5 PROBABILITY

5.1 What is Probability?

Probability theory is the branch of mathematics that studies the possible outcomes of given events together with the outcomes' relative likelihoods and distributions. In common usage, the word “probability" is used to mean the chance that a particular event (or set of events) will occur expressed on a linear scale from 0 (impossibility) to 1 (certainty). Factually, It is the study of random or indeterministic experiments eg tossing a coin or rolling a die. If we roll a die, we are certain it will come down but we are uncertain which face will show up. Ie the face showing up is indeterministic. Probability is a way of summarizing the uncertainty of statements or events. It gives a numerical measure for the degree of certainty (or degree of uncertainty) of the occurrence of an event.

We often use P to represent a probability Eg P(rain) would be the probability that it rains. In other cases Pr(.) is used instead of just P(.).

Definitions

- Experiment: A process by which an observation or measurement is obtained. Eg tossing a coin or rolling a die.
- Outcome: Possible result of a random experiment. Eg a 6 when a die is rolled once or a head when a coin is tossed.
- Sample space: Also called the probability space and it is a collection or set of all possible outcomes of a random experiment. Sample space is usually denoted by S or Ω or U
- Event: it’s a subset of the sample space. Events are usually denoted by upper case letters.

5.2 Approaches to Probability

There are three ways to define probability, namely classical, empirical and subjective probability.

5.2.1 Classical probability

Classical or theoretical probability is used when each outcome in a sample space is equally likely to occur. The underlying idea behind this view of probability is symmetry. Ie if the sample space contains n outcomes that are fairly likely then P(one outcome)=1/n.

The classical probability for an event A is given by

$$P(A) = \frac{\text{Number of outcomes in A}}{\text{Total number of outcomes in S}} = \frac{n(A)}{n(S)}$$

Eg Roll a die and observe that $P(A) = P(\text{rolling a } 3) = \frac{1}{6}$.

Example A fair die,is rolled once, write down the sample space S hence find the probability that the score showing up is ; a) a multiple of 3 b) a prime number.
Solution

$S = \{1, 2, 3, 4, 5, 6\}$ Multiples of 3 are 3 and 6 while prime numbers are 2, 3 and 5

Thus $P(\text{Multipleof } 3) = \frac{2}{6} = \frac{1}{3}$ and $P(\text{prime number}) = \frac{3}{6} = \frac{1}{2}$

5.2.2 Frequentist or Empirical probability

When the outcomes of an experiment are not equally likely, we can conduct experiments to give us some idea of how likely the different outcomes are. For example, suppose we are interested in measuring the probability of producing a defective item in a manufacturing process. The probability could be measured by monitoring the process over a reasonably long period of time and calculating the proportion of defective items.

In a nut shell Empirical (or frequentist or statistical) probability is based on observed data. The empirical probability of an event A is the relative frequency of event A, that is

$$P(A) = \frac{\text{Frequency of event A}}{\text{Total number of observations}}$$

Example 1 The following are the counts of fish of each type, that you have caught before.

Fish Types	Blue gill	Red gill	Crappy	Total
No of times caught	13	17	10	40

Estimate the probability that the next fish you catch will be a Blue gill.

$P(\text{Blue gill}) = \frac{13}{40} = 0.325$

Example 2 A girl lists the number of male and female children her parent and her parent's brothers and sisters have. Her results were as tabulated below

	Males	Females
Her parents	2	5
Her mother's sisters	6	8
Her mother's brothers	4	8
Her father's sisters	5	8
Her father's brothers	7	7
Totals	24	36

d) Find the probability that, if the girl has children of her own, the 1^{st} born will be a girl.

e) If the girl eventually has 10 children, how many are likely to be males?

Solution

a) Following the family pattern, $P(1^{st} \text{ born will be a girl}) = \frac{36}{60} = 0.6$

b) 40% of the children will be males. Thus 4 out of 10 children are likely to be males.

Remark: The empirical probability definition has a weakness that it depends on the results of a particular experiment. The next time this experiment is repeated, you are likely to get a somewhat different result. However, as an experiment is repeated many times, the empirical probability of an event, based on the combined results, approaches the theoretical probability of the event.

5.2.3 Subjective Probability:

Subjective probabilities result from intuition, educated guesses, and estimates. For example: given a patient's health and extent of injuries a doctor may feel that the patient has a 90% chance of a full recovery. Subjectivity means two people can assign different probabilities to the same event.

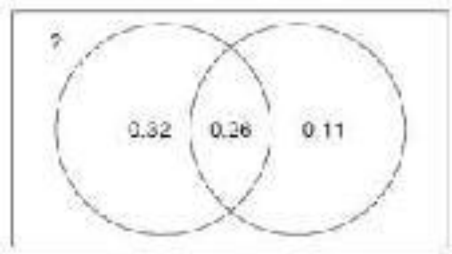
Regardless of the way probabilities are defined, they always follow the same laws, which we will explore in the following Section.

Exercise

1) What is the probability of getting a total of 7 or 11, when two dice are rolled?
2) Two cards are drawn from a pack, without replacement. What is the probability that both are greater than 2 and less than 8?
3) A permutation of the word "white" is chosen at random. Find the probability that it begins with a vowel. Also find the probability that it ends with a consonant.
4) Find the probability that a leap year will have 53 Sundays.
5) Two tetrahedral (4-sided) symmetrical dice are rolled, one after the other. Find the probability that; a) both dice will land on the same number. b) each die will land on a number less than 3 c) the two numbers will differ by at most 1.
   Will the answers change if we rolled the dice simultaneously?

Ways to represent probabilities:

1) *Venn diagram*; We may write the probabilities inside the elementary pieces within a Venn diagram. For example, P(AB') = 0.32 and $P(A) = P(AB) + P(AB') = 0.58$ [why?] The relative sizes of the pieces do not have to match the numbers.

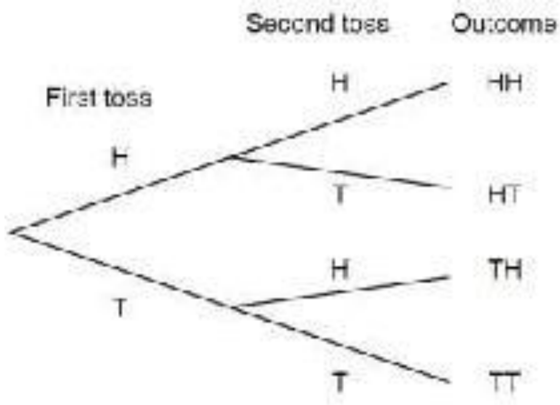


2) *Two-way table;* this is a popular way to represent statistical data. The cells of the table correspond to the intersections of row and column events. Note that the contents of the table add up across rows and columns of the table. The bottom-right corner of the table contains P(S) = 1

	B	B'	
A	0.26	0.32	0.58
A'	0.11	?	0.42
	0.37	0.63	1

3) *Tree diagram*; Tree diagrams or probability trees are simper clear ways of representing probabilistic information. A tree diagram may be used to show the sequence of choices that lead to the complete description of outcomes. For example, when tossing two coins, we may represent this as follows

   A tree diagram is also often useful for representing conditional probabilities



5.3 Review of set notation

Unions of Events: The event $A \cup B$ read as 'A union B' consists of the outcomes that are contained within at least one of the events A and B. The probability of this event $P(A \cup B)$; is the probability that events A and/or B occurs.

Intersection of events: The event $A \cap B$ (or simply AB) read as 'A intersection B' consists of outcomes that are contained within both events A and B. The probability of this event is the probability that both events A and B occur [but not necessarily at the same time]. Here after we will abbreviate intersection as AB.

Complement: The complement of event A, (denoted A'), is the set of all outcomes in a sample that are not included in the event A.

Set notation

Suppose a set S consists of points labelled 1, 2, 3 and 4. We denote this by $S = \{1, 2, 3, 4\}$. If $A = \{1, 2\}$ and $B = \{2, 3, 4\}$, then A and B are subsets of S, denoted by $A \subset S$ and $B \subset S$ (B is contained in S). We denote the fact that 2 is an element of A by $2 \in A$.

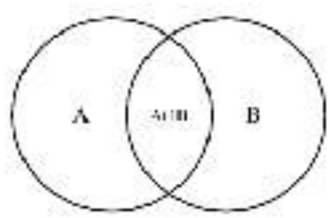
The union of A and B, $A \cup B = \{1, 2, 3, 4\}$. If $C = \{4\}$, then $A \cup C = \{1, 2, 4\}$. The intersection $A \cap B = AB = \{2\}$: The complement of A, is $A' = \{3, 4\}$.

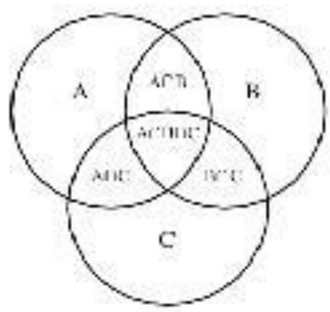
Distributive laws; $A \cap (B \cup C) = AB \cup AC$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

De Morgan's Law; $(A \cup B)' = A'B'$ and $(AB)' = A' \cup B'$

Venn diagram

Venn diagram is a diagram that shows all possible logical relations between a finite collection of sets. The sets A and B are represented as circles; operations between them (intersections, unions and complements) can also be represented as parts of the diagram.





Exercise

1) Use the Venn diagrams to illustrate Distributive laws and De Morgan's law.
2) Simplify the following (Draw the Venn diagrams to visualize)

   a) $(A')'$ b) $(AB)' \cup A$ c) $AB \cup AB'$ d) $(A \cup B \cup) \cap B$

3) Represent by set notation and exhibit on a Venn diagram the following events
   a) both A and B occur
   b) exactly one of A, B occurs
   c) A and B but not C occur s
   d) at least one of A, B, C occurs
   e) at most one of A, B, C occurs
4) The sample space consists of eight capital letters (outcomes), A, B, C ,...,H. Let V be the event that the letter represents a vowel, and L be the event that the letter is made of straight lines. Describe the outcomes that comprise ; a) VL b) $V \cup L'$ c) $V'L'$
5) Out of all items sent for refurbishing, 40% had mechanical defects, 50% had electrical defects, and 25% had both. Denoting A = fan item has a mechanical defect and B = fan item has an electrical defect, fill the probabilities into the Venn diagram and determine the quantities listed below. a) $P(A)$ b) $P(AB)$ c) $P(A'B)$ d) $P(A'B')$ e) $P(A \cup B)$ f) $P(A' \cup B')$ g) $P[(A \cup B)']$
6) A sample of mutual funds was classified according to whether a fund was up or down last year (A and A') and whether it was investing in international stocks (B and B'). The

probabilities of these events and their intersections are represented in the two-way table below. Fill in all the question marks hence find the probability of $A \cup B$

	B	B'	
A	0.33	?	?
A'	?	?	0.52
	0.64	?	1

5.4 Rules of Probability

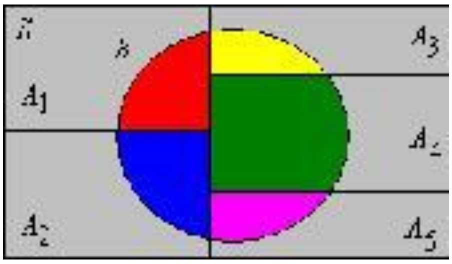
1) For any event A in a sample space S, $0 \le P(A) \le 1$. Consequently If A is empty (has no elements), then $P(A) = P(\phi) = 0$ and if A =S then $P(A) = P(S) = 1$
2) If A is an event in the sample space S, then A' (read as 'A complement') is an event in S but outside A. $P(A) + P(A') = 1 \Rightarrow P(A') = 1 - P(A)$
3) If the sample space S contains n disjoint events $E_1, E_2, \ldots, E_n$, then

$$P(E_1) + P(E_2) +, \ldots P(E_n) = \sum_{i=1}^{n} P(E_i) = 1$$

4) Let A and B be two events such that $A \subseteq B$, then $P(A) \le P(B)$
5) For any two events A and B, $P(A \cup B) = P(A) + P(B) - P(AB)$ where $P(AB) = P(A \cap B)$.

   Extension of this rule leads to the **Inclusion–Exclusion Principle.** This principle is a way to extend the general addition rule to 3 or more events. Here we will limit it to 3 events.
   $P(A \cup B \cup C) = P(A) + P(B) - P(AB) - P(AC) - P(BC) + P(ABC)$
6) **Law of Partitions:** The law of partitions is a way to calculate the probability of an event.
   Let $A_1, A_2, \ldots, A_k$ form a partition of the sample space S.then, for any events B

$$P(B) = P(A_1 B) + P(A_2 B) + \ldots + P(A_k B) = \sum_{i=1}^{k} P(A_i B)$$



Example 1 The Probability that John passes a Maths exam is 4/5 and that he passes a Chemistry exam is 5/6. If the probability that he passes both exams is 3/4, find the probability that he will pass at least one exam.

Solution

Let M be the event thet John passes Math exam, and C be the event thet John passes Chemistry exam.

$$P(\text{John passes at least one exam}) = P(M \cup C) = P(M) + P(C) - P(MC) = \frac{4}{5} + \frac{5}{6} - \frac{3}{4} = \frac{53}{60}$$

Example 2 A fair 6 sided die is rolled twice and the sum of the scores showing up noted. Define A, B and C to be the event that the sum of the scores is greater than 7, a multiple of 3 and a prime number respectively. Show that $P(A \cup B) = P(A) + P(B) - P(AB)$ and also find $P(A \cup C)$, $P(BC)$ and $P(BC')$

Solution

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$P(A) = \frac{15}{36} = \frac{5}{12} = P(C)$ and $P(B) = \frac{12}{36} = \frac{1}{3}$

$A \cap B$ means the set of all multiples of 3 that are greater than 7. Clearly $P(A \cap B) = \frac{5}{36}$

$A \cup B$ means the set of all values that are multiples of 3 and/or greater than 7. Clearly

$$P(A \cup B) = \frac{22}{36} = \frac{11}{18} = P(A) + P(B) - P(AB)$$

$A \cap C$ means the set of all multiples of 3 that are prime number. Clearly $P(A \cap C) = \frac{2}{36} = \frac{1}{18}$

$P(A \cup C) = P(A) + P(C) - P(AC) = \frac{5}{12} + \frac{5}{12} - \frac{1}{18} = \frac{7}{9}$

$B \cap C$ means the set of all greater than 7 that are prime numbers. Clearly $P(BC) = \frac{2}{36} = \frac{1}{18}$

$B \cap C'$ means the set of all greater than 7 that are not prime numbers. Clearly $P(BC') = \frac{11}{36}$

Exercise

1) Which of the following is a probability function defined on $S = \{E_1, E_2, E_3\}$

   a) $P(E_1) = \frac{1}{4}$, $P(E_2) = \frac{1}{3}$ and $P(E_3) = \frac{1}{2}$ b) $P(E_1) = \frac{1}{3}$, $P(E_2) = \frac{1}{6}$ and $P(E_3) = \frac{1}{2}$

   c) $P(E_1) = \frac{2}{3}$, $P(E_2) = -\frac{1}{3}$ and $P(E_3) = \frac{2}{3}$ d) $P(E_1) = 0$, $P(E_2) = \frac{1}{3}$ and $P(E_3) = \frac{2}{3}$

2) As a foreign language, 40% of the students took Spanish and 30% took French, while 60% took at least one of these languages. What percent of students took both Spanish and French?

3) In a class of 100 students, 30 are in mathematics. Moreover, of the 40 females in the class, 10 are in Mathematics. If a student is selected at random from the class, what is the probability that the student will be a male or be in mathematics?

4) The probability that a car stopped at a road brook will have faulty breaks is 0.23, the probability that it will have badly worn out tyres is 0.24 and the probability that it will have faulty brakes and/or badly worn out tyres is 0.38. Find the probability that a car which has just been stopped will have both faulty brakes and badly worn out tyres.

5) Given two events A and B in the sane sample space such that $P(A) = 0.59$, $P(B) = 0.3$ and $P(AB) = 0.21$. Find; a) $P(A \cup B)$ b) $P(A'B)$ c) $P(AB')$ d) $P(A \cup B')$

6) Let A and B be two events in the sane sample space such that $P(A \cup B) = \frac{3}{4}$, $P(B') = \frac{2}{3}$ and $P(AB) = \frac{1}{4}$. Find $P(B)$, $P(A)$ and $P(AB')$

7) Suppose that $P(A) = 0.4$, $P(B) = 0.5$ and $P(A \cap B) = 0.2$ Find; a) $P(A \cup B)$ b) $P(A'B')$ c) $P[A' \cap (A \cup B)]$ d) $P[A \cup (A'B)]$

8) A die is loaded such that even numbers are twice as likely as odd numbers. Find the probability that for a single toss of this die the spot showing up is greater than 3

9) A point is selected at random inside an equilateral triangle of sides 3 units. Find the probability that its distance to any corner is greater than 1 unit.

Definition: (Odds of an event) It's the ratio of the probability of an event occurring to that of the event not happening. If A is an event then the odds of A is given by $\frac{P(A)}{P(A')} = \frac{P(A)}{1-P(A)}$

Example Find *P(A)* and *P(A')* if the odds of event A is $\frac{5}{4}$

Solution

$\frac{P(A)}{1-P(A)} = \frac{5}{4} \Rightarrow 4(1 - P(A)) = 5P(A) \Rightarrow 4 = 9P(A) \Rightarrow P(A) = \frac{4}{9}$ and $P(A') = \frac{5}{9}$

Question: Find $P(E)$ and $P(E')$ if the odds of event E is (i) $\frac{3}{4}$ (ii) $\frac{a}{b}$

5.5 Relationship between Events

i) Mutually exclusive events: Two events A and B are said to be mutually exclusive if they cannot occur simultaneously. That is if the occurrence of A totally excludes the occurrence of B. Effectively events A and B are said to be mutually exclusive if they disjoint. *ie* $A \cap B = \phi \Rightarrow P(AB) = 0$

ii) Exhaustive events: Disjoint events whose union equals the sample space.

iii) Independent events: Two events A and B are said to be independent if the occurrence of A does not affect the occurrence of B. If events A and B are independent then $P(AB) = P(A) \times P(B)$

Remark: Three events A, B and C are said to be jointly independent if and only if

- $P(AB) = P(A) \times P(B)$, $P(AC) = P(A) \times P(C)$ and $P(BC) = P(B) \times P(C)$ (ie they are pairwise independent) and
- if $P(ABC) = P(A) \times P(B) \times P(C)$

Note it does not necessarily mean that if events A,B and C are pairwise independent then they are jointly independent

Example 1 Roll a fair die twice and define A to be the event that the sum of the scores showing up is greater than 7, B be the event that the sum of the scores showing up is a multiple of 3 and C be the event that the sum of the scores showing up is a prime number. Which of the events A, B and C are independent? Are the 3 events jointly independent?

Solution

From the above example, $P(A) = P(C) = \frac{5}{12}$, $P(B) = \frac{1}{3}$, $P(AB) = \frac{5}{36}$ and $P(AC) = P(BC) = \frac{1}{18}$

Since $P(AB) \neq 0$ events A and B are not mutually exclusive. Similarly events A & C and B and C are not mutually exclusive

$P(A) \times P(B) = \frac{5}{12} \times \frac{1}{3} = \frac{5}{36} = P(AB) \Rightarrow$ A and B are independent events.

$P(A) \times P(C) = \frac{5}{12} \times \frac{5}{12} = \frac{25}{144} \neq P(AC) \Rightarrow$ A and C are dependent events.

$P(B) \times P(C) = \frac{1}{3} \times \frac{5}{12} = \frac{5}{36} \neq P(BC) \Rightarrow$ B and C are dependent events.

The 3 events are not jointly independent since pairwise independence is not satisfied.

Example 2 Three different machines in a factory have the following probabilities of breaking down during a shift.

Machine	A	B	C
probability	$^4/_{15}$	$^3/_{10}$	$^2/_{11}$

a) All the machines will break down

b) None of the machines will break down.

Find the probability that in a particular shift,;

Solution

Since the events of breaking down of machines are independent, probability that all the machines will break down is given by $P(ABC) = \frac{4}{15} \times \frac{3}{10} \times \frac{2}{11} = \frac{4}{275}$

The probability that none of the machines will break down is $P(ABC) = \frac{11}{15} \times \frac{7}{10} \times \frac{9}{11} = \frac{21}{50}$

Exercises

1) In a game of archery the probability that A hits the target is $/_{13}$ and the probability that B hits the target is $^2/_5$. What is the probability that the target will be hit?
2) Toss a fair coin 3 times and let A be the event that two or more heads appears, B be the event that all outcomes are the same and C be the event that at most two tails appears. Which of the events A, B and C are independent? Are the 3 events jointly independent?
3) A fair coin and a fair die are rolled together once. Let A be the event that a head and an even number appears, B be the event that a prime number appears and C be the event that a tail and an odd number appears.
   a) Express explicitly the event that i) A and B occurs ii) Only B occurs iii) B and C occur

b) Which of the events A, B and C are independent and which ones are mutually exclusive?

4) The allocation of Mary's portfolio consists of 25% in bonds and 75% in stocks. Suppose in a one-year period, the probabilities for Mary to make profits in bonds and stocks are 0.9 and 0.4 respectively.
   a) Find the probability that Mary's portfolio turns out to be profitable in one year.
   b) Given that Mary has a loss in her portfolio in one year, find the relative proportion of the loss in stocks.

5) A die is loaded so that the probability of a face showing up is proportional to the face number. Write down the probability of each sample point. If A is the event that an even number appears, B is the event that a prime number appears and C is the event that an odd number appears.
   a) Find the probability that: i) A and/or B occurs ii) A but not B occurs iii) B and C occurs d) A and/or C occurs
   b) Which of the events A, B and C are independent and which ones are mutually exclusive?

Theorem 1: If events A and B are independent, then A and B' are also independent
Proof
Decomposing A into two disjoint events AB and AB'. We can write
$P(A) = P(AB)+ P(AB') \Rightarrow P(AB') = P(A) - P(AB) = P(A) - P(A)\times P(B)$ since events A and B are independent. Thus $P(AB') = P(A)[1- P(B)]= P(A)\times P(B') \Rightarrow$ A and B' are independent

Theorem 2: If events A and B are independent, then A' and B' are also independent
Proof
Decomposing B' into two disjoint events AB' and A'B'. We can write
$P(B') = P(AB') + P(AB') \Rightarrow P(A'B') = P(B') - P(AB') = P(B') - P(A)\times P(B')$ since events A and B' are independent. (From theorem 1 above) Thus
$P(A'B') = [1-P(A)]P(B') = P(A') \times P(B') \Rightarrow$ A' and B' are also independent

5.6 Counting Rules useful in Probability

In some experiments it is helpful to list the elements of the sample space systematically by means of a tree diagram,. In many cases, we shall be able to solve a probability problem by counting the number of points in the sample space without actually listing each element.
Theorem (Multiplication principle)
If one operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.

Eg How large is the sample space when a pair of dice is thrown?
Solution; The first die can be thrown in $n_1 = 6$ ways and the second in
$n_2 = 6$ Ways. Therefore, the pair of dice can land in $n_1 n_2 = 36$ possible ways.
The above theorem can naturally be extended to more than two operations: if we have
$n_1, n_2, \ldots, n_k$ consequent choices, then the total number of ways is $n_1 \times n_2 \times \ldots \times n_k$

Permutations
Permutations refer to an arrangement of objects when the order matters (for example, letters in a word).The number of permutations of n distinct objects taken r at a time is ${}_nP_r = \frac{n!}{(n-r)!}$

Example From among ten employees, three are to be selected to travel to three out-of-town plants A, B, and C, one to each plant. Since the plants are located in different cities, the order in which the employees are assigned to the plants is an important consideration. In how many ways can the assignments be made?

Solution;

Because order is important, the number of possible distinct assignments is $_{10}P_3 = 720$

In other words, there are ten choices for plant A, but then only nine for plant B, and eight for plant C. This gives a total of 10(9)(8) ways of assigning employees to the plants.

Combinations

The term combination refers to the arrangement of objects when order does not matter. For example, choosing 4 books to buy at the store in any order will leave you with the same set of books. The number of distinct subsets or combinations of size r that can be selected from n distinct objects, (r _ n), is given by $_nC_r = \frac{n!}{r!(n-r)!}$

Example 1 In the previous example, suppose that three employees are to be selected from among the ten available to go to the same plant. In how many ways can this selection be made?

Solution

Here, order is not important; we want to know how many subsets of size r = 3 can be selected from n = 10 people. The result is $_{10}C_3 = 120$

Example 2 In a poker consisting of 5 cards, find the probability of holding 2 aces and 3 jacks.

Solution

The number of ways of being dealt 2 aces from 4 is $_4C_2 = 6$ and the number of ways of being dealt 3 jacks from 4 is $_4C_3 = 4$

The total number of 5-card poker hands, all of which are equally likely is $_{52}C_5 = 2,598,960$

Hence, the probability of getting 2 aces and 3 jacks in a 5-card poker hand is

$$P(C) == \frac{6 \times 4}{2,598,960} \approx 0.00000923446$$

Example 3 A university warehouse has received a shipment of 25 printers, of which 10 are laser printers and 15 are inkjet models. If 6 of these 25 are selected at random to be checked by a particular technician, what is the probability that; a) exactly 3 of these selected are laser printers? b) at least 3 inkjet printers?

Solution

First choose 3 of the 15 inkjet and then 3 of the 10 laser printers.

There are $_{15}C_3$ and $_{10}C_3$ ways to do it, and therefore

$$\text{P(exactly 3 of the 6)} = \frac{_{15}C_3 \times {}_{10}C_3}{_{25}C_6} = 0.3083$$

$$P(\text{at least } 3) = \frac{_{15}C_3 \times {}_{10}C_3}{_{25}C_6} + \frac{_{15}C_4 \times {}_{10}C_2}{_{25}C_6} + \frac{_{15}C_5 \times {}_{10}C_1}{_{25}C_6} + \frac{_{15}C_6 \times {}_{10}C_0}{_{25}C_6} = 0.8530$$

Exercises

1) An incoming lot of silicon wafers is to be inspected for defectives by an engineer in a microchip manufacturing plant. Suppose that, in a tray containing 20 wafers, 4 are

defective. Two wafers are to be selected randomly for inspection. Find the probability that neither is defective.

2) A person draws 5 cards from a shuffled pack of cards. Find the probability that the person has at least; a) 3 aces. b) 4 cards of the same suit.

3) A California license plate consists of a sequence of seven symbols: number, letter, letter, letter, number, number, number, where a letter is any one of 26 letters and a number is one among *0, 1,… 9*. Assume that all license plates are equally likely. What is the probability that; a) all symbols are different? b) all symbols are different and the first number is the largest among the numbers?

4) A bag contains 80 balls numbered 1…. 80. Before the game starts, you choose 10 different numbers from amongst 1…. 80 and write them on a piece of paper. Then 20 balls are selected (without replacement) out of the bag at random. What is the probability that;
   a) all your numbers are selected?
   b) none of your numbers is selected?
   c) exactly 4 of your numbers are selected?

5) A full deck of 52 cards contains 13 hearts. Pick 8 cards from the deck at random (a) without replacement and (b) with replacement. In each case compute the probability that you get no hearts.

6) Three people enter the elevator on the basement level. The building has 7 floors. Find the probability that all three get off at different floors.

7) In a group of 7 people, each person shakes hands with every other person. How many handshakes did occur?

8) A marketing director considers that there's “overwhelming agreement" in a 5-member focus group when either 4 or 5 people like or dislike the product. If, in fact, the product's popularity is 50% (so that all outcomes are equally likely), what is the probability that the focus group will be in “overwhelming agreement" about it? Is the marketing director making a judgement error in declaring such agreement “overwhelming"?

9) A die is tossed 5 times. Find the probability that we will have 4 of a kind.

10) A tennis tournament has $2n$ participants, n Swedes and n Norwegians. First, n people are chosen at random from the $2n$ (with no regard to nationality) and then paired randomly with the other n people. Each pair proceeds to play one match. An outcome is a *set* of n (ordered) pairs, giving the winner and the loser in each of the n matches. (a) Determine the number of outcomes. (b) What do you need to assume to conclude that all outcomes are equally likely? (c) Under this assumption, compute the probability that all Swedes are the winners.

11) A group of 18 Scandinavians consists of 5 Norwegians, 6 Swedes, and 7 Finns. They are seated at random around a table. Compute the following probabilities: (a) that all the Norwegians sit together, (b) that all the Norwegians and all the Swedes sit together, and (c) that all the Norwegians, all the Swedes, and all the Finns sit together.

12) In a lottery, 6 numbers are drawn out of 45. You hit a jackpot if you guess all 6 numbers correctly, and get $400 if you guess 5 numbers out of 6. What are the probabilities of each of those events?

13) There are 21 Bachelor of Science programs at New Mexico Tech. Given 21 areas from which to choose, in how many ways can a student select:
   a) A major area and a minor area?
   b) A major area and first and second minor?

14) From a box containing 5 chocolates and 4 hard candies, a child takes ahandful of 4 (at random). What is the probability that exactly 3 of the 4 arechocolates?

15) If a group consist of 8 men and 6 women, in how many ways can a committee of 5 be selected if:

a) The committee is to consist of 3 men and 3 women.
b) There are no restrictions on the number of men and women on the committee.
c) There must at least one man.
d) There must be at least one of each sex.

16) Suppose we have a lot of 40 transistors of which 8 are defective. If we sample without replacement, what is the probability that we get 4 good transistors in the first 5 draws?
17) A housewife is asked to rank four brands A, B, C, and D of household cleaner according to her preference, number one being the one she prefers most, etc. she really has no preference among the four brands. Hence, any ordering is equally likely to occur.
    a) Find the probability that brand A is ranked number one.
    b) Find the probability that brand C is number one D is number 2 in the rankings.
    c) Find the probability that brand A is ranked number one or number 2.
18) How many ways can one arrange the letters of the word ADVANTAGE so
19) that the three As are adjacent to each other?
20) Eight tires of different brands are ranked 1 to 8 (best to worst) according to mileage performance. If four of these tires are chosen at random by a customer, find the probability that the best tire among the four selected by the customer is actually ranked third among the original eight.

5.7 Conditional Probability and Independence

Humans often have to act based on incomplete information. If your boss has looked at you gloomily, you might conclude that something's wrong with your job performance. However, if you know that she just suffered some losses in the stock market, this extra information may change your assessment of the situation. Conditional probability is a tool for dealing with additional information like this.

Conditional probability is the probability of an event occurring given the knowledge that another event has occurred. The conditional probability of event A occurring, given that event B has occurred is denoted by P(A/B) and is read “probability of A given B” and is given by

$$P(A/B) = \frac{P(AB)}{P(B)} \quad \text{provided} \quad P(B) > 0 \quad \text{Similarly} \quad P(B/A) = \frac{P(AB)}{P(A)} \quad \text{provided } P(A) > 0$$

$$\Rightarrow P(AB) = P(A/B) \times P(B) = P(B/A) \times P(A)$$

Remark: Another way to express independence is to say that the knowledge of B occurring does not change our assessment of P(A). This means that if A and B are independent then $P(A/B) = P(A)$ and $P(B/A) = P(B)$

Example In a large metropolitan area, the probability of a family owning a colour T.V , a computer or both 0.86, 0.35 and 0.29 respectively. What is the probability that a family chosen at random during a survey will own a colour T.V and/or a computer? Given that the family chosen at random during a survey owns a colour T.V, what is the probability that it will own a computer?

Solution

Let T and C be the event of owning a colour T.V and a computer respectively. Then

$$P(T \cup C) = P(T) + P(C) - P(TC) = 0.86 + 0.35 - 0.29 = 0.92$$

$$P(C/T) - \frac{P(TC)}{P(T)} = \frac{0.29}{0.86} \approx 0.337209$$

Reduced sample space approach

In case when all the outcomes are equally likely, it is sometimes easier to find conditional probabilities directly, without having to apply the above equation. If we already know that B