premium basis, was £3,071.40.

(i) Calculate the reserve required per policy at 31 December 2018, assuming that reserves are calculated on the same basis as the premiums. [3]

(ii) There were 197 policies in force on 1 January 2018. During 2018 there were 9 deaths, interest was earned at twice the rate expected and expenses were incurred at twice the rate expected. By considering the total reserve required at the start and end of the year, and all the cashflows during the year, calculate the profit or loss made by the insurer from all sources (not just from mortality) in respect of these policies for the 2018 calendar year. [6]

[Total 9]

X5.9 A special 3-year term assurance issued to a man aged exactly 62 pays 50,000 immediately on death within the policy term. On survival to the end of the term, or immediately on earlier surrender, half of the total premiums paid to date (without interest) will be returned to the policyholder. A level annual premium is paid at the start of each year.

The insurance company uses the following basis to calculate its premiums:

Mortality: independent probabilities as defined by the AM92 Select table

Surrender: forces of surrender of 5%, 2.5% and 1% in policy years 1, 2, and 3 respectively

Interest: 3% *pa* effective

Expenses: none

(i) Assuming that forces of decrement are constant over each year of age, construct a multiple decrement table that would be suitable for valuing the cashflows for this policy, using a radix of $(al)_{62} = 100{,}000$. [5]

(ii) Using the entries in the multiple decrement table, or otherwise, calculate the annual premium for this policy. [6]

[Total 11]

X5.10 A life insurance company issues a 3-year unit-linked endowment assurance contract to a female life aged 60 exact under which level premiums of £5,000 per annum are payable in advance. In the first year, 85% of the premium is allocated to units and 104% in the second and third years. The units are subject to a bid-offer spread of 5%, and an annual management charge of 0.75% of the bid value of the units is deducted at the end of each year.

If the policyholder dies during the term of the policy, the death benefit of £20,000 or the bid value of the units after the deduction of the management charge, whichever is higher, is payable at the end of the year of death. On survival to the end of the term, the bid value of the units is payable.

The company holds unit reserves equal to the full bid value of the units but does not set up non-unit reserves. It uses the following assumptions in carrying out profit tests of this contract:

Mortality:		AM92 Ultimate
Surrenders:		None
Expenses:	Initial:	600
	Renewal:	100 at the start of each of the second and third policy years
Unit fund growth rate:		6% per annum
Non-unit fund interest rate:		4% per annum
Risk discount rate:		10% per annum

Calculate the profit margin on this contract. [12]

X5.11 A 3-year unitised with-profits endowment is to be issued to a man aged exactly 55. The policy includes the following features:

- Allocation rate of 85% in year 1 and 100% thereafter.
- Premium of 5,000 paid at start of each year.
- Death benefit, paid at the end of the year of death, equal to the end year unit fund value plus terminal bonus, or 15,000, if higher.
- Maturity benefit equal to the end-year unit fund value plus terminal bonus.
- Surrender is permitted at the end of the first and second years, equal to the unit fund value plus terminal bonus less a surrender penalty of 80 per surrender.
- A policy fee is deducted at the start of each year except the first, equal to 1.5% of the unit fund value immediately **after** the premium for that year has been paid.

Calculate the net present value for this policy on the following assumptions:

- Initial expenses: 500
- Renewal expenses: 30
- Termination expenses: 50 per termination (death, surrender or maturity)
- Initial commission: 5% of the first year's premium
- Renewal commission: 1% of the second and third year's premiums
- Investment and actuarial management expenses:
  0.25% of the end-year unit fund value each year
- Mortality: 80% of AM92 Select
- Surrender probability: 10% of all policies in force at the end of each year
- Regular bonus interest: 4% per annum
- Terminal bonus rates: 1% of the unit fund value after 1 year
  3% of the unit fund value after 2 years
  6.5% of the unit fund value after 3 years
- Non-unit interest: 2% per annum
- Risk discount rate: 8% per annum
- All expected investment returns are assumed to be distributed to the policyholder through the regular and terminal bonuses.

[13]

X5.12 (i) Consider a policy issued t years ago to a policyholder then aged x. The policy provides a benefit of S at the end of the policy year of death (and no benefit on survival).

In relation to the policy year $(t,t+1)$, write down in the form of symbols, and also explain in words, the expressions 'death strain', 'death strain at risk', 'expected death strain' and 'actual death strain'. [6]

(ii) A life insurance company issues the following policies:

- 15-year term assurances with a sum assured of £150,000 where the death benefit is payable at the end of the year of death
- 5-year single premium temporary immediate annuities with an annual benefit payable in arrear of £25,000.

On 1 January 2015, the company sold 5,000 term assurance policies to male lives aged 45 exact and 1,000 temporary immediate annuity policies to male lives aged 55 exact. For the term assurance policies, premiums are payable annually in advance. During the first two years, there were fifteen actual deaths from the term assurance policies written and five actual deaths from the immediate annuity policies written.

(a) Calculate the death strain at risk for each type of policy during 2017.

(b) During 2017, there were eight actual deaths from the term assurance policies written and one actual death from the immediate annuity policies written.

Calculate the total mortality profit or loss to the office in the year 2017.

Basis: Interest: 4% *pa* effective

Mortality: AM92 Ultimate for term assurances

PMA92C20 for annuities

[9]
[Total 15]

END OF PAPER

For the session leading to the April 2019 exams – CS2 & CM1 Subjects

Marking vouchers

Subjects	Assignments	Mocks
CS2, CM1	13 March 2019	20 March 2019

Series X and Y Assignments

Subjects	Assignment	Recommended submission date	Final deadline date
CS2, CM1	**X1**	**21 November 2018**	9 January 2019
CS2, CM1	**X2**	**5 December 2018**	23 January 2019
CS2, CM1	**X3**	**19 December 2018**	30 January 2019
CS2, CM1	**Y1**	**9 January 2019**	6 February 2019
CS2, CM1	**X4**	**23 January 2019**	20 February 2019
CS2, CM1	**X5**	**6 February 2019**	27 February 2019
CS2, CM1	**Y2**	**20 February 2019**	13 March 2019

Mock Exams

Subjects	Recommended submission date	Final deadline date
CS2 (Paper A/B), CM1 (Paper A/B)	**6 March 2019**	20 March 2019

We encourage you to work to the recommended submission dates where possible.

If you submit your mock on the final deadline date you are likely to receive your script back less than a week before your exam.

For the session leading to the September 2019 exams – CS2 & CM1 Subjects

Marking vouchers

Subjects	Assignments	Mocks
CS2	21 August 2019	28 August 2019
CM1	28 August 2019	4 September 2019

Series X and Y Assignments

Subjects	Assignment	Recommended submission date	Final deadline date
CS2	**X1**	**22 May 2019**	3 July 2019
CM1		**29 May 2019**	10 July 2019
CS2	**X2**	**5 June 2019**	10 July 2019
CM1		**12 June 2019**	17 July 2019
CS2	**X3**	**12 June 2019**	17 July 2019
CM1		**19 June 2019**	24 July 2019
CS2	**Y1**	**26 June 2019**	24 July 2019
CM1		**3 July 2019**	31 July 2019
CS2	**X4**	**10 July 2019**	31 July 2019
CM1		**17 July 2019**	7 August 2019
CS2	**X5**	**17 July 2019**	7 August 2019
CM1		**24 July 2019**	14 August 2019
CS2	**Y2**	**31 July 2019**	14 August 2019
CM1		**7 August 2019**	21 August 2019

Mock Exams

Subjects	Recommended submission date	Final deadline date
CS2 (Paper A/B)	**14 August 2019**	28 August 2019
CM1 (Paper A/B)	**21 August 2019**	4 September 2019

We encourage you to work to the recommended submission dates where possible.

If you submit your mock on the final deadline date you are likely to receive your script back less than a week before your exam.

Assignment X1 – Solutions

Markers: This document sets out one approach to solving each of the questions. Please give credit for other valid approaches. In particular, in 'trial and error' questions, full marks should be awarded for obtaining the correct final answer whatever method is used (eg 'table mode' on a calculator), so long as sufficient working is given.

Solution X1.1

Perpetuities are covered in Chapter 8, Section 6.

$$PV = 50a_{\overline{\infty}|} = \frac{50}{i} = £833.33$$ [1]

[Total 1]

Solution X1.2

Simple interest is covered in Chapter 4, Section 1.

The length of time can be found from the equation:

$$800(1+0.04t) = 1{,}000$$ [1]

$$\Rightarrow \quad t = 6\tfrac{1}{4}\text{years}$$ [1]

[Total 2]

Solution X1.3

Deterministic and stochastic models are introduced in Chapter 2, Section 3.

A deterministic model uses one set of input parameters and gives the results of the relevant calculations for this single scenario. [1]

A stochastic model involves at least one input parameter varying according to an assumed probability distribution. As such, the output will vary along with the input and the model produces distributions of the relevant results for a distribution of scenarios. [1]

Often the output from a stochastic model is in the form of many thousands of simulated outcomes of the process. We can study the distributions of these outcomes. [1]

[Total 3]

Solution X1.4

Simple discount is covered in Chapter 4, Section 3.

The discount factor for 91 days under simple discount is:

$$(1-nd)=\left(1-\tfrac{91}{365}\times 0.10\right)=0.9751 \qquad [1]$$

Equating this to the effective discount factor:

$$v^{\frac{91}{365}}=0.975 \quad \Rightarrow \quad (1+i)^{-\frac{91}{365}}=0.975 \qquad [1]$$

$$\Rightarrow \quad (1+i)=0.975^{-\frac{365}{91}} \Rightarrow \quad i=10.66\% \qquad [1]$$

[Total 3]

Alternatively, using $\frac{91}{365.25}$ or even $\frac{1}{4}$ (since 91 days is approximately a quarter of a year) in both calculations will give the same answer to 4 SF.

Solution X1.5

This material appears in Section 1 of Chapter 1.

The three key forms of data analysis are:

- descriptive analysis: producing summary statistics (*eg* measures of central tendency and dispersion) and presenting the data in a simpler format [1]
- inferential analysis: using a data sample to estimate summary parameters for the wider population from which the sample was taken, and testing hypotheses [1]
- predictive analysis: extends the principles of inferential analysis to analyse past data and make predictions about future events. [1]

[Total 3]

Solution X1.6

Accumulations of level annuities are covered in Chapter 8, Sections 2 and 4.

The annual rate of interest, i , is given by:

$$(1+i)=1.03^2=1.0609 \quad \Rightarrow \quad i=6.09\%$$

The present value of the annuity is given by:

$$PV=300\ddot{a}_{\overline{7}|}+100v^7 a_{\overline{5}|}^{(4)} \qquad [1]$$

Now:

$$\ddot{a}_{\overline{7}|}=\frac{1-v^7}{d}=\frac{1-1.0609^{-7}}{1-1.0609^{-1}}=5.90345$$

and:

$$a_{\overline{5}|}^{(4)}=\frac{1-v^5}{i^{(4)}}=\frac{1-1.0609^{-5}}{4\left[1.0609^{¼}-1\right]}=\frac{1-1.0609^{-5}}{0.059557}=4.29685$$

So:

$$PV=300\times 5.90345+100\times 1.0609^{-7}\times 4.29685 \qquad [1]$$

$$=1771.04+284.07$$

$$=\$2,055.11 \qquad [1]$$

So the accumulated amount after 12 years is:

$$\$2,055.11\times 1.0609^{12}=\$4,178 \qquad [1]$$

[Total 4]

Alternatively, this could be calculated as:

$$300\ddot{s}_{\overline{7}|}(1+i)^5+100s_{\overline{5}|}^{(4)}$$

We could calculate the second annuity by working in quarters eg $25v_{6.09\%}^7 a_{\overline{20}|\,@1.4889157\%}$ *or* $25s_{\overline{20}|\,@1.4889157\%}$.

Solution X1.7

Level annuities payable pthly are covered in Chapter 8, Section 4.

(i) ***Level monthly annuity***

Let X be the monthly amount paid. Then the present value of the payments is:

$$12Xa_{\overline{10}|}^{(12)} = 12X\left(\frac{1-1.08^{-10}}{i^{(12)}}\right) = 83.4324X \qquad [1]$$

where $i^{(12)} = 12\left(1.08^{1/12} - 1\right) = 7.7208\%$.

Since the investor paid £10,000, $X = \dfrac{10{,}000}{83.4324} = £119.86$. [1]

[Total 2]

Alternatively, we could work in months: $Xa_{\overline{120}|@0.643403011\%}$

(ii) ***Level annuity payable half yearly***

Let Y be the half-yearly amount. Then the present value of the payments is:

$$2Yv^2\ddot{a}_{\overline{10}|}^{(2)} = 2Y\left(\frac{1}{1.08^2}\right)\left(\frac{1-1.08^{-10}}{d^{(2)}}\right) = 12.19154Y \qquad [1]$$

where $d^{(2)} = 2\left(1 - 1.08^{-1/2}\right) = 7.5499\%$.

Since the investor paid £10,000, $Y = \dfrac{10{,}000}{12.19154} = £820.24$. [1]

[Total 2]

Alternatively, we could calculate the annuity by working in half-years: $Yv_{8\%}^2\ddot{a}_{\overline{20}|@3.9230485\%}$.

Solution X1.8

Real and money rates of interest are covered in Chapter 6.

A real rate of interest is used when inflation needs to be taken into account. [1]

A money rate of interest is used when inflation does not need to be taken into account. [1]

An example for a real rate of interest: A person is saving to go on a round the world trip, which they have calculated will cost £10,000 today. [1]

An example for a money rate of interest: A person has a loan of £10,000, which needs to be paid back in full in 5 years' time. [1]

[Total 4]

Allow any suitable examples here.

Solution X1.9

Nominal interest and discount are covered in Chapter 5, Section 1.

(i) ***Rate of discount convertible quarterly***

Using the formula $1+i=\left(1+\dfrac{i^{(p)}}{p}\right)^p$ gives:

$$1+i=\left(1+\frac{0.1}{4}\right)^4=1.10381$$ [1]

Alternatively $(1+i)^{1/4}=1.025$.

Now using the formula $d^{(p)}=p\left[1-(1+i)^{-\frac{1}{p}}\right]$ gives:

$$d^{(4)}=4\left[1-(1+i)^{-1/4}\right]=4\left[1-1.025^{-1}\right]=9.76\%$$ [1]

[Total 2]

Note: Although $d=\dfrac{i}{1+i}$*, this formula does not work for nominal rates, ie:* $d^{(4)}\neq\dfrac{i^{(4)}}{1+i^{(4)}}$.

However it does work for effective rates so it is true that:

$$\frac{d^{(4)}}{4}=\frac{i^{(4)}/4}{1+i^{(4)}/4}$$

(ii) ***Accumulated amount***

Let i_1 and i_2 be the effective annual rates for the first and second years respectively. Then using the formula $\frac{1}{1+i} = \left(1 - \frac{d^{(p)}}{p}\right)^p$ gives:

$$1 + i_1 = \left(1 - \frac{0.06}{2}\right)^{-2} = 1.06281 \qquad [1]$$

Alternatively $(1+i_1)^{-\frac{1}{2}} = 0.97$.

Now using the formula $1 + i = \left(1 + \frac{i^{(p)}}{p}\right)^p$ gives:

$$1 + i_2 = \left(1 + \frac{0.06}{3}\right)^3 = 1.061208 \qquad [1]$$

Alternatively $(1+i_2)^{\frac{1}{3}} = 1.02$.

So accumulating the £500:

$$500(1+i_1)(1+i_2) = 500 \times 1.06281 \times 1.061208 = £563.93 \qquad [1]$$

[Total 3]

Alternatively, we could use the other factors that we calculated:

$$500(1+i_1)(1+i_2) = 500 \times 0.97^{-2} \times 1.02^3 = £563.93$$

Note: It is easy to become confused between a quarter (3 months) and the 4 months given in the question. Many students incorrectly use a quarterly time period.

Solution X1.10

This topic is covered in Chapter 2, Section 6.

When assessing the suitability of a model for a particular exercise, it is important to consider the following:

- The objectives of the modelling exercise. [½]
- The validity of the model for the purpose to which it is to be put. [½]
- The validity of the data to be used. [½]
- The validity of the assumptions. [½]
- The possible errors associated with the model or parameters used not being a perfect representation of the real world situation being modelled. [½]
- The impact of correlations between the random variables that 'drive' the model. [½]
- The extent of correlations between the various results produced from the model. [½]
- The current relevance of models written and used in the past. [½]
- The credibility of the data input. [½]
- The credibility of the results output. [½]
- The dangers of spurious accuracy. [½]
- The ease with which the model and its results can be communicated. [½]
- Short-term and long-term properties of a model. [½]
- Regulatory requirements. [½]

[Maximum 5]

Solution X1.11

The interest and discount rates used in this question are covered in Chapter 4, Sections 1 and 3, and in Chapter 5, Sections 1 and 2.

(i) ***Effective rate***

$$(1+i)^2 = 1.127 \quad \Rightarrow \quad i = 1.127^{\frac{1}{2}} - 1 = 6.16026\%$$

So the equivalent effective annual rate of interest is 6.16%. [1]

[Total 1]

*Don't get confused. This is an **effective** rate paid every two years not a nominal rate convertible every two years as in part (v).*

(ii) ***Effective rate***

$v = 1 - d = 0.9425 \quad \Rightarrow \quad (1+i) = 0.9425^{-1} \quad \Rightarrow \quad i = 6.1008\%$

So the equivalent effective annual rate of interest is 6.10%. [1]

[Total 1]

Very often it is easier to turn a rate of discount into a rate of interest by first finding v.

Alternatively, we could use $i = \dfrac{d}{1-d}$.

(iii) ***Effective rate***

$i = e^{12 \times 0.005} - 1 = 6.18365\%$

So the equivalent effective annual rate of interest is 6.18%. [1]

[Total 1]

(iv) ***Effective rate***

Using $\dfrac{1}{1+i} = \left(1 - \dfrac{d^{(4)}}{4}\right)^4$, we have:

$$i = \left(1 - \frac{0.06}{4}\right)^{-4} - 1 = 6.23193\%$$

So the equivalent effective annual rate of interest is 6.23%. [1]

[Total 1]

(v) ***Effective rate***

Using the formula $1 + i = \left(1 + \dfrac{i^{(p)}}{p}\right)^p$, we have:

$$i = \left(1 + \frac{0.14}{\frac{1}{2}}\right)^{\frac{1}{2}} - 1 = 13.1371\%$$

So the equivalent effective annual rate of interest is 13.14%. [1]

[Total 1]

Solution X1.12

Compound increasing annuities are covered in Chapter 9, Section 2.2.

We will work in years, with $v_1 = \dfrac{1}{1.06}$ denoting the one-year discount factor applicable in each of the first 10 years, and $v_2 = \dfrac{1}{1.04}$ denoting the one-year discount factor applicable in each of the final 10 years.

The times, payments and discount factors are:

Time in years	Time in half-years	Payment	Discount factor
0.5	1	250	$v_1^{0.5}$
1	2	250×1.02	v_1
1.5	3	250×1.02^2	$v_1^{1.5}$
10	20	250×1.02^{19}	v_1^{10}
10.5	21	250×1.02^{20}	$v_1^{10} v_2^{0.5}$
11	22	250×1.02^{21}	$v_1^{10} v_2$
20	40	250×1.02^{39}	$v_1^{10} v_2^{10}$

The total present value of the payments received is therefore:

$$250v_1^{0.5} + 250(1.02)v_1 + 250(1.02)^2 v_1^{1.5} + \cdots + 250(1.02)^{19} v_1^{10}$$
$$+250(1.02)^{20} v_1^{10} v_2^{0.5} + 250(1.02)^{21} v_1^{10} v_2 + \cdots + 250(1.02)^{39} v_1^{10} v_2^{10} \qquad [2]$$

The terms in the first line of the present value expression form a geometric progression of 20 terms, with first term $250v_1^{0.5}$ and common ratio $(1.02)v_1^{0.5}$, so:

$$250v_1^{0.5} + 250(1.02)v_1 + 250(1.02)^2 v_1^{1.5} + \cdots + 250(1.02)^{19} v_1^{10}$$

$$= 250v_1^{0.5} \frac{\left(1 - \left(1.02v_1^{0.5}\right)^{20}\right)}{1 - 1.02v_1^{0.5}}$$

$$= 4{,}450.86 \qquad [1½]$$

The terms in the second line of the present value expression form a geometric progression of 20 terms, with first term $250(1.02)^{20}v_1^{10}v_2^{0.5}$ and common ratio $(1.02)v_2^{0.5}$, so:

$$250(1.02)^{20}v_1^{10}v_2^{0.5}+250(1.02)^{21}v_1^{10}v_2+\cdots+250(1.02)^{39}v_1^{10}v_2^{10}$$

$$=250(1.02)^{20}v_1^{10}v_2^{0.5}\frac{\left(1-\left(1.02v_2^{0.5}\right)^{20}\right)}{1-1.02v_2^{0.5}}$$

$$=4{,}075.60 \qquad [1½]$$

So the total present value of the payments is:

$$4{,}450.86+4{,}075.60=£8{,}526.46 \qquad [1]$$

[Total 6]

Alternatively, we could work in half-years, with $v_3=\dfrac{1}{1.06^{0.5}}$ *denoting the half-year discount factor applicable in each of the first 10 years, and* $v_4=\dfrac{1}{1.04^{0.5}}$ *denoting the half-year discount factor applicable in each of the final 10 years.*

Then the total present value of the payments received is:

$$250v_3+250(1.02)v_3^2+250(1.02)^2v_3^3+\cdots+250(1.02)^{19}v_3^{20}$$
$$+250(1.02)^{20}v_3^{20}v_4+250(1.02)^{21}v_3^{20}v_4^2+\cdots+250(1.02)^{39}v_3^{20}v_4^{20}$$

The terms in the first line of the present value expression form a geometric progression of 20 terms, with first term $250v_3$ *and common ratio* $(1.02)v_3$*, giving a total of:*

$$250v_3\frac{\left(1-\left(1.02v_3\right)^{20}\right)}{1-1.02v_3}=4{,}450.86$$

The terms in the second line of the present value expression form a geometric progression of 20 terms, with first term $250(1.02)^{20}v_3^{20}v_4$ *and common ratio* $(1.02)v_4$*, giving a total of:*

$$250(1.02)^{20}v_3^{20}v_4\frac{\left(1-\left(1.02v_4\right)^{20}\right)}{1-1.02v_4}=4{,}075.60$$

So the total present value of the payments is:

$$4{,}450.86+4{,}075.60=£8{,}526.46$$

as before.

Alternatively, the sum:

$$250v_3 + 250(1.02)v_3^2 + 250(1.02)^2 v_3^3 + \cdots + 250(1.02)^{19} v_3^{20}$$

can be evaluated as an annuity as:

$$\frac{250}{1.02} a_{\overline{20|}@j_1} \quad where \;\; j_1 = \frac{1.06^{0.5}}{1.02} - 1 = 0.93755\%$$

and the sum:

$$250(1.02)^{20} v_3^{20} v_4 + 250(1.02)^{21} v_3^{20} v_4^2 + \cdots + 250(1.02)^{39} v_3^{20} v_4^{20}$$

can be evaluated as an annuity as:

$$250(1.02)^{19} v_3^{20} a_{\overline{20|}@j_2} \quad where \;\; j_2 = \frac{1.04^{0.5}}{1.02} - 1 = -0.019225\%$$

Solution X1.13

The force of interest as a function of time is covered in Chapter 5, Section 4.

(i) ***Equivalent constant force of interest***

The accumulation factor from $t = 0$ to $t = 10$ is:

$$A(0,10) = e^{\int_0^{10} 0.03 - 0.005t + 0.001t^2 \, dt}$$

$$= e^{\left[0.03t - 0.0025t^2 + \frac{0.001}{3}t^3\right]_0^{10}} \qquad [1]$$

$$= e^{0.383} \qquad [1]$$

Let δ be the constant force of interest, then:

$$e^{10\delta} = e^{0.383} \;\Rightarrow\; 10\delta = 0.383 \;\Rightarrow\; \delta = 3.83\% \qquad [1]$$

[Total 3]

(ii) ***Accumulated value***

The accumulated value is:

$$250e^{\int_0^7 0.03-0.005t+0.001t^2\,dt}+150e^{\int_5^7 0.03-0.005t+0.001t^2\,dt} \qquad [1]$$

$$=250e^{\left[0.03t-0.0025t^2+\frac{0.001}{3}t^3\right]_0^7}+150e^{\left[0.03t-0.0025t^2+\frac{0.001}{3}t^3\right]_5^7}$$

$$=250e^{0.20183}+150e^{0.20183-0.12917} \qquad [1]$$

$$=250e^{0.20183}+150e^{0.07267}$$

$$=305.91+161.31 \qquad [1]$$

$$=£467.22 \qquad [1]$$

[Total 4]

Alternatively, we could accumulate 250 to time 5, then add 150, then accumulate further to time 7.

Solution X1.14

Simple increasing annuities are covered in Chapter 9, Section 1.

(i) ***Proof***

The symbol $(Ia)_{\overline{n}|}$ represents the present value of a payment of 1 at time 1, 2 at time 2, and so on up to a final payment of n at time n. So:

$$(Ia)_{\overline{n}|}=v+2v^2+3v^3+\cdots+nv^n \qquad \text{(Equation 1)} \qquad [1]$$

Multiplying through by $1+i$:

$$(1+i)(Ia)_{\overline{n}|}=1+2v+3v^2+\cdots+nv^{n-1} \qquad \text{(Equation 2)} \qquad [½]$$

Subtracting Equation 1 from Equation 2 gives:

$$i(Ia)_{\overline{n}|}=1+v+v^2+\cdots+v^{n-1}-nv^n \qquad [½]$$

Noting that:

$$1+v+v^2+\cdots+v^{n-1}=\ddot{a}_{\overline{n}|} \qquad [½]$$

gives: $$(Ia)_{\overline{n}|}=\frac{\ddot{a}_{\overline{n}|}-nv^n}{i} \qquad [½]$$

[Total 3]

(ii) ***Present value***

As the payments decrease each month, we work in months with a monthly effective interest rate of:

$$1.06^{1/12}-1=0.486755\% \qquad [½]$$

A timeline showing the payments is as follows:

	300	290	...	70	Cashflow
0	1	2	...	24	Time (months)

The present value of the payments (working in months) is therefore:

$$310a_{\overline{24}|}-10(Ia)_{\overline{24}|} \qquad [1]$$

Alternatively, the present value of the payments may be written as:

$$300a_{\overline{24}|}-10v(Ia)_{\overline{23}|}$$

Now:

$$a_{\overline{24}|}=\frac{1-1.00486755^{-24}}{0.00486755}=22.5994 \qquad [1]$$

and:

$$(Ia)_{\overline{24}|}=\frac{\left(\dfrac{1-1.00486755^{-24}}{0.00486755/1.00486755}\right)-24\times1.00486755^{-24}}{0.00486755}=277.2351 \qquad [1]$$

So the present value is:

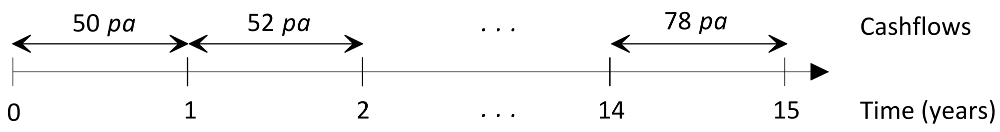
$$310\times22.5994-10\times277.2351=£4,233.45 \qquad [½]$$

[Total 4]

Solution X1.15

Simple increasing annuities are covered in Chapter 9, Section 1. Compound increasing annuities are covered in Chapter 9, Section 2.2.

For Option 1, a timeline showing the payments made is as follows:



The present value of the annuity under Option 1 is therefore:

$$48\bar{a}_{\overline{15}|} + 2(I\bar{a})_{\overline{15}|} \qquad [1]$$

Alternatively, this can be expressed as:

$$50\bar{a}_{\overline{15}|} + 2v(I\bar{a})_{\overline{14}|}$$

Now:

$$\bar{a}_{\overline{15}|} = \frac{1-v^{15}}{\delta} = \frac{1-1.07^{-15}}{\ln(1.07)} = 9.42310 \qquad [½]$$

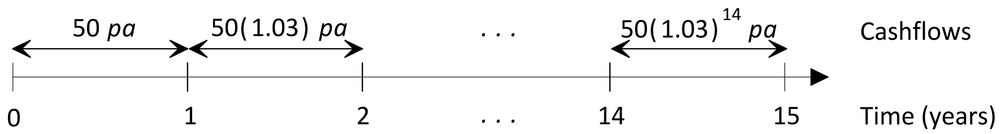
and:

$$(I\bar{a})_{\overline{15}|} = \frac{\left(\dfrac{1-1.07^{-15}}{0.07/1.07}\right) - 15\times 1.07^{-15}}{\ln(1.07)} = 63.68406 \qquad [1]$$

So the present value of the annuity under Option 1 is:

$$48\times 9.42310 + 2\times 63.68406 = £579.68 \qquad [½]$$

For Option 2, a timeline showing the payments made is as follows:



The present value of the annuity under Option 2 is therefore:

$$50\bar{a}_{\overline{1}|} + 50\times 1.03\,v\,\bar{a}_{\overline{1}|} + \cdots + 50\times 1.03^{14}v^{14}\bar{a}_{\overline{1}|} = 50\bar{a}_{\overline{1}|}\left(1 + 1.03\,v + \cdots + 1.03^{14}v^{14}\right) \qquad [1½]$$

The terms in brackets form a geometric progression of 15 terms with first term 1 and common ratio $1.03v$, so:

$$1+1.03\,v+\cdots+1.03^{14}v^{14}=\frac{1-(1.03v)^{15}}{1-1.03v}=\frac{1-(1.03\times1.07^{-1})^{15}}{1-1.03\times1.07^{-1}}=11.64483 \qquad [1]$$

Alternatively, this summation can be calculated as:

$$\ddot{a}_{\overline{15}|@j} \quad \textit{where} \;\; j=\frac{1.07}{1.03}-1=3.88350\%$$

Also:

$$\bar{a}_{\overline{1}|}=\frac{1-v}{\delta}=\frac{1-1.07^{-1}}{\ln(1.07)}=0.96692 \qquad [½]$$

So the present value of the annuity under Option 2 is:

$$50\times0.96692\times11.64483=£562.98 \qquad [½]$$

Since the present value of the payments is higher under Option 1, it provides the better deal for the investor. [½]

[Total 7]

Solution X1.16

The force of interest as a function of time is covered in Chapter 5, Section 4. Payment streams are covered in Chapter 7, Sections 1 and 2.

Let $A(t_1,t_2)$ denote the accumulated value at time t_2 of 1 unit paid at time t_1.

(i) ***Accumulated amount***

For $0\le t<6$:

$$\int_0^t \delta(s)\,ds=\int_0^t (0.04+0.005s)\,ds=[0.04s+0.0025s^2]_0^t=0.04t+0.0025t^2 \qquad [1]$$

So:

$$A(0,t)=e^{0.04t+0.0025t^2} \quad (0\le t<6) \qquad [1]$$

and:

$$A(0,6)=e^{0.33}$$

For $6 \le t < 8$:

$$\int_6^t \delta(s)\,ds = \int_6^t (0.16 - 0.015s)\,ds$$

$$= [0.16s - 0.0075s^2]_6^t$$

$$= 0.16(t-6) - 0.0075(t^2 - 6^2)$$

$$= -0.69 + 0.16t - 0.0075t^2 \qquad [1]$$

So:

$$A(0,t) = e^{0.33} e^{-0.69+0.16t-0.0075t^2} = e^{-0.36+0.16t-0.0075t^2} \quad (6 \le t < 8) \qquad [1]$$

and:

$$A(0,8) = e^{0.44}$$

For $8 \le t$:

$$\int_8^t \delta(s)\,ds = \int_8^t 0.04\,ds = 0.04(t-8) \qquad [1]$$

So:

$$A(0,t) = e^{0.44} e^{0.04(t-8)} = e^{0.12+0.04t} \quad (8 \le t) \qquad [1]$$

In summary:

$$A(0,t) = \begin{cases} e^{0.04t+0.0025t^2} & 0 \le t < 6 \\ e^{-0.36+0.16t-0.0075t^2} & 6 \le t < 8 \\ e^{0.12+0.04t} & 8 \le t \end{cases}$$

[Total 6]

(ii)(a) ***Present value***

The present value is:

$$\frac{100}{A(0,9)} = \frac{100}{e^{0.12+0.04\times 9}} = \frac{100}{e^{0.48}} = 100e^{-0.48} = £61.88 \qquad [1½]$$

Alternatively, we could calculate this from first principles:

$$PV = 100e^{-\int_0^6 0.04+0.005t\,dt}\, e^{-\int_6^8 0.16-0.015t\,dt}\, e^{-\int_8^9 0.04\,dt}$$

$$= 100e^{-\left[0.04t+0.0025t^2\right]_0^6}\, e^{-\left[0.16t-0.0075t^2\right]_6^8}\, e^{-\left[0.04t\right]_8^9}$$

$$= 100e^{-0.33}e^{-(0.8-0.69)}e^{-(0.36-0.32)}$$

$$= 100e^{-0.48} = £61.88$$

(ii)(b) ***Equivalent annual effective rate of discount***

Letting d denote the equivalent annual effective rate of discount:

$$100e^{-0.48} = 100(1-d)^9 \;\Rightarrow\; e^{-0.48} = (1-d)^9 \qquad [1]$$

Therefore:

$$d = 1-\left(e^{-0.48}\right)^{1/9} = 0.051936 \;\; ie \;\; 5.1936\% \qquad [½]$$

[Total 3]

Alternatively, we could calculate the equivalent annual effective interest rate first:

$$100e^{-0.48} = 100v^9 \;\Rightarrow\; i = \left(e^{0.48}\right)^{1/9} - 1 = 0.054781$$

We can then use this to calculate the annual effective rate of discount:

$$d = \frac{i}{1+i} = \frac{0.054781}{1.054781} = 0.051936 \;\; ie \;\; 5.1936\%$$

(iii) ***Present value of payment stream at time 4***

The present value at time 10 is:

$$PV_{t=10} = \int_{10}^{15} 45e^{0.01t} e^{-\int_{10}^{t} 0.04\,ds}\, dt \qquad [1]$$

Carrying out the integration:

$$PV_{t=10} = \int_{10}^{15} 45e^{0.01t}e^{-\left[0.04s\right]_{10}^{t}}\,dt = \int_{10}^{15} 45e^{0.01t}e^{-0.04t+0.4}\,dt = 45e^{0.4}\int_{10}^{15} e^{-0.03t}\,dt$$

$$= 45e^{0.4}\left[\frac{e^{-0.03t}}{-0.03}\right]_{10}^{15} = \frac{45e^{0.4}}{-0.03}\left(e^{-0.03\times 15} - e^{-0.03\times 10}\right) = 230.912 \qquad [2]$$

We now need to discount this back to time 4.

The present value at time 4 is:

$$PV_{t=4} = \frac{230.912}{A(4,10)}$$

where:

$$A(4,10) = \frac{A(0,10)}{A(0,4)} = \frac{e^{0.12+0.04\times 10}}{e^{0.04\times 4+0.0025\times 4^2}} = \frac{e^{0.52}}{e^{0.2}} = e^{0.32} \qquad [1]$$

So the present value of this payment stream at time 4 is:

$$PV_{t=4} = \frac{230.912}{A(4,10)} = 230.912e^{-0.32} = 167.677 \qquad [1]$$

[Total 5]

Alternatively, the discount factor from time 10 to time 4 could be calculated from first principles:

$$e^{-\int_4^6 0.04+0.005t\,dt}\,e^{-\int_6^8 0.16-0.015t\,dt}\,e^{-\int_8^{10} 0.04\,dt}$$

$$= e^{-\left[0.04t+0.0025t^2\right]_4^6}\,e^{-\left[0.16t-0.0075t^2\right]_6^8}\,e^{-\left[0.04t\right]_8^{10}}$$

$$= e^{-(0.33-0.2)}e^{-(0.8-0.69)}e^{-(0.4-0.32)}$$

$$= e^{-0.32}$$

Alternatively, the answer could be calculated by first calculating the present value at time 0, using the integral:

$$PV_{t=0} = \int_{10}^{15} 45e^{0.01t}\frac{1}{A(0,t)}\,dt = \int_{10}^{15} 45e^{0.01t}\,e^{-0.12-0.04t}\,dt = 137.282$$

and then accumulating this forward to time 4:

$$PV_{t=4} = PV_{t=0} \times A(0,4) = 137.282e^{0.04\times 4+0.0025\times 4^2} = 137.282e^{0.2} = 167.677$$

where the expressions for $A(0,t)$ *are taken from part (i).*

Assignment X2 – Solutions

Markers: This document sets out one approach to solving each of the questions. Please give credit for other valid approaches. In particular, in 'trial and error' questions, full marks should be awarded for obtaining the correct final answer whatever method is used (eg 'table mode' on a calculator), so long as sufficient working is given.

Solution X2.1

APR is covered in Chapter 11, Section 6.

The APR is the interest rate that solves the equation of value:

$$20{,}000 = 12 \times 427.90 a_{\overline{5}|}^{(12)}$$ [1]

Alternatively, $a_{\overline{5}|}^{(12)} = 3.89499$.

As a first guess, the APR is approximately twice the flat rate of interest:

$$\text{Flat Rate} = \frac{\text{total interest}}{\text{loan} \times \text{years}} = \frac{5 \times 12 \times 427.90 - 20{,}000}{20{,}000 \times 5} = 5.67\%$$

So a good first guess for the APR is 11%.

$$i = 0.11 \quad \Rightarrow \quad 12 \times 427.90 a_{\overline{5}|}^{(12)} = £19{,}916$$

$$i = 0.10 \quad \Rightarrow \quad 12 \times 427.90 a_{\overline{5}|}^{(12)} = £20{,}342$$ [1]

Markers: award this mark for any correct evaluation on the way to 10.8% and 10.9%.

Alternatively:

$$i = 0.11 \quad \Rightarrow \quad a_{\overline{5}|}^{(12)} = 3.87872$$

$$i = 0.10 \quad \Rightarrow \quad a_{\overline{5}|}^{(12)} = 3.96154$$

Using interpolation:

$$i = 10\% + \frac{20{,}000 - 20{,}342}{19{,}916 - 20{,}342}(11\% - 10\%) = 10.8\%$$

Try $i = 10.8\%$ in the equation of value:

$$12 \times 427.90 a_{\overline{5}|}^{(12)} = 20{,}000.2$$ [½]

Alternatively, $a_{\overline{5}|}^{(12)} = 3.89503$.

Try $i = 10.9\%$ in the equation of value:

$$12 \times 427.90 a_{\overline{5}|}^{(12)} = 19{,}958.3$$ [½]

Alternatively, $a_{\overline{5}|}^{(12)} = 3.88686$.

Since the value obtained using 10.8% is closer to the actual value than that obtained using 10.9%, the APR is 10.8%. [1]

[Total 4]

Markers: in order to obtain full marks, students must show the expression evaluated at 10.8% and 10.9%, before stating the APR correct to the nearest 0.1%.

Solution X2.2

Valuing equities is covered in Chapter 13, Section 2, and the relationship between money rates of interest and real rates of interest is covered in Chapter 13, Section 3.3.

The effective annual money rate of return required is:

$$i = 1.02 \times 1.03 - 1 = 0.0506 \ \ \textit{ie} \ 5.06\%$$ [1]

Using the given growth assumption, future dividends will be:

- £0.25 in 2 months' time
- $£0.25 \times 1.015$ in 8 months' time
- $£0.25 \times 1.015^2$ in 14 months' time

and so on.

The price that should be paid by the investor is the present value of the future dividends:

$$\frac{0.25}{1.0506^{2/12}} + \frac{0.25 \times 1.015}{1.0506^{8/12}} + \frac{0.25 \times 1.015^2}{1.0506^{14/12}} + \cdots$$ [1½]

This is the sum to infinity of a geometric progression with $a = \dfrac{0.25}{1.0506^{2/12}}$ and $r = \dfrac{1.015}{1.0506^{1/2}}$.

Using the formula $\dfrac{a}{1-r}$, we see that the price for one share is:

$$\frac{\dfrac{0.25}{1.0506^{2/12}}}{1 - \dfrac{1.015}{1.0506^{1/2}}} = £25.45$$ [1½]

[Total 4]

Alternatively, the present value of the future dividends can be evaluated as:

$$\frac{0.25}{1.0506^{2/12}}\ddot{a}_{\overline{\infty}|@j} \quad \textit{where} \quad j=\frac{1.0506^{1/2}}{1.015}-1=0.98402\%$$

Solution X2.3

This question is based on Subject CT1, April 2006, Question 4.

Optional redemption dates are covered in Chapter 13, Section 1.7.

Carrying out the capital gains test:

$$i^{(2)}@5\%=2\left((1+i)^{\frac{1}{2}}-1\right)=4.939\%$$

$$(1-t_1)\frac{D}{R}=(1-0.2)\frac{6.25}{100}=5\% \qquad [1]$$

Since $i^{(2)}<(1-t_1)\frac{D}{R}$, there is a capital loss for the investor. In order to ensure that the investor always makes a return of at least 5% *pa*, we need to consider the investor's worst case scenario, which is that the bond is redeemed (*ie* the loss is incurred) at the earliest possible date, *ie* after 10 years. [1]

Letting P be the price for £100 nominal, the equation of value for P is:

$$P=6.25\times 0.8a_{\overline{10}|}^{(2)}+100v^{10} \qquad [1]$$

Calculating the necessary annuity:

$$a_{\overline{10}|}^{(2)}=\frac{1-1.05^{-10}}{2(1.05^{0.5}-1)}=7.817079 \qquad [1]$$

So the price per £100 nominal is:

$$P=39.085+61.391=£100.48 \qquad [1]$$

[Total 5]

Solution X2.4

Repayment loans are covered in Chapter 11, Sections 2, 3 and 5.

(i) ***Capital content***

The capital outstanding immediately after the 5th quarterly repayment is given by the present value at that time of the future repayments:

$$4\times 170.07a_{\overline{3.75}|}^{(4)}=£2,317 \qquad [1]$$

The interest content of the sixth repayment (quarter of a year later) is:

$$2{,}317 \times (1.05^{0.25} - 1) = £28.44$$

The capital content of the sixth repayment can be found by subtraction:

$$170.07 - 28.44 = £141.63$$ [1]

[Total 2]

Alternatively, working in quarters gives the capital outstanding after 5th repayment of:

$$170.07 a_{\overline{15}|@1.2272\%} = 170.07 \times 13.624 = £2{,}317$$

Interest content of sixth repayment:

$$1.2272\% \times 2{,}317 = £28.44$$

Capital content of sixth repayment:

$$170.07 - 28.44 = £141.63$$

(ii) ***Interest***

The capital outstanding at the end of the first year (*ie* with 4 years still to run) is:

$$4 \times 170.07 a_{\overline{4}|}^{(4)} = 4 \times 170.07 \times 3.6118 = £2{,}457.01$$ [½]

Similarly, the capital outstanding at the end of the second year (*ie* with 3 years still to run) is:

$$4 \times 170.07 a_{\overline{3}|}^{(4)} = 4 \times 170.07 \times 2.7738 = £1{,}886.95$$ [½]

So the capital repaid during the second year is:

$$2{,}457.01 - 1{,}886.95 = £570.06$$ [1]

Subtracting this from the total payment made during the year gives the total interest paid during the second year:

$$4 \times 170.07 - 570.06 = £110.22$$ [1]

[Total 3]

Alternatively, we could work in quarters:

$$170.07 a_{\overline{16}|@1.2272\%} = 170.07 \times 14.447 = £2{,}457.01$$

$$170.07 a_{\overline{12}|@1.2272\%} = 170.07 \times 11.095 = £1{,}886.95$$

Solution X2.5

The net present value of a project is covered in Chapter 12, Section 1, and the discounted payback period is covered in Chapter 12, Section 2.

(i) ***Net present value***

The net present value is:

$$NPV = -10{,}000 + 1{,}500 a_{\overline{5}|}^{(2)} + 3{,}000 v^5 a_{\overline{5}|}^{(2)} = -10{,}000 + 1{,}500 a_{\overline{5}|}^{(2)} \left(1 + 2v^5\right) \quad @8\% \qquad [1]$$

Now:

$$a_{\overline{5}|}^{(2)} = \frac{1 - 1.08^{-5}}{2(1.08^{0.5} - 1)} = 4.07103 \qquad [½]$$

So:

$$NPV = -10{,}000 + 1{,}500 \times 4.07103\left(1 + 2 \times 1.08^{-5}\right) = £4{,}418.56 \qquad [½]$$

[Total 2]

(ii) ***Discounted payback period (DPP)***

Since the total income in the first 5 years ($5 \times 1{,}500 = £7{,}500$) is less than the initial investment of £10,000, the DPP must fall in the final 5 years of the project.

Let n be the number of years of income payments needed to reach the DPP, *ie* n is the number of years of income payments until the net present value is greater than or equal to 0. So we have:

$$-10{,}000 + 1{,}500 a_{\overline{5}|}^{(2)} + 3{,}000 v^5 a_{\overline{n-5}|}^{(2)} \geq 0 \quad @8\% \qquad [2]$$

Using the value of $a_{\overline{5}|}^{(2)} = 4.07103$ from part (i), this becomes:

$$3{,}000 v^5 a_{\overline{n-5}|}^{(2)} \geq 3{,}893.46 \quad \Rightarrow \quad a_{\overline{n-5}|}^{(2)} \geq 1.90692$$

Therefore:

$$\frac{1 - 1.08^{-(n-5)}}{2(1.08^{0.5} - 1)} \geq 1.90692 \quad \Rightarrow \quad 1.08^{-(n-5)} \leq 0.85038$$

Taking logs, and reversing the inequality when we divide by $-\ln 1.08$, gives:

$$-(n-5)\ln 1.08 \leq \ln 0.85038 \quad \Rightarrow \quad n - 5 \geq 2.106 \quad \Rightarrow \quad n \geq 7.106 \qquad [2]$$

So, the project will move into profit with the first income payment received after time 7.106 years, *ie* the payment at time 7.5 years. So the DPP is 7.5 years. [1]

[Total 5]

Alternatively, we can consider the time when the accumulated profit is first greater than or equal to 0, ie:

$$-10{,}000(1.08)^n + 1{,}500(1.08)^{n-5} s_{\overline{5}|}^{(2)} + 3{,}000 s_{\overline{n-5}|}^{(2)} \geq 0$$

Solution X2.6

Index-linked bonds are covered in Chapter 13, Section 4.

(i) ***Index value on 1 July 2009***

The annual nominal coupon paid under this index-linked bond is £4 per £100 nominal. The coupon actually paid on 1 January 2014 was £4.31 per £100 nominal. So, taking into account the time lag of 6 months:

$$4.31 = 4 \times \frac{\text{Inflation index on 1 July 2013}}{\text{Inflation index on 1 July 2009}}$$

where 1 July 2013 is 6 months before the date of the coupon payment, and 1 July 2009 is 6 months before the date of issue of the bond. [½]

So the inflation index on 1 July 2009 was:

$$\frac{4}{4.31} \times \text{Inflation index on 1 July 2013} = \frac{4}{4.31} \times 125.0 = 116.0$$ [½]

[Total 1]

(ii) ***Amounts of coupon and redemption payments received***

Based on an annual nominal coupon of £4 per £100 nominal and a time lag for indexation of 6 months, the coupon payment received on 1 January 2015 was:

$$4 \times \frac{\text{Inflation index on 1 July 2014}}{\text{Inflation index on 1 July 2009}} = 4 \times \frac{128.2}{116.0} = £4.42$$ [1]

Similarly, the coupon payment received on 1 January 2016 was:

$$4 \times \frac{\text{Inflation index on 1 July 2015}}{\text{Inflation index on 1 July 2009}} = 4 \times \frac{131.3}{116.0} = £4.53$$ [1]

and the coupon payment received on 1 January 2017 was:

$$4 \times \frac{\text{Inflation index on 1 July 2016}}{\text{Inflation index on 1 July 2009}} = 4 \times \frac{134.8}{116.0} = £4.65$$ [1]

The redemption payment before indexation was £105 per £100 nominal, so the redemption payment actually received on 1 January 2017 was:

$$105\times\frac{\text{Inflation index on 1 July 2016}}{\text{Inflation index on 1 July 2009}}=105\times\frac{134.8}{116.0}=\pounds122.02$$ [1]

[Total 4]

(iii) ***Price paid***

Since we are given the investor's real rate of return, in order to calculate the price paid for the bond we need to work out the real (*ie* inflation-adjusted) cashflows as at 1 July 2014, the date of purchase.

The real (*ie* inflation-adjusted) value as at 1 July 2014 of the coupon payment of £4.42 on 1 January 2015 is:

$$4.42\times\frac{128.2}{130.9}$$ [½]

The real value as at 1 July 2014 of the coupon payment of £4.53 on 1 January 2016 is:

$$4.53\times\frac{128.2}{132.6}$$ [½]

The real value as at 1 July 2014 of the coupon payment of £4.65 and the redemption payment of £122.02 on 1 January 2017 is:

$$(4.65+122.02)\times\frac{128.2}{136.0}=126.67\times\frac{128.2}{136.0}$$ [1]

The price paid by the investor on 1 July 2014 is given by:

$$4.42\times\frac{128.2}{130.9}\times v^{0.5}+4.53\times\frac{128.2}{132.6}\times v^{1.5}+126.67\times\frac{128.2}{136.0}\times v^{2.5}$$ [1]

Using a real return of 5% *pa* effective, the price paid by the investor is:

$$4.42\times\frac{128.2}{130.9}\times1.05^{-0.5}+4.53\times\frac{128.2}{132.6}\times1.05^{-1.5}+126.67\times\frac{128.2}{136.0}\times1.05^{-2.5}$$

$$=\pounds113.99$$ [1]

[Total 4]

Solution X2.7

Repayment loans are covered in Chapter 11, Sections 2, 3 and 5.

(i) ***Amount of loan***

The loan amount is equal to the present value of the repayments made. Letting L denote the loan amount:

$$L = 1{,}100 a_{\overline{20}|} + 300 (Ia)_{\overline{20}|}$$ [1½]

Using values from the *Tables* at an annual effective interest rate of 7%, this gives:

$$L = 1{,}100 \times 10.5940 + 300 \times 88.1031 = £38{,}084.33$$ [1½]

[Total 3]

(ii) ***Capital outstanding immediately after the third instalment***

Working prospectively, the capital outstanding immediately after the third instalment has been paid is equal to the present value at time 3 years of the remaining instalments. The fourth instalment (due in one year's time) is £2,300 and thereafter the payments increase by £300 each year. So the capital outstanding immediately after the third instalment has been paid is:

$$2{,}000 a_{\overline{17}|} + 300 (Ia)_{\overline{17}|}$$ [1]

Using values from the *Tables* at an annual effective interest rate of 7%, this gives:

$$2{,}000 \times 9.7632 + 300 \times 72.3555 = £41{,}233.05$$ [1]

[Total 2]

Alternatively, working retrospectively, the capital outstanding immediately after the third instalment has been paid is equal to the accumulated value of the original loan amount minus the accumulated value of the past repayments:

$$38{,}084.33 \times 1.07^3 - 1{,}400 \times 1.07^2 - 1{,}700 \times 1.07 - 2{,}000 = £41{,}233.08$$

(iii) ***Explanation***

The capital outstanding after the third instalment has been paid is greater than the original loan amount. This is because the first few repayments are not enough to pay the interest accruing on the loan, so the capital outstanding initially increases. [1]

For example, the interest in the first year is 7% of £38,084.33, which is £2,665.90. This is greater than the first instalment of £1,400, so the unpaid interest is added to the capital outstanding, causing it to increase. [1]

[Total 2]

(iv) ***Quarterly payment***

The interest rate of 9% *pa* convertible half-yearly is equivalent to an annual effective interest rate of:

$$i=\left(1+\frac{i^{(2)}}{2}\right)^2-1=1.045^2-1=9.2025\% \qquad [1]$$

After the third instalment has been paid, the outstanding loan amount is £41,233.05 and 17 years of repayments remain. Letting Q denote the new quarterly repayment:

$$41{,}233.05=4Qa_{\overline{17}|}^{(4)} \ @9.2025\% \qquad [1]$$

Now:

$$a_{\overline{17}|}^{(4)}=\frac{1-1.092025^{-17}}{4\left(1.092025^{0.25}-1\right)}=8.71932$$

So:

$$Q=\frac{41{,}233.05}{4\times 8.71932}=£1{,}182.23 \qquad [1]$$

[Total 3]

Alternatively, we could work in quarters using a quarterly effective interest rate of $1.092025^{0.25}-1=2.22524\%$. *In this case:*

$$41{,}233.05=Qa_{\overline{68}|} \ @2.22524\% \ \Rightarrow \ Q=\frac{41{,}233.05}{34.87730}=£1{,}182.23$$

(v) ***Total interest paid***

The total interest paid is equal to the difference between the total repayments made and the total capital to be repaid.

There are 3 years of annual increasing repayments, followed by 17 years of level quarterly repayments, so the total of all the repayments made is:

$$1{,}400+1{,}700+2{,}000+17\times 4\times 1{,}182.23=£85{,}491.64 \qquad [1]$$

The capital to be repaid is the original loan amount of £38,084.33. So the total interest paid is:

$$85{,}491.64-38{,}084.33=£47{,}407.31 \qquad [1]$$

[Total 2]

Solution X2.8

Fixed-interest bonds are covered in Chapter 13, Section 1.

(i) ***Issue price***

The price (per £100 nominal) can be calculated using the equation of value:

$$P_1 = 0.65 \times 14 a_{\overline{10}|}^{(2)} + 110 v^{10} \quad @\ 9.5\%\ pa$$ [1]

$$= 0.65 \times 14 \times \frac{1-1.095^{-10}}{2(1.095^{0.5}-1)} + 110 \times 1.095^{-10}$$

$$= 0.65 \times 14 \times 6.4245 + 110 \times 1.095^{-10}$$

$$= £102.85$$ [2]

[Total 3]

Alternatively, we could work in half-years:

$$P_1 = 0.65 \times 7 a_{\overline{20}|} + 110 v^{20} \quad @\ 1.095^{1/2} - 1$$

(ii) ***Best price***

The remaining term is now 9 years. Carrying out a capital gains test:

$$i^{(2)} @10\% = 0.097618 \qquad (1-t_1)\frac{D}{R} = 0.75 \times \frac{14}{110} = 0.095455$$

Since $i^{(2)} > (1-t_1)\frac{D}{R}$ there *is* a capital gain. [1]

So the maximum price (per £100 nominal) this investor will be willing to pay is:

$$P_2 = 0.75 \times 14 a_{\overline{9}|}^{(2)} + 110 v^9 - 0.35(110 - P_2) v^9 \quad @ 10\%\ pa$$ [2]

$$= 10.5 a_{\overline{9}|}^{(2)} + 71.5 v^9 + 0.35 P_2 v^9$$

Rearranging gives:

$$P_2(1 - 0.35 v^9) = 10.5 a_{\overline{9}|}^{(2)} + 71.5 v^9$$

$$= 10.5 \times 5.8996 + 71.5 \times 0.42410$$

$$= 92.268$$ [1]

$$\Rightarrow P_2 = \frac{92.268}{0.85157} = £108.35 \text{ per } £100 \text{ nominal}$$ [1]

[Total 5]

Again, we could work in half-years.

(iii) ***Net running yield***

The net running yield is:

$$\frac{\text{net coupon rate}}{\text{price paid}} = \frac{0.75 \times 14}{108.35} = 9.69\% \qquad [1]$$

[Total 1]

Since the coupons are paid half-yearly, this running yield is convertible half-yearly.

(iv) ***Net yield***

For each £100 nominal, the original investor paid £102.85 at the outset, received coupon payments (taxed at 35%) for 1 year, then sold the stock for £108.35 (with no liability for CGT).

The investor's cashflows are shown on the timeline below:

		£108.35	
− £102.85	£4.55	£4.55	cashflows
0	½	1	years

So the equation of value is:

$$102.85 = 4.55v^{½} + 112.90v$$

ie $112.90v + 4.55v^{½} - 102.85 = 0$ [2]

This is a quadratic equation in $v^{½}$. Using the quadratic formula gives:

$$v^{½} = \frac{-4.55 \pm \sqrt{4.55^2 - 4 \times 112.90 \times -102.85}}{2 \times 112.90}$$

$$= \frac{-4.55 \pm 215.56}{225.80}$$

$$= 0.93452 \qquad \text{(or a negative value)} \qquad [1]$$

$$\Rightarrow \quad v = 0.93452^2 = 0.8733 \quad \Rightarrow \quad i = 14.5\% \qquad [1]$$

[Total 4]

Solution X2.9

This question is Subject CT1, September 2005, Question 11.

Equations of value are the subject of Chapter 10. The internal rate of return is covered in Chapter 12, Section 1, and the discounted payback period is covered in Chapter 12, Section 2.

(i)(a) ***Equation of value***

The equation of value equates the value of the payments made to the value of the payments received at an appropriate rate of interest. [1]

The values can be determined at any point in time, but it is common to equate present values, *ie* the values at time 0. [1]

(i)(b) ***Discounted payback period***

An investment project usually requires expenditure (or investment) in the early years before returns (or profits) emerge in the later years. So during the early years the 'project account' will be overdrawn and interest will be incurred at the borrowing rate. [1]

The discounted payback period is the time from the beginning of the project until the 'project account' balance first becomes positive. [1]
[Total 4]

Markers: As an alternative to this final point, students should be awarded one mark if they state that the discounted payback period ends at the first point in time such that the net present value (or accumulated value) of project cashflows up to that time is positive.

(ii) ***Checking the investment criteria***

Is the internal rate of return > 9% pa effective?

If the net present value at 9% is positive then the internal rate of return will be greater than 9%, and similarly if the net present value at 9% is negative then the internal rate of return will be less than 9%.

Cash outflows

The present value of the research and associated costs is:

$$1.5\bar{a}_{\overline{3}|9\%} = £3.96535m \qquad [1]$$

The present value of the rent is:

$$0.3v_{9\%}^{3}\ddot{a}_{\overline{12}|9\%}^{(4)} = £1.75112m \qquad [1]$$

The present value of the staff costs is:

$$v_{9\%}^3\left\{1+1.05v_{9\%}+1.05^2v_{9\%}^2+1.05^3v_{9\%}^3+\cdots+1.05^{11}v_{9\%}^{11}\right\}$$

$$=v_{9\%}^3\frac{1-\left(\frac{1.05}{1.09}\right)^{12}}{1-\frac{1.05}{1.09}}=£7.60691m \qquad [2]$$

Alternatively, the present value of the staff costs may be calculated using an annuity:

$$v_{9\%}^3\ddot{a}_{\overline{12}|@I\%} \quad \textit{where} \quad I=\frac{1.09}{1.05}-1=3.8095\%$$

Total present value of cash outflow is £13.32338m.

Cash inflows

The present value of the net sales income is:

$$v_{9\%}^3\bar{a}_{\overline{3}|9\%}+1.9v_{9\%}^6\bar{a}_{\overline{3}|9\%}+2.5v_{9\%}^9\bar{a}_{\overline{6}|9\%}=£10.42887m \qquad [2]$$

The present value of the proceeds from the sale of branch operation is:

$$8v_{9\%}^{15}=£2.19630m \qquad [1]$$

Total present value of cash inflow is £12.62517m.

So the net present value of the project at 9% *pa* is:

$$12.62517-13.32338=-£0.698m \qquad [1]$$

Since the net present value is negative at 9% interest, the $IRR<9\%$. So the criterion is *not* satisfied. [1]

Is the discounted payback period less than 12 years?

If the net present value of the cashflows up to time 12 (excluding the payments then due) is negative the discounted payback period is ≥ 12 years, and similarly if this is positive the discounted payback period is <12 years.

Present value of cash outflows up to time 12

Research *etc*:

$$1.5\bar{a}_{\overline{3}|7\%}=£4.07270m \qquad [1]$$

Rent:

$$0.3v_{7\%}^3\ddot{a}_{\overline{9}|7\%}^{(4)}=£1.66472m \qquad [1]$$

Staff costs:

$$v_{7\%}^3\left\{1+1.05v_{7\%}+1.05^2v_{7\%}^2+1.05^3v_{7\%}^3+\cdots+1.05^8v_{7\%}^8\right\}$$

$$=v_{7\%}^3\frac{1-\left(\frac{1.05}{1.07}\right)^9}{1-\frac{1.05}{1.07}}=£6.82069m \qquad [2]$$

Alternatively, the present value of the staff costs may be calculated using an annuity:

$$v_{7\%}^3\ddot{a}_{\overline{9}|@I\%} \quad \textit{where} \quad I=\frac{1.07}{1.05}-1=1.9048\%$$

Total present value of cash outflow up to time 12 is £12.55811m.

Present value of cash inflows up to time 12

Net sales income:

$$v_{7\%}^3\bar{a}_{\overline{3}|7\%}+1.9v_{7\%}^6\bar{a}_{\overline{3}|7\%}+2.5v_{7\%}^9\bar{a}_{\overline{3}|7\%}$$

$$=2.216356+3.437493+3.69213=£9.34598m \qquad [2]$$

So the net present value of the cashflows up to time 12 is:

$$9.34598-12.55811=-£3.212m\,. \qquad [1]$$

Since this is negative, the $DPP\geq 12$ years. The criterion is *not* satisfied. [1]

[Total 17]

Note that if you include the cash outflows at time 12, the net present value is –£3.934m, and the conclusion is the same.

Alternatively, if we work in terms of accumulated values:

$$AV(research)=9.17248$$

$$AV(rent)=3.74928$$

$$AV(staff)=15.36153$$

$$AV(sales)=21.04883$$

The accumulated value of cashflows just before time 12 is $-£7.234m$*, giving the same conclusion.*

Assignment X3 – Solutions

Markers: This document sets out one approach to solving each of the questions. Please give credit for other valid approaches. In particular, in 'trial and error' questions, full marks should be awarded for obtaining the correct final answer whatever method is used (eg 'table mode' on a calculator), so long as sufficient working is given.

Solution X3.1

Continuous-time forward rates are covered in Chapter 14, Section 2.

Let y_t denote the t-year spot rate, $f_{t,r}$ denote the annual forward rate of interest from time t to time $t+r$ and $F_{t,r}$ denote the continuous-time forward rate from time t to time $t+r$.

Then:

$$(1+f_{2,2})^2 = \frac{(1+y_4)^4}{(1+y_2)^2} = \frac{1.06^4}{1.04^2} = 1.16723$$

$\Rightarrow \quad f_{2,2} = 8.0385\%$ [1]

The continuous-time forward rate is the force of interest equivalent to the annual forward rate of interest:

$$F_{2,2} = \ln(1+f_{2,2}) = \ln 1.080385 = 7.73\%$$ [1]

[Total 2]

Solution X3.2

The formula for the EPV of a term assurance policy is covered in Chapter 16, Section 2.

The EPV of the benefits from this term assurance policy is:

$$EPV = 100{,}000 v\, q_{75} + 200{,}000 v^2\, p_{75}\, q_{76}$$ [1]

Using PFA92C20 mortality and an interest rate of 5% *pa* effective, this gives:

$$EPV = \frac{100{,}000}{1.05} \times 0.019478 + \frac{200{,}000}{1.05^2} \times (1-0.019478) \times 0.022127 = 5{,}790.83$$ [1]

[Total 2]

Solution X3.3

Guaranteed annuities-due are covered in Chapter 17, Section 8, and life annuities payable more frequently than annually are covered in Chapter 18, Section 4.

The EPV of the annuity payments is:

$$17{,}000\ddot{a}^{(12)}_{\overline{60:10|}} = 17{,}000\left(\ddot{a}^{(12)}_{\overline{10|}} + v^{10}\,{}_{10}p_{60}\,\ddot{a}^{(12)}_{70}\right) \quad [1]$$

Now:

$$\ddot{a}^{(12)}_{\overline{10|}} = \frac{1-1.06^{-10}}{12(1-1.06^{-1/12})} = 7.5972 \quad [½]$$

Alternatively, we could use values from the Tables, and the formula $\ddot{a}^{(12)}_{\overline{10|}} = \dfrac{i}{d^{(12)}}a_{\overline{10|}}$.

Also:

$$\ddot{a}^{(12)}_{70} = \ddot{a}_{70} - \frac{11}{24} = 9.140 - \frac{11}{24} = 8.6817 \quad [½]$$

and:

$${}_{10}p_{60} = \frac{l_{70}}{l_{60}} = \frac{8{,}054.0544}{9{,}287.2164} = 0.86722 \quad [½]$$

So, the EPV of the annuity payments is:

$$17{,}000\left(7.5972 + 1.06^{-10} \times 0.86722 \times 8.6817\right) = 200{,}620 \text{ (to 5 SF)} \quad [½]$$

[Total 3]

Solution X3.4

Volatility and convexity are covered in Chapter 14, Section 4.

The present value of the payments can be found from:

$$PV(i) = 1{,}000(1+i)^{-5} + 5{,}000(1+i)^{-10} \quad [½]$$

Evaluating this when $i = 0.1$ gives £2,548.64. [½]

The volatility is found from:

$$vol(i) = -\frac{PV'(i)}{PV(i)} = -\frac{-5\times 1{,}000(1+i)^{-6} - 10\times 5{,}000(1+i)^{-11}}{PV(i)} \quad [1]$$

Evaluating this when $i = 0.1$ gives:

$$vol(0.1) = \frac{20{,}347.06}{2{,}548.64} = 7.9835 \qquad [½]$$

Alternatively, we could calculate the DMT and then use $vol(i) = v \times DMT(i)$ *:*

$$DMT(i) = \frac{5 \times 1{,}000v^5 + 10 \times 5{,}000v^{10}}{PV(i)}$$

Evaluating this when $i = 0.1$ *gives:*

$$DMT(0.1) = \frac{22{,}381.77}{2{,}548.64} = 8.78186 \quad \Rightarrow \quad vol(0.1) = 1.1^{-1} \times 8.78186 = 7.9835$$

However, this method is not advisable here since we need to calculate the convexity by differentiation, so it is more efficient to calculate the volatility by differentiation, too.

The convexity is found from:

$$conv(i) = \frac{PV''(i)}{PV(i)} = \frac{(-5)(-6) \times 1{,}000(1+i)^{-7} + (-10)(-11) \times 5{,}000(1+i)^{-12}}{PV(i)} \qquad [1]$$

Evaluating this when $i = 0.1$ gives:

$$conv(0.1) = \frac{190{,}641.69}{2{,}548.64} = 74.8014 \qquad [½]$$

[Total 4]

Solution X3.5

Deferred annuities are introduced in Chapter 17, Section 6, and increasing term assurances are covered in Chapter 19, Section 3.

There are two parts to the benefit. Let's consider the deferred annuity first.

$$\text{EPV annuity benefit} = 15{,}000 \,{}_{25|}\bar{a}_{65} = 15{,}000 \frac{D_{65}}{D_{[40]}} \bar{a}_{65} \qquad [½]$$

The D/D factor is calculated using AM92 Select mortality. So:

$$\frac{D_{65}}{D_{[40]}} = \frac{689.23}{2{,}052.54} = 0.33579 \qquad [½]$$

The annuity factor is calculated using PMA92C20 mortality. So:

$$\bar{a}_{65} \approx \ddot{a}_{65} - 0.5 = 13.666 - 0.5 = 13.166 \qquad [½]$$

Hence:

$$\text{EPV annuity benefit} = 15{,}000 \times 0.33579 \times 13.166 = 66{,}315.90$$ [½]

Let's now consider the death benefit:

$$\begin{aligned}\text{EPV death benefit} &= 4{,}500(IA)^{\;1}_{[40]:\overline{25}|} \\ &= 4{,}500\left\{(IA)_{[40]} - \frac{D_{65}}{D_{[40]}}\left[(IA)_{65} + 25A_{65}\right]\right\} \\ &= 4{,}500\left[7.95835 - 0.33579(7.89442 + 25 \times 0.52786)\right] \\ &= 4{,}500 \times 0.87615 \\ &= 3{,}942.68\end{aligned}$$ [1½]

So the total EPV of the benefits provided is:

$$66{,}315.90 + 3{,}942.68 = £70{,}259$$ [½]

[Total 4]

Solution X3.6

The present value random variable and variance of a whole life assurance with benefits payable immediately on death are covered in Chapter 16, Section 6.

(i) ***Present value random variable***

The present value random variable is $5{,}000v^{T_{42}}$. [1]

[Total 1]

(ii) ***Variance formula***

For the variance we can write:

$$\text{var}\left(5{,}000v^{T_{42}}\right) = 5{,}000^2\,\text{var}\left(v^{T_{42}}\right)$$ [½]

Applying the standard formula for the variance of $\text{var}(X) = E(X^2) - (E(X))^2$ gives:

$$\begin{aligned}\text{var}\left(v^{T_{42}}\right) &= E\left(\left(v^{T_{42}}\right)^2\right) - \left(E\left(v^{T_{42}}\right)\right)^2 \\ &= E\left(\left(v^2\right)^{T_{42}}\right) - \left(E\left(v^{T_{42}}\right)\right)^2 \\ &= {}^2\bar{A}_{42} - \left(\bar{A}_{42}\right)^2\end{aligned}$$ [1]

where ${}^2\bar{A}_{42}$ is calculated at the interest rate $(1+i)^2 - 1$. [½]

Putting these results together gives the variance formula stated in the question:

$$\text{var}\left(5{,}000v^{T_{42}}\right) = 5{,}000^2\left({}^2\bar{A}_{42} - \left(\bar{A}_{42}\right)^2\right)$$

[Total 2]

(iii) ***Variance calculation***

Evaluating this using an interest rate of 4% gives:

$$5{,}000^2\left((1+i_1)^{0.5} \times {}^2A_{42} - \left(1.04^{0.5} \times A_{42}\right)^2\right)$$ [1]

where $i_1 = 1.04^2 - 1$.

So the variance is:

$$5{,}000^2\left(1.04 \times 0.07758 - \left(1.04^{0.5} \times 0.24787\right)^2\right) = 419{,}652$$ [1]

[Total 2]

Solution X3.7

Life annuities that increase at a constant compound rate are covered in Chapter 19, Section 2.

(i) ***Present value random variable***

If, for example, the policyholder dies in the fourth policy year (*ie* if $K_{65} = 3$), the present value of the benefits is:

$$X = 10{,}000\left(v + 1.03v^2 + 1.03^2v^3\right) = 10{,}000v\left(1 + 1.03v + 1.03^2v^2\right)$$

With $i = 0.03$, this simplifies to:

$$X = 10{,}000v \times 3$$

So, in general, the present value random variable is:

$$X = 10{,}000\, v\, K_{65} = \frac{10{,}000K_{65}}{1.03}$$ [2]

[Total 2]

(ii) ***Expected present value***

$$E(X) = \frac{10{,}000 \times E(K_{65})}{1.03} = \frac{10{,}000e_{65}}{1.03} = \frac{10{,}000 \times 16.645}{1.03} = £161{,}602$$ [1]

[Total 1]

(iii) ***Probability***

From (i), the present value random variable is $X = \dfrac{10{,}000 K_{65}}{1.03}$. So:

$$P(X > 250{,}000) = P\left(K_{65} > \frac{250{,}000 \times 1.03}{10{,}000}\right)$$
$$= P(K_{65} > 25.75)$$
$$= P(K_{65} \geq 26) \qquad [1]$$

since K_{65} can only take non-negative integer values.

Assuming AM92 Ultimate mortality:

$$P(K_{65} \geq 26) = {}_{26}p_{65} = \frac{l_{91}}{l_{65}} = \frac{1{,}376.1906}{8{,}821.2612} = 0.15601 \qquad [1]$$

[Total 2]

Solution X3.8

Calculating probabilities for non-integer ages is covered in Chapter 15, Section 3.

The survival probability can be written as:

$${}_{2}p_{63.25} = {}_{0.75}p_{63.25} \times p_{64} \times {}_{0.25}p_{65} \qquad [½]$$

From the *Tables*:

$$p_{64} = 1 - q_{64} = 1 - 0.02199 = 0.97801 \qquad [½]$$

We can alternatively calculate p_{64} *as* $\dfrac{l_{65}}{l_{64}}$.

(a) ***Uniform distribution of deaths (UDD)***

Under the UDD assumption:

$${}_{0.25}p_{65} = 1 - 0.25 q_{65} = 1 - 0.25 \times 0.02447 = 0.99388 \qquad [1]$$

and:

$${}_{0.75}p_{63.25} = \frac{p_{63}}{{}_{0.25}p_{63}} = \frac{p_{63}}{1 - 0.25 q_{63}} = \frac{0.98035}{1 - 0.25 \times 0.01965} = 0.98519 \qquad [1]$$

So:

$${}_{2}p_{63.25} = 0.98519 \times 0.97801 \times 0.99388 = 0.95763 \qquad [½]$$

Alternative solutions

We could alternatively have used the UDD formula ${}_{t-s}q_{x+s} = \dfrac{(t-s)q_x}{1-sq_x}$ *to give:*

$${}_{0.75}q_{63.25} = \frac{0.75q_{63}}{1-0.25q_{63}} = \frac{0.75 \times 0.01965}{1-0.25 \times 0.01965} = 0.01481$$

and hence:

$${}_{0.75}p_{63.25} = 1 - 0.01481 = 0.98519$$ *[1]*

Another alternative is to use:

$${}_{2}p_{63.25} = \frac{l_{65.25}}{l_{63.25}}$$ *[1]*

and interpolate between the life table values from ELT15 (Males). We have:

$$\begin{aligned} l_{63.25} &= 0.25 \times l_{64} + 0.75 \times l_{63} \\ &= 0.25 \times 81{,}076 + 0.75 \times 82{,}701 \\ &= 82{,}294.75 \end{aligned}$$ *[½]*

$$\begin{aligned} l_{65.25} &= 0.25 \times l_{66} + 0.75 \times l_{65} \\ &= 0.25 \times 77{,}353 + 0.75 \times 79{,}293 \end{aligned}$$

and: $= 78{,}808$ *[½]*

So: $${}_{2}p_{63.25} = \frac{78{,}808}{82{,}294.75} = 0.95763$$ *[½]*

(b) ***Constant force of mortality***

Under the constant force of mortality assumption:

$${}_{0.25}p_{65} = (p_{65})^{0.25} = (1-q_{65})^{0.25} = 0.97553^{0.25} = 0.99383$$ [1]

and:

$${}_{0.75}p_{63.25} = (p_{63})^{0.75} = (1-q_{63})^{0.75} = (0.98035)^{0.75} = 0.98523$$ [1]

So:

$${}_{2}p_{63.25} = 0.98523 \times 0.97801 \times 0.99383 = 0.95761$$ [½]

[Total 6]

Solution X3.9

Deferred probabilities of death are covered in Chapter 15, Section 2; life annuities payable more frequently than annually are covered in Chapter 18, Section 4, and increasing life annuities are covered in Chapter 19, Section 3.

(i) ***Deferred probability***

${}_{3|}q_{[55]+1}$ is the probability that a life aged exactly 56, who entered the select population 1 year ago, dies between the ages of 59 and 60. [1]

Using AM92 mortality, we have:

$$ {}_{3|}q_{[55]+1} = \frac{d_{59}}{l_{[55]+1}} = \frac{66.7876}{9{,}513.9375} = 0.00702 $$ [1]

[Total 2]

Alternatively: $ {}_{3|}q_{[55]+1} = {}_{3}p_{[55]+1}\, q_{59} = \frac{l_{59}}{l_{[55]+1}} q_{59} = \frac{9{,}354.0040}{9{,}513.9375} \times 0.007140 = 0.00702 $

(ii)(a) $\ddot{a}^{(4)}_{[40]:\overline{5|}}$

We can calculate this as follows:

$$ \begin{aligned} \ddot{a}^{(4)}_{[40]:\overline{5|}} &= \ddot{a}^{(4)}_{[40]} - v^5\, {}_{5}p_{[40]}\, \ddot{a}^{(4)}_{45} \\ &\approx \left(\ddot{a}_{[40]} - \tfrac{3}{8}\right) - v^5 \frac{l_{45}}{l_{[40]}}\left(\ddot{a}_{45} - \tfrac{3}{8}\right) \\ &= \left(15.494 - \tfrac{3}{8}\right) - 1.06^{-5} \times \frac{9{,}801.3123}{9{,}854.3036}\left(14.850 - \tfrac{3}{8}\right) \\ &= 4.361 \end{aligned} $$ [2]

(ii)(b) $(I\bar{a})_{70:\overline{10|}}$

We can calculate this as follows:

$$ \begin{aligned} (I\bar{a})_{70:\overline{10|}} &= (I\bar{a})_{70} - v^{10}\, {}_{10}p_{70}\left((I\bar{a})_{80} + 10\bar{a}_{80}\right) \\ &\approx \left((I\ddot{a})_{70} - \tfrac{1}{2}\ddot{a}_{70}\right) - v^{10}\frac{l_{80}}{l_{70}}\left(\left((I\ddot{a})_{80} - \tfrac{1}{2}\ddot{a}_{80}\right) + 10\left(\ddot{a}_{80} - \tfrac{1}{2}\right)\right) \\ &= \left(67.198 - \tfrac{1}{2} \times 9.140\right) \\ &\qquad - 1.06^{-10} \times \frac{5{,}266.4604}{8{,}054.0544}\left(\left(32.860 - \tfrac{1}{2} \times 6.271\right) + 10\left(6.271 - \tfrac{1}{2}\right)\right) \\ &= 30.703 \end{aligned} $$ [2]

[Total 4]

Solution X3.10

Calculating probabilities of death and survival is covered in Chapter 15, Section 2.

There are three possible outcomes at the end of 20 years:

- both (30) and (40) alive
- exactly one of (30) and (40) alive
- neither of (30) and (40) alive.

The probabilities of these events and the present value of the amount due to the trust in each case are:

Event	Present Value	Probability
(30), (40) alive	$\frac{20{,}000}{3}v_{4\%}^{20} = 3{,}042.58$	${}_{20}p_{30} \times {}_{20}p_{40} = 0.9785 \times 0.9423$ $= 0.9220$
only one of (30), (40) alive	$\frac{20{,}000}{2}v_{4\%}^{20} = 4{,}563.87$	${}_{20}p_{30}(1 - {}_{20}p_{40}) + {}_{20}p_{40}(1 - {}_{20}p_{30})$ $= 0.9785 \times 0.0577 + 0.9423 \times 0.0215$ $= 0.0767$
(30), (40) both dead	$20{,}000v_{4\%}^{20} = 9{,}127.74$	$(1 - {}_{20}p_{30})(1 - {}_{20}p_{40})$ $= 0.0577 \times 0.0215 = 0.0012$

[1 in total for the 3 present values]
[1 mark for each probability – 3 marks in total]

So the expected value is:

$$3{,}042.58 \times 0.9220 + 4{,}563.87 \times 0.0767 + 9{,}127.74 \times 0.0012 = 3{,}166$$ [1]

and the variance is:

$$3{,}042.58^2 \times 0.9220 + 4{,}563.87^2 \times 0.0767 + 9{,}127.74^2 \times 0.0012 - 3{,}166^2 = 207{,}578$$ [2]

[Total 7]

Solution X3.11

Conventional with-profits policies are covered in Chapter 19, Section 4.

(i) ***Four bonus distribution methods***

There are three methods of declaring a regular reversionary bonus, whereby the sum assured is increased and, once increased, cannot be decreased. [½]

- Simple reversionary bonus: the rate of bonus each year is a percentage of the initial sum assured. [½]
- Compound reversionary bonus: the rate of bonus each year is a percentage of the initial sum assured plus previously declared reversionary bonuses. [½]
- Super-compound reversionary bonus: there are two rates of bonus. One is applied to the initial sum assured, the other is applied to the previously declared bonuses. [½]

In addition, there is the terminal bonus, whereby the sum assured is increased at maturity or on earlier claim. The terminal bonus rate is normally a percentage of final sum assured. [1]
[Total 3]

(ii) ***EPV of benefits***

Assuming that deaths occur uniformly over each year of age (so that the death benefit is payable on average, halfway through each year), the EPV of the benefits from this endowment assurance is:

$$50{,}000\left(v^{0.5}q_{50}+1.0192308\,v^{1.5}\,{}_{1|}q_{50}+\cdots+1.0192308^{14}\,v^{14.5}\,{}_{14|}q_{50}\right)$$
$$+50{,}000(1.0192308)^{15}\,v^{15}\,{}_{15}p_{50} \qquad [1]$$

where $v=\dfrac{1}{1.06}$.

We can rewrite this as:

$$\frac{50{,}000}{1.0192308v^{0.5}}\left(1.0192308\,v q_{50}+1.0192308^{2}\,v^{2}\,{}_{1|}q_{50}+\cdots+1.0192308^{15}\,v^{15}\,{}_{14|}q_{50}\right)$$
$$+50{,}000(1.0192308)^{15}\,v^{15}\,{}_{15}p_{50}$$

Now, making the substitution that:

$$1.0192308v = \frac{1.0192308}{1.06} = \frac{1}{1.04} = V$$

we have:

$$\frac{50{,}000}{1.0192308v^{0.5}}\left(Vq_{50} + V^2\,{}_{1|}q_{50} + \cdots + V^{15}\,{}_{14|}q_{50}\right) + 50{,}000V^{15}\,{}_{15}p_{50}$$

$$= \frac{50{,}000 \times 1.06^{0.5}}{1.0192308} A^{1}_{50:\overline{15|}}{}^{@4\%} + 50{,}000 A_{50:\overline{15|}}^{\;\;\;1}{}^{@4\%}$$ [1½]

Now:

$$A_{50:\overline{15|}}^{\;\;\;1}{}^{@4\%} = \frac{D_{65}}{D_{50}} = \frac{689.23}{1{,}366.61} = 0.50434$$ [½]

and: $$A^{1}_{50:\overline{15|}}{}^{@4\%} = A_{50:\overline{15|}} - A_{50:\overline{15|}}^{\;\;\;1} = 0.56719 - 0.50434 = 0.06285$$ [½]

Alternatively:

$$A^{1}_{50:\overline{15|}}{}^{@4\%} = A_{50} - \frac{D_{65}}{D_{50}} A_{65} = 0.32907 - \frac{689.23}{1{,}366.61} \times 0.52786 = 0.06285$$

So, the EPV of the benefits from this endowment assurance is:

$$\frac{50{,}000 \times 1.06^{0.5}}{1.0192308} \times 0.06285 + 50{,}000 \times 0.50434 = 28{,}391$$ [½]

[Total 4]

Solution X3.12

Immunisation is covered in Chapter 14, Section 4.

(i) ***Timing and amount of payments***

Let the liabilities consist of a lump sum of amount X payable at time t and a lump sum of amount $2X$ payable at time $t+5$. Then the present value is:

$$Xv^t + 2Xv^{t+5} = 75{,}000$$ [1]

The discounted mean term is given by:

$$\frac{Xtv^t + 2X(t+5)v^{t+5}}{75{,}000} = 8 \quad \Rightarrow \quad \frac{Xv^t\left[t + 2(t+5)v^5\right]}{75{,}000} = 8$$ [1]

Rearranging the present value equation gives:

$$Xv^t = \frac{75{,}000}{(1+2v^5)}$$

and substituting this into the DMT expression gives

$$\frac{t+2(t+5)v^5}{1+2v^5} = 8 \quad \Rightarrow \quad t = \frac{8+6v^5}{1+2v^5} = 5.0044 \text{ years}$$ [2]

Alternatively:

$$DMT = \frac{Xtv^t + 2X(t+5)v^{t+5}}{Xv^t + 2Xv^{t+5}} = 8 \quad \Rightarrow \quad \frac{t+2(t+5)v^5}{1+2v^5} = 8 \quad \Rightarrow \quad t = \frac{8+6v^5}{1+2v^5} = 5.0044 \text{ years}$$

or we could have calculated the volatility and used $DMT = (1+i) \times vol$.

Substituting back into the present value equation:

$$X = \frac{75{,}000}{(1+2v^5)v^{5.0044}} = £40{,}245$$ [1]

So, the payments are £40,245 after 5 years and £80,491 after 10 years. [1]

[Total 6]

(ii) ***Redington's conditions***

The present value of the assets is:

$$119{,}540v^8 = 75{,}001$$

ie (almost exactly) the same as the present value of the liabilities. [½]

We also know that the durations of the assets and the liabilities are the same (both 8 years). [½]

So Redington's first two conditions for immunisation are satisfied. [½]

However, the convexity (or spread) of the assets is less than the convexity of the liabilities because the single asset cashflow falls between the liability cashflows. [½]

Redington's third condition requires the opposite to be true for immunisation, so the fund is not immunised. [½]

In fact, this portfolio is 'reverse immunised', *ie* a small change in interest rates in either direction will lead to a *deficit*. [½]

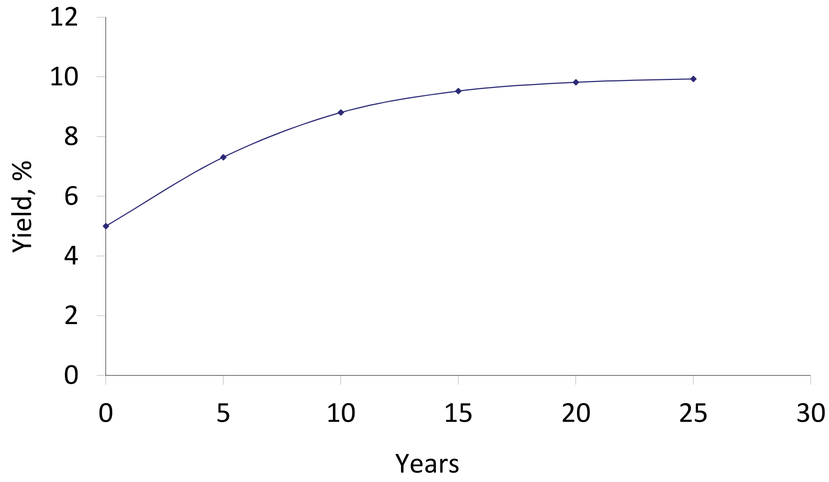
[Total 3]

Solution X3.13

Yield curve theories and par yields are covered in Chapter 14, Section 3.

(i) ***Yield curve***

The yield curve (showing the spot rates as a function of term) looks like this:



1 mark for general shape of yield curve
1 mark for 5% as the initial value
1 mark for 10% indicated as the asymptotic value
[Total 3]

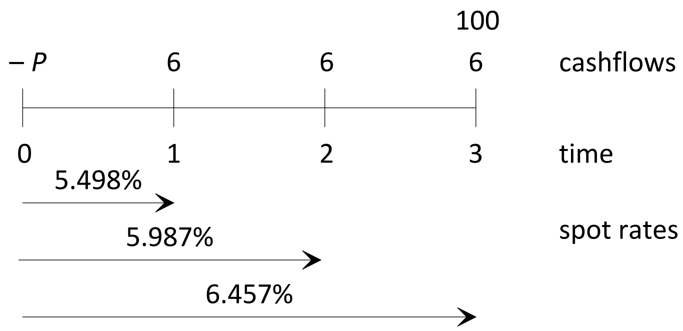
(ii) ***Liquidity preference theory***

The liquidity preference theory states that, as a general rule, investors prefer to hold stocks that they are not 'locked into' for a long period. Investors are therefore prepared to pay more for shorter, more liquid, stocks. [1]

This implies that the yields obtainable on these shorter stocks will be lower. Hence the yield curve will normally have an upward slope. [1]
[Total 2]

(iii)(a) ***Price of the bond***

The cashflow diagram with the spot rates for this bond is:



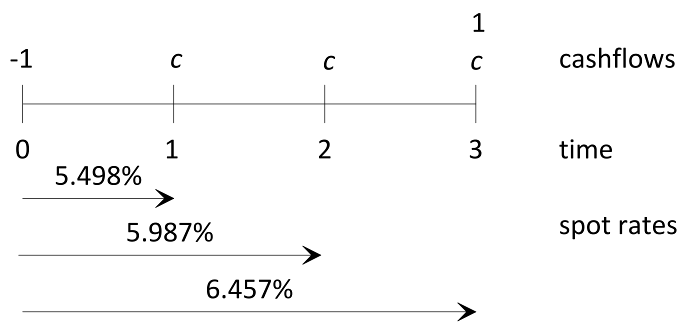
Award 1 mark for calculating the correct spot rates. [1]

The price for £100 nominal of the bond, P, is found from the equation of value:

$$P = 6 \times 1.05498^{-1} + 6 \times 1.05987^{-2} + 106 \times 1.06457^{-3} = £98.89 \quad [1]$$

(iii)(b) ***Par yield***

Let the 3-year par yield be *c*. Then the cashflow diagram is:



The equation of value is:

$$1 = c \times 1.05498^{-1} + c \times 1.05987^{-2} + (1+c) \times 1.06457^{-3} \quad [1]$$

So: $$c = \frac{1 - 1.06457^{-3}}{1.05498^{-1} + 1.05987^{-2} + 1.06457^{-3}} = 6.42\% \quad [1]$$

[Total 4]

Solution X3.14

Temporary annuities-due are covered in Chapter 17, Section 5, and premium conversion equations are covered in Chapter 18, Section 3.

(i) ***Description of benefit***

The benefit is an annuity of 10,000 *pa* payable annually in advance for a maximum of 10 years, but ceasing on earlier death (*ie* a temporary annuity-due). [1]

[Total 1]

(ii) ***Proof***

We can write:

$$\ddot{a}_{x:\overline{n}|} = E\left(\ddot{a}_{\overline{\min\{K+1,n\}}|}\right) = E\left(\frac{1-v^{\min\{K+1,n\}}}{d}\right) = \frac{1-E\left(v^{\min\{K+1,n\}}\right)}{d} = \frac{1-A_{x:\overline{n}|}}{d}$$ [1½]

Rearranging this gives the required result:

$$A_{x:\overline{n}|} = 1 - d\,\ddot{a}_{x:\overline{n}|}$$ [½]

[Total 2]

(iii) ***Expected present value and standard deviation***

The expected present value is:

$$\begin{aligned}10{,}000\,\ddot{a}_{x:\overline{10}|} &= 10{,}000\sum_{k=0}^{9} v^k\,{}_kp_x \\ &= 10{,}000\sum_{k=0}^{9} e^{-0.04k}\,e^{-0.02k} \\ &= 10{,}000\left(\frac{1-e^{-0.06\times 10}}{1-e^{-0.06}}\right) \\ &= 77{,}476.56\end{aligned}$$ [2]

Here we have recognised that the terms being summed form a geometric progression with first term $a=1$ and common ratio $r=e^{-0.06}$, and have used the result that the sum of the first n terms of a geometric series $a+ar+ar^2+\cdots+ar^{n-1}$ is:

$$\frac{a(1-r^n)}{1-r}$$

The variance of the present value random variable is:

$$\begin{aligned}\text{var}(W) &= 10{,}000^2\,\text{var}\left[\ddot{a}_{\overline{\min\{K+1,10\}}|}\right]\\ &= 10{,}000^2\,\text{var}\left[\frac{1-v^{\min\{K+1,10\}}}{d}\right]\\ &= \frac{10{,}000^2}{d^2}\text{var}\left[v^{\min\{K+1,10\}}\right]\end{aligned}$$ [1]

$v^{\min\{K+1,10\}}$ is the present value of a benefit of 1 paid at time 10 or at the end of the year of death of (x), whichever is sooner. So it is the present value of a 10-year endowment assurance on (x), and:

$$\text{var}\left[v^{\min\{K+1,10\}}\right] = {}^2A_{x:\overline{10}|} - \left(A_{x:\overline{10}|}\right)^2$$ [1]

We can calculate the values of these assurances using premium conversion:

$$A_{x:\overline{10}|} = 1 - d\,\ddot{a}_{x:\overline{10}|} = 1-\left(1-e^{-0.04}\right)\times 7.747656 = 0.696210$$ [1]

Since ${}^2A_{x:\overline{10}|}$ is equal to $A_{x:\overline{10}|}$ evaluated using a force of interest of $2\delta = 0.08$, and:

$$\ddot{a}_{x:\overline{10}|}@(\delta = 0.08) = \sum_{k=0}^{9} e^{-0.08k}\,e^{-0.02k} = \frac{1-e^{-0.1\times 10}}{1-e^{-0.1}} = 6.642533$$ [1]

it follows that:

$${}^2A_{x:\overline{10}|} = 1-\left(1-e^{-0.08}\right)\times 6.642533 = 0.489298$$ [1]

So:

$$\text{var}\left[v^{\min\{K+1,10\}}\right] = 0.489298 - 0.696210^2 = 0.004589$$

The standard deviation of the present value random variable is therefore:

$$\frac{10{,}000}{d}\times\sqrt{0.004589} = \frac{10{,}000}{\left(1-e^{-0.04}\right)}\times\sqrt{0.004589} = 17{,}277$$ [1]

[Total 8]

Assignment X4 – Solutions

Markers: This document sets out one approach to solving each of the questions. Please give credit for other valid approaches. In particular, in 'trial and error' questions, full marks should be awarded for obtaining the correct final answer whatever method is used (eg 'table mode' on a calculator), so long as sufficient working is given.

Solution X4.1

Net premium reserves are covered in Chapter 21, Section 6.

Using the formula for the net premium reserve for an endowment assurance (from page 37 of the *Tables*):

$$ {}_2V = 20{,}000\left(1-\frac{\ddot{a}_{62:\overline{3}|}}{\ddot{a}_{60:\overline{5}|}}\right) = 20{,}000\left(1-\frac{2.805}{4.390}\right) = £7{,}221 \qquad [2]$$

[Total 2]

Alternatively, we could approach this by first calculating the net premium, NP:

$$NP\,\ddot{a}_{60:\overline{5}|} = 20{,}000A_{60:\overline{5}|} \quad\Rightarrow\quad NP = \frac{20{,}000\times 0.75152}{4.390} = £3{,}423.78$$

and then calculating the net premium reserve at time 2:

$$ {}_2V = 20{,}000A_{62:\overline{3}|} - 3{,}423.78\,\ddot{a}_{62:\overline{3}|} = 20{,}000\times 0.84123 - 3{,}423.78\times 2.805 = £7{,}221$$

Solution X4.2

The integral expression for the EPV of benefits from a term assurance that makes a payment immediately on death appears in Chapter 16, Section 6.

The single premium, P, is equal to the EPV of the benefits, which we can write as an integral:

$$P = 200{,}000\int_0^1 {}_tp_{65}\,\mu_{65+t}\,e^{-\delta t}\,dt \qquad [1]$$

Since the force of mortality μ is assumed to be constant between ages 65 and 66, we can calculate it directly using the value of $p_{65} = 1 - q_{65}$ from the *Tables:*

$$\mu = -\ln\left(p_{65}\right) = -\ln\left(1-0.02447\right) = 0.0247744 \qquad [1]$$

We cannot simply take the value of μ_{65} direct from the Tables, as this is the value of the force of mortality at exact age 65, not the constant force of mortality that is equivalent to ELT15 (Males) mortality over the year.

Substituting these values into the first integral gives:

$$P = 200{,}000 \times \int_0^1 e^{-\mu t} \mu e^{-\delta t} dt = 200{,}000 \times 0.0247744 \int_0^1 e^{-(0.0247744+0.025)t} dt \qquad [1]$$

Carrying out the integration gives:

$$P = 200{,}000 \times \frac{0.0247744}{0.0497744}\left(1 - e^{-0.0497744}\right) = 4{,}834 \qquad [1]$$

[Total 4]

Solution X4.3

Calculating probabilities based on the order of death of two lives is covered in Chapter 23, Section 1.

Both lives must survive for the 5-year deferred period. So we can write:

$$_{5|4}q^{\ 1}_{[61]:[61]} = {}_5p_{[61]:[61]} \times {}_4q^{\ 1}_{66:66} \qquad [1]$$

Now, using the symmetry of the situation (both lives are the same age), we can write this in terms of single life functions as follows:

$$\begin{aligned} _{5|4}q^{\ 1}_{[61]:[61]} &= \left({}_5p_{[61]}\right)^2 \times \frac{1}{2}\,{}_4q_{66:66} \\ &= \left({}_5p_{[61]}\right)^2 \times \frac{1}{2}\left(1 - {}_4p_{66:66}\right) \\ &= \left({}_5p_{[61]}\right)^2 \times \frac{1}{2}\left[1 - \left({}_4p_{66}\right)^2\right] \\ &= \left(\frac{l_{66}}{l_{[61]}}\right)^2 \times \frac{1}{2}\left[1 - \left(\frac{l_{70}}{l_{66}}\right)^2\right] \end{aligned} \qquad [2]$$

Evaluating this using AM92 mortality gives:

$$_{5|4}q^{\ 1}_{[61]:[61]} = \left(\frac{8{,}695.6199}{9{,}184.9687}\right)^2 \times \frac{1}{2}\left[1 - \left(\frac{8{,}054.0544}{8{,}695.6199}\right)^2\right] = 0.06369 \qquad [1]$$

[Total 4]

Solution X4.4

This question is Subject CT5, September 2005, Question 6.

Temporary joint life annuities are covered in Chapter 23, Section 4.

The symbol $\ddot{a}^{(12)}_{60:50:\overline{20}|}$ represents the expected present value of an annuity of 1 *pa* payable monthly in advance while a life aged exactly 60 and a life aged exactly 50 are both alive, with payments being made for a maximum of 20 years from the outset. [2]

Using PMA92C20/PFA92C20 mortality and 4% *pa* interest:

$$\ddot{a}^{(12)}_{60:50:\overline{20}|} = \ddot{a}^{(12)}_{60:50} - v^{20}\ {}_{20}p^{(m)}_{60}\ {}_{20}p^{(f)}_{50}\ \ddot{a}^{(12)}_{80:70} \qquad [1]$$

$$\approx \left(\ddot{a}_{60:50} - \frac{11}{24}\right) - v^{20}\ {}_{20}p^{(m)}_{60}\ {}_{20}p^{(f)}_{50}\left(\ddot{a}_{80:70} - \frac{11}{24}\right) \qquad [1]$$

$$= \left(15.161 - \frac{11}{24}\right) - 1.04^{-20} \times \frac{6{,}953.536}{9{,}826.131} \times \frac{9{,}392.621}{9{,}952.697} \times \left(6.876 - \frac{11}{24}\right)$$

$$= 12.747 \qquad [1]$$

[Total 5]

Solution X4.5

This question is Subject CT5, April 2005, Question 7.

The expected present value and the variance of the present value of a joint life annuity are covered in Chapter 22, Section 3.

The present value random variable for this annuity is:

$$\bar{a}_{\overline{T_{xy}}|} = \bar{a}_{\overline{\min\{T_x, T_y\}}|} \qquad [½]$$

The expected present value is:

$$E\left(\bar{a}_{\overline{T_{xy}}|}\right) = \bar{a}_{xy} \qquad [1]$$

Alternatively, we could write:

$$\bar{a}_{\overline{T_{xy}}|} = \frac{1 - v^{T_{xy}}}{\delta} \quad \Rightarrow \quad E\left(\bar{a}_{\overline{T_{xy}}|}\right) = \frac{1 - \bar{A}_{xy}}{\delta}$$

The variance of the present value random variable is:

$$\text{var}\left(\bar{a}_{\overline{T_{xy}}|}\right)=\text{var}\left(\frac{1-v^{T_{xy}}}{\delta}\right)=\frac{1}{\delta^2}\text{var}\left(v^{T_{xy}}\right)=\frac{1}{\delta^2}\left[E\left(v^{2T_{xy}}\right)-\left[E\left(v^{T_{xy}}\right)^2\right]\right] \quad [1½]$$

Now:

$$E\left(v^{T_{xy}}\right)=\bar{A}_{xy} \quad [½]$$

and:

$$E\left(v^{2T_{xy}}\right)={}^2\bar{A}_{xy} \quad [1]$$

where the superscript of 2 to the left of the assurance symbol indicates that the assurance is evaluated using twice the standard force of interest, which is equivalent to evaluating using the rate of interest $i'=(1+i)^2-1$.

So the variance of the present value random variable is:

$$\text{var}\left(\bar{a}_{\overline{T_{xy}}|}\right)=\frac{1}{\delta^2}\left[{}^2\bar{A}_{xy}-\left(\bar{A}_{xy}\right)^2\right] \quad [½]$$

[Total 5]

Solution X4.6

Calculating premiums that satisfy probabilities, using the loss random variable, is covered in Chapter 20, Section 2.

Let K denote the curtate future lifetime of a new policyholder. Then the insurer's loss random variable for the policy is:

$$L=100{,}000v^{K+1}+0.05P\ddot{a}_{\overline{K+1}|}-P\ddot{a}_{\overline{K+1}|}$$

$$=100{,}000v^{K+1}-0.95P\ddot{a}_{\overline{K+1}|} \quad [1]$$

L will be positive if the policyholder dies 'too soon'. We want to find the value of t such that:

$$P(L>0)=P(T<t)=0.01$$

where T represents the policyholder's complete future lifetime.

In other words, we want to find t such that:

$$P(T\geq t)={}_tp_{[35]}=0.99 \quad [1]$$

In terms of life table functions, we have:

$$\frac{l_{[35]+t}}{l_{[35]}} \geq 0.99 \quad \Rightarrow \quad l_{[35]+t} \geq 0.99 l_{[35]} = 0.99 \times 9{,}892.9151 = 9{,}793.99$$ [½]

From the *Tables*: $l_{45} = 9{,}801.3123$ and $l_{46} = 9{,}786.9534$

So t lies somewhere between 10 and 11, and we set $K = 10$. [1]

So we need to find the 'break even' premium P, assuming the benefit is paid at the end of year 11 and using 6% *pa* interest. This is given by the equation:

$$0.95 P \ddot{a}_{\overline{11}|} = 100{,}000 v^{11}$$ [½]

Rearranging to find P:

$$P = \frac{100{,}000 v^{11}}{0.95 \ddot{a}_{\overline{11}|}} = \frac{100{,}000}{0.95 \ddot{s}_{\overline{11}|}} = \frac{100{,}000}{0.95 \times 15.86994} = £6{,}632.86$$ [1]

[Total 5]

Solution X4.7

This question is Subject CT5, April 2010, Question 5.

The formula for calculating survival probabilities from the force of mortality appears in Chapter 15, Section 1.

(i) ***Survival to age 70 exact***

We need:

$${}_{50}p_{20} = \exp\left[-\int_0^{50} \mu_{20+s}\, ds\right] = \exp\left[-\int_0^{50} \left(e^{0.0002(20+s)} - 1\right) ds\right]$$ [½]

The integral can be evaluated as follows:

$$\int_0^{50} \left(e^{0.0002(20+s)} - 1\right) ds = \left[\frac{e^{0.004+0.0002s}}{0.0002} - s\right]_0^{50} = \left(\frac{e^{0.014}}{0.0002} - 50\right) - \left(\frac{e^{0.004}}{0.0002}\right)$$

$$= 5{,}000\left(e^{0.014} - e^{0.004}\right) - 50 = 0.45224$$ [1]

So we have:

$${}_{50}p_{20} = \exp[-0.45224] = 0.63620$$ [½]

[Total 2]

(ii) ***Death between age 60 and age 70***

By analogy with the first part of the question, we can see that:

$$\int_0^{40}\left(e^{0.0002(20+s)}-1\right)ds=\left[\frac{e^{0.004+0.0002s}}{0.0002}-s\right]_0^{40}=\left(\frac{e^{0.012}}{0.0002}-40\right)-\left(\frac{e^{0.004}}{0.0002}\right)$$

$$=5{,}000\left(e^{0.012}-e^{0.004}\right)-40=0.32139 \qquad [1]$$

So:

$${}_{40}p_{20}=\exp\left[-0.32139\right]=0.72514 \qquad [½]$$

So the probability that a life now aged 20 exact dies between 60 and 70 is:

$${}_{40}p_{20}-{}_{50}p_{20}=0.72514-0.63620=0.08894 \qquad [1½]$$

[Total 3]

Solution X4.8

The equality of gross premium prospective and retrospective reserves is covered in Chapter 21, Section 4.

(i)(a) ***Gross premium***

Let P denote the gross annual premium. Then the equation of value is:

$$P\ddot{a}_{x:\overline{n|}}=A^1_{x:\overline{n|}}+I+e\ddot{a}_{x:\overline{n|}}$$

So:

$$P=\frac{A^1_{x:\overline{n|}}+I+e\ddot{a}_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}} \qquad [1]$$

(i)(b) ***Prospective gross premium reserve***

The prospective gross premium reserve at the end of the t th policy year is:

$${}_tV^{pro}=A^1_{x+t:\overline{n-t|}}+e\ddot{a}_{x+t:\overline{n-t|}}-P\ddot{a}_{x+t:\overline{n-t|}} \qquad [1]$$

(i)(c) ***Retrospective gross premium reserve***

The retrospective gross premium reserve at the end of the t th policy year is:

$${}_tV^{retro}=\left[P\ddot{a}_{x:\overline{t|}}-A^1_{x:\overline{t|}}-I-e\ddot{a}_{x:\overline{t|}}\right]\frac{(1+i)^t}{{}_tp_x} \qquad [1]$$

[Total 3]

(ii) ***Equality of reserves***

The premium equation is:

$$P\ddot{a}_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + I + e\ddot{a}_{x:\overline{n}|}$$

Splitting this up at time t , it is equivalent to:

$$P\left(\ddot{a}_{x:\overline{t}|} + v^t\,{}_tp_x\,\ddot{a}_{x+t:\overline{n-t}|}\right) = A^{1}_{x:\overline{t}|} + v^t\,{}_tp_x\,A^{\;1}_{x+t:\overline{n-t}|} + I + e\left(\ddot{a}_{x:\overline{t}|} + v^t\,{}_tp_x\,\ddot{a}_{x+t:\overline{n-t}|}\right) \qquad [1]$$

Now, rearranging so that all the terms containing $v^t\,{}_tp_x$ are on the same side of the equation, we get:

$$P\ddot{a}_{x:\overline{t}|} - A^{1}_{x:\overline{t}|} - I - e\ddot{a}_{x:\overline{t}|} = v^t\,{}_tp_x\left(A^{\;1}_{x+t:\overline{n-t}|} + e\ddot{a}_{x+t:\overline{n-t}|} - P\ddot{a}_{x+t:\overline{n-t}|}\right) \qquad [1]$$

Dividing both sides through by $v^t\,{}_tp_x$ then gives:

$$\left(P\ddot{a}_{x:\overline{t}|} - A^{1}_{x:\overline{t}|} - I - e\ddot{a}_{x:\overline{t}|}\right)\frac{(1+i)^t}{{}_tp_x} = A^{\;1}_{x+t:\overline{n-t}|} + e\ddot{a}_{x+t:\overline{n-t}|} - P\ddot{a}_{x+t:\overline{n-t}|} \qquad [½]$$

The LHS of this equation is the retrospective reserve at the end of the t th policy year and the RHS is the prospective reserve at that time. So the reserves are equal. [½]

[Total 3]

Solution X4.9

Calculating gross premiums is covered in Chapter 20, Section 3.

If the annual premium is P, then:

$$EPV\ premiums = P\ddot{a}_{[50]:\overline{10}|} = 8.318P \qquad [1]$$

$$\begin{aligned}EPV\ benefits &= 10{,}000A^{\;1}_{[50]:\overline{5}|} + 15{,}000v^5\,{}_5p_{[50]}A^{\;1}_{55:\overline{10}|}\\ &= 10{,}000\left(A_{[50]} - \frac{D_{55}}{D_{[50]}}A_{55}\right) + 15{,}000\frac{D_{55}}{D_{[50]}}\left(A_{55} - \frac{D_{65}}{D_{55}}A_{65}\right)\\ &= 10{,}000\left(0.32868 - \frac{1{,}105.41}{1{,}365.77}\times 0.38950\right)\\ &\qquad + 15{,}000\times\frac{1{,}105.41}{1{,}365.77}\left(0.38950 - \frac{689.23}{1{,}105.41}\times 0.52786\right)\\ &= 867.31\end{aligned} \qquad [3]$$

$$EPV\ expenses = 0.25P + 0.05P(\ddot{a}_{[50]:\overline{10}|} - 1)$$

$$= (0.25 + 0.05 \times 7.318)P$$

$$= 0.6159P \qquad [1]$$

So the premium equation is:

$$8.318P = 867.31 + 0.6159P$$

and:

$$P = \frac{867.31}{7.7021} = £112.61 \qquad [1]$$

[Total 6]

Note for markers: any candidate who uses ultimate instead of select mortality, but otherwise performs the calculations correctly, will get an answer of £113.36. Award 5 marks for this.

Solution X4.10

This question is Subject CT5, September 2005, Question 9.

The net future loss random variable is covered in Chapter 20, Section 2, and last survivor annuities are covered in Chapter 22, Section 3.

(i) ***Net future loss random variable***

The net future loss random variable at the outset for this policy is:

$$L = 10{,}000\ddot{a}_{\overline{K_{\overline{60:60}}+1}|} - P = 10{,}000\ddot{a}_{\overline{\max\{K_{60},K_{60}\}+1}|} - P \qquad [1½]$$

where P is the single premium and K_{60} is the curtate future lifetime of a life aged 60. [½]

[Total 2]

(ii) ***Single premium***

The premium is:

$$P = 10{,}000\ddot{a}_{\overline{60:60}}$$

$$= 10{,}000\left(\ddot{a}_{60}^{(m)} + \ddot{a}_{60}^{(f)} - \ddot{a}_{60:60}\right) \qquad [1]$$

$$= 10{,}000\left(15.632 + 16.652 - 14.090\right) \qquad [1]$$

$$= 10{,}000 \times 18.194$$

$$= 181{,}940 \qquad [1]$$

[Total 3]

(iii) ***Standard deviation***

The variance of the net future loss random variable is:

$$\text{var}(L) = 10{,}000^2 \left[\frac{{}^2A_{\overline{60:60}} - \left(A_{\overline{60:60}}\right)^2}{d^2} \right] \quad [1]$$

This formula is derived as follows:

$$\text{var}\left(L\right) = \text{var}\left(10{,}000\,\ddot{a}_{\overline{K_{\overline{60:60}}+1}|} - P\right) = 10{,}000^2\,\text{var}\left(\frac{1 - v^{K_{\overline{60:60}}+1}}{d}\right)$$

$$= \frac{10{,}000^2}{d^2}\text{var}\left(v^{K_{\overline{60:60}}+1}\right) = \frac{10{,}000^2}{d^2}\left[{}^2A_{\overline{60:60}} - \left(A_{\overline{60:60}}\right)^2\right]$$

Using premium conversion and the result $\ddot{a}_{\overline{60:60}} = 18.194$ at 4% interest from part (ii), we have:

$$A_{\overline{60:60}} = 1 - d\,\ddot{a}_{\overline{60:60}} = 1 - \frac{0.04}{1.04} \times 18.194 = 0.30023 \quad [1]$$

Also:

$${}^2A_{\overline{60:60}} = \left(1 - d\,\ddot{a}_{\overline{60:60}}\right) @ 8.16\% = 1 - \frac{0.0816}{1.0816} \times 11.957 = 0.09792 \quad [1]$$

So the standard deviation of L is:

$$\sqrt{10{,}000^2 \left[\frac{{}^2A_{\overline{60:60}} - \left(A_{\overline{60:60}}\right)^2}{d^2} \right]} = £22{,}933 \quad [1]$$

[Total 4]

The final answer is quite sensitive to rounding.

Solution X4.11

Life annuities that increase at a constant compound rate are covered in Chapter 19, Section 2, and calculating gross premiums is covered in Chapter 20, Section 3.

Premiums

If P is the annual premium, the EPV of the premiums is:

$$EPV\left[\text{premiums}\right] = P\ddot{a}_{[51]:\overline{14}|} = 10.697P$$ [½]

Annuity benefits

The EPV of the annuity benefits is:

$$EPV\left[\text{annuity benefits}\right] = 20{,}000 \times 1.04^{14} \times v^{14}\ {}_{14}p_{[51]}\,\ddot{a}'_{65}$$ [½]

where:

$$\ddot{a}'_{65} = 1 + 1.04v\,p_{65} + 1.04^2 v^2\ {}_2p_{65} + \cdots$$ [½]

But $v = 1.04^{-1}$ so:

$$\ddot{a}'_{65} = 1 + p_{65} + {}_2p_{65} + \cdots = 1 + e_{65}$$ [½]

So:

$$EPV\left[\text{annuity benefits}\right] = 20{,}000 \times \frac{l_{65}}{l_{[51]}}\left(1 + e_{65}\right)$$

$$= 20{,}000 \times \frac{8{,}821.2612}{9{,}680.8990} \times 17.645$$

$$= 321{,}563.43$$ [1]

Death benefits

The EPV of the death benefits is:

$$EPV\left[\text{death benefits}\right] = P(IA)^{1}_{[51]:\overline{14}|}$$ [½]

Now:

$$(IA)^{\,1}_{[51]:\overline{14}|} = (IA)_{[51]} - \frac{D_{65}}{D_{[51]}}\left[(IA)_{65} + 14A_{65}\right]$$

$$= 8.58624 - \frac{689.23}{1{,}309.83}\left[7.89442 + 14 \times 0.52786\right]$$

$$= 0.543587 \qquad [1½]$$

So:

$$EPV\left[\text{death benefits}\right] = 0.543587P$$

Expenses

The EPV of the expenses is:

$$EPV\left[\text{expenses}\right] = 300 + 0.03P\left(\ddot{a}_{[51]:\overline{14}|} - 1\right) + 150A'^{\,1}_{[51]:\overline{14}|}$$

$$+0.0025 \times EPV\left[\text{annuity benefits}\right] \qquad [2]$$

where:

$$\ddot{a}_{[51]:\overline{14}|} = 10.697 \text{ (as before)}$$

and:

$$A'^{\,1}_{[51]:\overline{14}|} = 1.04v\,q_{[51]} + 1.04^2v^2\,{}_{1|}q_{[51]} + 1.04^3v^3\,{}_{2|}q_{[51]} + \cdots + 1.04^{14}v^{14}\,{}_{13|}q_{[51]} \qquad [½]$$

But $v = 1.04^{-1}$ so:

$$A'^{\,1}_{[51]:\overline{14}|} = q_{[51]} + {}_{1|}q_{[51]} + {}_{2|}q_{[51]} + \cdots + {}_{13|}q_{[51]} = {}_{14}q_{[51]} \qquad [½]$$

$$= 1 - \frac{l_{65}}{l_{[51]}} = 1 - \frac{8{,}821.2612}{9{,}680.8990} = 0.088797 \qquad [½]$$

So:

$$EPV\left[\text{expenses}\right] = 300 + 0.29091P + 150 \times 0.088797 + 0.0025 \times 321{,}563.43$$

$$= 1{,}117.23 + 0.29091P \qquad [½]$$

The equation of value is:

$$EPV\left[\text{premiums}\right]=EPV\left[\text{benefits}\right]+EPV\left[\text{expenses}\right]$$

So:

$$10.697P=321{,}563.43+0.543587P+1{,}117.23+0.29091P$$

$$\Rightarrow \qquad P=\frac{322{,}680.66}{9.862503}=32{,}718$$ [1]

[Total 10]

Solution X4.12

This question is Subject CT5, September 2005, Question 8.

Reversionary annuities are covered in Chapter 23, Section 3.

The annuity is payable monthly and is guaranteed for 5 years. It is then paid throughout the lifetime of the male and continues to be paid to the female, at half the original annual amount, following the death of the male. However, the date of commencement of the payments to the female depends on when the male dies. If he dies before time 5, the payments to the female start at time 5, ie they just follow on from the guaranteed part. If he dies after time 5, the payments to the female start on the monthly payment date following his death.

EPV of the guaranteed annuity

The expected present value of the guaranteed annuity benefit is:

$$20{,}000\ddot{a}_{\overline{5}|}^{(12)}=20{,}000\times\frac{i}{d^{(12)}}\times a_{\overline{5}|}=20{,}000\times1.021537\times4.4518$$

$$=90{,}953.57$$ [1½]

Alternatively, $\ddot{a}_{\overline{5}|}^{(12)}=\dfrac{1-1.04^{-5}}{12(1-1.04^{-1/12})}=4.5477$.

EPV of the contingent benefit payable to the male (ie after the guarantee expires)

This is:

$$20{,}000v^5\ {}_5p_{65}^{(m)}\ \ddot{a}_{70}^{(12)}\approx20{,}000\times1.04^{-5}\times\frac{9{,}238.134}{9{,}647.797}\times\left(11.562-\frac{11}{24}\right)$$

$$=174{,}777.62$$ [1½]

EPV of the annuity payable to the female following death of the male, provided both are still alive at time 5

This is:

$$10{,}000v^5 \, {}_5p_{65}^{(m)} \, {}_5p_{62}^{(f)} \, \ddot{a}_{70(m)|67(f)}^{(12)} \qquad [1]$$

Now:

$$\begin{aligned}\ddot{a}_{70(m)|67(f)}^{(12)} &= \ddot{a}_{67(f)}^{(12)} - \ddot{a}_{70(m):67(f)}^{(12)} \\ &\approx \ddot{a}_{67(f)} - \frac{11}{24} - \left(\ddot{a}_{70(m):67(f)} - \frac{11}{24}\right) \\ &= \ddot{a}_{67(f)} - \ddot{a}_{70(m):67(f)} \\ &= 14.111 - 10.233 \\ &= 3.878 \qquad [1½]\end{aligned}$$

So:

$$\begin{aligned}&10{,}000v^5 \, {}_5p_{65}^{(m)} \, {}_5p_{62}^{(f)} \, \ddot{a}_{70(m)|67(f)}^{(12)} \\ &= 10{,}000 \times 1.04^{-5} \times \frac{9{,}238.134}{9{,}647.797} \times \frac{9{,}605.483}{9{,}804.173} \times 3.878 \\ &= 29{,}902.36 \qquad [1½]\end{aligned}$$

EPV of the annuity payable to female from time 5, provided she is alive and the male is dead

This is:

$$\begin{aligned}&10{,}000v^5 \, {}_5q_{65}^{(m)} \, {}_5p_{62}^{(f)} \, \ddot{a}_{67}^{(12)} \\ &\approx 10{,}000 \times 1.04^{-5} \times \left(1 - \frac{9{,}238.134}{9{,}647.797}\right) \times \frac{9{,}605.483}{9{,}804.173} \times \left(14.111 - \frac{11}{24}\right) \\ &= 4{,}668.29 \qquad [1½]\end{aligned}$$

Total EPV

Summing all the parts above, we get the total expected present value to be:

$$90{,}953.57 + 174{,}777.62 + 29{,}902.36 + 4{,}668.29 = £300{,}302 \qquad [1½]$$

[Total 10]

The final answer is quite sensitive to rounding.

Solution X4.13

This question is Subject CT5, September 2005, Question 10.

The gross future loss random variable is covered in Chapter 20, Section 2, and calculating the premium for a conventional with-profits contract is covered in Chapter 20, Section 3.6. Calculating the gross premium prospective reserve for a conventional with-profits policy is covered in Chapter 21, Section 2.3.

(i) ***Gross future loss random variable***

Suppose that:

- b is the level of simple bonus, expressed as a percentage of the sum assured
- I is the initial expense
- e is the renewal expense, payable at the start of each year, including the first
- f is the termination expense, payable at the time a claim is made
- P is the annual premium
- K is the curtate future lifetime of the policyholder
- T is the complete future lifetime of the policyholder

[1 for all notation defined, ½ mark if up to three items missing]

Then the gross future loss random variable at the outset is:

$$L = 100{,}000\left[1 + b(K+1)\right]v^T + I + e\,\ddot{a}_{\overline{K+1}|} + f\,v^T - P\,\ddot{a}_{\overline{K+1}|}$$ [2]

[Total 3]

(ii) ***Annual premium***

The rate of interest is 6% pa in this part of the question.

The expected present value of the premiums is:

$$P\ddot{a}_{[20]} = 16.877P$$ [½]

The expected present value of the expenses is:

$$200 + 0.05P\left(\ddot{a}_{[20]} - 1\right) = 200 + 0.79385P$$ [½]

The expected present value of the benefits is:

$$100{,}000\bar{A}_{[20]} + 3{,}000(I\bar{A})_{[20]}$$ [½]

Now:

$$\bar{A}_{[20]} \approx 1.06^{½} \times A_{[20]} = 1.06^{½} \times 0.04472 = 0.04604$$ [½]

and:

$$(I\bar{A})_{[20]} \approx 1.06^{½} \times (IA)_{[20]} = 1.06^{½} \times 2.00874 = 2.06812$$ [½]

So:

$$\text{EPV benefits} = 10{,}808.58$$ [½]

Using the principle of equivalence we have:

$$16.877P = 10{,}808.58 + 200 + 0.79385P \quad \Rightarrow \quad P = £684.48$$ [1]

[Total 4]

(iii) ***Reserve at time 3***

Note that the rate of interest is 4% pa in this part of the question.

The reserve at time 3 is:

$${}_3V = 110{,}000\bar{A}_{23} + 4{,}000(I\bar{A})_{23} - 0.95P\ddot{a}_{23}$$ [2]

$$= 110{,}000 \times 1.04^{½} \times 0.12469 + 4{,}000 \times 1.04^{½} \times 6.09644 - 0.95 \times 684.48 \times 22.758$$

$$= £24{,}057.70$$

or £24,058 to the nearest £1. [2]

[Total 4]

Solution X4.14

Evaluating contingent probabilities and assurances is covered in Chapter 23, Sections 1, 2 and 4. Prospective reserves are covered in Chapter 21, Section 2.

(i) ***Premium annuity factor***

If (x) dies within the 20 years, and (y) is already dead by that point, then the benefit will be paid at the moment of (x)'s death and the contract will terminate, so premiums would cease at that point. [½]

If (x) dies within the 20 years, and (y) is still alive at that point, then the sum assured becomes guaranteed to be payable at time 20. This payment would be made whether or not any more premiums were paid, so it is appropriate to assume that premiums would cease at the moment of (x)'s death. [½]

If (x) is still alive at time 20, regardless of what has happened to (y), the contract ceases without any benefit being paid out, so premiums would also cease at this point. [½]

Taken together, these mean that the premiums would be paid for 20 years, or until the death of (x), if earlier, irrespective of what happens to (y). [½]

So, given that premiums are payable continuously, the appropriate annuity factor would be:

$$\bar{a}_{x:\overline{20}|} \qquad [1]$$

[Total 3]

(ii) ***Annual rate of premium***

The assurance is payable immediately on (x)'s death if (x) dies after (y) and within the 20 years. The expected present value of this is:

$$20{,}000\bar{A}^{\;2}_{x:y:\overline{20}|} = 20{,}000\bar{A}^{\;1}_{x:\overline{20}|} - 20{,}000\bar{A}^{\;1}_{x:y:\overline{20}|} \qquad [1]$$

Now:

$$\bar{A}^{\;1}_{x:\overline{20}|} = \int_0^{20} e^{-\delta t}\,{}_tp_x\,\mu_{x+t}\,dt$$

$$= \int_0^{20} e^{-0.05t}\,e^{-0.005t}\,0.005\,dt$$

$$= \frac{0.005}{-0.055}\left[e^{-0.055\times 20} - e^0\right]$$

$$= \frac{0.005}{0.055}\left[1 - e^{-1.1}\right] = 0.060648 \qquad [1½]$$

and:

$$\bar{A}^{\;1}_{x:y:\overline{20}|} = \int_0^{20} e^{-\delta t}\,{}_tp_{x:y}\,\mu_{x+t}\,dt$$

$$= \int_0^{20} e^{-0.05t}\,e^{-0.01t}\,0.005\,dt$$

$$= \frac{0.005}{-0.06}\left[e^{-0.06\times 20} - e^0\right]$$

$$= \frac{0.005}{0.06}\left[1 - e^{-1.2}\right] = 0.058234 \qquad [1½]$$

The expected present value of this benefit is therefore:

$$20{,}000\times(0.060648 - 0.058234) = £48.28 \qquad [½]$$

In addition, the assurance is payable if (x) dies while (y) is alive, in which case the payment is made at the end of the twenty-year period. The value of this is:

$$20{,}000\,e^{-20\delta}\,{}_{20}q^{\;1}_{x:y} \qquad [1]$$

The probability factor is:

$$_{20}q^{1}_{x:y} = \int_0^{20} {}_tp_{x:y}\,\mu_{x+t}\,dt = 0.005\int_0^{20} e^{-0.01t}\,dt = \frac{0.005}{0.01}\left(1-e^{-0.2}\right) = 0.090635$$

This gives a value of $20{,}000 \times 0.367879 \times 0.090635 = 666.86$. [1]

So the total expected present value of the benefit is £715.14. [½]

If P is the annual rate of premium, the equation of value is:

$$P\bar{a}_{x:\overline{20}|} = 715.14$$ [½]

where:

$$\bar{a}_{x:\overline{20}|} = \int_0^{20} e^{-\delta t}\,{}_tp_x\,dt$$

$$= \int_0^{20} e^{-0.05t}\,e^{-0.005t}\,dt$$

$$= \frac{1}{-0.055}\left[e^{-0.055\times 20} - e^{0}\right]$$

$$= \frac{1}{0.055}\left[1-e^{-1.1}\right] = 12.1296$$ [1]

So:

$$P = \frac{715.14}{12.1296} = £58.96$$ [½]

[Total 9]

(iii) ***Prospective reserve at time 4***

(a) *Only (x) alive*

The benefit will be paid immediately on the death of (x+4) if (x+4) dies within the next 16 years; premiums would cease in 16 years or on the earlier death of (x+4). So the prospective reserve would be:

$$_4V^{(a)} = 20{,}000\bar{A}^{\,1}_{x+4:\overline{16}|} - 58.96\bar{a}_{x+4:\overline{16}|}$$ [1]

Now:

$$\bar{A}^{\;1}_{x+4:\overline{16|}} = \int_0^{16} e^{-\delta t} \; {}_tp_{x+4}\, \mu_{x+4+t}\, dt$$

$$= \int_0^{16} e^{-0.05t}\, e^{-0.005t}\, 0.005\, dt$$

$$= \frac{0.005}{-0.055}\left[e^{-0.055\times 16} - e^0\right]$$

$$= \frac{0.005}{0.055}\left[1 - e^{-0.88}\right] = 0.053202 \qquad [1]$$

and:

$$\bar{a}_{x+4:\overline{16|}} = \int_0^{16} e^{-\delta t} \; {}_tp_{x+4}\, dt$$

$$= \int_0^{16} e^{-0.05t}\, e^{-0.005t}\, dt$$

$$= \frac{1}{-0.055}\left[e^{-0.055\times 16} - e^0\right]$$

$$= \frac{1}{0.055}\left[1 - e^{-0.88}\right] = 10.640311 \qquad [1]$$

Hence the reserve is:

$${}_4V^{(a)} = 20{,}000 \times 0.053202 - 58.96 \times 10.640311 = £436.68 \qquad [½]$$

(b) *Only (y) alive*

This means that (*x*) must have died at some time in the past four years with (*y*) being alive at the time of (*x*)'s death. This means that the benefit will definitely be paid in 16 years' time, without any further premiums being payable. The reserve at time 4 is therefore:

$${}_4V^{(b)} = 20{,}000e^{-16\delta} = 20{,}000e^{-16\times 0.05} = £8{,}986.58 \qquad [1]$$

(c) *Both dead*

As the benefit has not yet been paid out, then (*x*) must have died before (*y*) in the last four years. This is the same situation as (b), and so the reserve value is the same, *ie*:

$${}_4V^{(c)} = £8{,}986.58 \qquad [½]$$

[Total 5]

(iv) ***Comment***

The reserve is much larger for (b) and (c) than for (a) because in (b) and (c) the benefit is certain to be paid, whereas in (a) it is not certain to be paid. [1]

[Total 1]

Assignment X5 – Solutions

Markers: This document sets out one approach to solving each of the questions. Please give credit for other valid approaches. In particular, in 'trial and error' questions, full marks should be awarded for obtaining the correct final answer whatever method is used (eg 'table mode' on a calculator), so long as sufficient working is given.

Solution X5.1

Calculating the profit emerging in a particular year is covered in Chapter 27, Section 1.

The expected profit for year 5 per policy in force at the beginning of the year is given by:

$$\left({}_4V+P-e\right)(1+i)-{}_5V(ap)_{x+4}-(S+f)(aq)^d_{x+4}-(5P+f)(aq)^s_{x+4} \qquad [1]$$

where e denotes the renewal expenses and f denotes the claims expenses.

Putting in the values gives an expected profit of:

$$\begin{aligned}&(5{,}000+1{,}100-40)(1.08)-6{,}500(1-0.01-0.07)\\&\qquad -(12{,}000+100)\times 0.01-(5\times 1{,}100+100)\times 0.07\\&=£51.80\end{aligned} \qquad [2]$$

[Total 3]

Solution X5.2

Calculating the profit emerging in a particular year is covered in Chapter 27, Section 1. The profit signature is introduced in Chapter 27, Section 2.

(i) ***Revised profit in first year***

The net premium reserve per policy in force at the end of the first year can be calculated using the formula on page 37 of the *Tables*:

$${}_1V^{net}=10{,}000\left(1-\frac{\ddot{a}_{62:\overline{3}|}}{\ddot{a}_{61:\overline{4}|}}\right)=10{,}000\left(1-\frac{2.805}{3.622}\right)=£2{,}256 \qquad [1½]$$

So the profit arising from surrenders in the first year is:

$$\begin{aligned}0.1\times p_{61}\times(2{,}256-1{,}500)&=0.1\times(1-q_{61})\times(2{,}256-1{,}500)\\&=0.1\times 0.990991\times(2{,}256-1{,}500) \qquad [1]\\&=£75\end{aligned}$$

So the profit in the first year will be increased by £75, from –£100 to –£25. [½]

[Total 3]

(ii) ***Impact on profit signature***

In subsequent years, the number of policies remaining in force will be reduced by a factor of 0.9, leading to a corresponding reduction in the profits in the profit signature. [1]

[Total 1]

Solution X5.3

This question based on Subject CT5, April 2005, Question 5.

Valuing cashflows using a multiple state model is covered in Chapter 25, Section 2.

The probability of a life aged 40, who is currently sick, staying in the sick state for at least *t* years is given by:

$$ {}_t p_{40}^{\overline{SS}} = \exp\left(-\int_0^t \left(\rho_{40+s} + \nu_{40+s}\right) ds\right) $$

Since the transition intensities are assumed to be constant, the expression simplifies to:

$$ {}_t p_{40}^{\overline{SS}} = e^{-t(\rho+\nu)} \qquad [1] $$

The expected present value of the sickness benefit is then:

$$ 2{,}000 \int_0^{20} e^{-\delta t} \, {}_t p_{40}^{\overline{SS}} \, dt = 2{,}000 \int_0^{20} e^{-(\delta+\rho+\nu)t} \, dt \qquad [1] $$

$$ = \left[-\frac{2{,}000}{\delta+\rho+\nu} e^{-(\delta+\rho+\nu)t} \right]_0^{20} \qquad [1] $$

$$ = \frac{2{,}000}{\ln 1.04 + 0.05} \left[1 - e^{-20(\ln 1.04 + 0.05)} \right] $$

$$ = £18{,}652.72 \qquad [1] $$

[Total 4]

Solution X5.4

Accumulating with-profits (AWP) contracts and unitised with-profits (UWP) contracts are covered in Chapter 26, Section 2.

(i) ***Features of accumulating with-profits***

The benefits take the form of an accumulating fund of premiums. [½]

The fund accumulates with interest, ... [½]

... where the interest rate may be partly guaranteed, ... [½]

... with the remainder being discretionary 'bonus' interest, which can vary over time. [½]

Interest rates cannot be negative, ... [½]

... they will reflect the underlying profits made by the insurer (including investment profits), ... [½]

... but will be smoothed over time so as to produce a more stable progression compared to the underlying asset returns. [½]

On death or maturity, a terminal bonus can be paid out in addition to the accumulated fund value, ... [½]

... at the discretion of the insurance company. [½]

[Maximum 4]

(ii) ***Fund value on 6th April***

First we need the daily effective interest rate. This is:

$$i = 1.0425^{1/365} - 1 = 0.0114\%$$ [½]

The fund on 15th March (in 4 days' time), after the deduction of the policy fee, will be:

$$65{,}292 \times (1+i)^4 - 3 = 65{,}318.79$$ [½]

The fund on 1st April (17 days later), after payment of the premium, will be:

$$65{,}318.79 \times (1+i)^{17} + 600 = 66{,}045.53$$ [½]

So by 6th April (5 days later) the fund value will be:

$$66{,}045.53 \times (1+i)^5 = 66{,}083.20$$ [½]

[Total 2]

Solution X5.5

Zeroising future negative cashflows for unit-linked policies is covered in Chapter 28, Section 2.2. Calculating a policy's net present value is covered in Chapter 27, Section 2.1.

(i) ***Required reserves***

The reserves required at the end of year 2 and year 1 are:

$$_2V = \frac{6}{1.08} = 5.56$$ [½]

and: $$_1V = \frac{1}{1.08}\left(12 + 0.99 \times \frac{6}{1.08}\right) = 16.20$$ [1½]

[Total 2]

(ii) ***NPV of profits before and after zeroisation***

Before zeroisation, the net present value (based on a risk discount rate of 10%) is:

$$-\frac{25}{1.10} - 12 \times \frac{0.99}{1.10^2} - 6 \times \frac{0.99^2}{1.10^3} + 25 \times \frac{0.99^3}{1.10^4} + 35 \times \frac{0.99^4}{1.10^5} = 0.48$$ [1]

After zeroisation the profit in year 1 becomes:

$$-25 - 0.99 \times {}_1V = -25 - 0.99 \times 16.20 = -41.04$$ [1]

So the profit vector becomes:

Year	*In-force profit*
1	–41.04
2	0
3	0
4	25
5	35

[1]

So the NPV after zeroisation is:

$$-\frac{41.04}{1.10} + 0 + 0 + 25 \times \frac{0.99^3}{1.10^4} + 35 \times \frac{0.99^4}{1.10^5} = 0.14$$ [1]

As expected, the NPV after zeroisation is smaller because losses are realised earlier and the risk discount rate is greater than the accumulation rate. [1]

[Total 5]

Solution X5.6

Valuing cashflows using a multiple state model is covered in Chapter 25, Section 2.

(i) $50{,}000\int_0^{10} e^{-\delta t}\left({}_tp_{50}^{aa}\,\mu_{50+t} + {}_tp_{50}^{ai}\,\nu_{50+t}\right)dt$ [2]

This can be reasoned as follows. Suppose that the life dies at age $50+t$. This can be from either the able state or the ill state – the probability density functions (PDFs) of these are in the brackets. A benefit of £50,000 is then paid at time t, and this is discounted back to time 0. Integrating over all possible times t gives the required expression.

(ii) $50{,}000\int_0^{9} e^{-\delta t}\,{}_tp_{50}^{aa}\,\sigma_{50+t}\left(\int_1^{10-t} e^{-\delta s}\,{}_sp_{50+t}^{\overline{ii}}\,\nu_{50+t+s}\,ds\right)dt$ [3]

This time, suppose that the life gets sick at time t. The PDF for this is ${}_tp_{50}^{aa}\,\sigma_{50+t}$. The life could get sick at any time, but if this happens after time 9, it will not lead to any benefit. So we integrate t between the limits of 0 and 9.

The life has to stay sick for a year before any benefit is paid. If the life remains sick for s (>1) years, and dies from the sick state at age $50+t+s$, then the benefit is paid at time $t+s$ and must be discounted back to time 0. The PDF of this happening is ${}_sp_{50+t}^{\overline{ii}}\,\nu_{50+t+s}$. Note that s must be at least 1 for any benefit to be paid, but the policy term is 10 years. So, given that the life falls sick at time t, the duration of sickness required for the payment of the benefit is between 1 and $10-t$. So we integrate s between these limits.

(iii) $5{,}000\int_0^{10} e^{-\delta t}\,{}_tp_{50}^{ai}\,dt$ [2]

If the life is sick at time t, which has probability ${}_tp_{50}^{ai}$, then benefits will be received at the rate of £5,000 pa. This is multiplied by the discount factor $e^{-\delta t}$. Then integrate over all points in time t where a benefit could be paid.

[Total 7]

Solution X5.7

Net premium reserves are covered in Chapter 21, Section 6. Calculating a policy's net present value is covered in Chapter 27, Section 2.1, and determining a premium using profit testing is covered in Chapter 27, Section 4.

Calculating the reserves

The calculations in this question are quite sensitive to rounding, and valid alternative methods can give answers that are up to ± 0.25 *different from the answers given.*

We need to calculate the reserves at times 1 and 2, which means that first we need to calculate the net premium, using the equivalence principle.

The expected present value of the benefits is:

$$\begin{aligned}150{,}000A^{1}_{62:\overline{3}|} &= 150{,}000\left(A_{62}-\frac{D_{65}}{D_{62}}A_{65}\right)\\ &= 150{,}000\left(0.48458-\frac{689.23}{802.40}\times 0.52786\right)\\ &= 4{,}675.36\end{aligned}$$ [½]

Alternatively, we could calculate the term assurance as:

$$A^{1}_{62:\overline{3}|} = A_{62:\overline{3}|}-\frac{D_{65}}{D_{62}} = 0.89013-\frac{689.23}{802.40} = 0.031169$$

So, if NP is the annual net premium:

$$NP = \frac{150{,}000A^{1}_{62:\overline{3}|}}{\ddot{a}_{62:\overline{3}|}} = \frac{4{,}675.36}{2.857} = 1{,}636.46$$ [½]

The net premium reserve is found by calculating the EPV of future benefits less the EPV of future net premiums:

$${}_1V = 150{,}000A^{1}_{63:\overline{2}|} - 1{,}636.46\ddot{a}_{63:\overline{2}|}$$ [½]

From the *Tables*, $\ddot{a}_{63:\overline{2}|} = 1.951$. Calculating the term assurance function:

$$A^{1}_{63:\overline{2}|} = A_{63}-\frac{D_{65}}{D_{63}}A_{65} = 0.49890-\frac{689.23}{763.74}\times 0.52786 = 0.022538$$ [½]

Again, we could alternatively calculate the term assurance as:

$$A^{1}_{63:\overline{2}|} = A_{63:\overline{2}|}-\frac{D_{65}}{D_{63}} = 0.92498-\frac{689.23}{763.74} = 0.022539$$

So we have:

$$ {}_1V = 150{,}000 \times 0.022538 - 1{,}636.46 \times 1.951 = 187.93 \qquad [½] $$

For the reserve at the end of year 2, we can work from first principles rather than try to use any assurance functions. The benefit will be 150,000 paid at the end of the year if death occurs, and the premium is payable immediately, so:

$$ {}_2V = 150{,}000 v\, q_{64} - 1{,}636.46 = 150{,}000 \times \frac{0.012716}{1.04} - 1{,}636.46 = 197.58 \qquad [½] $$

Profit test

Let the premium be P. The figures for the profit test are:

Year	1	2	3
Premium	P	P	P
Expenses	–400	–50	–50
Interest (on $P-E$) (@ 6%)	$0.06(P-400)$	$0.06(P-50)$	$0.06(P-50)$
Death benefit [1]	–1,074.60	–1,622.25	–1,907.40
Reserve at start of year	0	187.93	197.58
Interest on reserves (@ 6%)	0	11.28	11.85
Expected reserve at end of year [2]	–186.58	–195.44	0

[*½ mark in total for the premium, expenses and interest rows*]
[*½ mark for each death benefit – total 1½ marks*]
[1 *mark for the reserve figures*]

Notes on calculations:

(1) Death benefit

Year 1: $150{,}000 q_{[62]} = 150{,}000 \times 0.007164 = 1{,}074.60$

Year 2: $150{,}000 q_{[62]+1} = 150{,}000 \times 0.010815 = 1{,}622.25$

Year 3: $150{,}000 q_{64} = 150{,}000 \times 0.012716 = 1{,}907.40$

(2) Reserves

The reserve that needs to be set up at the end of year 1 is the reserve for the start of year 2 for those that survive the first year:

$$ 187.93 p_{[62]} = 187.93 \times (1 - 0.007164) = 186.58 $$

The reserve that needs to be set up at the end of year 2 is the reserve for the start of year 3 for those that survive the second year:

$$197.58 p_{[62]+1} = 197.58 \times (1 - 0.010815) = 195.44$$

Summing the columns, this gives us a profit vector of:

Year 1 $\quad 1.06(P-400) - 1{,}261.18 = 1.06P - 1{,}685.18$

Year 2 $\quad 1.06(P-50) - 1{,}618.48 = 1.06P - 1{,}671.48$

Year 3 $\quad 1.06(P-50) - 1{,}697.97 = 1.06P - 1{,}750.97$ [½]

To obtain the net present value, we can calculate the profit signature, which is the profit per policy in force at inception. This is obtained by multiplying the profit vector by the survival probability to the start of that year:

Year 1 $\quad 1.06P - 1{,}685.18$

Year 2 $\quad \left[1.06P - 1{,}671.48\right] p_{[62]} = \left[1.06P - 1{,}671.48\right] \times (1 - 0.007164)$

$$= 1.052406P - 1{,}659.51$$

Year 3 $\quad \left[1.06P - 1{,}750.97\right] {}_2p_{[62]} = \left[1.06P - 1{,}750.97\right] \dfrac{l_{64}}{l_{[62]}}$

$$= \left[1.06P - 1{,}750.97\right] \times \frac{8{,}934.8771}{9{,}097.7405} = 1.041024P - 1{,}719.62$$

[1]

The net present value is the discounted value of the profit signature using the risk discount rate:

$$\frac{1.06P - 1{,}685.18}{1.09} + \frac{1.052406P - 1{,}659.51}{1.09^2} + \frac{1.041024P - 1{,}719.62}{1.09^3}$$ [½]

Setting this equal to zero, we get:

$$2.662128P = 4{,}270.675 \quad \Rightarrow \quad P = 1{,}604.23$$ [1]

[Total 9]

Solution X5.8

The calculation of reserves is covered in Chapter 21, Sections 2 and 3. Calculating the profit arising in a particular year is first introduced in Chapter 21, Section 5 and is revisited in Chapter 27, Section 1.

(i) ***Reserve***

The prospective reserve at the end of the 5th policy year is:

$$_5V^{pro} = 60{,}000A^{\,1}_{70:\overline{20|}} + 120{,}000\frac{D_{90}}{D_{70}} + 0.01P\ddot{a}_{70:\overline{20|}} - P\ddot{a}_{70:\overline{20|}} \qquad [1]$$

We have:

$$\frac{D_{90}}{D_{70}} = \frac{48.61}{517.23} = 0.09398 \qquad [½]$$

Also:

$$A^{\,1}_{70:\overline{20|}} = A_{70} - \frac{D_{90}}{D_{70}}A_{90} = 0.60097 - 0.09398 \times 0.84196 = 0.52184 \qquad [½]$$

and:

$$\ddot{a}_{70:\overline{20|}} = \ddot{a}_{70} - \frac{D_{90}}{D_{70}}\ddot{a}_{90} = 10.375 - 0.09398 \times 4.109 = 9.9888 \qquad [½]$$

So:

$$\begin{aligned} _5V^{pro} &= 60{,}000 \times 0.52184 + 120{,}000 \times 0.09398 - 0.99 \times 3{,}071.40 \times 9.9888 \\ &= £12{,}215.36 \end{aligned} \qquad [½]$$

[Total 3]

Alternatively, we could have calculated the reserve retrospectively using the formula:

$$_5V^{retro} = \frac{D_{[65]}}{D_{70}}\left[P\ddot{a}_{[65]:\overline{5|}} - 60{,}000 \times A^{\,1}_{[65]:\overline{5|}} - 200 - 0.01P(\ddot{a}_{[65]:\overline{5|}} - 1)\right] \qquad [1]$$

We have:

$$\frac{D_{70}}{D_{[65]}} = \frac{517.23}{685.44} = 0.75460 \qquad [½]$$

$$\ddot{a}_{[65]:\overline{5|}} = \ddot{a}_{[65]} - \frac{D_{70}}{D_{[65]}}\ddot{a}_{70} = 12.337 - 0.75460 \times 10.375 = 4.5081 \qquad [½]$$

$$A^{\,1}_{[65]:\overline{5|}} = A_{[65]} - \frac{D_{70}}{D_{[65]}}A_{70} = 0.52550 - 0.75460 \times 0.60097 = 0.07201 \qquad [½]$$

Hence:

$$ {}_5V^{retro} = \frac{3{,}071.40 \times 4.5081 - 60{,}000 \times 0.07201 - 200 - 0.01 \times 3{,}071.40 \times 3.5081}{0.75460} $$

$$ = £12{,}215.42 $$ *[½]*

[Total 3]

The prospective and retrospective reserves are equal since they are both calculated on the same basis as the premium. The few pence difference is a result of using rounded values from the Tables.

(ii) ***Insurer's profit***

The reserve per policy in force at the end of 2017 is:

$$ {}_4V^{pro} = 60{,}000 A^{1}_{69:\overline{21}|} + 120{,}000 \frac{D_{90}}{D_{69}} + 0.01 P \ddot{a}_{69:\overline{21}|} - P \ddot{a}_{69:\overline{21}|} $$

We have:

$$ \frac{D_{90}}{D_{69}} = \frac{48.61}{550.14} = 0.08836 $$

$$ A^{1}_{69:\overline{21}|} = A_{69} - \frac{D_{90}}{D_{69}} A_{90} = 0.58638 - 0.08836 \times 0.84196 = 0.51198 $$ [½]

$$ \ddot{a}_{69:\overline{21}|} = \ddot{a}_{69} - \frac{D_{90}}{D_{69}} \ddot{a}_{90} = 10.754 - 0.08836 \times 4.109 = 10.3909 $$ [½]

and hence:

$$ {}_4V^{pro} = 60{,}000 \times 0.51198 + 120{,}000 \times 0.08836 - 0.99 \times 3{,}071.40 \times 10.3909 $$

$$ = £9{,}726.66 $$ [½]

The reserves required on 1 January 2018 total:

$$ 197 \times 9{,}726.66 = £1{,}916{,}152.02 $$ [½]

The premiums received on 1 January 2018 total:

$$ 197 \times 3{,}071.40 = £605{,}065.80 $$ [½]

Expenses incurred at the start of 2018 total:

$$ 2 \times 0.01 \times 197 \times 3{,}071.40 = £12{,}101.32 $$ [1]

Interest earned during 2018 was:

$$ 2 \times 0.04 \times (1{,}916{,}152.02 + 605{,}065.80 - 12{,}101.32) = £200{,}729.32 $$ [1]

There were 9 deaths during 2018. So the reserves required on 31 December 2018 (using the prospective reserve figure of £12,215.36) total:

$$(197-9)\times 12{,}215.36 = £2{,}296{,}487.68$$ [½]

So the profit earned in 2018 was:

$$1{,}916{,}152.02 + 605{,}065.80 - 12{,}101.32 + 200{,}729.32 - 9\times 60{,}000 - 2{,}296{,}487.68$$

$$= -£126{,}642$$

ie a loss of approximately £127,000. [1]

[Total 6]

Solution X5.9

The construction and use of multiple decrement tables are covered in Chapter 25, Sections 4 and 5.

(i) ***Multiple decrement table***

First we need to calculate the dependent probabilities of decrement, using the implied underlying forces.

The (assumed constant) force of mortality for the year of age beginning at age $62+t$ is obtained from:

$$\bar{\mu}^d_{62+t} = -\ln\left(p^d_{[62]+t}\right) = -\ln\left(1-q^d_{[62]+t}\right)$$ [½]

for $t=0,1,2$, where $q^d_{[62]+t}$ is the mortality probability from the AM92 Select table.

This gives the following values:

Age x	Duration t	$q^d_{[62]+t}$	$\bar{\mu}^d_x$
62	0	0.007164	0.0071898
63	1	0.010815	0.0108739
64	2	0.012716	0.0127975

[1]

We now construct the dependent probabilities using:

$$(aq)^j_x = \frac{\bar{\mu}^j_x}{\bar{\mu}^d_x + \bar{\mu}^s_x}\left[1-e^{-(\bar{\mu}^d_x+\bar{\mu}^s_x)}\right]$$ [1]

for $j=d,s$ and $x=62, 63$ and 64, where *d* represents death and *s* represents surrender.

This gives:

Age x	$(aq)_x^d$	$(aq)_x^s$
62	0.0069881	0.0485971
63	0.0106812	0.0245569
64	0.0126528	0.0098869

[1½]

The multiple decrement table is then constructed recursively using:

$$(ad)_x^j = (al)_x \, (aq)_x^j \qquad\qquad j = d, s$$

$$(al)_{x+1} = (al)_x - (ad)_x^d - (ad)_x^s$$

$$(al)_{62} = 100{,}000$$

The table is then:

x	$(al)_x$	$(ad)_x^d$	$(ad)_x^s$
62	100,000	698.81	4,859.71
63	94,441.48	1,008.75	2,319.19
64	91,113.55	1,152.84	900.83
65	89,059.88		

[1]
[Total 5]

(ii) ***Premium***

Assuming deaths occur on average half way through each year, the expected present value of the death benefit is:

$$EPV[D] = \frac{50{,}000}{(al)_{62}}\left[v^{½}(ad)_{62}^d + v^{1½}(ad)_{63}^d + v^{2½}(ad)_{64}^d\right]$$

$$= \frac{50{,}000}{100{,}000}\left[\frac{698.81}{1.03^{½}} + \frac{1{,}008.75}{1.03^{1½}} + \frac{1{,}152.84}{1.03^{2½}}\right]$$

$$= 1{,}362.13$$ [2]

The EPV of the surrender and survival benefit is:

$$EPV[S] = \frac{0.5P}{(al)_{62}}\left[v^{½}(ad)^{s}_{62} + 2v^{1½}(ad)^{s}_{63} + 3v^{2½}(ad)^{s}_{64} + 3v^{3}(al)_{65}\right]$$

$$= \frac{0.5P}{100{,}000}\left[\frac{4{,}859.71}{1.03^{½}} + \frac{2\times 2{,}319.19}{1.03^{1½}} + \frac{3\times 900.83}{1.03^{2½}} + \frac{3\times 89{,}059.88}{1.03^{3}}\right]$$

$$= 1.28121P$$ [2]

The EPV of the premiums is:

$$EPV[P] = P\left[\frac{(al)_{62}}{(al)_{62}} + v\frac{(al)_{63}}{(al)_{62}} + v^{2}\frac{(al)_{64}}{(al)_{62}}\right]$$

$$= P\left[1 + \frac{0.9444148}{1.03} + \frac{0.9111355}{1.03^{2}}\right]$$

$$= 2.77574P$$ [1]

The equation of value is therefore:

$$2.77574P = 1{,}362.13 + 1.28121P$$

$$\Rightarrow P = \frac{1{,}362.13}{2.77574 - 1.28121} = 911.41$$ [1]

[Total 6]

Alternatively, we could calculate the EPVs using the dependent probabilities directly, ie using:

$$EPV[D] = 50{,}000\left[v^{½}(aq)^{d}_{62} + v^{1½}(ap)_{62}(aq)^{d}_{63} + v^{2½}\,{}_{2}(ap)_{62}(aq)^{d}_{64}\right]$$

$$EPV[S] = 0.5P\left[v^{½}(aq)^{s}_{62} + 2v^{1½}(ap)_{62}(aq)^{s}_{63} + 3v^{2½}\,{}_{2}(ap)_{62}(aq)^{s}_{64} + 3v^{3}\,{}_{3}(ap)_{62}\right]$$

$$EPV[P] = P\left[1 + v\,{}_{1}(ap)_{62} + v^{2}\,{}_{2}(ap)_{62}\right]$$

Solution X5.10

This question is based on Subject CT5, September 2006, Question 9.

Profit testing of unit-linked contracts is covered in Chapter 27, Section 1.3. Calculating the profit margin is covered in Chapter 27, Section 2.1.

Unit fund

The expected cashflows in the unit fund are given in the table below. Cashflows out of the fund are shown as negative entries.

Year	Premium	Cost of allocation	Fund at start of year	End fund before charge	Management charge	Fund at end of year
1	5,000	4,037.50	4,037.50	4,279.75	– 32.10	4,247.65
2	5,000	4,940	9,187.65	9,738.91	– 73.04	9,665.87
3	5,000	4,940	14,605.87	15,482.22	– 116.12	15,366.10

[4, lose ½ for each incorrect value, subject to minimum of 0]

Markers, award full method marks here if the method has been followed through correctly, ie if one incorrect value early in the calculation has led to many incorrect values but the method is otherwise perfect, award 3.5 marks out of 4.

Non-unit fund

The expected cashflows in the non-unit fund are:

Year	Premium less cost of allocation	Expenses	Interest	Expected benefit cost	Management charge	Profit vector
1	962.50	–600	14.50	–126.37	32.10	282.73
2	60	–100	–1.60	–93.10	73.04	–61.66
3	60	–100	–1.60	–46.86	116.12	27.66

[3, lose ½ for each incorrect value, subject to minimum of 0]

Markers: the marks for the expected benefit cost column are awarded separately below.

Markers, award full method marks here if the method has been followed through correctly, ie if one incorrect value early in the calculation has led to many incorrect values but the method is otherwise perfect, award 2.5 marks out of 3.

The expected benefit cost figures are calculated as follows.

If the policyholder dies in year 1, a death benefit of £20,000 is payable at the end of the first year. The sum of £4,247.65 comes from the unit fund, and the remainder comes from the non-unit fund.

The expected death cost in year 1 is:

$$(20{,}000-4{,}247.65)q_{60}=15{,}752.35\times 0.008022=126.37 \qquad [1]$$

The entries for years 2 and 3 are calculated in a similar way:

$$(20{,}000-9{,}665.87)q_{61}=10{,}334.13\times 0.009009=93.10 \qquad [½]$$

and: $$(20{,}000-15{,}366.10)q_{62}=4{,}633.90\times 0.010112=46.86 \qquad [½]$$

If the policyholder survives to the end of year 3, she receives the full bid value of the units. This comes from the unit fund, so there is no cashflow from the non-unit fund.

The net present value of the profit, discounted at the risk discount rate, is:

$$\begin{aligned}\text{NPV}&=\frac{282.73}{1.10}+p_{60}\frac{-61.66}{1.10^2}+{}_2p_{60}\frac{27.66}{1.10^3}\\&=\frac{282.73}{1.10}+(1-0.008022)\frac{-61.66}{1.10^2}+\frac{9{,}129.7170}{9{,}287.2164}\times\frac{27.66}{1.10^3}\\&=226.91\end{aligned} \qquad [1]$$

The expected present value of the premiums, discounted at the risk discount rate, is:

$$\begin{aligned}\text{EPV premiums}&=5{,}000\left(1+\frac{p_{60}}{1.10}+\frac{{}_2p_{60}}{1.10^2}\right)\\&=5{,}000\left(1+(1-0.008022)\times\frac{1}{1.10}+\frac{9{,}129.7170}{9{,}287.2164}\times\frac{1}{1.10^2}\right)\\&=13{,}571.14\end{aligned} \qquad [1]$$

So the profit margin on the contract is:

$$\frac{226.91}{13{,}571.14}=1.7\% \qquad [1]$$

[Total 12]

Solution X5.11

Profit testing of unitised with-profits contracts is covered in Chapter 27, Section 1.4.

The unit fund values, plus the terminal bonus amounts, are shown in the following table:

Year	Allocated premium (1)	Total unit fund at start of year (2)	Policy fee (3)	Total unit fund at end of year (4)	Terminal bonus (5)	Unit fund plus TB at end of year (6)
1	4,250	4,250	0	4,420	44.20	4,464.20
2	5,000	9,420	141.30	9,649.85	289.50	9,939.34
3	5,000	14,649.85	219.75	15,007.30	975.48	15,982.78

[3]

Key: $(2) = (1) + (4 \text{ from previous year})$ in years 2 and 3

$(3) = (2) \times 0.015$ in years 2 and 3

$(4) = [(2) - (3)] \times 1.04$

$(5) = (4) \times (\text{TB rate})$

The non-unit cashflows and net present value are calculated in the following tables:

Year	Profit on allocation (7)	Policy fee (8)	Expenses and commission (9)	Interest (10)	AM92 select (11)	Mortality probability (12)
1	750	0	750	0	0.003358	0.002686
2	0	141.30	80	1.23	0.004903	0.003922
3	0	219.75	80	2.79	0.005650	0.004520

[3]

Year	Expected death cost (13)	Dependent surrender probability (14)	Expected surrender profit (15)	Investment and actuarial expenses (16)	Termination expenses (17)	Cashflow (18)
1	28.30	0.099731	7.98	11.05	5.12	– 36.50
2	19.85	0.099608	7.97	24.12	5.18	21.34
3	0	0	0	37.52	50	55.02

[4]

Year	Probability in force (19)	Profit signature (20)	Discount (21)	NPV (22)
1	1	– 36.50	0.925926	– 33.79
2	0.897582	19.16	0.857339	16.42
3	0.804655	44.28	0.793832	35.15
			Total	**17.78**

[3]

Key: $(7) = 5{,}000 - (1)$

$(8) = (3)$

$(9)_1 = 500 + 0.05 \times 5{,}000;$ $\qquad (9)_2 = (9)_3 = 30 + 0.01 \times 5{,}000$

$(10) = [(7) + (8) - (9)] \times 0.02$

$(12) = (11) \times 0.8$

$(13) = \max\left[15{,}000 - (6), 0\right] \times (12)$

$(14) = [1 - (12)] \times 0.1$

$(15) = 80 \times (14)$

$(16) = (4) \times 0.0025$

$(17)_1 = (17)_2 = 50 \times [(12) + (14)]$

$(18) = (7) + (8) - (9) + (10) - (13) + (15) - (16) - (17)$

$(19)_t = (19)_{t-1} \times \left[1 - (12)_{t-1} - (14)_{t-1}\right], \; t = 2, 3$

$(20) = (18) \times (19)$

$(21)_t = 1.08^{-t}$

$(22) = (20) \times (21)$

[Total 13]

Solution X5.12

This question is based on Subject CT5, April 2005, Question 14.

Mortality profit is the topic of Chapter 24.

(i) ***Definitions***

The death strain in the policy year $(t, t+1)$ is the random variable:

$$DS = \begin{cases} 0 & \text{if the policyholder survives to time } t+1 \\ S - {}_{t+1}V & \text{if the policyholder dies in the year } (t, t+1) \end{cases}$$ [1]

The maximum death strain is called the death strain at risk:

$$DSAR = S - {}_{t+1}V$$ [1]

The death strain at risk is the amount of money, over and above the year-end reserve, that has to be paid in respect of each death during the policy year $(t, t+1)$. [1]

The expected death strain for such a policy is:

$$EDS = q_{x+t}\left(S - {}_{t+1}V\right)$$ [1]

This is the amount that the life insurance company expects to pay over and above the year-end reserve for the policy. [1]

For a group of identical policies, the expected death strain is given by:

expected number of deaths $\times DSAR$ [½]

The actual death strain is the observed value at time $t+1$ of the death strain random variable:

$$ADS = \begin{cases} 0 & \text{if the policyholder survived to time } t+1 \\ S - {}_{t+1}V & \text{if the policyholder died in the year } (t, t+1) \end{cases}$$ [1]

For a group of identical policies, the actual death strain is given by:

actual number of deaths $\times DSAR$ [½]

[Maximum 6]

(ii)(a) ***Death strain at risk for each type of policy for calendar year 2017***

The end of calendar year 2017 is time 3, when time is measured in years from the start of the policies.

Term assurance

To calculate the reserve at time 3, we first need to calculate the annual premium for the policy. If we denote this by P, then:

$$P\ddot{a}_{45:\overline{15}|} = 150{,}000 A^{1}_{45:\overline{15}|} \qquad [½]$$

From the *Tables*:

$$\ddot{a}_{45:\overline{15}|} = 11.386 \qquad [½]$$

Also:

$$A^{1}_{45:\overline{15}|} = A_{45:\overline{15}|} - \frac{D_{60}}{D_{45}} = 0.56206 - \frac{882.85}{1{,}677.97} = 0.03592 \qquad [½]$$

Alternatively, we can calculate this as:

$$A^{1}_{45:\overline{15}|} = A_{45} - \frac{D_{60}}{D_{45}} A_{60} = 0.27605 - \frac{882.85}{1{,}677.97} \times 0.45640 = 0.03592$$

So:

$$P = \frac{150{,}000 \times 0.03592}{11.386} = £473.21 \qquad [½]$$

The reserve at time 3 is:

$$\begin{aligned} {}_3V &= 150{,}000 A^{1}_{48:\overline{12}|} - 473.21 \ddot{a}_{48:\overline{12}|} \\ &= 150{,}000 \left(0.63025 - \frac{882.85}{1{,}484.43} \right) - 473.21 \times 9.613 \\ &= £777.52 \end{aligned} \qquad [1]$$

Alternatively, we could calculate the term assurance value as:

$$A^{1}_{48:\overline{12}|} = A_{48} - \frac{D_{60}}{D_{48}} \times A_{60} = 0.30695 - \frac{882.85}{1{,}484.43} \times 0.45640 = 0.035511$$

The death strain at risk for each term assurance policy is then:

$$DSAR = S - {}_3V = 150{,}000 - 777.52 = £149{,}222 \qquad [½]$$

Temporary annuity policies

Watch out here – these policyholders are aged 55 at entry and have PMA92C20 mortality.

The reserve at time 3 for the temporary annuity is:

$$\begin{aligned}{}_3V &= 25{,}000\,a_{58:\overline{2}|} \\ &= 25{,}000\left(v\,p_{58} + v^2\,{}_2p_{58}\right) \\ &= 25{,}000\left(\frac{1-0.001814}{1.04} + \frac{(1-0.001814)(1-0.002110)}{1.04^2}\right) \\ &= £47{,}018.15 \end{aligned}$$ [1]

There is no death benefit for this policy. However, if the policyholder survives to time 3, there is an annuity payment of £25,000, which is not included in the reserve at time 3.

So the death strain at risk for each temporary annuity is:

$$DSAR = 0 - \left({}_3V + 25{,}000\right) = -£72{,}018$$ [1]

This calculation is quite sensitive to rounding and to the method of calculation used. For example, if we calculate the annuity as:

$$a_{58:\overline{2}|} = a_{58} - v^2\,{}_2p_{58}\,a_{60} = 15.356 - 1.04^{-2} \times \frac{9{,}826.131}{9{,}864.803} \times 14.632 = 1.881$$

then we obtain:

$${}_3V = 47{,}023.16 \quad \textit{and} \quad DSAR = -72{,}023$$

(ii)(b) ***Total mortality profit or loss***

Term assurance policies

There are $5{,}000 - 15 = 4{,}985$ term assurance policies in force on 1 January 2017.

The expected death strain for this group of policies is:

$$EDS = 4{,}985\,q_{47} \times 149{,}222 = 4{,}985 \times 0.001802 \times 149{,}222 = £1{,}340{,}457$$ [½]

The actual death strain for this group of policies is:

$$ADS = 8 \times 149{,}222 = £1{,}193{,}776$$ [½]

So the mortality profit from this group of policies is:

$$MP = EDS - ADS = £146{,}681$$ [½]

Temporary annuity policies

There are $1,000-5=995$ temporary annuity policies in force on 1 January 2017.

The expected death strain for this group of policies is:

$$EDS = 995q_{57} \times (-72,018) = 995 \times 0.001558 \times (-72,018) = -£111,643$$ [½]

The actual death strain for this group of policies is:

$$ADS = 1 \times (-72,018) = -£72,018$$ [½]

So the mortality profit from this group of policies is:

$$MP = EDS - ADS = -£39,625$$ [½]

Total mortality profit

The total mortality profit is then:

$$146,681 - 39,625 = £107,056$$ [½]

[Total 9]